The Alignment of Student Conceptions With Course Learning Outcomes: A Case Study on the Definite Integral

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Students pursuing science, technology, engineering, and mathematics (STEM) degrees require some proficiency in Calculus. The Fundamental Theorem of Calculus is arguably the most important theorem taught in a Calculus I course, yet it is not covered until the last weeks of the semester in a traditional three-part Calculus sequence. As a result, students tend not to hold deep conceptual understandings around integration, and the conceptions that they do hold are procedurally based. This paper illustrates the disconnection between student conceptions around integration (primarily area-based) and the general emphasis (from mathematics education literature) on accumulation. In addition, our work highlights how course learning objectives may drive a primarily area-based interpretation of integration. Our preliminary findings from interview data with Calculus III students confirm prior research, suggesting that students hold strong procedural and area-based conceptions, rather than accumulation conceptions.

Keywords: accumulation, definite integral, course learning objectives

Background

Extensive research has examined the diversity of student conceptions and challenges in understanding integration in introductory calculus (Jones et al., 2017; Serhan, 2015; Thompson & Silverman, 2008). As Thompson (1994) highlighted, "the realization that the accumulation of a quantity and the rate of change of its accumulation are tightly related - is one of the intellectual hallmarks in the development of the calculus." Some researchers have pushed for teaching integration with a pedagogical focus on "adding up pieces" to support productive conceptual development among learners (Jones & Ely, 2023). Even with this emphasis on accumulation, students often leave Calculus courses with a conception of integration as an antiderivative or a process for computing areas under or between curves (Jones, 2015; Nguyen & Rebello, 2011; Serhan, 2015; Wagner, 2017). In fact, students often learn about the Riemann sum as a means for calculating the definite integral, and later forget this procedure once they have learned the Evaluation Theorem (Jones et al., 2017).

It is not until the Calculus II course, at our institution, that students are exposed to applications of integral calculus. It is in these applications where having a conception about accumulation is more productive than simply holding an area conception. In Calculus III, students have opportunities to explore accumulation across multiple dimensions, though they may not always recognize it explicitly. Building on prior literature that documents this phenomenon, our research team aims to explore whether similar patterns emerge among students at our institution. We pose the following question:

1. What conceptions do Calculus III students hold on the definite integral?

Preliminary Work

The first research question was the primary subject of an undergraduate's (third author) summer research project. Upon completion of the data analysis for this work, the research team wondered about the relationship between the findings and the actual content that students learned in their previous Calculus I and II courses at our institution. Thus, we pose the following question:

2. Based on the stated course learning objectives (CLOs) in Calculus I and II, what conceptions do we expect students to develop around integration after completing the first two courses in the three-part Calculus sequence?

In this preliminary report, we share findings from both research questions and pose further questions to explore. We intend to continue this work by systematically reviewing CLOs from various Calculus programs across the United States to observe how CLOs align with established perspectives around accumulation.

Methods

Author 1 and Author 3 participated in an independent study project where we set out to explore student conceptions around integration. During the first phase of the project, the research team performed a literature review and constructed an a priori codebook. The codebook was based on the work of Grundmeier et al. (2006) and Jones (2013), highlighting student conceptions that we anticipated students to hold around integration (see Table 1). Afterwards, the team developed a semi-structured interview protocol to engage with Calculus III students.

Table 1. A Priori Codebook of Student Conceptions of the Definite Integral.

Codes	Elaboration
Opposite	The student conceives the integral as the opposite of the derivative.
Area	The student conceives the integral as the area underneath a curve.
Infinite Sum	The student conceives the integral as an infinite sum of quantities.
Accumulation	The students discuss the idea of adding up, totaling, or accumulating areas. Without explicit reference to Riemann sums.
Procedure	The student only discusses procedures for evaluating a definite integral.

During the second phase of the project, Author 2 joined the research team. With IRB approval, the research team recruited ten students from our institution who were currently enrolled in Calculus III. Each participant participated in an hour-long video-recorded semi-structured interview, for which they were compensated with a $25 gift card to the campus bookstore. Each participant was asked eight Calculus I questions, where five were explicitly about integrals. Two questions were informal questions, posed as a means of capturing students' general ideas about integration (see Figure 1). Three traditional Calculus I questions that involved integrals were then posed to the participants (see Figure 2).

3. When you think about integrals, what comes to mind? (Draw a map and draw arrows connecting different ideas that are related and state how they are related.)

4. How are integrals used by scientists?

Figure 1. Informal questions about integrals posed to the participants.

The data from the semi-structured interviews were transcribed and coded using the a priori codebook. Each member of the research team coded the ten transcripts independently and met to discuss codes. Any disagreements around coding were discussed and resolved during research meetings.

5. Explain this equation.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

What do the limits signify in this equation?

7. A truck is dumping sand onto a scale. At each time t (in seconds) the sand is dumping out at a rate of $f(t)$ tons/sec. What do each of these quantities signify?

a. $f(4)$

b. $\frac{f(4)-f(0)}{4}$

c. $f'(4)$

d. $\int_0^4 f(t)\,dt$

8. a. If $\int_1^5 f(x)\,dx = 25$, then write down the value of $\int_1^5 (f(x)+2)\,dx$

b. If $\int_2^4 f(t)\,dt = 10$, then write down the value of $\int_3^5 f(t-1)\,dt$

Figure 2. Formal questions involving integrals posed to the participants.

Considering preliminary findings from the student interviews, the research team decided to then explore and code the CLOs for the Calculus I and II courses that the student participants had taken in prior semesters. The CLOs were coded using the same codebook in order to make comparisons between the findings.

Preliminary Findings

In this section, we present findings in response to each research question.

Student Conceptions on the Definite Integral

We noticed several key observations from the student data. First, and consistent with the literature, all participants referenced the concept of area at least once during the interview. For instance, one participant sketched a picture of an arbitrary function $f(x)$ as they interpreted question 7d, "So because we have infinitely many rectangles, because that n approaches infinity… essentially it's like this whole area filled out." Second, all students demonstrated procedural understanding in solving at least one of the given problems (e.g., question 8).

Third, a majority of students articulated that integration is related to the notion of being the opposite of a derivative (7/10). One participant stated, "I think integrals are like antiderivatives…same thing as derivative just opposite." In addition, 7/10 students associated the integral with accumulation. Many students discussed an idea about totals. One student mentioned "total electric charge," and another mentioned "total work done" over some interval. In discussing question 7d, another student stated the following:

> If you're looking at the line, that's the output of the function. That's how much … sand got dumped. But, for an integral … it's kind of like saying, it's this much sand. It's this much sand. Like it's-it makes me think of saying like, that's all the sand ever between 0 seconds and 4 seconds.

This participant used gestures and repetition to emphasize the addition to a pile of sand as a representation of the integral in 7d.

Lastly, only 6/10 of the participants accurately discussed the idea of an infinite sum either within the definition of an integral or while explaining the process of integration. This was surprising given that each participant was presented with the Riemann sum in question 5.

Expected Learning as Outlined in CLOs

The Calculus I course at our institution covers the foundations of differential and integral calculus, with 12 CLOs. Six of these learning goals, as displayed in Table 2, pertain to student learning about integration. We noticed that the predominant emphasis in this Calculus I course was to facilitate the development of an area conception around integration. The Calculus II course covers techniques and applications of integration, sequences and series, and polar and parametric equations. There are 15 Calculus II CLOs, and only four were explicitly related to integration. All four of these CLOs were coded as Procedural. For instance, one CLO was, "to learn how to apply the definite integral and integration techniques to the problems of finding the arc length of a curve, the area of a surface of revolution, moments and centroids, and hydrostatic pressure and force." While it may occur in Calculus II, we do not speculate on whether an emphasis on an accumulation interpretation occurs during instruction.

Table 2. Six Relevant Calculus I learning goals Pertaining to Integration

CLO	Code
7. To study and understand the concept of an antiderivative of a function, and to find families of antiderivatives for some simple functions.	-
8. To grasp the meaning of the "area under the curve" problem, to be able to approximate such areas with left and right hand Riemann sums, and to evaluate the limit of these sums using summation formulas provided.	Area, Infinite Sum
9. To understand the meaning of the definite integral, to learn to use the notation correctly, and to be able to interpret its meaning in terms of areas.	Area
10. To learn how to use the Fundamental Theorem of Calculus to evaluate definite integrals and to find derivatives of definite integrals as a function of the upper limit.	Procedural
11. To learn how and when to apply the method of substitution to evaluate a definite integral, and to be able to interpret a definite integral using the Net Change Theorem.	Procedural
12. To understand how to apply the definite integral to find the area between two curves.	Area

Discussion and Conclusion

Though the sample size was small, when compared to the CLOs, these findings provide insight into potential conceptions students may develop about integration following their coursework in Calculus I and II at our institution. Findings indicated that students' conceptions of integration encompassed expected procedural methods, as well as connections to area under a curve and, in some cases, volume. Notably, students referenced accumulation (including "totaling" and "adding up") more frequently than anticipated. Overall, their conceptions were consistent with the stated CLOs.

To foster an understanding of accumulation, it is essential to identify concrete ways to *integrate* it into the course learning objectives. With that said, we pose the following questions for the audience to consider:

1. What CLOs would need to change if we want students to develop a robust conception of accumulation?
2. What barriers exist to shifting from a primarily area and procedural focus to an accumulation focus on integration? And, in what ways could we better leverage the introduction of the Riemann sum to cultivate an accumulation conception?

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