Students' Embodied Reinvention of the Bijectivity of Invertible Functions

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We present a preliminary local instructional theory of students' embodied guided reinvention of the bijectivity theorem of invertible functions. We leveraged the design principles of Realistic Mathematics Education (Freudenthal, 1991) and Embodied Design (Abrahamson et al., 2020). We present our task sequence, describe implementation in a classroom, and provide an analysis of student reasoning. We discuss our plans for reimplementation and implications for teaching.

Keywords: Realistic Mathematics Education, Embodied Cognition, Invertible Functions

Introduction and Literature Background

It is essential for students to understand properties of functions and inverse functions. The extensive literature on functions documents various student difficulties such as inattention to defining properties (e.g., Markovits et al., 1986) and issues with domain and codomain (e.g., Even, 1990; Hitt, 1998). Recent work has identified productive techniques for approaching defining properties of functions (e.g., Uscanga et al., 2024). Even when students attend to defining properties, they may conflate well-defined and one-to-one (e.g., Clement, 2001; Even 1993), and everywhere-defined and onto (e.g., Melhuish et al., 2020; Vinner & Dreyfus, 1989).

Reasoning about inverse functions is multifaceted. It may entail conceptualizing an inverse function as an "undoing" (Cook et al., 2023, p. 6), "manipulated element" (p. 7), or "coordination of the binary operation, identity, and set" (p. 8). This may involve recognizing the composition of a function and its inverse as the identity function, which acts as a "do-nothing function" (Bagley et al., 2015; Serbin, 2023). Reasoning about inverse functions may involve using analytic, graphical, or covariational approaches (e.g., Oehrtman et al., 2008; Paoletti et al., 2018; Paoletti, 2020). Students must also make sense of why a function needs to be bijective (i.e., one-to-one and onto) for its inverse relation to be a function (i.e., well-and everywhere-defined) (Bansilal et al., 2017), which is challenging for them.

Here, we explicate our local instructional theory (LIT; Gravemeijer, 2004) to engage students in reasoning about the properties of well- and everywhere-defined, one-to-one, and onto to reinvent a direction of this target theorem: *Let* $f: A \to B$ *be a function. An inverse relation* $f^{-1}: B \to A$ *is everywhere- and well-defined iff* f *is onto and one-to-one.* An LIT is "the description of and rationale for the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). We provide the rationale for our preliminary instructional task sequence and evidence of students' progress. We merge the design principles from Realistic Mathematics Education (RME; Gravemeijer, 1999) and embodied cognition (Abrahamson et al., 2020) to inform the design of our task sequence. We address the research question: *What learning route can students follow to reinvent the target theorem?*

Theoretical Background: RME × Embodied Design Principles

Design research (Gravemeijer et al., 2003) involves a cycle of designing instructional task sequences, implementing them with students, analyzing students' reasoning, and revising the task sequences accordingly. We leveraged RME (Freudenthal, 1973; 1991), an instructional design theory in which, "students are expected to develop formal mathematics by way of mathematizing their own informal activities" (Gravemeijer, 1999, p. 160). RME has three

heuristics that guide researchers' design of an LIT. The first is using problems with "contexts that are *experientially real* for the students and can be used as starting points for progressive mathematization" (p. 158). The second is *guided reinvention*, by which instructors leverage students' ways of reasoning to guide them to reinvent formal mathematics. The third is the opportunity for students to develop an *emergent model*, an object or representation that students project meaning onto as they model the mathematics in play. We also use embodied design principles (Abrahamson et al., 2020; Nathan, 2021) to frame students' emergent models as *multiparty* enactments (Kelton & Ma, 2018) in which students use their bodies, physical materials/props, and space to carry their personal meanings for the mathematics (e.g., Soto, 2022; Lajos & Katz, 2025). Embodied design principles suggest the instructor should leverage a variety of students' embodiments (e.g., Soto et al., 2024) to support their guided reinvention.

As students reinvent mathematical concepts and transition from using their emergent model as a "model of" to a "model for," they engage in *horizontal* and *vertical mathematizing* (Rasmussen et al., 2005; Treffers, 1987). *Horizontal mathematizing* refers to "formulating a problem situation in such a way that it is amenable to further mathematical analysis" (p. 54) and entails a shift towards symbolizing via inscriptions and taking stock of what is noticed across a variety of emergent models. *Vertical mathematizing* is defined as performing mathematical activities that are built on and are grounded in one's own horizontal mathematical activities. For our context, this entails the act of moving from an embodied model to a formal definition, conjecturing, and arriving at classroom generated theorems using refined language.

Methods

We conducted a teaching experiment (Cobb, 2012) in a Fundamentals of Algebraic Structures course at a public university in the southern US. The course was taught by a teacher researcher (TR) who is a coauthor. The students were junior and senior prospective secondary math teachers. The task sequence implementation took place in one 75-minute class. Before this lesson, the course had addressed the definitions of relations, well- and everywhere-definedness of functions, and inverse relations, but not one-to-one and onto, which were focal properties in this LIT. We note that one group of students used "one-to-one" language as they worked on the tasks, but they did not recall the term or meaning of onto. We video-recorded two groups of four students each to capture their multiparty embodied models and class discussions. Each author watched the class videos and wrote content logs (Lesh & Lehrer, 2012) on students' reasoning, evidence of mathematization, embodied models used, and how the TR leveraged the students' embodiments to guide their reinvention. The authors compared and discussed content logs.

Results

In Activity 1 of our LIT, students embodied the functions in Figure 1 and their inverse relations. Students leveraged their embodiments (e.g., multi-party enactments, inscriptions) to make sense of everywhere- and well-definedness issues of the inverse relations. In Activity 2, students used their embodied models to formulate definitions for one-to-one and onto, make conjectures relating properties of functions and inverse relations, and arrive at the target theorem.

Activity 1: Enacting Puddle Diagrams of Functions and Their Inverse Relations

Activity 1 prompts students to physically act out puddle diagrams for functions and their inverse relations and determine whether each inverse relation is a function. Students played the role of elements in the domain and codomain for various functions (Figure 1). TR provided element name tags labeled 1, 2, 3, 4, 5, and 6, strings, and domain and codomain labels. We

present Group 2's (G2) embodied models for inverse relations. Next, students symbolized their embodied models using sets of ordered pairs and puddle diagram inscriptions. They leveraged their body formations to discuss whether their inverse relations were functions. This task was designed to give students an opportunity to model functions' inverse relations, which they could later use to make conjectures about the relationships between function properties.

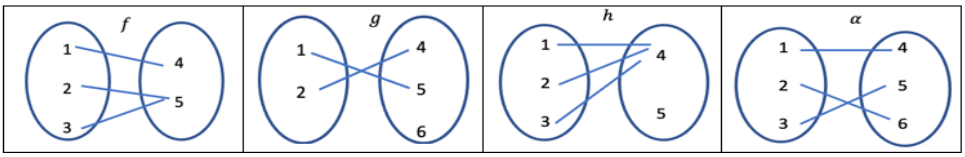
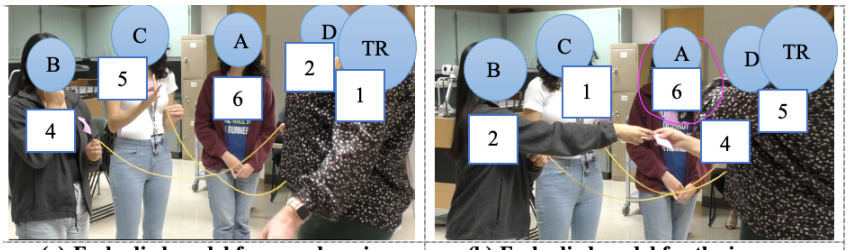


Figure 1. Parent functions given to students in Activity 1.

G2's embodied model for the inverse relation of g: "Switching (element) name tags." G2 made an embodied formation for the puddle diagram of g, and student C playing element 5 gestured a swap with her fingers (Figure 2a). To embody the inverse relation of g, the students playing elements 2 and 4 switched element name tags, and the students playing elements 1 and 5 switched tags (Figure 2). They were perturbed when the student playing 6 had no one to switch tags with, and their "swapping x and y" procedure was abrupted. The instructor remarked, "That's weird [glanced at 6 who was still standing in codomain area], should 6 be in the domain or codomain?" (Figure 2b). B questioned, "if 6 doesn't have a partner, can it still switch? No, right? because there is no one there." TR probed further: "What do we do with 6?" The students' strategy of switching name tags among paired elements in the domain and codomain did not work. They recognized that this embodied breakdown occurred because student A (6) did not have someone to switch tags with. The group next began to mix features of their embodied model (e.g., no one to swap with, kick her out, no partner) with formal property language.



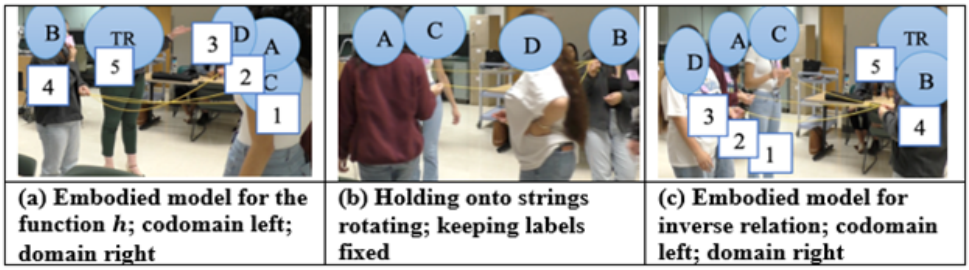
(a) Embodied model for g; codomain left; domain right

(b) Embodied model for the inverse relation; codomain left; domain right

Figure 2. Embodied model for function g and its inverse relation "switching element name tags."

G2's adjustment and symbolization of their embodied model: "Kicking out" a peer in the codomain of g^{-1}. The group continued to focus on A (6) questioning where A belonged, the codomain or domain side. B (1) and C (2) suggested, "kick her [6] out." TR leveraged A's eviction and body location to invite her to stand on the domain side. She asked, "Why do we not want her in our domain set?" B replied, "Because she didn't have a connection in the first place." TR asked what function property was violated. B replied "Every x has to have a y," and D said, "Everywhere-defined." The group concluded g^{-1} is not a function, unless they restrict its domain. G2 confirmed g^{-1} was well-defined and concluded it was not a function because it was not everywhere-defined. Following their multi-body enaction, B symbolized g and g^{-1} as the sets, $\{(1,5), (2,4)\}$ and $\{(5,1), (4,2)\}$. Student B claimed to not know how to symbolize that "6 did not have a partner," so she wrote "Either remove or switch to domain without partner."

G2's embodied model for the inverse relation of h: "Holding onto strings and rotating." G2 embodied h as B (4 in the codomain) held three strings, TR (5 in the codomain) held no strings, and students C (1), A (2), and D (3) (in the domain) each held one string emanating from B (4) (Figure 3a). After looking at her peers across from her, student B exclaimed, "I only have one heart guys. I can't split it." G2 then decided their prior tag switching model would not work. At best, B (4) could only switch tags with one peer on the other side. This foreshadowed an issue that h^{-1} is not well-defined because h is not one-to-one. G1 suggested that they "just switch the [domain and codomain] papers on the table." G2 seemed puzzled and fell silent when TR asked, "is this [h^{-1}] a function?" TR offered a change in embodied perspective and motioned for them to rotate their entire embodied model for h (Figure 3b). After arriving at their new locations, C (1), A (2), and D (3) looked down at the codomain label on their side (Figure 3c). D stated, "It would be the same situation as [g] where it's well-defined but not everywhere-defined" and pointed to TR (5) in the domain who was not holding a string. TR noted D's partial meaning of "not everywhere-defined" and revisited issues of "well-definedness" during the class discussion.



(a) Embodied model for the function h; codomain left; domain right | (b) Holding onto strings rotating; keeping labels fixed | (c) Embodied model for inverse relation; codomain left; domain right

Figure 3. Group 2's embodied model for h and inverse relation "rotating multi-body string formation"

Activity 2: Mathematizing Embodied Models to Inform Definition and Conjecture Drafts

Activity 2 was designed for students take stock of their embodied models. The first part prompted students to "make conjectures about what properties f must have for f^{-1} to be a function." This task was intended to engage students in horizontal mathematizing by explicating their informal ways of reasoning about one-to-one and onto grounded in their embodied models. TR could then leverage their embodied reasoning and introduce formal mathematical language to refer to their ideas. Next, students had to "fill in the blanks" in these definitions with either one-to-one or onto: (1) Let $f: X \to Y$ be a function. We say f is ___ when we have if $f(x_1) = f(x_2)$, then $x_1 = x_2$; (2) Let $f: X \to Y$ be a function. We say f is ___ if and only if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. This encouraged students to further mathematize their embodied models as they identified mathematical symbols to represent their enacted properties. The last prompt asked students to formalize their conjectures as theorems. This engaged students in vertical mathematizing as they explicated relationships between a function's properties of one-to-one and onto and the inverse relation's properties of well- and everywhere-definedness. The class's negotiation of the theorem drafts concluded the guided reinvention of the target theorem.

Horizontal mathematizing: Taking stock of prior embodied models. During class discussion, TR asked about issues with the inverse relations. G1 explained that f^{-1} is not a function because "5 had two outputs" which violates well-definedness. G2 stated "our 6 doesn't have a partner [for g]," so g^{-1} is not a function. For h, G2 explained, "When we did the inverse to it, we got it was not well- or everywhere-defined, which meant the inverse relation was not a function." For α, TR mentioned that α^{-1} was a function, and G2 described this as "everyone has a partner and only one partner." TR encouraged more explicit language, "that's what well-

defined and everywhere-defined means, but now it's more special than that." Students drew from their embodied formations to formalize their language. They had a spatial sense for one-to-one and onto but were not familiar with the formal definitions. As they attempted to formalize their language, TR revoiced students' contributions: "We cannot have repeated partners." She posed the question, "What about the issue of everybody having a partner?" Revoicing again TR stated, "Everyone has a partner, and it's a unique partner" and introduced the term, one-to-one.

Vertical mathematizing: Defining, conjecturing, and reinventing target theorem. TR shared G1's conjecture that a function being one-to-one is sufficient for the inverse relation to be a function. Student E in G1 shared that g is a counterexample because it is one-to-one, but g^{-1} is not everywhere-defined. Drawing on their embodied model [gestured three fingers on each hand], E revised the conjecture and added that there "has to be the same amount of inputs as there are outputs." TR rephrased their contribution, "everything in the codomain needs to have a buddy." TR then introduced the term onto. To encourage symbolizing and formalizing, TR provided a fill-in-the-blank definition box. Students matched their notions of one-to-one and onto to the formal symbolic definition to fill in the boxes. Student F in G1 explained, "if it's one-to-one, there would be one input for an output and if it's onto, everything in the codomain has to have a partner, so that would be everywhere-defined." TR asked students what the "it" referred to. She wrote templates of students' conjectures and asked them to fill in the blanks: *If _ is one-to-one, then _ is well-defined. If _ is onto, then _ is everywhere-defined. If _ is one-to-one and onto, then _ is a function.* Student B said the first blank "has to be the function." TR followed with, "if f is one-to-one, then what is well-defined?" Student D replied "f inverse." TR then transitioned to the onto conjecture. She wrote student G's conjecture that *If f^{-1} is onto, then f is everywhere-defined*, and asked for input. Student B said, "I think he's okay, but the last one, would it be *If f is one-to-one and onto, then the f inverse is a function.*" TR validated this theorem draft and asked about the puddle diagram of f, "is that onto? [Students: "yes"] and is f inverse everywhere-defined?" [Students: "yes"]. She validated G's conjecture, "We could have f inverse or f, but from what we noticed in our little diagrams if f is onto, f inverse is everywhere-defined." TR wrote the class's target theorem: *If f is one-to-one and onto, then f^{-1} is a function.* This concluded the class's reinvention of the target theorem.

Discussion of Preliminary Findings and Conclusion

We shared our preliminary LIT on students' guided reinvention of the target theorem (refer to intro). Students' progressive mathematization of their embodied models supported them to reinvent the target theorem. Thus, we recommend the use of embodied enactments of puddle mapping diagrams to reason about function properties. Transitioning from horizontal to vertical mathematizing seemed difficult for students in this study. We suggest that more research should identify how tasks and instructors can scaffold that transition for students. A contribution is crossing design principles (RME and embodied cognition) to arrive at an instructional design framework for embodied reinvention. By leveraging this hybrid framework, our LIT contributes to the literature on instructional design and the teaching and learning of (invertible) functions. Future research could test the efficacy of this intersection in other design research studies. This is a preliminary task sequence which we plan to implement again as a new iteration of the design cycle to refine the tasks. Our revisions involve incorporating examples of functions with infinite domains/codomains and having students refine their conjectures to capture the bidirectional implication the target theorem could have. We seek advice from the audience on the task sequence and our merging of RME and embodied cognition.

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