Descriptive Interpretations of Equivalence in Measurement Contexts: The Case of Emily

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Proportional reasoning is a central concept for students to learn, yet they often struggle when solving proportionality-type problems. While prior work has examined students' solution strategies, little attention has been given to the meanings of equivalence that support proportional and multiplicative reasoning. This study investigates how students reason about equivalence in measurement contexts with tasks adapted from Cook et al. (2021). Using task-based clinical interviews with four preservice secondary teachers, we analyzed responses to tasks involving relationships such as "1 lunar cycle = 28 days." Focusing on one participant, we identified the use of two ways of reasoning about equivalence which proved productive for her. Findings highlight the critical role of descriptive reasoning for connecting quantities and magnitudes in proportional situations. We argue for expanding research on equivalence in measurement-based proportional contexts.

Keywords: proportional reasoning, multiplicative reasoning, equivalence, measurement problems

Introduction & Literature Review

Studies using the students-and-professors problem (S&P problem), as well as others like it, highlight the nuance and complexity that can be involved when using proportional reasoning to successfully reason about quantities involved in such problems (Christianson et al., 2012; Clement et al., 1981; Cohen & Kanim, 2005; Fisher, 1988; Van Dooren et al., 2009). Proportional (and multiplicative) reasoning has been well-studied (e.g. Clark et al., 2003; Lesh et al., 1988; Modestou & Gagatis, 2010; Vergnaud, 1983, 1988) and for good reason- such reasoning has been referred to as a critical connector between elementary-level math and higher-level math (Lesh et al, 1988). Despite the importance of proportional reasoning in the mathematics curriculum, students' experiences have not been sufficient in supporting them in developing such reasoning (Tjoe & de la Torre, 2014); this highlights the need for research continuing to explore students' proportional reasoning.

Although much scholarship has been done on students' solution processes in S&P-type problems, little attention has been paid to the meanings of equality and equivalence that support students in reasoning productively about these scenarios. More broadly, we observe that, while interpretations of the equals sign have received a large amount of attention (Jones, 2008; Jones & Pratt, 2006; Kieran, 1981), very few studies have examined the interpretations that support productive reasoning about equivalence in proportional, quantitative scenarios. Additionally, while much of the research into the S&P problem (and those like it) has focused on producing the correct algebraic equation (e.g., $S = 6P$), very little attention has been paid to how students might reason about the magnitudes involved in such problems. This leads us to pose the following research question: In what ways might students productively reason with equivalence as they engage in tasks involving proportional quantities?

Theoretical Perspective

Equivalence

For a context in which to discuss students' ways of reasoning about equivalence in proportional contexts, consider the following tasks originally described by Cook et al. (2021):

> Task A: In Glenn Junior High school, there are 28 students for every 1 teacher. Determine the number of students if there are 37 teachers. Then write an equation that generally relates the number of teachers, *T*, to the number of students, *S*, in Glenn Junior High.
>
> Task B: There are 28 days in a lunar cycle. Determine the number of days that pass in 37 lunar cycles. Then write an equation that generally relates a number of lunar cycles, *L*, and the number of days that pass during those lunar cycles, *D*. (p. 36)

Both tasks are variations of the S&P problem described by Clement et al. (1981). In the spirit of inquiring about solutions to these tasks, Cook et al. (2021) developed a conceptual analysis to answer questions a student might ask when approaching Task A and Task B. The solutions to and interpretation of both tasks require flexible use of equivalence. The authors hypothesized three ways of reasoning that could support students' productive engagement with these measurement/proportionality tasks: *numerical*, *transformational*, and *descriptive*.

A *numerical* way of reasoning identifies two expressions as "equivalent when they have the same numerical value" (Cook et al., 2021, p. 36). In the context of expressions such as $2(x + 1) = 2x + 2$, a numerical way of reasoning is a natural default because, for all values of the variable x, the left-hand and right-hand sides evaluate to the same number. However, numerical ways of reasoning can fail when referring to a statement such as '28 days = 1 lunar cycle' (as 28 is not equal to 1). These kinds of measurement equations require a different kind of reasoning. To that end, Cook et al. (2021) identified two other ways of reasoning about equivalence, a *transformational* way of reasoning and a *descriptive* way of reasoning. The transformational way of reasoning involves some manner of algebraically appropriate modification of the expression or equation (e.g., distributing the 2 in $2(x +1)$ to obtain $2x +2$). A descriptive way of reasoning, adapted from the work of Prediger & Zwetzschler (2013), "treats two objects as equivalent if they describe, represent, or model the same quantity or situation" (Cook et al., 2021, p. 37). For example, a student might engage in descriptive reasoning when referring to a clock to grapple with modular arithmetic or referring to area under a curve to justify why two integral expressions might be the same. Descriptive ways of reasoning are particularly important for proportionality tasks, as attention to measurement and magnitude can help students reason through the task.

Quantities and Magnitudes

Undergirding the solution to proportionality problems is attention to the quantities involved. Thompson (1994) defined *quantity* as a scheme consisting of three components: quality of object, unit or dimension, and the ability to quantify the object. An individual, for Thompson, engages in quantitative reasoning when working with all three components of the scheme. Thompson et al. (2014) establish that *magnitude* can be thought of as a scheme at several levels. Of relevance to this report, one of Thompson's levels refers to Wildi's (1991) definition of magnitude of a quantity *Q* as "the measure of *Q* with respect to a unit *u*, times the magnitude of *u*" (Thompson et al., 2014, p. 5). The invariance of magnitude with respect to change in measure is highlighted by using this definition; this creates a powerful mental image for magnitude which can be used by the student in proportionality contexts. In the context of Task B, 28 days and 1 lunar cycle are equal because 28 and 1 both describe measurement of the same duration of time, albeit in different units (the length of one day, |day|, and the length of one lunar cycle, |lunar cycle|, respectively). In the language of Thompson and Wildi, 28*|day| = 1*|lunar cycle|; this setup helps lead to the algebraic solution to the task, $28L = D$. The solution equation treats the variables L and D as measurements, while |lunar cycle| and |day| are units of measure. We associate this approach with a descriptive way of reasoning. We hypothesize that attending to these features can be productive for reasoning about proportional quantities; it is unclear, however, whether and how these ideas might emerge when working with students.

Methods

We conducted task-based clinical interviews (Clement, 2000) with four students at a large public research university in the United States. All four participants were pre-service secondary teachers, completing the third year of their coursework in the Spring semester of 2025. Individual interviews (one participant per interview) centering on Tasks A and B from the previous section were completed in Spring 2025 and were approximately 40 minutes in length. We recorded student words and utterances, as well as their written work, via iPad and the ExplainEverything app. We presented participants first with Task A before presenting Task B. As the participants worked on both tasks, the interviewer (third author) asked them to explain their reasoning via clarifying follow-up questions, focusing on key proportional and quantitative relationships in the two tasks (e.g., does 28 students = 1 teacher, and why? Does 28 days = 1 lunar cycle, and why?). We also asked participants to describe similarities or differences between the two tasks to elicit more details from their line of reasoning. Our analysis in this proposal focuses primarily on one participant's responses to Task B. We selected Emily (pseudonym) because her responses were particularly vivid and illustrative of the types of reasoning conjectured by Cook et al. (2021).

Transcripts were produced for each interview session. The first three authors analyzed the transcripts and video recordings of the participants' work, employing Clement's (2000) cycles of interpretive analysis with the ways of reasoning hypothesized by Cook et al. (2021) serving as initial codes. Our goal was to develop models of the student's reasoning that provided a rational description of her language and actions; models were discussed and negotiated until all three researchers were in agreement. The transcripts were coded for instances of each way of reasoning about equivalence in Cook et al. (2021)— numeric, transformational, and descriptive—and screen captures of relevant portions of the video recording were used to augment the transcripts.

Results

For Task A, Emily began by finding the answer and using numbers to work backward in an attempt to find the relationship that made sense ("if there's one teacher, then there's 28 students, there's two teachers, then there's 56 students"), leading her to the correct algebraic solution of $S = 28T$. Further, she justified her choice by using a numerical way of reasoning to eliminate the reversal error as a possible answer ("If you did it the other way…you have five students, you do not have 140 teachers"). Emily used this line of reasoning for Task B as well:

> *Emily*: So we have 37 cycles that have 28 days in each of them [*writes 37*28*]... So that would be [1036] days that pass in 37…days. Um, then write an equation that generally relates the number of lunar cycles, L, the number of days that passed during those lunar cycles. Um, so if we have one lunar cycle, it's gonna be the same way that it was before [referring to her work in Task A]. If we have, I think it's gonna be 28 L equals D [*writes* $28L = D$]. 'cause if you had it the other way, if you had 28 D, would still be like after 28 days it should equal one lunar cycle [*writes* $28D = L$]. But if you have, if you put in 28

for D, you get 28 squared, which is 784 [*writes 784 L*]. And that is not one lunar cycle…So I think that this would have to be the general equation for L and D [*boxes 28L = D*].

We infer Emily relied on numerical ways of reasoning to justify her equation by arguing that the rule written the other way would provide an incorrect numeric relationship between the number of days and lunar cycles.

The interviewer continued by asking how Emily might respond to an assertion that one does not equal 28, so therefore one lunar cycle cannot equal 28 days. Her response pivoted to using a descriptive way of reasoning about equivalence:

Emily: Well, I would ask them if one foot equals 12 inches [*writes 1 ft = 12 in*]. So I think that's a pretty common…conversion… so one and 12 are not equal, but our units are different [*underlines 'ft' on the left and 'in' on the right*]...your units are different, but these two things [*referring to 1 ft and 12 inches*] are equal. But just because the units are different does not mean that in this case [*scrolls up to her work on Task A, which has 1 teacher ≠ 28 student*] I can say one teacher [*underlines 'teacher'*] equals 28 students [*underlines 'student'*]. Yeah, our units are different, but the value of those units is not the same.

The interviewer then asked her to elaborate about what she meant by the displayed equation, 1 LC = 28 days. Emily responded by drawing the number lines in Figure 1 and partitioning it into pieces which she claimed represented the 28 days ("28 is the number of subsections contained inside of one lunar cycle"). She articulated:

Emily: We can't cut one teacher up into 28 even pieces and say that those are the students… each of these [tick marks] is not equal to one student…one teacher's not made up of 28 students, but one lunar cycle is made up of 28 days.

Emily argued that one lunar cycle is composed of 28 days (28*|day| = |lunar cycle|). Hence, she leveraged a meaning for magnitude consistent with Thompson et al.'s (2014) description of a Wildi magnitude to justify the former equation; Emily stated that the same magnitude of time could be measured in terms of 28 times the magnitude of a day or 1 times the magnitude of one lunar cycle. Further, Emily contended it cannot be said that 28*|student| = |teacher| as they did not represent the same magnitude.

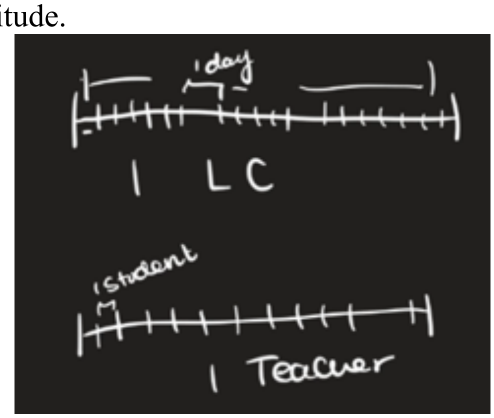

Figure 1. Emily's number lines comparing a partition of one lunar cycle and one teacher.

We infer that here Emily attended specifically to the idea that there is a measurement comparison that can be made between days and lunar cycles, while the same cannot be said for students and teachers; as a result, the latter magnitudes are incommensurate.

She continued to focus on the measure, consistent with understanding that 28*|day| = |lunar cycle| when asked to identify what is true for Task B which is not true for Task A:

Emily: So here we're saying the amount of time that has passed is the same. So if you're talking about, um, the length of one day multiplied by 28 [*writes length (day) * 28 = 1 LC*]...that will give you one lunar cycle. And you can say the same thing the other way…you can divide a lunar cycle up into 1/28 of a lunar cycle, and that's one day [*writes 1/28 LC = 1 day*]. But if we come down here and we say, "Okay, well one student times 28, does that equal one teacher? How many teachers does 28 students make up?"... 1/28 of a teacher that does not pop out one student. Like I can very confidently say that that is not one student.

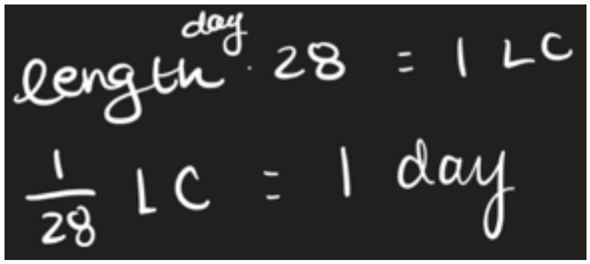

Figure 2. Emily's comparison of units for Task B.

In this excerpt, Emily explicitly sets off a distinction between days and length of days, indicating an attention to the magnitudes at play. Importantly, she identifies that (1/28)*|lunar cycle| = 1*|day|, which is a feature distinguishing the Wildi magnitude scheme from less sophisticated magnitude schemes (Thompson et al., 2014).

The interviewer continued by later asking Emily to respond to a student who wrote both *D = 28L* and *28D = L*. First, Emily says "so I guess that's the difference. I'm talking about length here [*refers to L in 28D = L*], but here I'm talking, if we're talking number of days [*refers to D in 28D = L*], which is what the problem states and the number of days that pass…". Here, she clearly distinguishes the measurement of the number of days and lunar cycles ("if we're talking number of days") versus the magnitude of each unit ("I'm talking about length here"); this is again consistent with the descriptive way of reasoning of the equation as proffered by Cook et al. (2021) (i.e.., "they describe, represent, or model the same quantity or situation" (p. 37)).

As she continued to reason why *D = 28L* and *L = 28D* could not both be true, she paralleled the initial reasoning she gave for why *D = 28L* was the correct answer; Emily used numbers to justify why four days would not equal 112 lunar cycles, as illustrated in Figure 3 below.

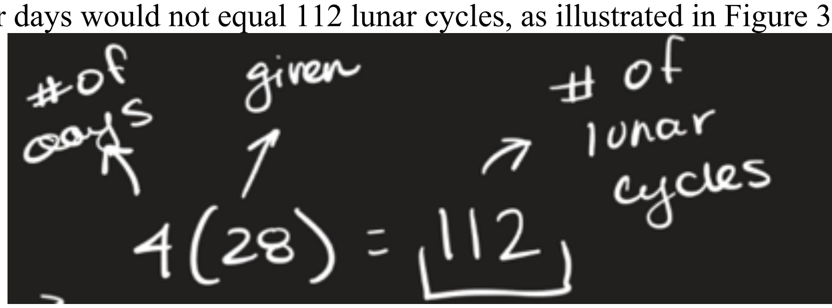

Figure 3.Emily's use of numerical ways of reasoning justifying why D = 28L.

She immediately referred to her initial line of reasoning ("the way I reasoned with it to begin with was with 37 times 28 gives me [1036] days…so then either that [*referring to her original reasoning*] would be wrong or this reasoning [*referring to the claim that 28D = L*] would be wrong"). Immediately after reestablishing the falsehood of the equation *28D = L*, Emily used both numerical and descriptive ways of reasoning by saying "if we're gonna think about it in practical terms, [one] lunar cycle, we can say that's a month…we have four days. That's not 112 months". We infer that Emily used a descriptive way of reasoning to support the numerical way of reasoning she had established earlier.

The distinction between measurement and units of measure became more evident as the interview continued:

Emily: This [*referring to D = 28L*] would be the number of lunar cycles . But here [*referring to her work in Figure 2b*], this is the number of days and this is the number of lunar cycles that have passed, which is not equivalent. These two things are not equivalent. Length of a lunar cycle and number of lunar cycles that have passed.

Int: Okay. And so those are, those are, those are different things, but clearly they're related in some way. [*Emily agrees.*] Um, so how…are they related?

Emily: Mm. Well, if you have three lunar cycles that have passed, right? So we're talking about the length of three lunar cycles… that is the length of those three lunar cycles. Each one of these is a lunar cycle. [*draws the number line in Figure 4a*] The length is not three…or I guess, um, 'cause if we're talking about like, each of these have 28 days [*produces Figure 4b*]...So I would say yellow is the length of one lunar cycle, and then blue is saying that three lunar cycles have passed [*produces Figure 4c*]….Because this is three lunar cycles, this is one, the length of one lunar cycle. Within here you have 28 [*produces Figure 4d*].

Int: So are you thinking about, about the, say the space between the blue brackets?...Is that the number of, are you thinking about that as the number of lunar cycles or the length of three lunar cycles? Or both, or neither?

Emily: Well, I guess the length, this is, this would be saying three lunar cycles have passed….We have three lunar cy- three full lunar, lunar cycles that have gone by, but the length of each one is the length of a day, times 28. So that's like each of these have 28 little, little markings in them [*produces Figure 4e*].

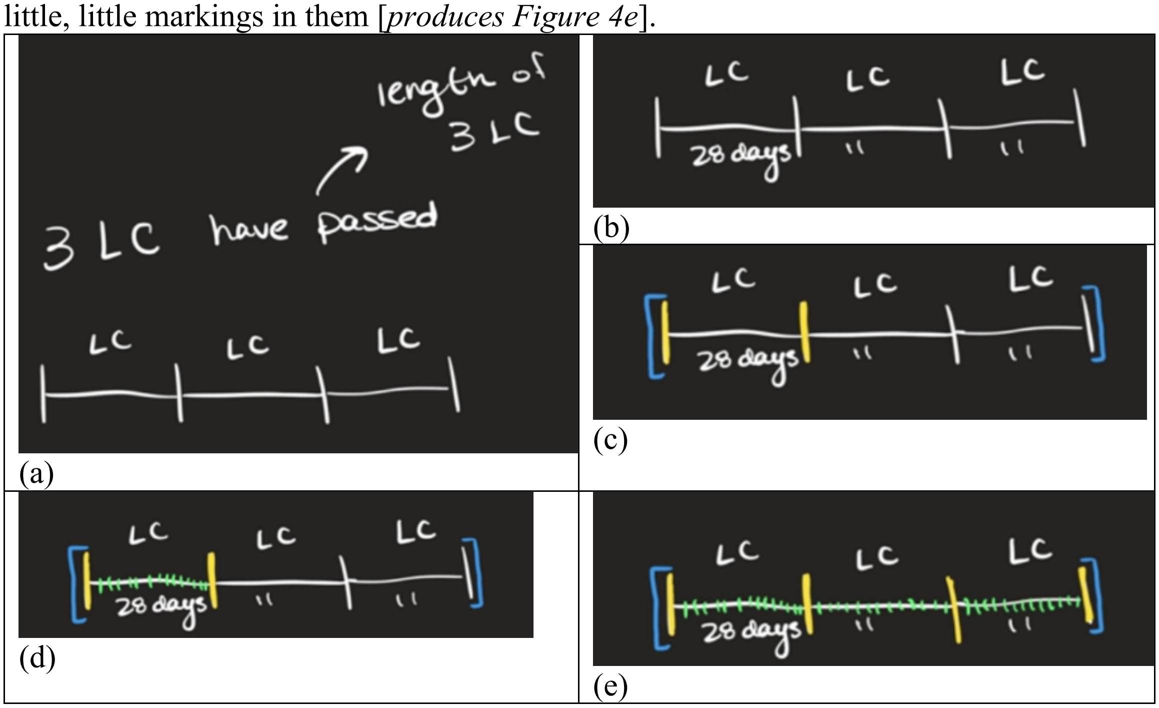

Figure 4. The evolution of Emily's number line diagram for 3 lunar cycles.

From this interaction, we infer that Emily argued 3*28*|day| = 3*|lunar cycle|; we particularly highlight Emily's careful attention to the distinction between the length of a lunar cycle (in yellow) and the accumulated duration of three lunar cycles (in blue). She then described how each lunar cycle consists of 28 days to draw a final representation. The final argument is an extension of her prior use of the Wildi magnitude accompanied by a clear representation of her reasoning. Namely, a single duration of time, as represented by the total line segment in Figure 4a, could be measured using two different units. With lunar cycles as her unit, the duration comprised 3*|lunar cycle|. With days as her unit, the duration comprised 3(28*|day|), which was an extension of her earlier reasoning with Wildi magnitudes. Combining these two arguments Emily concluded 3*28*|day| = 3*|lunar cycle|.

Discussion

Emily successfully used both numerical and descriptive ways of reasoning about equivalence when solving Tasks A and B. We claim that both the numerical and descriptive ways of reasoning about equality were essential to her productive approaches to these tasks. Whether she used one or the other, we observe, depended on the nature of the objects she was equating. When attempting to derive and verify the correct algebraic equation to Task B (relating, for example, D and L via $D = 28L$), Emily relied on numerical way of reasoning. On the other hand, when equating magnitudes of time (e.g., 1 lunar cycle and 28 days), Emily relied primarily on descriptive ways of reasoning. Put another way, Emily demonstrated numerical ways of reasoning to relate the *measurements*, and a descriptive way of reasoning to relate *magnitudes* (durations of time). Moreover, we note that equating durations of time (such as 1 lunar cycle and 28 days) do not lend themselves to a numerical way of reasoning because, indeed, the numerical values (measurements) are different. Taken together, these insights underscore that a descriptive way of reasoning can not only be useful but *critical* for students' reasoning about proportionality and measurement, especially considering that a numerical way of reasoning is predominant when working with most other types of algebraic expressions.

This paper contributes to the field by establishing that a descriptive way of reasoning about equivalence can be critical to students' thinking and provide a detailed example. We highlight that her work provides a clear and explicit empirical example of a student reasoning with Wildi magnitudes as described by Thompson et al. (2014). Specifically, by viewing the equality of 28 days and one lunar cycle in terms of being different measurements of the same magnitude, Emily was able to productively reason through the apparent contradictions of the solution process and extend the equality to larger periods of time than the one initially proposed by the problem. We note that while the students-and-professors problem has been studied extensively, we are unaware of research examining measurement conversion proportional situations such as Task B. We call for additional research exploring such scenarios and how students may rely on different ways of reasoning about equivalence (and magnitudes) when reasoning about such situations.

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