Defining a "Differentials-within-limits" Framework for First-year Calculus

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Work in first-year calculus has clearly shown the power that "differentials" afford for first-year calculus learning, but it also speaks of the need to build ideas of limits. While some researchers have loosely described informal differentials and limits within the same paradigm, their actual conceptual definitions and relationships have not always been explicitly constructed. Continued progress in this area could benefit from theoretical consideration of: (a) providing clearer intuitive definitions for differentials inside a limits-based approach to calculus, and (b) defining more exactly the relationships between differentials and limits. We provide here a theoretical framework for building a simultaneous differentials-and-limits approach to first-year calculus.

Keywords: calculus, differentials, limits, differentials-within-limits

There is a persistent theoretical issue facing the teaching and learning of first-year calculus. On the one hand, the research literature in recent years has painted a very clear picture of the need for first-year calculus to be heavily grounded in quantities and quantitative reasoning (Jones & Ely, 2023; Kertil & Gülbağcı Dede, 2022; Oehrtman & Simmons, 2023; Thompson & Dreyfus, 2016; Yu, 2024). These efforts are often centered on the need for *differentials* (e.g., *dx, dt*) to have quantitative meaning, which can then also interact with other quantities to, in turn, construct meanings for derivatives, *dy/dx*, and integrals, $\int_a^b f(x)dx$ (Hu & Rebello, 2013; López-Gay et al., 2015; Roundy, Dray, et al., 2015). Further, most STEM fields utilize free-standing differentials to capture quantitative relationships at small scales, such as $dI = r^2 dm$ or $dS = dQ/T$. Consequently, these paradigms have proven powerful for enabling students to develop deeper understandings of calculus and to become proficient users of calculus beyond the mathematics classroom (Amos & Heckler, 2018; Pina & Loverude, 2019; Roundy et al., 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012; Weber et al., 2012).

On the other hand, there has been a historical tension in the mathematics community between differentials and limits (Katz, 2009). The historical outcome was to discard differentials in favor of limits, leaving typical modern calculus based on limits exclusively, without differentials. It has taken considerable effort by researchers to redeem differentials to a "legitimate" status within teaching (Dray & Manogue, 2010; Ely, 2017; Martínez-Torregrosa et al., 2006; Rogers, 2005). In an interesting pendulum swing, however, some of these differentials-based reform efforts have now gone so far as to do away with limits, using (essentially) the hyperreal number system in their place. Because much of modern mathematics is based on the real numbers and on limits, this could create baggage for students in underdeveloping limits per se and in using a number system not generally accepted in the broader mathematical community.

What should calculus education do? Reasoning with differentials as quantitative objects is essential, but the limit is a necessary concept for the general mathematical community. The obvious answer seems to be to create instructional paradigms that coherently leverage *both* limits and differentials, yielding the benefits of each. In fact, some calculus education researchers have more-or-less done this, using approaches that in essence contain both differentials and limits (Jones, 2015; Oehrtman & Simmons, 2023; Weber et al., 2012). The problem is that some of this work has been done without careful attention to exactly *what* a differential is within a limit structure, and what its *properties* and its *relationship* to a limit are. We admit that our own prior work falls into this category, and we are guilty of this same laxity. Additionally, other paradigms that have tried to define differentials more explicitly (Thompson et al., 2013; Thompson & Dreyfus, 2016) tend to not cohere very well with the growing body of work on differentials in math and STEM education. Similarly, paradigms that view differentials only as "linear approximation" tend to miss out on imbuing expressions like *dy/dx* with direct quantitative meaning – a major benefit of differentials. We find that continued progress could benefit from building clearer theoretical frameworks on a differentials-and-limits approach that is also built on and compatible with the body of work in this area. Thus, in this *theoretical* paper, we contribute by supplying one possible set of definitions and relationships. To be clear, our framework is focused on *intuitive, conceptual* definitions for use in first-year calculus, rather than on *formal mathematical* definitions used for proof in the mathematics community. We anticipate that one could disagree with our specific definitions and properties, and we welcome discourse in this area, and we welcome others providing alternate definitions or improving upon ours.

Background

While we aim to add theoretical clarity to the simultaneous use of differentials and limits, we most certainly want our framework to stay true to the important work that has been done on differentials so far. The major theme in the literature is that a differential is an informal construct, representing a "tiny bit" or an "infinitesimal bit" of a quantity (Amos & Heckler, 2018; Hu & Rebello, 2013; Jones & Ely, 2023; López-Gay et al., 2015; Von Korff & Rebello, 2012). In STEM education, *dt* might refer to a tiny bit of time or *dA* might refer to a tiny bit of area. Many STEM educators intentionally leave differentials as an intuitive idea, without providing any formal mathematical definition. For example, Von Korff and Rebello (2014) state that they use "*d*" to mean something "vanishingly small, without giving a mathematical definition of what it means to be… vanishingly small" (p. 698). The main reason for doing so is that a formal definitions from within non-standard analysis can be as problematic for first-year calculus as the formal ε-δ definition of the limit has been. Rather, it is the informal notion that affords so much reasoning power (Jones & Ely, 2023). Simply put, differentials for first-year calculus work best when kept *intuitive* and *informal*, and we adopt this stance for our framework.

Based on the power of differentials, a natural question is: Why not use only differentials and abandon limits altogether? Some have actually taken this approach, using what is essentially a less formal version of non-standard analysis and its hyperreal number system (Dray & Manogue, 2010; Ely, 2021; Keisler, 2011). In this camp, a differential is a literal "infinitesimal": a non-zero number smaller than all real numbers. While this approach certainly has solid mathematical footing thanks to the work of Abraham Robinson's group in the 1960s (Robinson, 1961), the issue for us with this approach is that non-standard analysis is not generally utilized in the mathematical community and most modern mathematics is founded on the real numbers instead. We are concerned with developing a number system in our students' minds that can, and probably will, clash with subsequent mathematics study. It also makes using differentials a much harder "sell" to the broader mathematical community, impeding progress toward adopting differentials in instruction. Thus, our framework does *not* use a hyperreal version of differentials. Rather, out intent is for differentials to be a construct that can readily cohere with limits.

Next, we note that the research literature on differentials describes two distinct possible conceptualizations of differentials. Von Korff and Rebello (2014) described that some differentials have a "change" nature, such as *dt* representing a tiny change in time. On the other hand, they also described that some differentials have an "amount" nature, such as *ds* representing a tiny length on a curve. This distinction is important, because both types are used across STEM fields. Integrals such as $\int_R \rho\, dV$ and $\int_R r^2 dm$ make use of differentials that can be thought of as a tiny bit of volume (*dV*) and a tiny bit of mass (*dm*).

Finally, as quantitative objects, differentials can interact with other differentials or quantities to produce the key constructs of derivatives and integrals. A derivative can be seen as the ratio of tiny changes represented by two differentials, *dy* and *dx*, to produce *dy/dx* (Roundy, Dray, et al., 2015). An integral can be viewed as summing the products between a differential quantity and a "macroscopic" quantity, $f \cdot dx$, to produce $\int_a^b f(x)dx$ (Jones & Ely, 2023) – though other relationships beyond products are also possible (Ely, 2017; Simmons & Oehrtman, 2017).

Situating Our Theoretical Contribution

As mentioned, several researchers have already discussed approaches to calculus that essentially use both limits and differentials in the same paradigm (Jones, 2015; Oehrtman & Simmons, 2023; Weber et al., 2012). In these approaches, limits are examined in their own right, and derivatives and integrals are still defined as limits: $\lim_{\Delta x \to 0} \Delta y/\Delta x$ and $\lim_{\Delta x \to 0} \sum f(x_i)\Delta x$. But once the differential forms are provided, *dy/dx* and $\int_a^b f(x)dx$, the differentials are allowed to have meaning as tiny changes or amounts of a quantity. These approaches have proven powerful, yet the issue is that the co-existing nature of differentials and limits has not been carefully and explicitly articulated. On the other hand, when attempts have been made at articulating their interrelations (Thompson et al., 2013; Thompson & Dreyfus, 2016), the definitions have not been very compatible with the growing body of STEM research on differentials described in the previous section. Additionally, another attempt we have seen at describing the relationship was given as $dx = \lim_{\Delta x \to 0} \Delta x$, but such a definition has a major flaw. The value of that limit is unambiguously zero, meaning that all differentials would be size zero. Such thinking is akin to the problematic "collapse metaphor" described by (Oehrtman, 2009), wherein all differentials collapse to nothing and essentially disappear from the quantitative relationships altogether.

We wish to produce a framework for differentials and limits that (a) provides theoretical clarity, while (b) still aligning with and strengthening the important work done in this area in recent years. This theoretical paper contributes by describing one such possible theoretical framework to cohere differentials with limits. To be clear, we reiterate that our framework is meant to propose *intuitive, conceptual* definitions for first-year calculus education, and not *formal mathematical* definitions intended for proof.

The Basis of the Framework: Processes and Objects in the Limit Concept

Our theoretical framework for differentials and limits is based on Sfard's (1991, 1992) idea of processes and objects. Of course, many researchers have discussed processes and objects, including Arnon et al. (2014), Gray and Tall (1994), and others, but for our current purposes we only need the general idea of these processes and objects, as found in Sfard's work.

A *process* is defined as a dynamic action that can be imagined as happening. For example, one potential *process* for a limit, $\lim_{x \to a} f(x)$, is that of *x* sweeping through numbers as it moves closer and closer to *a* (Cornu, 1991; Fernández-Plaza et al., 2013; Oehrtman, 2009) (see Table 1). Yet, a different *process* that can be conceptualized for a limit is graphically zooming over and over toward a section in the neighborhood of *x* = *a* (Ely & Ellis, 2018; Ely & Samuels, 2019) (see Table 1). Notice that these two intuitive processes are both, essentially, "never-ending" (Davis & Vinner, 1986), because one cannot literally enact *x* passing through infinite real numbers on its way to a, nor can one zoom in a true infinite amount to a single point within the real numbers (see Table 1).

Next, a process can result in some kind of outcome that can be thought of as an entity in its own right. For limits, as x gets closer and closer to a, the function $f(x)$ might hone in on a single numeric value, L (or remain steady at L). This outcome can be conceptualized as its own entity, and when it is, it becomes an *object* existing in the person's mind (Sfard, 1991) (see Table 1). The entire symbolic expression $\lim_{x \to a} f(x)$ can now be seen as referring to this object, "L" (Gray & Tall, 1994). These objects can then be used in further processes, creating even more objects.

Table 1. Pre-existing definitions from the literature regarding limit processes and objects.

Idea	Definition
Limit Process, Approach	One type of limit process is dynamically *approaching*. For example, in $\lim_{x \to a} f(x)$ we can hold an image of x getting continuously closer to a.
Limit Process, Zoom	A distinct limit process is dynamically *zooming*. In $\lim_{x \to a} f(x)$, we can imagine of zooming forever into a portion of a graph around $x = a$.
Limit Process as Never-ending	There is never a moment at which the limit process "completes." While it results in an object, the process itself never attains a completed state.
Limit Object	The limit object is the outcome that results from the limit process. It is defined as the value we assign to the limit, as in $\lim_{x \to a} f(x) = L$.

Defining Differentials *Within* Limits

With these conceptual ideas for limits in place, we can now *conceptually* define a differential as a secondary object that also is produced from *within* the limit process. In other words, we do not simply see differentials *and* limits together, but we see differentials coming into existence *within* the limit process. Table 2 contains our definitions, which are intentionally kept informal for first-year calculus. They are not meant to be treated as formal mathematical definitions.

Table 2. Definitions of differentials as secondary objects from within the limit process..

Idea	Definition
Differential, based on approaching	Consider the approaching imagery for the limit process in $\lim_{x \to a} f(x) = L$. Once someone perceives that the input variable x has become incredibly close to its target a (even if never reaching it), they can "pause" the limit process at that moment and consider the remaining difference between x and a. In other words, the differential is the *object* produced by "pausing" the approaching process once x is extremely close to a, and by considering the remaining discrepancy.
Differential, based on zooming	Consider the zooming imagery for the limit process instead. As the zooming continues to tiny scales, one can again "pause" the limit process at some moment and consider the very tiny "amount" of a quantity remaining in the zooming window. Just as before, the differential is the *object* produced by "pausing" the zooming process once it has reached very small scales, and by considering the tiny bit in the zooming window.
Output Differential	One can also create a differential for the *output* variable. In $\lim_{x \to a} f(x) = L$, as x gets close to a, or as one zooms into a graph near $x = a$, one can also imagine a differential exiting for the output variable, df or dy.

Properties of Differentials in This Paradigm

Having provided our baseline conceptual definitions, we now provide specific properties that informal differentials have in our framework (Table 3).

Table 3. Properties of differentials in our framework.

Idea	Definition
Subjectivity of Closeness	What is "close enough" in the limit process for it to be paused, and for a differential to come into existence, is subjective both on context and the reasoner's perspective. In a scientific context, a differential often comes into existence once reaching the threshold of measurement tools, or when further measurement yields a "linear regime" (Roundy, Weber, et al., 2015). For math functions, a differential might exist once the error is in such remote decimal places that they contribute very little to the variable's value.
Lack of a Specific Size	As a direct consequence of the previous property, differentials do not have a single, specific size nor value. That is, the differential object is *not* defined by a numeric value, as opposed to the limit object. Rather, the differential object is defined by the conceptual idea of a tiny remaining change/amount.
Non-hyperreal	While a differential does not have an inherent size, it still resides in the realm of very small real numbers, and is *not* a non-standard hyperreal.
Contextual Assignment of Values	While differentials have no inherent size, for pragmatic reasons one can still assign a numeric value to a differential in a given situation, when context dictates, by "pausing" the limit process at a point deemed close enough or small enough. Upon pausing the process, one can imagine using that current state to assign a temporary value to a differential. This is actually a common and important practice in many STEM fields, which use measurements at small scales to produce contextual values for differentials (Roundy, Weber, et al., 2015). How small is small enough is context-dependent; in one situation a time interval of 0.01 seconds might be a "differential" amount of time based on that context, where in another situation a time interval of 0.01 seconds might be too big to be considered a "differential" amount of time.
Zero Value for Output Only	For $\lim_{x \to a} f(x) = L$, if $f(x)$ attains and remains at L at some point in the limit process, the differential df (or dy) is zero, a legitimate value for the *output* variable differential only. The *input* variable differential (dx) cannot be zero
Inheriting Quantitative Properties	Differentials automatically inherit all the quantitative properties of the quantity represented by that variable. For example, if F represents a force measured in Newtons, then the differential dF is automatically also a force, presumably also measured in Newtons (Hu & Rebello, 2013).

How is This Framework Different From Previous Work?

One could ask: Since this framework is also based on informal, conceptual ideas, does it add anything to what already exists in the literature? The main difference we see is that previous work tends to define differentials as "tiny bits" without giving much more theoretical clarity on how such tiny bits connects coherently to limits. Our framework clearly states one possibility where differentials come into existence as objects and what their relationship to limits is. By articulating them explicitly, we are better positioned to construct instructional activities to simultaneously build limits and differentials in the same calculus classroom.

Differentials-within-limits in Action: Differentials as "Models"

Our framework used the notions of limit processes and objects to provide a conceptual definition for differentials, and to lay out several conceptual properties of differentials. The next big question for our framework is what implications these definitions and properties have for first-year calculus teaching. These implications can be encapsulated through the idea of a mathematical *model*. In brief, a model is a mathematical description of an object or system, even if the model is not a perfect representation of the object or system being studied (for more on modeling, see Blum et al., 2007; Cai et al., 2014). We consider a "differential" to be precisely this – a *conceptual model* of what happens at small scales. To illustrate, imagine a string that produces a pitch when plucked, where a shorter length (L) corresponds to a higher pitch (f). Then consider the limit, $\lim_{L \to 0+} f$. In the limit process, we can think of the string becoming increasingly short and imagine "pausing" the process once we've reached some incredibly small length. This tiny remaining length is what we have defined to be a differential, dL, and it would correspond to a very high pitch. While we can never truly make the string be of length zero, because there would no longer be a string nor a pitch, we can model the phenomenon of the limit process by considering what happens as the string length becomes differential-sized. The dL is a *model* of an ever-shrinking string length, paused at a specific moment, and its relation to the output variable (f) can be examined within the differential model to generate expectations of what the limit value might be (i.e., ∞ or a finite value). The limit process can then be resumed and paused again, and the model dL can be reexamined.

Importantly, differentials are commonly used in this way across the sciences. Differentials such as dm, dI, dF, and dU are models of systems dealing with small masses or small energies, which can be given values through small-scale measurements (Roundy, Weber, et al., 2015).

Derivatives and Integrals as "Best Models" Within This Framework

Because derivatives and integrals are also defined as limits and use differential notation, our framework, including the idea of models, also extends to these concepts, though in a specific way. Derivatives and integrals are not only models, but are what we term *best models*.

To explain, in our definitions and properties for differentials, dy/dx would be a literal ratio of two tiny changes or amounts. But, there is a key theoretical issue to resolve. Take, for example, $y = x^2$. The derivative evaluated at $x = 3$ should be $dy/dx = \lim_{x \to 3} (x^2 - 3^2)/(x - 3) = 6$. The issue is that no matter where the limit process is paused, there is *no* specific point at which a literal ratio of tiny changes gives exactly "6." Even if the limit is paused when x is a distance of 0.0001 of 3, the tiny changes at that moment give 0.00060001/0.0001 = 6.0001, which is not exactly 6. How do we resolve this problem in our framework? First, differentials in our framework have no specific inherent size, and assigning a specific value like this would be done only for pragmatic, contextual reasons. Thus, we would not necessarily claim that dy/dx = 0.00060001/0.0001, unless the temporary assignment of those values is done for some practical reason, such as in experimental measurement. Even so, it is still a good model for what the derivative ultimately would be. Yet, there remains an uncomfortable mathematical issue in how we can state that $dy/dx = 6$, rather than $dy/dx \approx 6$, when we define a differential through pausing the limit process. This is where the idea of a *best model* comes in.

The derivative is fundamentally meant to answer the question: "What is the *single* numeric value that best represents the idea of an instantaneous rate of change at a given point?" In other words, we want a *best model* for the tricky "instantaneous rate" question. The answer certainly isn't 6.0001, because if we un-pause the limit process and re-pause it later, we would have a new, different discrepancy between x and a, and a different momentary ratio, say 6.0000001. Thus, none of these numbers functions as a single *best model* for the instantaneous rate, because unpausing the limit process will always refine our ratio past it. Since the differentials *dy/dx* are meant to *conceptually model* any arbitrary pause in the limit process, there is only one single number that encapsulates the whole set of ratios we could get by pausing at different moments – the value "6." That is, the limit object (the other object produced in the limit process) serves as this best model. Here is where the lack of inherent size for differentials is a benefit in our framework. We can say *dy/dx* = 6, because there are no specific values that *dy* and *dx* each have, and 6 represents the single *best model* we can construct for ratios of very tiny changes (i.e., an instantaneous rate). This also allows us to make statements such as *dy* = 6*dx*, because this statement is a *best model* for what to predict for an arbitrarily small change in *x*. If one imagined some miniscule change in *x* (*dx*) and tried to predict how much the output variable changed, then without specifying any particular size or value for *dx,* the single *best model* is that *dy* would be 6 times as large as dx (Yu, 2020).

Integrals function in a similar way. The definite integral fundamentally addresses the question: "How much of a quantity do we have, when variation in one of the quantity's constituents prevents using a simple relationship?" Take the example of water flowing through a pipe. If it had a constant rate, then given some elapse in time, we could calculate the amount that flowed through the pipe as $A = r \cdot t$. This is what Oehrtman and Simmons (2023) call a *basic model*. But, suppose the rate is not constant. We might then imagine the limit process of zooming into smaller and smaller time intervals, until the rate appears essentially constant. If we pause at that point, we create a differential time interval, *dt*, and a differential amount of water in that time interval, $dA = r \cdot dt$. Oehrtman and Simmons (2023) call this a *local model*. This local model is precisely the *best model* for how much water we have in that differential time segment. Of course, zooming can never reach a point where the variation in the rate is totally eliminated, but since differentials have no specific size, they are a *conceptual representation* of the best model for constructing a tiny bit of A in each piece. No other model functions better for capturing the amount of A. This is why we assign an equals sign for an integral value, $\int_a^b f(x)dx = I$, rather than $\int_a^b f(x)dx \approx I$, because no other number is a better model for what we should get for the total amount of the quantity over the given interval.

Conclusion

To recap, we had noticed that there was a need in calculus education research to construct an explicit framework for using informal differentials within a limits-based approach that coheres with the large body of work that is being done in differentials, integrals, and derivatives. In other words, we did not want a framework that would run counter to this important work. Using the literature as our launching point, we laid out both conceptual definitions and conceptual properties of differentials *within* the broader limit architecture. We then elaborated on the implications of this framework for instruction, based on the idea of *models* (for differentials, generally) and of *best model* (for derivatives and integrals, specifically). Of course, ours is one possible framework and we welcome discourse in this area. We believe that others' ideas could contribute to theoretically refining our framework, or creating alternate theoretical frameworks. Regardless, we believe that continued progress in calculus education research will benefit from having these explicit frameworks to work with. They can lead to instruction that better articulates for students what a differential is and how it fits within limits, so that students are able to learn *both* limits *and* differentials within a single course, leveraging the benefits of each.

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