Is Equivalence Just "Same Number"? College Students' Numerical vs Non-Numerical Conceptions of Equivalence

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As coursework extends beyond arithmetic, students encounter forms of equivalence beyond numerical equality. This study examines how college students characterize equivalence in mathematics generally, and specifically for equations and expressions. Responses from 198 students were analyzed and coded as numerical or non-numerical. Across questions, most responses appealed to numerical criteria, treating equivalence as equality to a single number, with higher prevalence of non-numerical criteria correlating with higher-level courses. We observed multiple variants of both numerical and non-numerical criteria, including converting algebraic expressions to numbers to justify equivalence. Findings highlight the need for educators to support students in distinguishing types of equivalence and their associated criteria.

Keywords: equivalence, equality, algebra, expressions, equations, solution set

Equivalence has been identified as a "big idea" in mathematics and is formalized via the notion of an equivalence relation (Charles, 2005; Cook et al., 2022; Siemon, 2022; Tay, 2019; Toh & Yeo, 2019). Student understanding of equivalence is crucial for their *inferential knowledge*–their understanding of the validity of mathematical procedures (Mirin et al., 2025). As students progress from arithmetic to algebra, and then to more advanced mathematics, they move beyond numerical equality and regularly encounter equivalence in many contexts, involving different types of objects and equivalence criteria. For example, equivalence criteria for algebraic expressions and equations go beyond determining whether two numbers are equal.

Here we consider students' characterizations of equivalence, both generally and in the context of expressions and equations—attending to the extent their equivalence criteria link to numerical equality or non-numerical equivalence. This provides insight into how students' criteria for the familiar notion of numerical equality intersect with their equivalence criteria in other contexts. Better understanding how college students conceptualize equivalence is critical to improving instruction that supports understanding of equivalence across mathematical contexts.

Equality and Equivalence in Mathematics

The terms "equality" and "equivalence" have been used in many ways in research and instruction (Asghari, 2019). To clarify the focus of this research, we define our use of this terminology (while recognizing that others might use terms differently). Here, we define *numerical equality* as a relationship where two objects are equivalent if (and only if) they are each equal to the same numerical quantity. Numerical equality is one type of equivalence relation where every equivalence class could be represented by a canonical element that is a single numerical value. Equivalence of numerical expressions (e.g., $6(2)$ and $9 + 3$) is one type of numerical equality. For example, one way to determine equivalence is to conclude that both $6(2)$ and $9 + 3$ can be represented by the canonical element, 12, of the equivalence class.

When equivalence involves something other than numerical equality—i.e., when it compares objects that cannot be represented by single numerical values—we call this *non-numerical equivalence*. In algebra, students encounter non-numerical equivalence in the context of equivalent algebraic expressions and equations. Two *algebraic expressions are equivalent* if they are equal to the same numerical output for every domain input (i.e., for every possible combination of substitution of numerical values in for the variables; *insertion equivalence* [Zwetzschler & Prediger, 2013])[1]. For example, $3x - 2x$ and x are equivalent because they are equal for all x-values; however, neither $3x - 2x$ nor x are always equal to a single numerical value. Relatedly, two *algebraic equations are equivalent* if they have the same solution set; i.e., they have the same truth values for every possible domain input. For example, $|3x - 2x| - 2 = 12$ and $|x| = 14$ are equivalent because they are true for the same values of x (only $x = \pm 14$).

Before algebra, numerical equality is often the most prevalent type of equivalence.[2] As a result, many K-12 studies have investigated whether students conceptualize the equals sign from an *operational* (as announcing the answer to a calculation) vs. a *relational* (as describing an equivalence relationship) view (e.g., Kieran, 1981; Knuth et al., 2008; McNeil et al., 2006; Stephens et al., 2013; Stylianou et al., 2024); studies with college students show they may also hold operational views of the equal sign (Fyfe et al., 2020; Weinberg, 2010). However, much less research has explored how students conceptualize non-numerical equivalence criteria.

Some studies (e.g., Cook et al., 2022; Jones et al., 2012; Mirin, 2019; Tondorf & Prediger, 2022; Wladis et al., 2022; Zwetzschler & Prediger, 2013) have considered how students relate *transformational* activity (i.e., equivalent symbolic representations can be transformed into one another) or *substitutive* activity (i.e., equivalent symbolic representations can be substituted for one another) to other equivalence criteria that justify this activity (e.g., *descriptive* criteria: representations describe the same situation; *insertion* criteria for algebraic expressions). The literature on students' work with algebraic expressions and equations suggests that they tend to focus on transformation without attending to underlying descriptive, insertional, or other criteria that are a *reason* for the validity of transformation (e.g., Cook et al., 2022; Sfard & Linchevski, 1994). It is thus unsurprising that for many students view an equation's "solution" as the result of transformation rather than a property of the equation (e.g., Frost, 2015; Tsamir & Tirosh, 2023).

While there is substantial research on students' thinking on numerical equality, there is much less on non-numerical equivalence. What literature does exist has thus far tended to focus on describing types of equivalence that students attend to or invoke in specific contexts, such as while solving equations or transforming expressions (e.g., Cook et al., 2022; Tondorf & Prediger, 2022). Our work complements this by describing what students think "equivalence" means, how they characterize "equivalent expressions" and "equivalent equations", and by explicitly considering how this relates to numerical vs. non-numerical equivalence criteria.

Method

Open-ended items on equivalence were administered to 198 students at a U.S. community college in 18 different courses, ranging from developmental elementary to linear algebra. Courses also included statistics, discrete mathematics, mathematics for elementary teachers, intermediate algebra, precalculus, and Calculus I-III. We operationalized course level based on algebra prerequisites: "low" = no algebra prerequisite: "mid" = elementary algebra prerequisite; "high" = intermediate algebra prerequisite. Items were administered during class. Some students

[1] We note that, with n variables, the "inputs" involves n-tuples.

2 We note that non-numerical equivalence relations do exist in earlier grades (e.g., shape similarity or congruence).

were given all three questions (40), some only 1 and 2 (77), and some only 1 and 3 (81).

Q1. What do you think equivalence means in mathematics?
Please give an example or two to show what you mean.
Can mathematical equivalence mean anything else?
Circle one: Yes No I'm not sure
Please explain how you know.
Please give an example or two to show what you mean.

Q2. How could you check whether two mathematical **expressions** are equivalent?
(An expression is a mathematical phrase that does **not** contain an equals or inequality sign.)
Please give an example or two to show what you mean:
Are there any other approaches you could take to check whether two mathematical expressions are equivalent?
Circle one: Yes No I'm not sure
Please give an example or two to show what you mean:

Q3. How could you check whether two mathematical **equations** are equivalent?
(An equation is a mathematical phrase that **contains an equals sign**.)
Please give an example or two to show what you mean:
Are there any other approaches you could take to check whether two mathematical equations are equivalent?
Circle one: Yes No I'm not sure
Please give an example or two to show what you mean:

Six researchers collaboratively employed *reflexive thematic analysis* (Braun & Clarke, 2006). This led to the emergence of several themes, one of which was "equivalence as numerical equality vs. non-numerical equivalence relationship" (details of other themes are discussed in other work, e.g., Wladis et al., 2022c, 2022b). After themes were developed, they were used as the basis for generating a codebook for formal coding, including codes indicating whether students cited numerical equality (i.e., examples or explanations describing numerical equality criteria) and/or non-numerical equivalence (i.e., examples or explanations appeared to illustrate non-numerical equivalence criteria). Not all student work merited one code or the other (e.g., because of ambiguity). Some students showed evidence of both; thus, codes were not mutually exclusive.

Results

First, we consider frequencies indicating whether students, when asked to describe and give examples of what equivalence means in mathematics (Q1), provided examples of numerical equality and/or of non-numerical equivalence relationships (Table 1).

Table 1. General Equivalence Item (Q1) Results (subset of students given Q1 and Q3).

	course level				p-value		
Numerical equality vs. non-numerical equivalence	low	mid	high	total	mid vs. low	high vs. mid	high vs. low/mid
cited numerical equality	91%	87%	91%	89%	0.2491	0.6862	
cited non-numerical equivalence	0%	0%	14%	2%		0.0015**	0.0001***
n	37	62	22	121	$^{**}p \leq 0.01$; $^{***}p \leq 0.001$		

Roughly 9 of 10 students cited numerical equality in Q1, while only 2% referred to non-numerical equivalence (those that did were all enrolled in higher-level courses). Just because students did not volunteer non-numerical equivalence examples does not mean they can't provide them; however, these patterns do suggest that college students' primary equivalence associations may be numerical.

Conceptualizing equivalence as numerical equality was also common in response to Q2 (equivalent expressions) and Q3 (equivalent equations); coding frequencies are given in Table 2. We then proceed to characterize students' examples in more detail. Due to space constraints, we illustrate just a few subcategories with a few brief examples of student work.

Table 2. Type of Equivalence Criteria Applied to Q2/Q3 (Some students provided more than one type.)

Numerical equality vs non-numerical equivalence criteria	Q2: expression item				Q3: equation item			
	low	mid	high	total	low	mid	high	total
numerical equality, arithmetic calculation	56%	41%	33%	44%	65%	34%	21%	40%
numerical equality, transform to numeric	44%	19%	56%	32%	20%	40%	50%	37%
other type of numerical equality	0%	22%	0%	12%	0%	9%	7%	6%
any kind of numerical equality	**94%**	**78%**	**89%**	**84%**	**80%**	**81%**	**79%**	**80%**
same solution set					0%	0%	7%	1%
transformation equivalence	6%	16%	22%	14%	5%	11%	0%	7%
other possible non-numerical equivalence	0%	3%	11%	4%	0%	4%	7%	4%
total possible non-numerical equivalence	**6%**	**19%**	**33%**	**18%**	**5%**	**15%**	**14%**	**12%**
"same solution" but ambiguous whether numerical/non-numerical					15%	2%	7%	6%
n (provided examples to compare)	16	32	9	57	20	47	14	81

Numerical Equality Criteria

Students attended to numerical equality criteria in different ways. In some cases students provided entirely numerical examples. Iota, a Calculus I student, stated for Q1 that equivalence means "when a answer is equal to each other", for Q2 wrote "we can substitute or solve the math expression and see if it is similar" and for Q3 wrote "solve the problem on both sides and if *they equal to each other* [emphasis added] they are equivalent". Figure 1 shows examples they gave.

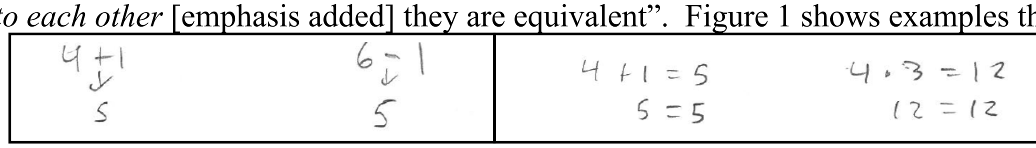

Figure 1. Examples given by Iota for Q2 (left) and Q3 (right)

In Q2, Iota gave numerical expressions with arrows to show numerical equality (Figure 1, left); whereas in Q3 they wrote $4 + 1 = 5$ and $4 \cdot 3 = 12$ and on the next line showed that the left and right side of each equation are equal. They write their work differently in Q3 vs. Q2, suggesting that they might view expressions and equations as different. Although it's somewhat ambiguous which objects Iota is comparing in Q3, their criteria for equivalence involves *comparing individual numbers*, suggesting a focus on numerical equality.

In some cases, students provided algebraic rather than numerical expressions or equations but then *converted* them to numerical values. Lambda, a student in elementary algebra, answered Q2 by writing "plug in numbers in the expressions" and then gave the examples shown in Figure 2.

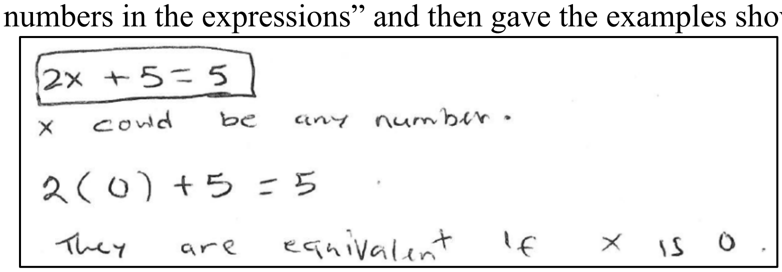

Figure 2. Examples given by Lambda for Q2

Lambda appears to have substituted a single value for x into an algebraic expression $2x + 5$ and, performed numerical calculation to demonstrate that $2x + 5$ is equal to 5 if x is 0. Although Lambda provided an algebraic expression, their demonstrated *criterion* for equivalence is consistent with numerical equality because it involves comparing single numerical values. This example, though numerical, incorporates some aspects of insertional equivalence; they assess equivalence by checking that expressions are equal for a *particular* value of x. The extent to which Lambda might conceptualize the more generalized notion of insertion equivalence of algebraic expressions can't be determined from this answer alone. However, their focus on a single numerical value (vs. considering all values in the domain) does suggest that they may be more comfortable using numerical equality criteria over non-numerical equivalence criteria.

Some students seemed to impose numerical equality on algebraic expressions differently—by setting them equal to a fixed value (e.g., Mu, linear algebra student, Figure 3). In Q1, Mu stated "Two things are equal to each other. Whatever is on the left side is equal to the right side."

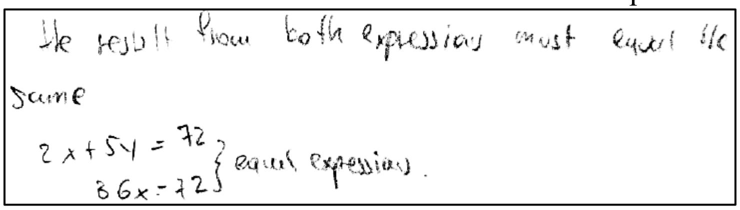

Figure 3. Example given by Mu for Q2

In Q2, Mu describes that the "result from both expressions must equal the same", and then they set two algebraic expressions equal to 72, calling them "equal". Like Lambda, Mu has taken two algebraic expressions and made them numerically equal, rather than considering how algebraic expressions represent larger sets of values and therefore are related by non-numerical equivalence criteria. As with prior students, Mu may be able to produce examples of non-numerical equivalence criteria in other contexts; however, their focus on setting algebraic expressions equal to a single number to illustrate equivalence suggests that they may be more comfortable with numerical equality criteria than non-numerical equivalence criteria.

When students answered Q3, they often talked about "solving" equations to determine equivalence, but it was also often ambiguous exactly which objects they were comparing. For example, one student in a combination intermediate algebra/precalculus course wrote for Q3, "You can check by solving the equations and see if they are equal to each other"; in this case, it is unclear what is "equal" (equations? x-values? solution sets?) and thus this response was coded as ambiguous. In other cases, there was evidence that students appeared to conceptualize equations as representing a single numerical value, suggesting an orientation around numerical equality criteria. Xi, a Calculus I student, wrote for Q3 "If they both add up to the same number" and then wrote the work in Figure 4.

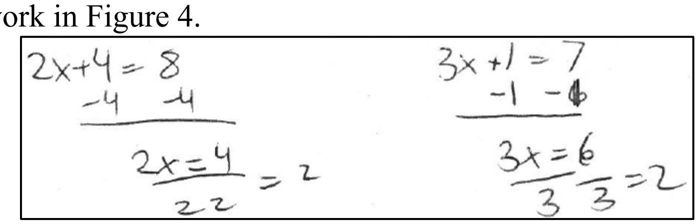

Figure 4. Example given by Xi for Q3

Xi solves two algebraic equations correctly, but their framing of equations "add[ing] up to the *same number* [emphasis added]", and their use of equals signs suggest a possible operational view of the equals sign, or even perhaps that the equations themselves are conceptualized as equal to the number 2 (it is ambiguous from their writing whether this is the case—taken together, this suggests a likely orientation towards numerical equality rather than non-numerical equivalence when considering the equivalence relationships between these algebraic equations.

Non-numerical Equivalence Criteria

While it was not always completely clear when students were using non-numerical equivalence criteria, we did observe various responses that suggested students may be attending to these criteria. For example, one student (Rho, in Calculus III) explicitly referred to equivalent equations having the "same solution set" in Q3 and provided the work given in Figure 5.

$3 = 6x$ $\quad\quad$ $3 = 4x + 1$

$x = \frac{1}{2}$ solves both, therefore the equations are equivalent

Figure 5. Example given by Rho for Q3

Rho uses the term "solution set" and provides the "solution" in equation form rather than representing it as only a single numerical value (e.g., like Xi in Figure 4). This suggests that Rho is using non-numerical equivalence criteria when considering the equivalence of equations, even though they, like other students, focus on a solving process that generates a single solution.

As expected, some students described *transformation* of algebraic expressions/equations as a type of non-numerical equivalence criteria. For example, Sigma, a student in a combined introductory algebra/statistics course, wrote in response to Q2 "find the value of each expression and see if they are the same value", giving examples in Figure 6.

$9x + 7x = {}^{-}7x + 23x$ $\downarrow \quad\quad \downarrow$ $16x \quad\quad 16x$	If the equations are the same, they should plot the same points on a graph.

Figure 6. Sigma's Examples in Q2

Unlike students who converted algebraic expressions to numerical examples (e.g. Lambda, Mu), Sigma did not convert expressions to single numerical values; instead, for them, getting the "same value" appears to mean transforming expressions algebraically into the same algebraic expression representing a generic value (e.g., achieving the same "simplest form", $16x$, through transformation). While Sigma does not explicitly describe how these expressions have the same outputs for all the same inputs (i.e., non-numerical insertion equivalence for algebraic equations), they also provide a second example (Figure 6, right), describing "equations" as "plot[ting] the same points on a graph", which may be referring to insertion equivalence in the sense that $y = 9x + 7x$ and $y = -7x + 23x$ represent the same line (i.e., have the same solution set; i.e., $9x + 7x$ and $-7x + 23x$ have the same outputs for all the same inputs).

Some students illustrated another potential type of non-numerical equivalence criteria by describing equations as equivalent when they represent the same relationship, perhaps appealing to a type of descriptive equivalence (Figure 7).

You could check whether two mathematical equations are equivalent by doing the opposite way when you check back. For example: Original answer $12 \times 5 = 60$ / check back answer $60 \div 12 = 5$	If $F = \frac{V}{T}$ then $V = F \cdot T$
	$a+b=c$ equivalent to $a = c - b$

Figure 7. Examples of "Same Relationship". Upsilon (Q3, left), Phi (Q3, upper right), Chi (Q1, lower right)

Upsilon, an elementary algebra/statistics student, Phi, an elementary algebra/math for elementary education student, and Chi, an intermediate algebra and precalculus student, all provided examples of equations that appeared to represent the same "relationship" between the numbers/letters. While Upsilon provides arithmetic equations, we note that if they are comparing the two equations ($12 \times 5 = 60$ and $60 \div 12 = 5$), their equivalence criteria (whether the numbers in the equation represent the same multiplicative relationship) are not about comparing individual numerical values. (As with other students, it is not completely clear which objects Upsilon is comparing—they may be comparing the numerical equality of the expressions 12×5 and 60 by performing division; however, their use of the term "equation" and similar patterns in examples given by Phi and Chi suggest that some students may be using this type of "same relationship" non-numerical equivalence criteria to determine the equivalence of equations, even if it is not completely clear whether Upsilon is personally making this comparison in this excerpt). Both Phi and Chi appear to be comparing equivalent algebraic equations where the equations represent the same multiplicative or additive "relationship" between the same variables. This type of criteria is not typically called "equivalence" in the curriculum, but it does represent an important non-numerical equivalence relation that could be productively leveraged during equation transformations. The work of these students may illustrate an opportunity for the mathematics community to consider alternate ways of characterizing certain types of equivalent equations using non-numerical equivalence criteria that some students may already be noticing. This could be a potentially productive route to explore for improving instruction.

Discussion

Across our sample, students predominantly described numerical equality criteria. The frequency of non-numerical equivalence criteria increased with course level; however, even in higher-level courses, numerical equality criteria still dominate. This occurred even in contexts where students were explicitly asked if equivalence could have other meanings, when they were asked specifically about equations and expressions, and when they themselves provided examples of algebraic expressions/equations. This is somewhat surprising, given the centrality of non-numerical equivalence (e.g., of expressions and equations) in algebra and beyond. On the other hand, the broad use of a single term, "equivalence", to describe different types of objects and criteria across the curriculum could lead to ambiguity in these distinctions for students. In particular, if existing instruction does not address the ways that the term "equivalence" is used to describe *different* equivalence criteria in different contexts, this may predispose students to try to make sense of all equivalence criteria under the lens of numerical equality (the one type of criteria that may be the most overtly addressed in existing curriculum and instruction).

One limitation of this study is that it asked students about their meanings for the term "equivalence". Although students encounter many examples of equivalence relations in the K–16 curriculum, the term "equivalence" may not always be used, and therefore, even when students recognize non-numerical equivalence relationships, they may not associate them with the term "equivalence". A helpful area for future research could be to investigate how students enact equivalence criteria in comparison to what they themselves call "equivalence".

This research demonstrates a critical need to explore more carefully how instruction shapes the development of student conceptions of non-numerical equivalence criteria, especially those for equivalent algebraic expressions and equations. In particular, it is important to better understand which kinds of instructional approaches support students in understanding the context-dependent nature of equivalence criteria, including distinctions among numerical expressions, algebraic expressions, and algebraic equations.

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