Conceptual Foundations for Connecting School and Abstract Algebra: A Historical Analysis

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This paper uses a conceptual analysis of histories of algebra to identify a kind of reasoning, combinatorial reasoning, that was essential to the solution of polynomial equations. Historically, mathematicians' increased reliance on combinatorial reasoning in the solution of polynomial equations co-occurred with the rise of symbolic algebra. Algebraic symbols were often the context in, and on, which they engaged in combinatorial reasoning. Combinatorial reasoning, however, is often in the background of proofs or arguments—present but not framed as an essential kind of reasoning that could be developed to understand the proofs or arguments. The purpose of foregrounding this reasoning in this paper is to argue that it can serve as a conceptual foundation for designing coherent algebra curricula across grades 9-16—curricula that allow students and teachers to see deeper connections between school and abstract algebra.

Keywords: School Algebra, Abstract Algebra, Combinatorial Reasoning, History of Mathematics

Historically, mathematicians' work on the algebraic solution of the general polynomial equation[1] played an important role in the genesis of abstract algebra (Kleiner, 1986; Nový, 1973; Pesic, 2003; Stedall, 2012). However, one is hard pressed to see how the solution of polynomial equations, as presented in secondary curricula, is connected to ideas in abstract algebra, as presented in many modern-day textbooks. Secondary algebra curricula emphasize a range of techniques, often disconnected from one another, for finding the roots of polynomial equations (e.g., factoring, quadratic formula, rational root theorem, complex conjugate theorem) (Burch et al., 2021; Sangwin & Jones, 2017; Thompson, et al., 2012). Post-secondary abstract algebra curricula often begin with an abstract treatment of groups, making limited connections to the solution of polynomial equations in early parts of texts (e.g., Fraleigh, 2003; Herstein, 1997). This curricular structure can lead to a disconnect, particularly for future secondary teachers, between what they will teach to their future high school students, and what they are learning in their collegiate level abstract algebra courses (Wasserman, 2018).

For this reason, mathematics educators have recommended examining secondary curricula from an advanced perspective (Àlvarez & White, 2018; e.g., Lee & Heid, 2018; Serbin, et al., 2024), and using the history of algebra (Suoinmen, 2018; e.g., Barnett, 2017; Edwards, 1984; Blum-Smith & Coskey, 2017) to develop more coherent algebra curricula and experiences across grades 9-16. The purpose of this paper is to build on these recommendations. We do so by presenting a conceptual analysis (Thompson, 2008) that highlights how combinatorial reasoning with and on algebraic symbolism played a crucial role in advances in the algebraic solution of the general polynomial equation (Derbyshire, 2006; Stedall, 2012; Tignol, 1988/2016). Our examples start with Viète's formulas, and end with one of LaGrange's contributions to the solution of the quartic. These examples precede Ruffini's and Abel's work on the impossibility of the solution of the general polynomial equation of degree five (Ayoub, 1980; Pesic, 2003),

[1] Throughout, I focus on algebraic methods for solving polynomial equations and differentiate it from numerical methods (Rider Hamburg, 1976). Numerical methods are various approximation methods that give a numerical value for a root of a polynomial. Algebraic methods are ones that structurally relate the roots, factors, and coefficients of a polynomial to determine if the roots of the general polynomial equation can be expressed in terms of known coefficients.

and Galois's identification of general conditions for the solvability of polynomial equations (Edwards, 2012; Tignol, 1988/2016). We focus on this historical time-period (16th-18th century) because it laid the groundwork for the transition from viewing algebra as the solution of polynomial equations to viewing algebra as the study of structure. By identifying combinatorial reasoning as a foundational kind of reasoning during this period of time, we aim to make apparent a kind of reasoning that if emphasized in curriculum and instruction would allow students and teachers to experience a greater sense of connection between, and coherence across, secondary and post-secondary algebra experiences.

Background Literature: Context for Our Analysis

We use Kaput's (2008/2017) framework for algebraic reasoning as an organizational tool for discussing the algebra literature, which we use to motivate the reason for our analysis. Within Kaput's framework, secondary algebra research has tended to focus most heavily on strand 2, functions (Kieran, 2007; Warren et al., 2016), where mathematics education researchers have identified co-variational reasoning as a foundational kind of reasoning for this strand (e.g., Thompson & Carlson, 2017; Confrey & Smith, 1996). Less attention has been given to strand 1 at the secondary level (Kieran, 2007; Warren et al., 2016), a strand that includes the algebraic solution of polynomial equations. The studies that do attend to strand 1 often focus on students' development of structure sense as they engage in syntactically guided transformations on conventional symbol systems (core aspect B, Kaput's framework) (e.g., Hoch & Dreyfus, 2005).

Core Aspects and Strands
Two Core Aspects
(A) Algebra as systematically symbolizing generalizations of regularities and constraints. (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbolic systems.
Core Aspects A & B are Embodied in Three Strands
1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning. 2. Algebra as the study of functions, relations, and joint variation. 3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.

Figure 1. Kaput's (2008, p. 11) framework for algebraic reasoning

One common focus of this work has been to investigate the role computer algebra systems (CAS) can have in supporting students to develop structure sense, particularly in relation to expanding and factoring algebraic expressions and solving polynomial equations (e.g., Saldanha & Kieran, 2006). Early work in this area emphasized that a skilled teacher's use of CAS could support students to develop a better understanding of equivalence (Nicaud, et al., 2004; Kieran & Saldanha, 2005), and of the relationship between a polynomial and its factors (Artigue, 2002), including relationships that might otherwise be too complex for secondary students to handle outside of CAS (LaGrange, 2003). As CAS have become more widely used by teachers, and the tools more powerful, researchers have tended to present fewer positive results (Artigue, 2010). More recent findings, for example, include that CAS can support students to develop non-reversible equation solving techniques (Jankvist, et al., 2019), that CAS tasks in textbooks tend to involve low procedural complexity (Davis & Fonger, 2015), and tend not to foster opportunities for students to engage in proofs (Jankvist & Misfeldt, 2019).

At the post-secondary level, there have been recent studies with practicing or future secondary teachers that have focused on understanding relationships between a polynomial and its factors (e.g., Burch, 2023; Tillema & Burch, 2022). One focal point of these studies has been to support secondary teachers to use concepts they are learning in abstract algebra to unify ideas related to factoring that they will teach at the secondary level (Serbin, et al., 2024). Often these studies, like the studies at the secondary level, have been framed around secondary teachers' development of structure sense (e.g., Lee & Heid, 2018; Novotná & Hoch, 2008). While many studies related to the algebraic solution of polynomials have been framed around structure sense, there has been less attention to foundational forms of reasoning that produce it (Linchevski & Linveh, 1999; Novotná & Hoch, 2008) as compared to the much more extensive literature on functions and co-variation. The primary purpose of this paper, then, is to explicitly name an important form of reasoning, combinatorial reasoning, from which students could develop their structure sense related to the solution of polynomial equations.

Conceptual Framework

Thompson (2008, p. 1-31) "propose[s] a method by which mathematics educators can make tacit meanings explicit and thereby address problems of instruction and curricula in a new light." The method he proposes is conceptual analysis. Conceptual analyses involve specifying the mental operations that a person must carry out to understand a situation in the way they are understanding it (Glasersfeld, 1995, p. 78). This definition of conceptual analysis makes historical sources of mathematics ripe for conceptual analyses for at least two reasons. First, historical sources often capture mathematical understandings as they emerged, developed, and changed over time. It is rare, for example, the way an understanding is initially expressed is the way that it is used or understood at later points in time. This evolutionary process allows one to analyze differences in understanding, along with what features of an understanding maybe advantageous to aim for in current curricula. Second, often features of an understanding are implicit or taken as shared at the time an understanding is expressed (i.e., in a particular historical time-period). These implicit understandings may support the development of new understandings where the new understandings may ultimately be what gets encoded in curricula, while the implicit understandings fall away. For example, many of the concepts related to group theory (e.g., a group, a normal subgroup, a conjugacy class, factor group, etc.) were initially intimately tied to the solution of polynomial equations (Cajori, 1912; Barnett, 2017; Dehn, 1930). However, over time this connection has become much more hidden in curricula (e.g., Fraleigh, 2003). The benefit, then, of examining historical sources is that one can make implicit understandings explicit, especially, if implicit understanding or reasoning that was important to the genesis of later understandings is lost.

Elsewhere, I have given accounts of the mental operations that are involved in combinatorial reasoning, and which produce combinatorial structure, along with the consequences this has for differences in secondary students' or teachers' understandings (e.g., Tillema, 2018; Tillema, 2020; Tillema et al., 2024; Tillema et al., 2025). I summarize that work here by saying that these operations (along with schemes, coordination among schemes, and abstraction) produce "psychological slots," and a structure on these slots. From this perspective, some of the main differences among reasoners has to do with the psychological slot structure that they use to interpret situations. For example, a reasoner might interpret a statement like, $(a+b)^2$, using a "psychological slot structure" that is represented as $(\underline{\underline{\quad}+\underline{\quad}})(\underline{\underline{\quad}+\underline{\quad}})$. This is a 2-slot structure where each slot contains two slots—the larger slots are structurally connected by multiplication and the smaller slots by addition. The structure implies a recursive embedding of slots where the additive slot structure is embedded in the multiplicative one. On the other hand, a student may interpret a statement like $(a+b)^2$ with the following psychological slot structure, $(\underline{\qquad})(\underline{\qquad})$. This is a 2-slot structure where the slots are connected by multiplication only. This 2-slot structure yields non-normative interpretations of the expression, $(a+b)^2$. We provide this brief analysis to indicate what we mean by combinatorial reasoning and structure, but in our analysis below we do not map out in full detail the psychological slot structure that is implied by the analysis. We simply point out more informally where combinatorial reasoning can be inferred in the context of examining histories of algebra.

Examples from Histories of Algebra

The historical period between Cardano's (1545) *Ars Magna* and Lagrange's (1771) *Réflexions sur la résolution algébrique des équations* is often framed as a period of slow or no growth in histories of algebra (Stedall, 2012). This framing, in part, is because the discovery of methods for solving cubic and quartic equations, published by Cardano, occurred within a short timeframe. The short timeframe primed mathematicians to expect that the solution of higher degree equations might follow quickly (Bashmakova, 2000, van der Waerden, 1985). However, that did not happen; the discovery of whether and when higher degree equations could be solved had to await Ruffini (Ayoub, 1980), Abel (Pessic, 2003; Rosen, 1995), and Galois (Edwards, 1984, 2012) in the early 19th century. Thus, if growth is measured from the perspective of finding a solution to higher degree equations, then, indeed, little progress was made between Cardano and Lagrange.

However, Stedall (2012) convincingly argues that this time-period has both been neglected in histories of algebra, and that it was a period of *essential* growth (see also, Nový, 1973; Tignol, 1988/2016); essential in that the growth opened the way for the transformation that occurred to algebra in the 19th century (Kiernan, 1971). The developments in the 19th century changed the focus of algebra from the study of the solution of polynomial equations to the creation and study of structure more broadly (Derbyshire, 2006). The developments in algebra during the 16th to 18th century, then, should be of significant interest to mathematics educators who are aiming to design coherent algebra curricula across grades 9-16; that is, developments during the 16th to 18th century offer insight into the kinds of reasoning that supported the transformation of algebra in the 19th century. Moreover, the fact that historians of mathematics have given minimal attention to the 16th to 18th centuries means that studying it has the potential to provide new insight.[2]

One important transformation that occurred in the 16th to 18th century was mathematicians' perspective on equations and equation solving. At the beginning of the period, equations, including the solution of cubic and quartic equations, were solved within a mathematical framework of proportions. This perspective began to shift by the middle of the 17th century in the work of Girard, Descartes, and Harriot, all of whom began to treat polynomial equations as a product of factors (Funkhouser, 1930; Stedall, 2012, p. 50). This shift in perspective, coupled with mathematicians' growing use of symbolic algebra, supported mathematicians to make increasingly complex computations with symbols. As numerous mathematicians at the time commented, one impediment to progress on finding a general solution to higher order polynomials was the length and complexity of the symbolic computations (Stedall, 2012). Frequently, the structure that could be used to both simplify and organize this complexity was combinatorial structure (e.g., Cajori, 1912; St. George, 2003).

To illustrate this claim, I present three examples. The examples I provide are not intended to be a historical account of specific mathematicians' contributions to the development of algebra.

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[2] Plenty has been written about Cardano's and Lagrange's contributions to algebra (e.g., Branson, 2013). It is largely the period between the two that is the focus of Stedall's (2012) book.

Rather, the purpose of the examples is to make explicit the forms of reasoning that emerged during this period, that, if developed with modern students, could serve as a basis for the design of more coherent algebra experiences across grades 9-16.

The first example is a combinatorial argument for Viète's formulas. A combinatorial argument relies on treating binomial factors as a choice between an indeterminant and parameter. So, for example, in the quartic case one can expand the binomials, $(x+a)(x+b)(x+c)(x+d)$, using the following reasoning. There is 1 way to choose the indeterminant from each binomial (blue, Figure 2), 4 ways to choose the indeterminant from three of the four binomials (grey, Figure 2), 6 ways to choose the indeterminant from two of the binomials (green, Figure 2), 4 ways to choose the indeterminant from one of the binomials (yellow, Figure 2), and 1 way to choose the indeterminant from none of the binomials (orange, Figure 2). Using this structure, the

Selection Process	Outcome of Selection Process	Selection Process	Outcome of Selection Process
$(x+a)(x+b)(x+c)(x+d)$	x^4	$(x+a)(x+b)(x+c)(x+d)$	bcx^2
$(x+a)(x+b)(x+c)(x+d)$	ax^3	$(x+a)(x+b)(x+c)(x+d)$	bdx^2
$(x+a)(x+b)(x+c)(x+d)$	bx^3	$(x+a)(x+b)(x+c)(x+d)$	cdx^2
$(x+a)(x+b)(x+c)(x+d)$	cx^3	$(x+a)(x+b)(x+c)(x+d)$	$abcx$
$(x+a)(x+b)(x+c)(x+d)$	dx^3	$(x+a)(x+b)(x+c)(x+d)$	$abdx$
$(x+a)(x+b)(x+c)(x+d)$	abx^2	$(x+a)(x+b)(x+c)(x+d)$	$acdx$
$(x+a)(x+b)(x+c)(x+d)$	acx^2	$(x+a)(x+b)(x+c)(x+d)$	$bcdx$
$(x+a)(x+b)(x+c)(x+d)$	adx^2	$(x+a)(x+b)(x+c)(x+d)$	$abcd$

Figure 2. Combinatorial expansion of four binomials

final polynomial can be written as: $(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+dc)x^2 + (abc+abd+acd+bcd)x + abcd$. This combinatorial structure can, then, be used to link the roots and coefficients of a polynomial because $r_1 = -a$; $r_2 = -b$; $r_3 = -c$; $r_4 = -d$, which yields the following: $x^4 - (r_1+r_2+r_3+r_4)x^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2 - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x + r_1r_2r_3r_4 = x^4 + Ax^3 + Bx^2 + Cx + D$. This yields that the negative sum of the roots is the coefficient of the cubic term (i.e., $A = -(r_1+r_2+r_3+r_4)$); the positive sum of all possible combinations of two of the roots is the coefficient of the quadratic term (i.e., $B = (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)$), etc.

One important observation about Viète's formulas is that the expressions in the roots that make up the coefficients (i.e., $-(r_1+r_2+r_3+r_4)$) are all symmetric where a symmetric expression is one whose value is invariant under all permutations of the roots. For example, swapping r_1 and r_3 so that $-(r_1+r_2+r_3+r_4)$ becomes $-(r_3+r_2+r_1+r_4)$ does not change the value of the expression. The expressions in Viète's formulas are called the elementary symmetric polynomials, when they are expressed in *n* indeterminants (Figure 3). During the 17th

$$\sum_{1\le i_1\le n} x_{i_1} = x_1 + x_2 + \cdots + x_n$$
$$\sum_{1\le i_1<i_2\le n} x_{i_1}x_{i_2} = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$$
$$\sum_{1\le i_1<i_2<i_3\le n} x_{i_1}x_{i_2}x_{i_3} = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_{n-2}x_{n-1}x_n$$
$$\vdots$$
$$\sum_{1\le x_{i_1}<x_{i_2}\ldots\le x_{i_n}} x_1x_2\ldots x_n = x_1x_2\cdots x_n$$

Figure 3. The elementary symmetric polynomials

century, mathematicians worked on problems in which they generated other symmetric expressions, e.g., determining when two cubic equations shared a common root (St. George, 2003). Such problems led to the fundamental theorem of symmetric polynomials (FTSP) (Blum-

Smith & Coskey, 2017; Tignol, 1988/2016) and the Newton-Girard identities[3] (Cajori, 1912; Funkhouser, 1930). The FTSP states that any symmetric polynomial can be expressed as a polynomial in the elementary symmetric polynomials.

The second example I use to illustrate combinatorial structure is a case of the FTSP. This example continues with a quartic polynomial, $x^4 + Ax^3 + Bx^2 + Cx + D$. For convenience, I introduce the notation $\sum x_1^2 x_2 x_3$ to mean the sum of all possible combinations that take the form $x_1^2 x_2 x_3$ in four indeterminants (i.e., $(x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + \cdots + x_4^2 x_2 x_3)$). The FTSP states that it is possible to express this sum in terms of the elementary symmetric polynomials. Before accomplishing that goal, the sum itself can be understood through combinatorial structure; there are 4 choices for the indeterminant that will be squared, and $\binom{3}{2}$ choices for the remaining two indeterminants, which yields a total of $4 \cdot \binom{3}{2} = 12$ summands in $\sum x_1^2 x_2 x_3$.

Having identified a combinatorial structure of the sum, the goal is to find an expression in the elementary symmetric polynomials for the symmetric expression $\sum x_1^2 x_2 x_3$. We start with a product of elementary symmetric functions, $(x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)$, that produces partial products that take the appropriate form, $x_1^2 x_2 x_3$. Here again combinatorial structure can help to identify the number and form of the partial products. There are 4 summands in each of two polynomials so the product of the two polynomials will have $4^2 = 16$ partial products. Moreover, there are, for example, 3 expressions in the second polynomial that contain the indeterminate, x_1, and 1 expression that does not contain, x_1. Therefore, when we multiply the second polynomial by the indeterminate x_1 we will get these 3 expressions, $x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3$, and this 1 expression $x_1x_2x_3x_4$, a structure that will be repeated for each of the four indeterminants, giving the 12 summands of the form, $x_1^2 x_2 x_3$ and 4 summands of the form, $x_1x_2x_3x_4$. Thus, we can write $\sum x_1^2 x_2 x_3 = (x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) - 4(x_1x_2x_3x_4) = \sum x_1 \cdot \sum x_1x_2x_3 - 4\sum x_1x_2x_3x_4$. This example illustrates multiple ways that combinatorial structure is useful to reasoning about a case of the FTSP.

Lagrange put the FTSP to work in his *Réflexions* where he provided an analysis of known methods for the solution of cubic and quartic equations (Barnett, 2017; Stedall, 2012). All of the methods he analyzed relied on using a resolvent equation, a secondary equation that allowed for resolution of the first (e.g., Cardano's method relied on a sextic equation that was quadratic in the indeterminant). Lagrange's central insight in this process was to determine the relationship between the roots of the resolvent equation and the roots of the primary equation to be solved (Edwards, 1984). From this analysis, he showed that the solution of a quartic equation could be reduced to the solution of a cubic equation precisely because there was an expression in the roots of the quartic equation that took on *three* values (Tignol, 1988/2016, p. 127-128). One way to understand this observation is that any expression in the roots of a quartic equation that takes on three values under all permutations of the expression can be used to reduce a quartic equation to a cubic equation.[4] As part of this reduction the FTSP must be used to write the coefficients of the cubic equation in terms of the quartic equation.

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[3] The Newton-Girard identities are recursive formulas that can be used to express the sum of powers of roots/indeterminates (e.g., $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_1r_3 + r_2r_3)$. Practically, one use for these identities is that it is possible to express the sum of powers of the roots in terms of known coefficients without having to find the roots.

[4] Ruffini seems to have taken this observation as a starting point for his proofs of the insolvability of the quintic. With this as a starting point, he was able to show that no function in 5 indeterminants could take on 3 or 4 values

The third example, then, that I use to illustrate the role of combinatorial structure is the reduction of a quartic to a cubic equation, using the expression, $x_1x_2 + x_3x_4$, where the four indeterminants are assumed to be the roots of the quartic equation, $x^4 + Ax^3 + Bx^2 + Cx + D = 0$ (Cajori, 1912, p. 90-91). The expression $x_1x_2 + x_3x_4$ can take on the following three values under permutation of the indeterminants, $t_1 = x_1x_2 + x_3x_4$; $t_2 = x_1x_3 + x_2x_4$; $t_3 = x_1x_4 + x_2x_3$, an observation that we consider to be a form of combinatorial structure. It is, then, possible to write the following cubic equation, $(x - t_1)(x - t_2)(x - t_3) = x^3 - (t_1 + t_2 + t_3)x^2 + (t_1t_2 + t_1t_3 + t_2t_3)x - t_1t_2t_3$. Doing so uses a similar combinatorial structure to the first example. The expressions in the t_i are now symmetric, which means the expressions in the x_i are symmetric as well, a conclusion that involves a recursion of combinatorial structure. Figure 4 shows the relationship between the t_i and x_i. The second example we provided demonstrated how the FTSP could be used to express $t_1t_2 + t_1t_3 + t_2t_3 = \sum x_1^2x_2x_3$ in terms of the elementary symmetric polynomials. The FTSP can be used to derive the other two relationships shown in Figure 4, and doing so entails using combinatorial structure. Once each of the elementary symmetric expressions in the t_i are written as symmetric expressions in the x_i the following cubic equation can be written in terms of the coefficients of the quartic equation, $x^3 - Bx^2 + (AC - 4D)x - (A^2D + C^2 - 4BD) = 0$. The solvability of the original quartic equation, then, can be reduced to the solvability of the cubic equation with the addition of a fourth equation, namely that $D = x_1x_2x_3x_4$ because that yields a system of four equations in four unknowns.

$$t_1 + t_2 + t_3 = \sum x_1x_2$$
$$t_1t_2 + t_1t_3 + t_2t_3 = \sum x_1^2x_2x_3$$
$$t_1t_2t_3 = \sum x_1^3x_2x_3x_4 + \sum x_1^2x_2^2x_3^2$$

Figure 4. The relationship between the t_i and x_i

Discussion, Contribution, and Conclusion

In Lagrange's *Reflexions*, he commented that the FTSP was "self-evident," and provided no proof of it (translated in Tignol, 1988/2016, p. 95; Lagrange, 1770, 1771, Art. 98, p. 372). Tignol interprets this statement in two ways—both that the proof is not difficult and that among 18th century mathematicians the FTSP was commonly used in the solution of a range of problems. The latter interpretation is of importance to the argument in this paper; namely, that combinatorial reasoning with algebraic symbols was a foundational form of reasoning, even if taken for granted, that became commonplace during the 16th to 18th century as mathematicians sought the solution of higher degree polynomial equations. Moreover, an analysis of historical sources shows that this kind of reasoning was much broader than its use in Viéte's formulas, and the FTSP (e.g., Derbyshire, 2006; Stedall, 2012; Tignol, 1988/2016). This foundational form of reasoning contributed to the more obvious developments that occurred in 19th century algebra. Therefore, I position it as one form of connective tissue between "algebra" as the search for a general solution to polynomial equations, and "modern algebra," as the study of structure. It defines a *kind of reasoning and corresponding structure* that could be emphasized much more intentionally in modern secondary and post-secondary textbooks. Such an emphasis has the potential to lend coherence across students' algebra experiences.

(Ayoub, 1980; Bewersdorf, 2006). This starting point was one of the flaws with his proof—taking it as a starting point assumes the roots have a certain form without proving that they must be of this form. Abel's proof of the insolvability of the quintic addresses this flaw (Pesic, 2003).

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