A Quantities-based Task for Simultaneously Constructing Limits-at-a-point and Differentials

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The limit is a key calculus concept, but research has clearly shown how important differential-based reasoning is, as well. Thus, calculus education would benefit from paradigms that construct both in a way that they can coherently co-exist. In this preliminary report, we discuss a small piece of a larger project meant to build one possible paradigm. Here, we describe a single quantities-based task that is meant to develop both limits and differentials together, in the same activity. We also provide preliminary results on two pairs of students' emerging understandings.

Keywords: calculus, limits, differentials, quantities

Much work in calculus has examined the limit as a fundamental concept both for calculus, and for further mathematical study (Cornu, 1991; Davis & Vinner, 1986; Oehrtman, 2009; Viirman et al., 2022). Yet, much work has also demonstrated the benefits that reasoning about differentials, such as *dx* and *dt*, as legitimate quantitative objects affords for first-year calculus (Ely, 2021; Jones & Ely, 2023; Oehrtman & Simmons, 2023). Based on this research record, it appears that the best approach to first-year calculus may be to construct paradigms that can simultaneously, and coherently, include both limits *and* differentials. While a couple of potential approaches exist (Thompson & Dreyfus, 2016), these particular approaches tend to be at odds with the way that differentials are described in the research literature, and how differentials are typically used in STEM fields. Thus, continued efforts are needed in this area, to build differentials-and-limits paradigms that better align with the research record and calculus' client STEM fields. We are currently engaged in building one such possibility, though we readily state others could create distinct paradigms. In this *preliminary* report, we focus on a piece of this larger project, namely a single task meant to introduce limits and differentials simultaneously, in the same activity. Our contribution is to begin a conversation on how these ideas might be constructed in tandem, but in ways that they still match the growing body of research in this area.

"Preliminary" Nature of this Report

Our project has previously constructed a theoretical framework for defining differentials inside of a broader limits structure. However, the tasks we have constructed for learning differentials-within-limits are only beginning to be examined. At the very start of our learning trajectory, we have two quantities-based tasks meant to build both ideas simultaneously, and we are only now testing these with students. We discuss one of these tasks in this report. Our analysis plan and our results are therefore still preliminary, and we seek feedback in these areas.

Background Literature

A major theme in limits research is that a limit consists of both a *process* and an *object* (Cottrill et al., 1996; Dubinsky et al., 2005; Gray & Tall, 1994; Kidron & Tall, 2015; Sfard, 1991). A process is a dynamic action that one can imagine happening. There are actually two main distinct processes one can imagine for limits. First, for the limit $\lim_{x \to a} f(x)$, one can imagine *x* sweeping through the real numbers on its way to becoming closer and closer to *a* (Cornu, 1991; Davis & Vinner, 1986). Second, one could instead imagine indefinitely zooming in on the graph of *f* in the neighborhood of $x = a$ (Ely & Ellis, 2018; Ely & Samuels, 2019). These processes can lead to some outcome that can now be perceived as its own, self-existing entity, which is called the limit *object* (Sfard, 1991). That is, if we see that $f(x)$ hones in on (or remains at) some value L, we can return that value as the outcome and consider that value as its own object. In fact, we can even see the limit expression $\lim_{x \to a} f(x)$ as representing that value (Gray & Tall, 1994). One problem that students tend to have with the limit process and object, however, is thinking that the quantities in the limit process "collapse" to nothing as the limit "reaches" the object (Oehrtman, 2009). For example, a student might think that a shrinking time interval has vanished to nothing.

Next, a growing body of research has identified many benefits in reasoning with differentials (e.g., *dx* or *dt*) (Dray & Manogue, 2010; Ely, 2021; Jones, 2015; Oehrtman & Simmons, 2023). In this work, a differential is a quantitative object that represents a "tiny bit" of a quantity. While this definition is fairly informal, it is precisely its informal nature that is powerful for calculus education, and it is typically how differentials are understood and used across STEM fields (Amos & Heckler, 2018; Hu & Rebello, 2013; Nguyen & Rebello, 2011; Von Korff & Rebello, 2012). Such understandings allow improved quantitative reasoning for derivatives and integrals, and are also necessary for making sense of scientific relationships stated directly in terms of differentials, such as $dU = TdS - PdV$. Attempts at formally defining differentials can bring in too much theoretical baggage for first-year calculus, just as the formal ε-δ definition of limits has done. Consequently, our framework is based on these *intuitive, informal* ideas for differentials.

Research on differentials also describes that there are two distinct types of differentials. Von Korff and Rebello (2014) described that some differentials have a "change" nature, such as the differential *dt* representing some tiny change in time. On the other hand, some differentials have an "amount" nature, such as *dm* representing a tiny bit of mass (and *not* a change in mass).

Because of the power of differentials in calculus and related STEM fields, some researchers have discarded limits altogether, using only differentials (Dray & Manogue, 2010; Ely, 2021; Keisler, 2011). The issue for us here is that these paradigms essentially require non-standard hyperreals, which are not generally utilized in the mathematics community. Further, limits are an important foundation of further mathematics study, and we think calculus students should develop an understanding of them. Also, doing away with limits makes it harder to convince the wider undergraduate teaching community to adopt differentials in curriculum. Thus, we seek to build a paradigm where limits and differentials comfortably coexist in the same classroom. While some paradigms do exist that treat both limits and differentials (Oehrtman & Simmons, 2023; Stevens & Jones, 2023; Thompson et al., 2013; Thompson & Dreyfus, 2016; Weber et al., 2012), there are two ways in which work is still needed. Some of these paradigms have not explicitly articulated exactly how a differential should be defined in relation to the limit process, nor what the exact relationships between a limit and a differential is. Other paradigms have defined differentials in a way that does not align well with the large and growing research work on differentials. We want a paradigm that not only allows differentials and limits to co-exist, but where the differentials have the meaning and reasoning power described in the research record. Our project is meant to supply one possible approach for doing so.

Theoretical Framework: Differentials-within-limits

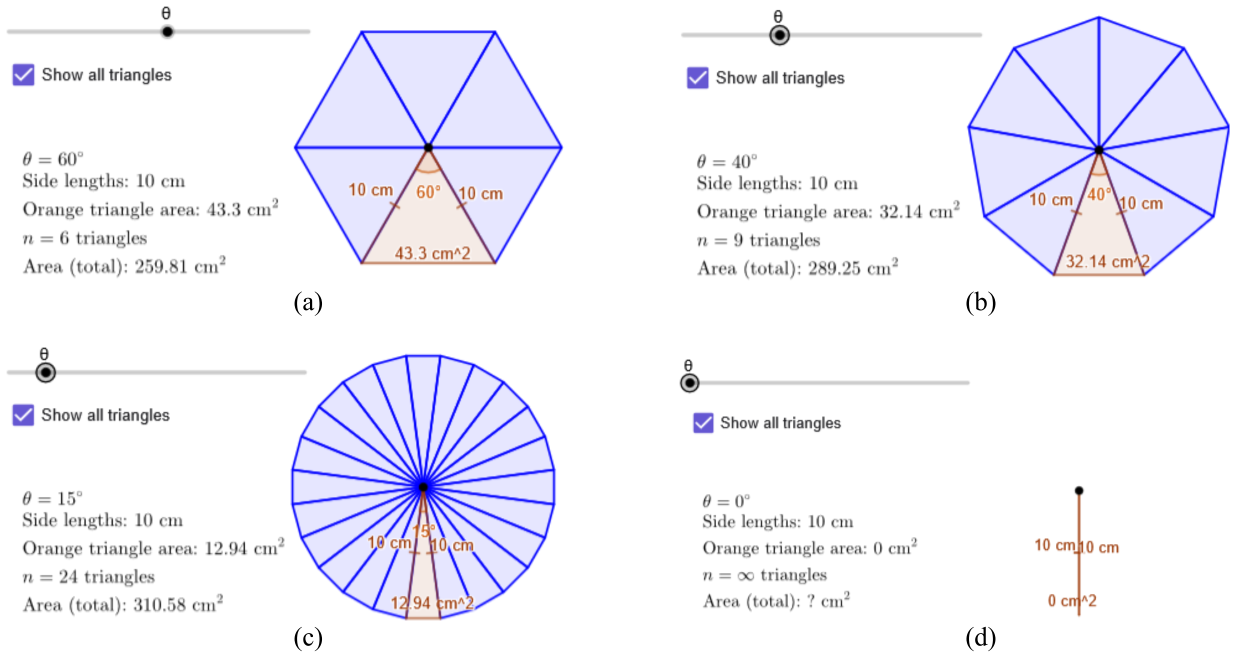
Our project previously developed a theoretical framework for conceptually defining differentials within a broader limits architecture that matches the research on differentials. In our framework, a differential is defined as a secondary object that results from *within* the limit *process*. In other words, we do not simply see our approach as differentials-*and*-limits, we see it as differentials-*within*-limits. Table 1 provides an abbreviated version of our framework, including some basic definitions and some properties that are relevant to this particular report.

Table 1. Abbreviated version of the framework, with basic definitions and properties.

Differential, based on approaching	In the "approaching" limit process, if one perceives that a variable has become incredibly close to its target (i.e., x approaches close to a), they can "pause" the limit process at that moment and consider the remaining difference between x and a. This difference is defined as a *differential, dx*.
Differential, based on zooming	In the "zooming" limit process, if one zooms to tiny scales, they can again "pause" the limit process at some moment and consider the very tiny amount of a quantity remaining in the zooming window. This tiny bit is an alternate way to produce a *differential, dx*.
Definition #3: Output Differential	One can also create a differential for the *output* variable. In $\lim_{x \to a} f(x) = L$, as x gets close to a, or as one zooms into a graph near $x = a$, one can also imagine a change/amount differential for the output variable, *df* or *dy*.
Property #1: Subjectivity of Closeness	What is "close enough" in the limit process for it to be paused, and for a differential to come into existence, is subjective both on context and the reasoner's perspective. There is no specific point where this must occur. This subjectivity allows differentials to retain the desired informal nature.
Property #2: Lack of Specific Size	As a consequence of the previous property, differentials do not have a single, specific size or value. The differential object is *not* defined by a numeric value, as opposed to the limit object. Rather, the differential object is defined by the conceptual idea of a tiny change/amount.
Property #3: Non-hyperreal	While a differential does not have an inherent size, it still resides in the realm of very small real numbers, and is not a non-standard hyperreal.
Property #4: Contextual Assignment of Values	While differentials have no inherent size, for pragmatic reasons one can still assign a numeric value to a differential in a given situation, when context dictates, by "pausing" the limit process at a point deemed close enough or small enough and using that current state to assign a temporary value to a differential. This is a common practice in STEM fields.
Property #5: Differential as a "model"	A differential is a *conceptual model* of what happens at small scales. They represent what we might expect in the relationship between quantities, when measured or observed at small scales.

The Task

This preliminary report focuses on a single quantities-based task toward the beginning of a larger learning trajectory. The task uses a dynamic Geogebra applet in which the areas of regular polygons converge to the area of a circle (Figure 1). Of course, this context is certainly not new, and has a long history in mathematics (Katz, 2009). Yet, the use of it to simultaneously build both the limit-at-a-point concept and the differential concept is new, as far as we are aware. The task starts with a regular polygon made up of triangles with a certain interior angle, θ (Figure 1a). The students can change θ with a slider to see that smaller internal angles create polygons with more sides (Figure 1b, 1c). The main question is: "What happens to the area as we let θ go toward zero?" It is true that because θ can only take on certain values to create regular polygons, this task technically is the limit-at-a-point of a *sequence*. However, we believe that the ideas built in this task readily extend to the limit of a continuous function.



The tasks allows the students to slide θ all the way to zero, to see that the area is undefined there, since $\theta = 0$ correspond to no triangles, and hence no polygon (Figure 1d). Doing so helps address the collapse metaphor. Once the students identify that the area should converge to the area of a circle (100π cm^2), they are asked to explain the relationship between the shrinking θ and the converging area, A. Afterward, the specific limit notation is introduced, $\lim_{\theta \to 0} A = 100\pi$ cm^2. While this is technically a right-hand limit, we do not attend to that detail at this point.

Next, the students are asked: "When does the area actually become equal to 100π cm^2?" This question is meant to lead to the construction of differentials. When the students identify that there is no such point, they are asked how they decided on the limit. As the students describe very small angles, and an area very close to 100π, we explain that such small quantities can be termed "differentials." The students are then asked to describe the differentials they see in the task. The intention is for them to identify (a) a tiny θ as the (amount) differential $d\theta$ and (b) the discrepancy between the area given by a tiny angle and the assumed "limit" area as the (change) differential, dA. The students are asked how these differentials relate to the limit, and the limit value, and how they serve as a *model* for the context, even though we cannot reach $\theta = 0$.

We enacted this task within a larger set of interviews with two pairs of students recruited from the end of a Pathways College Algebra course that focused on quantitative reasoning. Thus, none of the students had taken calculus before, nor seen limits before, but they were familiar with reasoning about quantities. One pair consisted of two males and the other of two females. This task was one of two that began the interview sequence. While data analysis is ongoing, our initial analysis has focused on using our framework's definitions and properties as starting codes. We are in the process of identifying other themes to investigate further. The analysis and possible resulting outcomes is what we hope to receive feedback on in this preliminary report.

Preliminary Results

Because our initial analysis has centered on our framework's definitions and properties, our preliminary results have been organized around those themes. The following are early pieces of data that provide some evidence of understandings related to aspects of our framework.

Defining a differential: Several statements suggested an understanding of a differential as tiny bits of quantities that exist within the limit process, as exemplified in Claire's statements.

Claire: If the angle is really close to zero, then it would be a differential. And that's just how we would like… that's how we'd like represent it. Because, like, you can't have an angle of zero, but you can have something really, really close to zero.

Claire: So I think it would be, like, the differential of the area would be like the space around the circle that we haven't filled yet.

Change versus amount differentials: Notice that Claire's statements also potentially suggest both an "amount" differential for the angle and a "change" differential for the area. Her imagery of the angle seemed to be based on zooming to small angles, while her imagery for the area seemed to be based on the difference between the polygon's area and the circle's. As another example, Austin stated, "You make the angle so small," suggesting a zoom into tiny angles. His partner, Logan, later said, "So you take 100π minus whatever that [polygon] area is, and that's our differential for area," suggesting a difference between the two areas.

Avoiding collapse: The quantities-based nature of the task seemed to be extremely helpful in avoiding the collapse metaphor. While some students, at the start, thought of the triangles becoming straight lines, they rather quickly realized the issue in doing so. For example, Austin stated, "Because if it hits zero, then… you don't have, like you don't have anything." In the other group, Claire explained, "You can't make a triangle with a zero degree angle."

Subjectivity of closeness and lack of specific size: The students themselves appeared to build meanings for the differential that matches well with these two properties in our framework.

Claire: Uh, yeah, just that we were talking about a differential being anything that's like approaching, or like a number really close to zero. I don't know if we ever went like, how, said how close "close" would be. But if the angle is really close to zero, then it would be a differential… Because, like, you can't have an angle of zero, but you can have something really, really close to zero.

Contextual assignment of values: The students appeared comfortable with occasionally using a very small number to represent a differential as a model for the limit process.

Logan: I mean $d\theta$ would be like 0.0001 or something like that, kind of small number… Which isn't zero, but it's close to zero.

Conclusion and Questions for the Audience

Our early results favorably indicate that this task is successfully helping students construct differentials within a broader limits architecture. They constructed both types of differentials, and gave evidence of thinking of the several of the intended properties for differentials. Because this work is still in its early stages, the following questions will be used to solicit feedback:

1. What are your thoughts on this task as a way to simultaneously construct both limits-at-a-point and differentials? Are there any issues with it you see?
2. Do you agree that these student excerpts are evidence of the definitions/properties?
3. What other aspects of the students' thinking or understanding would you want to see?
4. What kind of disconfirming evidence would you want us to be looking for?

References

Amos, N. R., & Heckler, A. F. (2018). Mediating relationship of differential products in understanding integration in introductory physics. *Physical Review Special Topics: Physics Education Research, 14*, article #010105. https://doi.org/10.1103/PhysRevPhysEducRes.14.010105

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153-166). Kluwer Academic Publishers.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *The Journal of Mathematical Behavior, 15*(2), 167-192.

Davis, R., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *The Journal of Mathematical Behavior, 5*(3), 281-303.

Dray, T., & Manogue, C. A. (2010). Putting differentials back into calculus. *The College Mathematics Journal, 41*(2), 90-100. https://doi.org/10.4169/074683410X480195

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis: Part 1. *Educational Studies in Mathematics, 58*, 335-359.

Ely, R. (2021). Teaching calculus with infinitesimals and differentials. *ZDM – The International Journal on Mathematics Education, 53*(3), 591-604. https://doi.org/10.1007/s11858-020-01194-2

Ely, R., & Ellis, A. B. (2018). Scaling-continuous variation: A productive foundation for calculus reasoning. In A. Weinberg, C. Rasmussen, J. Rabin, & M. Wawro (Eds.), *Proceedings of the 21st annual Conference on Research in Undergraduate Mathematics Education* (pp. 1180-1188). SIGMAA on RUME. http://sigmaa.maa.org/rume/Site/Proceedings.html

Ely, R., & Samuels, J. (2019). "Zoom in infinitely": Scaling-continuous covariational reasoning by calculus students. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd annual Conference on Research in Undergraduate Mathematics Education* (pp. 180-187). SIGMAA on RUME. http://sigmaa.maa.org/rume/Site/Proceedings.html

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140.

Hu, D., & Rebello, N. S. (2013). Understanding student use of differentials in physics integration problems. *Physical Review Special Topics: Physics Education Research, 9*(2), article #020108. https://doi.org/10.1103/PhysRevSTPER.9.020108

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior, 38*, 9-28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S. R., & Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education, 9*(1), 8-35. https://doi.org/10.1007/s40753-022-00203-x

Katz, V. J. (2009). *A history of mathematics* (3rd ed.). Pearson Education.

Keisler, H. J. (2011). *Elementary calculus: An infinitesimal approach* (2nd ed.). Dover Publications.

Kidron, I., & Tall, D. O. (2015). The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics, 88*(2), 183-199.

Nguyen, D., & Rebello, N. S. (2011). Students' difficulties with integration in electricity. *Physical Review Special Topics: Physics Education Research, 7*(1), article #010113. https://doi.org/10.1103/PhysRevSTPER.7.010113

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426. https://doi.org/10.5951/jresematheduc.40.4.0396

Oehrtman, M., & Simmons, C. (2023). Emergent quantitative models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education, 9*(1). https://doi.org/10.1007/s40753-022-00209-5

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Stevens, B. N., & Jones, S. R. (2023). Learning integrals based on adding up pieces across a unit on integration. *International Journal of Research in Undergraduate Mathematics Education, 9*(1), 118–148. https://doi.org/10.1007/s40753-022-00204-w

Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*, 124-147. https://doi.org/10.1080/07380569.2013.768941

Thompson, P. W., & Dreyfus, T. (2016). A coherent approach to the Fundamental Theorem of Calculus using differentials. In R. Göller, R. Biehler, & R. Hochsmuth (Eds.), *Proceedings of the Conference on Didactics of Mathematics in Higher Education as a Scientific Discipline* (pp. 355-359). KHDM.

Viirman, O., Vivier, L., & Monaghan, J. (2022). The limit notion at three educational levels in three countries. *International Journal of Research in Undergraduate Mathematics Education, 8*(2), 222-244. https://doi.org/10.1007/s40753-022-00181-0

Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics: Physics Education Research, 8*(1), article #010125. https://doi.org/10.1103/PhysRevSTPER.8.010125

Von Korff, J., & Rebello, N. S. (2014). Distinguishing between "change" and "amount" infinitesimals in first-semester calculus-based physics. *American Journal of Physics, 82*(7), 695-705. https://doi.org/10.1119/1.4875175

Weber, E., Tallman, M., Byerley, C., & Thompson, P. W. (2012). Understanding the derivative through the calculus triangle. *The Mathematics Teacher, 106*(4), 274-278. https://doi.org/10.5951/mathteacher.106.4.0274