The Case for Complexity Theory and Formal Science to Study the Structure of Mathematics

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Mathematical structure refers to the characteristics, topology, and relationships within and between mathematical concepts invariant to external contexts and interpretations which instantiate the concept. Research on mathematical structure is typically guided by the post-positivist and interpretivist research paradigms, which respectively emphasize the impact of human observations and interpretations on mathematical concepts, both of which are external to the intrinsic structure of mathematical concepts. This report proposes a shift to complexity theory, a research paradigm standard to modern physics and engineering which emphasizes physically necessary mathematical principles over observation and interpretation, to faithfully study structure intrinsic to mathematics itself. I propose the Tornado Theory as a theoretical framework emerging from the tenets of complexity theory and current mathematics education literature which could guide future research on mathematical structure. Finally, potential analytical techniques for the Tornado Theory based on network- and information-theoretic principles are introduced.

Keywords: Mathematical structure, complexity theory, nature of mathematics, network theory, reification

Why is the Structure of Mathematics Important for Mathematics Education Research?

Mathematical structure is a frequently occurring concept in mathematics education research which cuts across numerous domains of mathematics education literature (Gronow et al., 2022). Mason (2003) defined *structural* features as the internal characteristics, topology, and dynamics of a system which are invariant to factors external to the system. This is in contrast to *surface-level* features, which refer to the effect external contexts, interpretations, and examples have on a system (Mason, 2003). When applied to mathematical content, *mathematical structure* refers to the topological and dynamical properties of a mathematical concept independent of the contexts, interpretations, and examples which instantiate the concept (Alcock & Simpson, 2004, 2005; Dawkins, 2012; Dive, 2003; Martin, 2013; Sfard, 1991, 2003). For example, the powers of *i* and rotation by 90 degrees are two mathematical concepts with different surface-level features but identical structural features, because these two concepts represent the same mathematical idea (the cyclic group, i.e. oscillation, with a period of 4) but are instantiated in different contexts and facilitate different contextual interpretations.

The importance of student understanding of mathematical structure has been well-documented in mathematics education literature. Skemp (1976) differentiated between relational and instrumental understanding of mathematics, the former referring to connections between concepts and the topology of these connections and the latter referring to isolated mathematical concepts and skills. Baroody et al. (2007) and Star (2005) expanded upon Skemp's ideas, claiming deep conceptual and procedural knowledge build upon each other to generate highly connected and meaningful schema. Relational understanding of a concept therefore requires coordination of conceptual and procedural knowledge and attending to the *meaning of a mathematical concept for* connections to other mathematical ideas rather than the *meaning of* concepts in isolation (Baroody et al., 2007; Brownell, 1947). This focus on connections requires a shift in attention from surface-level features to structural features (Alcock & Simpson, 2004, 2005; Dawkins, 2012; Lehrer & Schauble, 2020; Martin, 2013; Mason, 2003; Sfard, 1991, 2003). Therefore, student attention to mathematical structure is critical in developing a relational understanding of mathematics.

In order to shift students' attention from surface-level to structural features, mathematics instructors need to consider both the structure of mathematical concepts and its relations with surface-level features which students may be inclined to focus on (Mowat, 2010; Strom et al., 2001). This enables mathematics instructors and mathematics education researchers (MERs) to develop hypothetical learning trajectories which scaffold students to shift from surface-level interpretations to structural thinking while remaining faithful to the structure of mathematical concepts (Ekici et al., 2020; Leikin & Dinur, 2007; Simon, 1995). Mathematical structure is therefore a key component to frameworks characterizing pedagogical content knowledge (PCK) (Ball et al., 2008; Gronow et al., 2022; Rowland, 2013; Vale et al., 2011).

Thus, attention to mathematical structure is critical to relational understanding of mathematics for both students and instructors. Given the importance of mathematical structure, it is valuable for the MER community to consider mathematical structure from a theoretical perspective to guide MERs in answering questions such as: what is mathematical structure? What topological properties emerge from and govern relationships between mathematical ideas and domains? How can mathematical structure be characterized and used by teachers to inform their instruction? How does mathematical structure evolve over time as new mathematical ideas are discovered or created? The *Structure of Mathematics (SOM)* refers to these general meta-mathematical explications of mathematical structure. In this report, I will first review existing literature on SOM and discuss a philosophical dilemma surrounding current SOM research. Next, I will suggest a shift in research paradigm which mitigates this dilemma by invoking complexity theory from modern physics and engineering. Finally, I will propose the Tornado Theory, a framework based on complexity theory, which could guide future research in SOM.

Literature Review

MERs have begun studying SOM by using qualitative techniques and graph theory to construct and analyze graphs representing connections between mathematical concepts (Hensley, 2025; Lapp et al., 2010; Mowat, 2010; Strom et al., 2001). These studies typically did not provide an explicit theoretical framework for SOM and implicitly assumed SOM could be extracted from student concept maps. Strom et al. (2001) created directed graphs modeling concept maps of second graders' mathematical arguments involving the area of shapes. Lapp et al. (2010) studied linear algebra concept maps constructed by undergraduate students as an undirected graph. Hensley (2025) used topological data analysis (TDA) to study concept maps created by calculus students and identified topological gaps as missed connections in students' understanding of calculus.

Mowat (2008, 2010) criticized the use of concept maps to characterize SOM, asserting concept maps are architectural structures designed by external observers of mathematics rather than originating from the connections between mathematical concepts themselves. Instead, Mowat (2008) described an explicit theoretical framework drawing on Lakoff and Núñez's (2000) theory of embodied cognition. Mowat's (2008, 2010) framework models SOM as a nested network where nodes represent conceptual domains and links represent conceptual metaphors. Most nodes contain few connections, but a few nodes (known as hubs) contain many connections, making these networks display a hub-and-spoke topology. Mowat (2008) posited this hub-and-spoke topology has a high risk of cascading failure. Thus, learners need to construct links between non-hub nodes to gain a deeper understanding of mathematical concepts. Mowat

(2010) provided a proof-of-concept of her framework by using data from semi-structured interviews with mathematicians, teachers, and MERs to extract conceptions about exponentiation from semantic connections, creating a network of concepts related to exponentiation. This framework will be discussed in further detail in the Tornado Theory section.

An alternative approach involves extracting emergent connections between concepts from texts to form a network, which has been used to study both SOM and the structure of other STEM disciplines. In particular, this approach was conducted by Chai et al. (2020) on biomedical research papers, Christianson et al. (2020) on linear algebra textbooks, and Tschisgale et al. (2023) on German Physics Olympiad questions. These studies typically used collocation analysis, where the strength of connections were determined by the location of words rather than implicit meaning. Although some of these studies are not in mathematical contexts, they leveraged network-based approaches similar to Mowat (2008, 2010) to characterize the structure of knowledge.

Mowat's (2008, 2010) critique of the usage of concept maps reflects an issue underlying current research on SOM. Because SOM fundamentally studies *structure* as internal properties independent of external contexts and interpretations, frameworks and methodologies used in SOM research should reflect the focus on internal rather than external aspects. Similarly, Hensley (2025) posited existing methodologies used in SOM research such as qualitative coding, frequency analysis, and adjacency matrix analysis focus on contextual, surface-level components of mathematical concepts rather than topological and structural relationships between mathematical concepts. In other words, current SOM frameworks and methodologies focus on the *meaning of* mathematical concepts rather than the *meaning of* mathematical concepts *for* the construction of the larger structure of mathematics (Brownell, 1947; Hensley, 2025).

In particular, Strom et al. (2001) and Lapp et al. (2010) explicitly referenced interpretivism as their high-level research paradigm, which situates knowledge and reality with respect to cognitive interpretations and social factors (Heath & Bleiler-Baxter, 2025; Stinson & Walshaw, 2017), both of which are external to the structure inherent in mathematics (Dive, 2003). Despite Mowat's (2010) and Hensley's (2025) urges for a focus on structure over context, their methodology drew upon the interpretation of exponentiation (Mowat, 2010) and calculus (Hensley, 2025) made by students, teachers, mathematicians, and MERs, which are again external to the structure inherent in mathematics. Literature outside mathematics education (Chai et al., 2020; Christianson et al., 2020) typically used post-positivist techniques such as collocation analysis to reduce the interpretive influence of textual analysis, but the post-positivist paradigm introduces factors external to mathematics in another way. In post-positivism, empirical observations hold authority on reality and knowledge (Stinson & Walshaw, 2017). Empirical observations, however, are influenced by the interpretation of the observer and thus also external to the structure inherent in mathematics itself and an unreliable source of knowledge for SOM (Davis & Sumara, 2008). Collocation analysis has a further drawback of assuming connections between words could be fully represented by their spatial relationships on a page, which is an oversimplification (Borge-Holthoefer & Arenas, 2010).

Therefore, there exists a disconnect between the ontological and epistemological foundations of the current research on SOM. By definition, SOM requires extraction of characteristics, dynamics, and topology internal to a mathematical system. On the other hand, SOM studies are informed by worldviews which extract information from unreliable sources such as empirical observations, cognitive interpretations, and social structures. This begs the question: ***how can we directly study internal properties of mathematical systems when the very nature of research is***

external? In the next section, I propose an alternative high-level theoretical worldview, complexity theory, which provides an approach to answer this question.

A High-Level Paradigmatic Shift: Complexity Theory and Formal Science

Complexity theory is a high-level paradigm arising from the scientific and engineering movements in the post-war twentieth century (Davis & Sumara, 2006, 2008). At this time, the shift from Newtonian physics to modern physics inspired scientists and engineers to study the universe and its objects as chaotic systems defined by irreducibly complex ensembles of moving parts rather than deterministic systems constructed from components. Complexity theory therefore posits the irreducibly complex nature of individual systems make it futile to study the theories and laws governing individual systems. Rather, learning about the universe entails learning about the mathematical relationships governing nonlinear, chaotic, and complex systems (Davis & Sumara, 2008). For example, Newtonian physics is grounded by observations of the natural world made by scientists before the twentieth century. Mathematical relationships were created to model and explain these observations. The mathematics of Newtonian physics could therefore be considered as *artificial interpretations* of the physical world. On the other hand, although quantum mechanics was initially inspired by observations made in the late nineteenth and early twentieth centuries, mathematicians and physicists in the mid- to late-twentieth century found quantum mechanics follow directly from the mathematical relationships of linear algebra, functional analysis, category theory, and braid theory (Baez & Lauda, 2011). Mathematical relationships are therefore *physically necessary* for quantum mechanics to exist in its current form (Baez & Lauda, 2011; Dive, 2003; Halvorson, 2011; Townsend, 2012) rather than *artificially necessary* to explain human interpretations of the universe.

In other words, in Newtonian physics, observations precede mathematical relationships, which are artificially created to explain observations. In quantum mechanics and complexity theory, mathematical relationships precede observations, and observations serve as evidence of the physical necessity of mathematical relationships for the universe and the human experience to exist (Dive, 2003; Lakoff & Núñez, 2000; Thagard, 2020). Schelling's (1971) seminal work applying complexity theory to the social sciences summarized complexity theory as follows, "[the] rich will get richer, the advantaged will gain more advantage – not because of intention, but because of the laws of nonlinear dynamics" (Davis & Sumara, 2008, p. 170). Such a worldview places the physically necessary system of mathematical relationships at the forefront of research. To a complexity theorist, external sources of authority such as empirical observations, cognitive interpretations, and social structures are peripheral to the physically necessary, internal mathematical relationships which govern complex systems (Davis & Sumara, 2008; Dive, 2003). Thus, complexity theory suggests investigations of SOM should leverage physically necessary mathematics to study internal structure to avoid external interpretations.

Formal science (not directly related to the formalist view of the nature of mathematics) is an example of applying complexity theory to research. Dive (2003), Franklin (1994), Ramoo (2024), and Thompson (2007) described the formal scientific process as based on abstraction and mathematical reasoning rather than empirical observation and statistical testing. In formal science, researchers do not test hypotheses by designing experiments and collecting data, which is the case in traditional, empirical science (Ramoo, 2024; Thompson, 2007). Rather, researchers convert the system to be studied into an abstraction using mathematical frameworks. This abstraction ideally removes external, surface-level features, enabling the researcher to focus solely on internal, structural features. Then, researchers study the system and formulate solutions by analyzing the abstraction using mathematical relationships. It is critical the mathematical relationships used are physically necessary. This ensures the analysis process does not introduce additional surface-level features which may corrupt intrinsic structural relationships (Dive, 2003; Ramoo, 2024; Thompson, 2007). Complexity theory and the formal sciences are commonly used in engineering and industrial mathematics, where the goal is to design solutions whose effect cannot be observed at the time of design. Formal science is also commonly used in modern physics, where many phenomena such as quantum entanglement and the Higgs boson cannot be observed without precise and expensive instrumentation. Therefore, physically necessary mathematical frameworks must first be applied to formulate the initial engineering design or discover physical phenomena prior to experimentation or testing (Dive, 2003; Franklin, 1994).

Although it is impossible to confirm the physical necessity of mathematical frameworks, the robust coherence between mathematics derived from category theoretic principles (Baez & Lauda, 2011; Halvorson, 2011), the physical world (Thagard, 2020), and the human experience (Lakoff & Núñez, 2000) suggest mathematical frameworks based on category theory is, to the best of human understanding, physically necessary (Baez & Lauda, 2011; Dive, 2003; Franklin, 1994). Common mathematical frameworks used in the formal sciences include game theory, control theory, information theory, and network theory (Dive, 2003; Franklin, 1994).

In particular, network theory provides a promising direction for applying complexity theory to SOM for two reasons. First, existing literature on SOM typically leverage network theory and its derivatives such as TDA to model structure (Chai et al., 2020; Christianson et al., 2020; Hensley, 2025; Mowat, 2008, 2010; Lapp et al., 2010; Strom et al., 2001; Tschisgale et al., 2023). Furthermore, a set of network-theoretic tools have been developed to extract structural features from social scientific data such as semantic texts and social networks (Barrat et al., 2008; Borge-Holthoefer & Arenas, 2010; Scott & Carrington, 2014). Therefore, leveraging network theory situates SOM in existing literature. Second, network theory extracts topological and structural relationships, aligning with the goals of studying SOM. In the next section, I will develop an emergent theoretical framework used to abstract SOM with network theory, the Tornado Theory. Other mathematical frameworks may reveal additional information related to SOM such as dynamic evolution of mathematical structure (control theory) or the impact of human activity on the discovery, creation, and teaching of mathematical structure (game theory). A full picture of SOM and its impact on the discovery, creation, teaching, and learning of mathematics involves integrating abstractions and implications from a variety of mathematical frameworks. Thus, a framework purely based on network theory is intrinsically limited, and future theoretical work should develop SOM frameworks from other mathematical perspectives.

Mid- and Ground-Level Framework: The Tornado Theory

The Tornado Theory (TT) describes SOM from a network theory perspective by synthesizing the theoretical frameworks of Mowat (2008) and Sfard (1991). Sfard's (1991) reification theory described the structure of mathematical knowledge as emerging from an iterative process of condensation and reification. Any mathematical concept can be represented as a concrete, embodied *process* or an abstract, decontextualized *object*. The process representation of a concept, due to its concreteness, is easier to construct, but the object representation of a concept is required for the concept to be transferred to contexts independent of its original construction pathway (Dive, 2003; Sfard, 1991; Sfard & Linchevski, 1994). *Condensation* refers to the initial construction of a mathematical concept from old concepts, which requires old concepts to be transferred to the context of the new concept. Thus, old concepts can only be used in condensation if they are represented as objects decontextualized from how they were initially constructed. On the other hand, because the new concept is constructed in the context of old concepts, the new concept is initially represented as a process (Sfard, 1991, 2003; Sfard & Linchevski, 1994). For the new concept to be used in novel situations, it must be re-represented in a form independent of the context of its construction. This decontextualization is known as *reification*, and the reified concept could then be used in the condensation process of further mathematical ideas (Sfard, 1991). Therefore, mathematical concepts could be characterized through an iterative cycle of condensation and reification known as the *Tornado*, which Sfard (2003) described as “the driving force behind the incessant growth of knowledge” (p. 359).

For instance, consider integration, which is tied to area and differentiation. During its initial construction, integration is constructed as a process in the context of area and differentiation. Integration, however, has other meanings, such as smoothing and frequency domain division. In order for integration to be used to construct further concepts such as low-pass filters (based on smoothing) and Fourier Transforms (based on frequency domain division) (McClellan et al., 2015; Oppenheim et al., 1994; Rønning, 2021; Wang, 2025), integration must be reified into an object, i.e. represented independently from the contexts of area and differentiation. Continuing the cycle, although Fourier Transforms are initially constructed as a process in the context of integration, Fourier Transforms have other meanings such as an inner product on the L_2 Hilbert Space (Ekici et al., 2020; Debnath & Mikusinski, 2005; Rønning, 2021). In order for Fourier Transforms to be used in the context of Hilbert Spaces, it must be reified as an object, i.e. decontextualized from its initial construction contexts of integration and the frequency domain. This condensation-reification cycle known as the Tornado continues iteratively to drive the discovery of new mathematical concepts.

Mowat’s (2008) network theory approach to SOM could be used to operationalize the topology of mathematical concepts formed by the Tornado. In particular, mathematical concepts may be arranged as a network with nodes representing individual concepts and links representing connections between concepts. Each node contains a network structure within itself representing the structure of the concept, with nodes of this sub-network representing sub-concepts building up the greater concept. Each sub-concept is represented by another network describing the structure of the sub-concept (Mowat, 2008). This nested network continues like a Russian doll.

Inspired by existing research on network theory characterizations of semantic networks (Albert & Barabási, 2002; Barrat et al., 2008; Borge-Holthoefer & Arenas, 2010; Newman, 2003), Mowat (2010) leveraged the Barabási-Albert Model (BAM) to describe the topology of each network layer. BAM posits the number of nodes N with exactly k links in a network can be approximated with the relationship $N[k] = \psi k^{-\gamma}$, where ψ is an arbitrary constant and γ characterizes the overall structure of the network (Albert & Barabási, 2002; Mowat, 2010). If $\gamma > 1$, the number of nodes quickly decreases for higher numbers of links, meaning most nodes have few connections and few nodes have many connections. Thus, the network displays a hub-and-spoke structure, which is easily constructed but has a high probability of cascading failure. If $\gamma < 1$, the number of nodes decreases slowly for higher number of links, meaning there are a significant number of nodes with many links. Thus, the network displays a distributed structure, which are difficult to construct or interpret but are robust and adaptable in novel situations due to the low probability of cascading failure (Mowat, 2010; Strom et al., 2001).

The BAM and Russian doll structure described by Mowat (2008, 2010) reflects and provides a mathematical abstraction for the Tornado. Condensation refers to the construction of a new layer of networks from old network layers in the Russian doll. Mowat (2010) claimed these new layers follow a hub-and-spoke structure (i.e. $\gamma > 1$) due to their ease of construction. This hub-and-spoke structure is analogous to the process representation of the Tornado. A hub-and-spoke network, however, is difficult to apply to contexts outside its initial construction process due to its high probability of cascading failure. The hub-and-spoke network needs to be converted into a distributed network (i.e. decreasing γ), analogous to the object representation of the Tornado (Mowat, 2010; Sfard, 1991, 2003; Sfard & Linchevski, 1994). Reification, which Sfard (2003) claimed to be “the driving force behind the incessant growth of knowledge” (p. 359), could therefore be represented by BAM as a decrease in γ of the network representation of the concept.

Although BAM was used by Mowat (2010), other formal scientific approaches could also be leveraged to demonstrate TT. For instance, Ferrer-i-Cancho (2005, 2006) established an information-theoretic approach to network analysis. Network construction is dependent on a fundamental tradeoff between “maximizing information transfer and saving the cost of [construction]” (Borge-Holthoefer & Arenas, 2010, p. 1285). The tradeoff is represented via the energy function $\Omega = \lambda I(N; C) - (1-\lambda)H(N)$, where $I(N; C)$ represents the amount of information intrinsic to concept C transferred via its network representation N, and $H(N)$ represents the entropy, or complexity, of the network itself. The key parameter is $\lambda \in [0,1]$, which represents the nature of the network: $\lambda = 1$ represents a fully faithful representation of the concept at the expense of ease of network construction, and λ close to 0 represents an easy-to-construct network at the expense of faithfulness to the intrinsic structure of the concept (Borge-Holthoefer & Arenas, 2010; Ferrer-i-Cancho, 2005, 2006; Ferrer-i-Cancho & Solé, 2003). Lower λ values correspond to higher γ values in BAM, a hub-and-spoke network, and a process representation of a concept. Higher λ values correspond to lower γ values in BAM, a distributed network, and an object representation of a concept. Thus, reification can be represented in Ferrer-i-Cancho’s framework as an increase in λ value. Using techniques developed in information theory (Krippendorff, 1986; Shannon, 1948), it is possible to extract values for $I(N; C)$ and $H(N)$ and demonstrate reification via variation in λ values.

Conclusions and Implications for Future Research

The frameworks developed in this report provide a starting point for researchers seeking to study SOM. Complexity theory was proposed as a high-level paradigm relevant to studying SOM because post-positivism and interpretivism places observations and interpretations external to the structural components of mathematics at the forefront of research (Heath & Bleiler-Baxter, 2025; Stinson & Walshaw, 2017). The need to focus on internal structure rather than external observations and interpretations could be accomplished through complexity theory and formal science. The Tornado Theory, as proposed in this report, represents one approach to ground research on SOM based on both formal scientific frameworks (Borge-Holthoefer & Arenas, 2010) and existing mathematics education literature (Sfard, 1991, 2003).

Future research should develop and apply sub-theories of the TT such as BAM and Ferrer-i-Cancho’s framework to analytically uncover topological and structural relationships between various mathematical concepts and domains. To remain faithful to complexity theory, all steps of the research process should reduce the impact of external observations and interpretations. The sub-theories should be based on physically necessary mathematical relationships such as the principles of network theory and information theory. For example, interviews and surveys may not be appropriate, as these methodologies focus on participants’ interpretation of mathematical structure rather than mathematical structure itself, even if the participants are experts. Rather, researchers should use mathematical relationships to analyze the abstractions provided by these sub-theories to arrive at conclusions about SOM. Finally, frameworks based on other formal scientific theories such as control theory and game theory should be leveraged to arrive at a holistic view of SOM.

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