Connecting Multiplicative Structures Across Contexts: Students' "Product Layer" Reasoning in Water Pressure and Electricity Tasks

Olha Sus
Tufts University

Andrew Izsák
Tufts University

*Students often struggle to interpret the multiplicative structure within integrals—what we call the "product layer" ($f(x) \cdot dx$)—when reasoning about applied problems. In this study, we analyze the reasoning of Melinda, an undergraduate in a calculus-based physics course, as she works on two tasks: estimating the total force on a tank wall (water pressure) and the total voltage across a capacitor (electricity). Using conceptual blending theory, we examine how Melinda connects symbolic, mathematical, and physical ideas when setting up integrals. Melinda frequently interprets integration as summation of small pieces or as area under a curve but struggles to coordinate multiplicative relationships among varying quantities (e.g., pressure· area, current· time). These findings suggest that challenges with product layer reasoning persist across contexts, highlighting the need for instruction that explicitly supports blended multiplicative reasoning.*

*Keywords:* definite integral, conceptual blending, introductory physics, knowledge resources, product layer

Reasoning about integrals in applied contexts is central to success in mathematics and physics, yet students often struggle to connect symbolic procedures with physical meaning (Brahmia et al., 2016; Wilcox & Corsiglia, 2019). These difficulties are especially pronounced when interpreting the product layer—the multiplicative structure within an integral (Hu & Rebello, 2013a; Sealey, 2014). Applied contexts such as fluid mechanics and electricity require students to coordinate two simultaneously varying quantities and relate them to physical phenomena, rather than merely computing antiderivatives.

Despite its importance, students' reasoning about the product layer remains poorly understood, particularly across different contexts. The present study addresses this gap by examining how an undergraduate student constructs and coordinates symbolic, mathematical, and physical ideas when working with integrals in two applied settings. By comparing reasoning patterns across contexts, we aim to identify both persistent challenges and context-specific features that influence product layer reasoning.

This study is guided by two research questions:

1. *How do students reason about the product layer within integrals solving water pressure and electricity tasks?*
2. *How do symbolic, mathematical, and physical spaces interact across these contexts, and what roles do blended spaces play in shaping students' reasoning?*

## Background and Literature Review

Students frequently struggle with applied integration, particularly in understanding the multiplicative structure of infinitesimal elements ($f(x) \cdot dx$), known as the product layer (Hu & Rebello, 2013a; Sealey, 2014). Mastery of this structure is critical for modeling quantities that depend on multiple varying measures, such as *pressure· area* or *current· time*.

Research consistently shows that students often interpret integrals as summing small pieces without attending to the multiplicative relationships among quantities (Jones, 2013; Sealey, 2014). These difficulties appear in a variety of physics contexts. In electricity, learners may fail to conceptualize $\lambda \cdot dx$ in continuous charge distributions, relying instead on memorized formulas (Nguyen & Rebello, 2011), while interpreting $dq$ or $dE$ requires coordination of symbolic, mathematical, and physical ideas (Meredith & Marrongelle, 2008). In fluid mechanics, students often decompose integrals into additive pieces, rarely interpreting $pressure \cdot area$ as a meaningful product, revealing fragile connections between notation, reasoning, and accumulation (Sus & Izsák, 2024).

Frameworks such as Multiplicatively Based Summation (MBS) (Jones, 2015) and Emergent Quantitative Models (Oehrtman & Simmons, 2023) describe productive ways of reasoning about integrals. Conceptual blending theory offers tools for understanding how students dynamically integrate symbolic, mathematical, and physical ideas (e.g., Hu & Rebello, 2013b). Despite these advances, the coordination of product layer reasoning across multiple applied contexts remains underexplored.

## Theoretical Perspectives

In this section, we present theoretical perspectives and a methodological approach which best spans the results obtained in the study. It includes two components—the knowledge-in-pieces cognitive perspective and the modified dynamic conceptual blending diagram (Sus & Izsák, 2025), coming from the dynamic conceptual blending perspective (e.g., Van den Eynde et al., 2020).

### The Knowledge-in-Pieces Epistemological Perspective

The perspective was first developed in science education on conceptual change (diSessa, 1993, 2006) and has since been applied to topics in mathematics, such as whole-number multiplication (Izsák, 2005), functions (Izsák, 2004; Moschovich, 1998), and integrals (Jones, 2013). It posits that reasoning is supported by diverse, fine-grained knowledge resources, which evolve from novice to expert through the construction, refinement, and reorganization of these resources. In this study, we focused on the knowledge resources students use when coordinating multiplication with a definite integral in physics contexts.

### Conceptual Blending Perspective, Modified Dynamic Conceptual Blending Diagram

Fauconnier and Turner (1998) introduced conceptual blending to model how people create new meaning by combining information from different experiential domains. Studies (e.g., Bing & Redish, 2007; Hu & Rebello, 2013b; Wittmann, 2010) have applied this framework to describe students' use of mathematics in physics. In this study, we take the modified dynamic conceptual blending diagram (Sus & Izsák, 2025) to analyze Melinda's dynamic reasoning about the product layer in applied integration contexts. We construct the modified dynamic conceptual blending diagram (MDCBD) first by identifying and coding knowledge resources, listing selected resources in apparent order of activation, and assigning resources to one of four mental spaces—*mathematics* (mathematical concepts without new physical input, including drawings and equations), *physics* (physical concepts and quantities without introducing new mathematical input, including drawings), *symbolic* (technical roles of mathematics), and *blended* (combinations of symbolic, mathematical, and physical ideas, including equations and drawings). We then connect selected elements with coded links. Resources and connections are coded as more confident (e.g., "I remember") or more tentative (e.g., "I guess"), with more confident elements/connections shown as bold text/green solid lines and more tentative as regular text/green dotted lines. Activated/deactivated resources are marked with squared blue/red boxes, while connections are marked with blue/red solid or dashed lines, respectively (Figure 1). By analyzing the elements and connections of these mental spaces, and their dynamic changes, we captured Melinda's moment-to-moment reasoning about the product layer as she blended ideas from symbolic, mathematical, and physical input spaces.

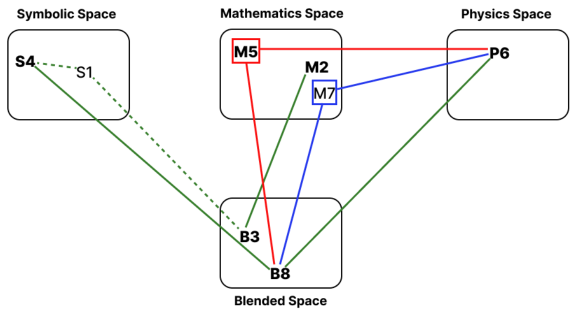



*Figure 1. Schematic representation of MDCBD.*

## Methods

We conducted three 1-hour, one-on-one, semi-structured think-aloud interviews (e.g., Bernard, 1994; Ginsburg, 1997) with three students in a second-semester calculus-based physics course at a selective university in Fall 2023 to examine how they reasoned about integrals and constructed a product layer. The first interview focused on multiplication in applied problems that required summing products, without mentioning integrals. The second and third interviews explored connections between definite integrals and applied problems. Interviews were recorded with two cameras to capture both the student-interviewer interaction and the student's written work. We then synchronized the two videos, transcribed verbatim, and analyzed the knowledge resources students used in building the product layer.

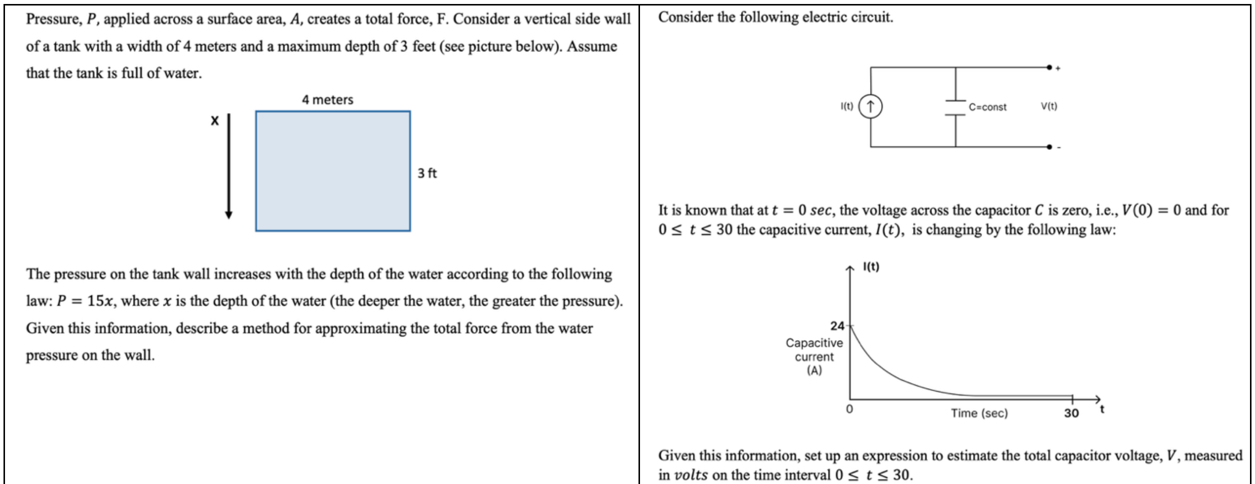
Melinda was selected as an ideal case for studying conceptual blending and product layer reasoning across contexts. She reported an A- in Precalculus, a 5 on AP Calculus AB, and a 4 on AP Calculus BC, indicating strong preparation. In both tasks, she consistently attempted to link symbolic, mathematical, and physical representations, even when her reasoning was incomplete. In the Water Pressure task, she progressed from approximating sums to constructing integral notation, reflecting developing ideas of accumulation and infinitesimals. In the Electricity task, she articulated relationships among voltage, current, and charge, built expressions from definitions, and worked through integral and multiplicative relationships. Her thinking demonstrated productive strategies (e.g., visualizing contributions of small increments) alongside challenges in coordinating two varying quantities within an integral.

## Results

This section presents an analysis of how Melinda reasoned about the product layer when calculating the total force exerted from the water pressure on a tank wall and determining the total voltage across a capacitor. The analysis focuses on how she constructed and coordinated symbolic, mathematical, physical, and blended spaces highlighting both productive reasoning strategies and challenges.

### Water Pressure Task

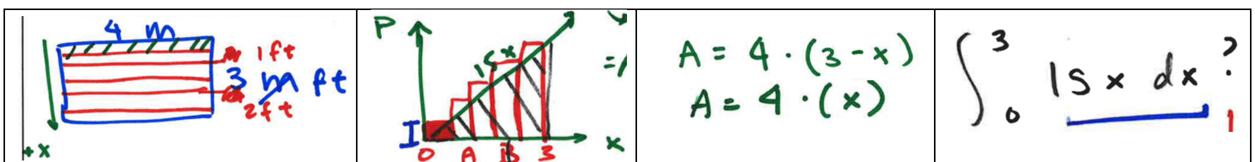
**Data.** Melinda began by reading the problem aloud (Figure 2a) and paraphrasing the key details—the pressure increases with depth according to the law $P = 15x$, where $x$ is the depth of the water—and described an approximation method using sampled points. Melinda then produced work presented in Figure 3 and explained that she would add pressures at different depths to get a total pressure, not a total force, on the wall.




| (a) | (b) |
|---|---|

*Figure 2. (a) Water Pressure task statement, Interview 2; (b) Electricity task statement, Interview 3.*

Melinda then drew the red horizontal lines in Figure 3a that correspond to the red vertical lines in Figure 3b. She continued by stating that the area under the curve in Figure 3b represented the total pressure over the entire tank wall. Shortly after, she admitted uncertainty about finding the force from the pressure: "I don't really know how to get the force from the pressure though." She reiterated that to estimate the total pressure, one would sum all pressure values from 0 to 3 meters, represented by rectangles in Figure 3b.




| (a) | (b) | (c) | (d) |
|---|---|---|---|

*Figure 3. Melinda's reasoning about the definite integral structure in the Water Pressure task.*

When asked to clarify her representation of the tank wall in Figure 3a, Melinda explained that she used the labels A and B on her diagram (erased red letters in Figure 3a) as arbitrary points along the depth rather than as areas. These letters A and B in Figure 3a correspond to letters A and B in Figure 3b. Melinda continued by explaining her interpretation of the surface area (Figure 3c):

> Surface area though … is like four, because the width doesn't change as you increase the depth, but the depth changes at three minus $x$ because three is like the total three feet is the total depth minus wherever you are on the $x$ axis or how far down you are.

The shaded green strip in Figure 3a represents the piece of area described by the first formula in Figure 3c. Melinda then changed labels A and B to $1ft$ and $2ft$ and corrected herself: "Your area would be four times $x$, you don't need the three minus. That's really counterintuitive actually." Ultimately, she represented the area at a given depth with 4 • (x).

Melinda produced two sketches during the task: one representing the tank wall with depths labeled at one-foot intervals (Figure 3a), and another showing pressure as a function of depth

(Figure 3b). She observed that both drawings share the variable $x$: "as you increase $x$, your pressure also increases. And here, as you increase $x$, your surface area also increases." She further suggested that the two quantities could be related by eliminating $x$: "If you do like $x = P/15$ and then $x = A/4$ and set them equal to each other, you could find the pressure as a function of area instead of $x$." Melinda clarified that the area of a shaded green strip in Figure 3a represented the surface area and was different from the area of a rectangle in Figure 3b where the height represents pressure.

As the discussion continued, Melinda reasoned about the integral notation $\int_0^3 15x\,dx$ (Figure 5d). She stated, "I kind of interpret it as from 0 to 3 … and you have the function of pressure as it relates to the depth, and then $dx$, which is like a super small amount of change in $x$." She explained further that the integral represented summing many small pieces: "Because you are taking the integral of it, the piece becomes like super small instead of like being as large." Finally, Melinda articulated the arithmetic operations involved in the integral expression:

> For the integral, I think it's like broken down into addition and multiplication.…you multiply like the height times the change in $x$, and then you add all of those like really small pieces together.

She related this multiplication to finding the area of a rectangle and recognized that the base and height corresponded to $dx$ and $P(x)$, respectively."

**Analysis.** Melinda's reasoning included elements from four mental spaces (Figure 5a). She began by noting that pressure increases with depth (P1) and linking the $P = 15x$ relation (S2) to the physical behavior of the fluid. To quantify this, she approximated the total pressure by summing contributions at sampled points along the tank wall (B3) and interpreted the corresponding rectangles in Figure 3b as small pieces contributing to the total pressure, blending symbolic and physical understanding (B4). She clarified the calculation of surface area, noting the width of the wall is constant while the depth varies, so that area equals $width \cdot depth$ (P5). Using sketches, she visualized the tank wall and the pressure function, observing correspondence of the $x$-axis between the two (B6). She then expressed the integral symbolically $\int_0^3 15x\,dx$ (S7) and interpreted $dx$ as a small change in depth (B8), describing the arithmetic operations involved: multiplying height by $dx$ and summing contributions (M9).

### Electricity Task

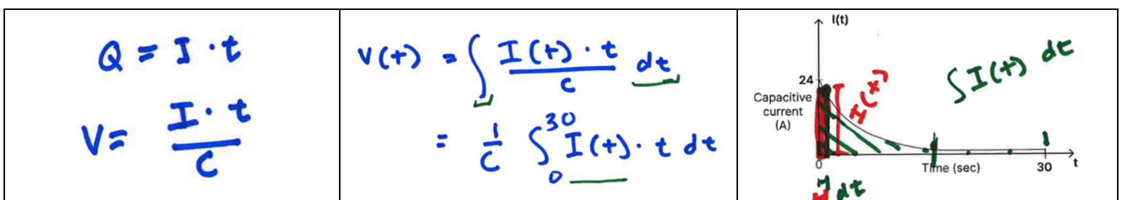
**Data.** Melinda began by reading the Electricity problem (see Figure 2b) and paraphrasing the key details, emphasizing that the voltage across the capacitor is zero at $t = 0$ and that the current varies with time. At first, she expressed uncertainty about how to connect the given quantities: "I don't really know how to relate voltage and current." Drawing on familiar ideas from resistive circuits, she considered Ohm's law but quickly dismissed it as irrelevant: "$V$ equals $IR$ … that really has nothing to do with the capacitor, that has to do with resistors."

To move forward, Melinda recalled the capacitor equation, $V = Q/C$, and explained: "That means the voltage is equal to the charge that is pushed through … by the battery and then over the $C$, which is a capacity, like a property of the capacitor." She then attempted to connect current and charge, reasoning, "I don't know how to relate the charge $Q$, $I$ current. Is it $I = Q/t$?" She elaborated, "I feel like the current is like the amount of charge that's moving through a wire at a certain time $t$," but admitted, "I don't know how $Q$, $I$, and $t$ relate." She then tentatively expressed $Q$ in terms of current and time and substituted into the voltage formula, arriving at $V = \frac{I \cdot t}{C}$ (Figure 4a). She noted that both current and voltage depend on time, signaling awareness of the dynamic nature of the situation.

At this point, Melinda suggested integration as a tool to account for the varying current: “Then you could, to estimate the total capacitor voltage, you could take the integral from 0 to 30 seconds t …” (Figure 4b). She interpreted the integral of $I(t)$ only as the area under the curve of $I(t)$ (green shaded region in Figure 4c) and stated: “That would be the total current that passes through between the times of 0 and 30.”

When asked about the visualization of the expression $I(t) \cdot t\, dt$ on the graph Figure 4c, Melinda responded that it was easier to visualize $I(t)$, the given line. She expressed uncertainty about the $t$ term, which built on her uncertainty about $Q = I \cdot t$. Regarding $dt$, she said: “$dt$ is like a small change in $t$” and indicated it on the drawing (Figure 4c). Explaining the full integral notation (Figure 4b), she stated: “The [green] shaded region would be the integral of $I(t)$ only. But I don’t know how to multiply that if there is an additional $t$ on the inside.”

To interpret the expression $I(t)dt$, Melinda reasoned: “If this is $dt$, then $I(t)$ is here [width and height of the red rectangle in Figure 4c] … multiplying them together is like this little rectangle … that would represent the total current between zero and a small amount of $t$ … then using that current you could, I guess, do like $V$ equals $I$ times $t$ over $C$.” Finally, she reflected on the meaning of the differential in the integral, noting: “$dt$ would be in changing seconds because $t$ is in seconds … $dt$ kind of falls out of the equation when you take the integral” and added that the total voltage is measured in volts.




(a) (b) (c)

*Figure 4. Melinda’s reasoning about the definite integral structure in the Electricity task.*

**Analysis.** Melinda’s reasoning unfolded across four interrelated mental spaces—symbolic (S), physics (P), mathematics (M), and blended (B)—revealing both tentative explorations and confident connections (Figure 5b). Initially, she expressed uncertainty about how voltage and current were related, briefly recalling Ohm’s law (S1) before dismissing it as irrelevant. This tentative symbolic recall illustrates her search for applicable prior knowledge. She then recalled the canonical capacitor relation, $V = Q/C$ (S2), and connected it to the physical property of the capacitor (P3), noting that $C$ characterizes the capacitor’s ability to store charge. Tentatively, she attempted to relate charge and current as $Q = I \cdot t$ (S4), combining this with the capacitor formula in blended reasoning (B5). To account for the time-varying current, Melinda proposed using an integral from 0 to 30 seconds (M6) to estimate the total capacitor voltage. She visualized the integral of $I(t)$ only (S7), not $I(t) \cdot t$, as the area under the current-time curve (B8), translating a dynamic physical process into a symbolic mathematical form. She interpreted $dt$ as a small change in time (M9) and conceptualized $I(t) \cdot dt$ as a small rectangle representing current over an infinitesimal interval (B11), while expressing difficulty visualizing $I(t) \cdot t\, dt$ and its integral (B10). She also recalled the units for voltage (P12) and noted that after finding the total current, one could plug that value into $V = \frac{I \cdot t}{C}$ to compute voltage.

## Discussion and Conclusion

This study examined how Melinda reasoned about the product layer—the multiplicative structure within an integral—while working on two applied problems. Across both tasks, Melinda demonstrated productive strategies, such as interpreting differentials as small changes and recognizing that integration involves combining contributions over an interval, but she also showed persistent difficulties in constructing the infinitesimal products that define the product layer.

In the Water Pressure task (Figure 5a), Melinda initially treated pressure values as directly summable and interpreted the area under the pressure-depth curve as "total pressure," overlooking the role of surface area. Although she later expressed the integral symbolically and described multiplication of height and a small change in depth, her interpretation remained primarily additive, with limited emphasis on how pressure and area together form an elemental force. In the Electricity task (Figure 5b), she represented $\int I(t)dt$ graphically as the area under a current-time curve and identified $dt$ as a small change in $t$, but she struggled to connect this representation to the capacitor voltage formula.

These patterns indicate that interpreting the integral as summing contributions or finding an area does not guarantee an understanding of how multiple varying quantities combine within an infinitesimal element. The Water Pressure problem offered geometric anchors and visual cues that supported partial blending of symbolic and physical ideas, while the Electricity problem relied on abstract symbolic relationships and some visual representation, resulting in more tentative and fragmented blending.

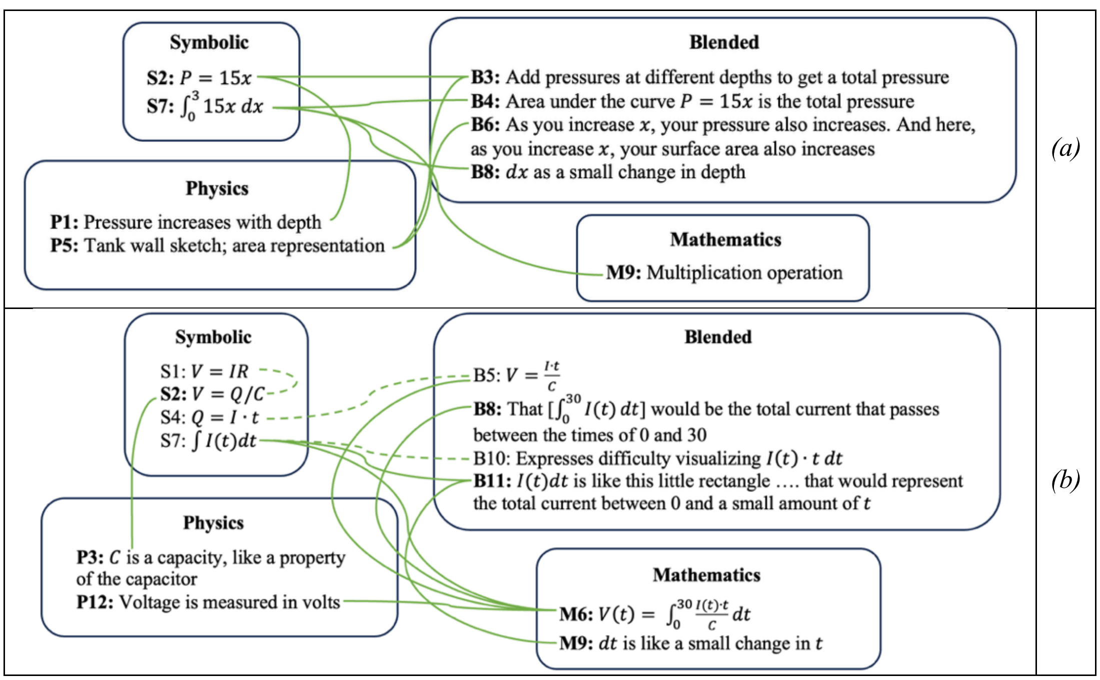



*Figure 5. MDCBD representation for (a) Water Pressure and (b) Electricity tasks.*

These findings suggest that instruction should explicitly foreground the blended meaning of the multiplicative structure in applied integrals and support students in coordinating symbolic, mathematical, and physical representations. Emphasizing how these quantities combine to form infinitesimal elements may enable students to move beyond additive interpretations toward a robust understanding of the product layer across contexts.

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