GTA Perceptions of Precalculus Students' Graphing via Covariational Reasoning

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The success of any educational change requires the support of instructors, and at the introductory undergraduate mathematics level, up to 30% of those courses are taught by graduate teaching assistants (GTAs). Research has shown the prevalence and importance of covariational reasoning at the precalculus/calculus level, and this study examines how GTAs interpret and respond to students' attempts at graphing via covariational reasoning. Through engaging in a version of Carlson et al.'s (2002) Bottle Problem, and then watching and responding to two precalculus students' video responses to the problem, 11 GTAs had the opportunity to express their teacher decision making around student thinking. I analyzed their responses using Herbst et al.'s (2023) categories of perception teachers use when considering framed student contributions: normativity, serviceability, and responsiveness. The results indicate that GTAs primarily focus on serviceable and non-normative aspects of students' work. I detail these results and discuss implications for GTA training.

Keywords: GTA training, covariational reasoning, instructor decision making

Graduate Teaching Assistants (GTAs) are often each responsible for teaching hundreds of undergraduate students over the course of their graduate program, particularly in introductory level courses. Building programs and compiling resources support these instructors, but understanding how these GTAs think about and respond to their students' thinking is both a critical and needed component to ensure that these resources meet GTAs' needs. In this study, I focus on GTA's perceptions of undergraduate precalculus students' work on a graphing task – an adaptation of Carlson et al.'s (2002) *Bottle Problem*. GTAs received student work with both quantitative and nonquantitative attempts at covariational reasoning. I analyzed 11 GTAs' responses to the work using Herbst et al.'s (2023) categories of perception, a framework enabling a focus on GTA's attention to the normativity, serviceability, and responsiveness of the student work. Analysis was guided by the research question: What do GTAs perceive in undergraduate precalculus students' work on a task aimed to support graphing via covariational reasoning?

Literature Review & Theoretical Frameworks

In this section, I discuss literature linking covariational reasoning and graphing, literature linking instructional noticing and decision making, and I describe the state of GTA training in the U.S. The first two include descriptions of the theoretical frameworks used in this study: covariational reasoning (Carlson et al., 2002) and categories of perception (Herbst et al., 2023).

Covariational Reasoning and Graphing

Covariational reasoning – reasoning how two quantities change together – is a construct built from research on students' quantitative reasoning – reasoning about and operations with measurable attributes (Carlson et al., 2002; Thompson & Carlson, 2017). Researchers have demonstrated the importance of students' covariational reasoning to support productive meanings for a variety of topics in K-12 and beyond. Relevant to this study its importance in precalculus meanings for rate of change relevant to graphing linear, quadratic, and exponential functions (e.g., Ellis et al., 2015; Johnson, 2015; Moore, 2014; Tasova, 2021; Yu, 2022).

The *Bottle Problem* (Carlson et al., 2002), in particular, was the impetus for a covariational reasoning framework; it involved students reasoning about the relationship between the height and volume of water in a spherical bottle as water was being poured into it. Their study distinguished students' mental actions based on how students described quantities changing. For example, *directional covariational reasoning* involves students describing one quantity's values increasing/decreasing/staying constant as another quantity's values are increasing/decreasing, and *amounts of change covariational reasoning* involves students describing changes in a quantity's values as increasing/decreasing/staying constant as another quantity's values changes by equal amounts. Students can use graphs to represent these relationships and use covariational reasoning to justify their graph by attending to (changes in) quantities' magnitudes. For example, for a linear relationship, students can justify a Cartesian graph of a straight line as representing a linear relationship between quantities by highlighting equal magnitude changes in one quantity and identifying corresponding equal magnitude changes in the second quantity.

As researchers have attended to students' covariational reasoning with graphs, they have uncovered just how difficult this reasoning can be. For example, students may struggle to conceptualize changing quantities as varying *smoothly*, reasoning instead with *chunks* (Castillo-Garsow, 2013). Reasoning with *chunks* can lead to attempts at *amounts of change* covariational reasoning that consider intervals of change that are too big (e.g., the relationship between quantities may shift from an increasing to a decreasing rate of change within the chosen interval) (e.g., Stevens et al., 2023). Students may also have meanings for graphs that make it impossible for them to represent dynamic relationships traced out from right-to-left or straight up and down (e.g., the graph would no longer represent a function), even if doing so would appropriately represent the relationship they conceive in their context (Moore et al., 2019).

Research on students' graphing has also identified several non-quantitative ways of reasoning. *Iconic translations* (Monk, 1992) and *thematic associations* (Thompson, 2015), for example, involve students attending to shapes and phenomenon to justify shapes in their graphs. For example, a student might conclude a graph representing a rider going around a Ferris wheel is curved because a Ferris wheel is circular, regardless of the quantities under consideration. Or a student might focus on the phenomenon that a rider traveling at a decreasing speed results in a concave down graph, even if the graph is relating two different distances (rather than distance-time graph). Called pseudo-analytical behavior (Vinner, 1997), these types of reasoning are classified when particular mental actions are indicated without providing adequate evidence they "possessed an understanding that supported that behavior" (Carlson et al., 2002, p. 358).

Altogether, this research indicates a wide variety of students' reasoning with graphs. This study aims to begin researching what GTAs, not researchers, attend to when interpreting some of the same student work that inspired these researchers.

Instructor Noticing & Decision Making

What instructors do with their knowledge of student thinking depends on what instructors notice and what their instruction makes available to be noticed. Several researchers have attended to mathematics teacher noticing (see an overview in Sherin et al., 2011), but in this article, I adopt a framework designed to categorize teachers noticing around student contributions to a problem-based task. Herbst et al. (2023) describe three categories of perception: serviceability, normativity, and responsiveness. These categories, which also rely on the idea of framing, serve as the analysis framework for this study (heretofore SNR framework).

Herbst et al. (2023) used a task-based geometry lesson as the context for their categories' examples. This context is important in determining how the other categories are interpreted. That is, framing is “the organization of events and individual experiences according to certain principles” (p. 403), and thus identifying an event as “task-based”, “geometry”, and a “lesson” results in certain expectations. Herbst (2006) has developed the notion of instructional situations to describe how teachers frame students’ work on problems (e.g. “construction” and “proof” for Geometry). Thus, if a teacher frames a task-based geometry lesson as involving construction, based on their expectations of what that entails, it presents the teacher with opportunities to anticipate and observe students’ actions on the task, and moreover, opportunities to scaffold students’ work accordingly. With this lens in mind, the following constructs are the SNR categories of perception available for a teacher to notice student contributions.

Normativity. Based on how a problem has been framed, a student’s contribution is deemed as more or less normative based on the extent to which it complies with or breeches what the teacher deems to be appropriate within the frame of an instructional situation. Thus, while a correct solution to a problem would certainly be seen as normative, a student who has an incorrect solution to a “construction” problem but who used construction tools (e.g., a compass) would still have a more normative contribution than a student who sketched a diagram correctly.

Serviceability. Serviceability attends to the extent to which a student’s contribution fits within the teacher’s instructional goal. For example, a student with a more serviceable solution may not have a complete solution or may have overgeneralized an observation, but they contributed an observation that is relevant to the teacher’s instructional goal for the lesson. In contrast, a less serviceable contribution from a student could be one in which a student provided a method to solve a task that is not the same as the method included in the instructional goal.

Responsiveness. Responsiveness attends to the extent to which a student contribution relates to the question asked in the problem. For instance, the aforementioned student who sketched a diagram correctly responded to the question asked, whereas students who change the givens or do not entirely address the question (e.g., are missing components of the solution) would be providing contributions that are less responsive.

As mentioned, Herbst et al. (2023) framework detail these constructs using examples from teachers discussing student contributions for a task-based lesson involving construction in a high school geometry class. A teacher does not necessarily perceive any one of these during a given lesson. In this study, the focus will be on GTAs’ perceptions when responding to student contributions of a task-based problem for precalculus students involving covariational reasoning and graphing.

Instructor (GTA) Training

Up to 30% of introductory undergraduate mathematics courses are taught by graduate teaching assistants (GTAs). Over the course of their graduate student career, GTAs may be responsible for instructing hundreds of undergraduate students. Training for these GTAs is often less extensive than traditional K-12 instructor training programs, and moreover, faculty members responsible for developing instructional materials for GTA training often find it challenging to locate instructional materials. Nevertheless, Kung and Speer (2009) have reported that GTA instructors want to place an emphasis on building GTA’s skills in probing and using student thinking. Recent efforts by the *Mathematical Association of America* (MAA), such as the *CoMInDS* resource suite and the *Instructional Practices Guide* (Abell et al., 2022) have begun to address the need for instructional materials for college instructor/GTA training. Both also provide insights into how to support understanding and using student thinking in teaching, but more work is needed to understand how GTAs currently think about student thinking and how their classroom context influences their noticing and decision making.

Methods

In this section, I describe the population and context of this study, the process of adapting the *Bottle Problem Task* into a GTA training task, and the analyses I conducted.

Methodology

This study required two phases of data collection: precalculus student responses to the Bottle Problem and GTA responses to those students' responses. First, the *Bottle Problem* was adapted into a task given to 39 students in an Applied Precalculus class at a mid-size university in the Northeast. This task included students submitting video responses to the task. From the 33 students who consented to share their classroom data, we de-identified their videos and created transcripts of each of the students' responses. These responses were collected and analyzed for correctness and level of covariational reasoning. A subset of two student videos and transcripts were then provided a group of 11 graduate students who were enrolled in a one-credit hour seminar on teaching undergraduate mathematics. These 11 graduate students were in their first or second year of teaching, and some of these graduate students either had or were currently teaching different sections of the same Applied Precalculus course. These graduate students submitted their own video responses to (a) the same adapted Bottle Problem task as the precalculus students and (b) how they would respond to the two students from the subset of student videos and transcripts. All GTAs in the course consented to sharing their classroom data.

Task Design

In this section, I describe the process of adapting a research task using for assessing students' covariational reasoning to a classroom task and then again to a GTA training task (Figure 1).

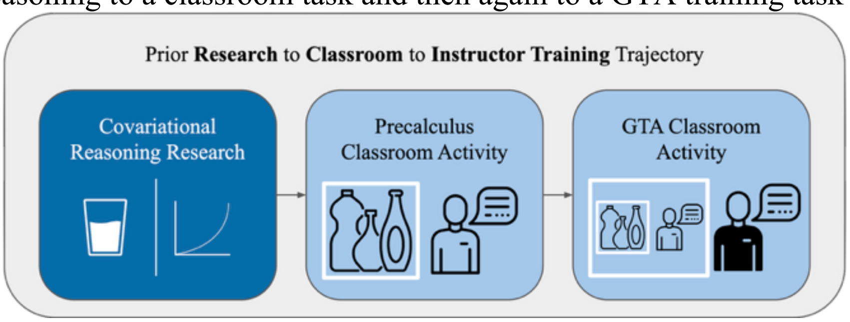

Figure 1. Transition from research to classroom to GTA training.

Adapted Bottle Problem Task. The original *Bottle Problem* Task asked students to graph the relationship between the height and volume of water poured into a spherical bottle. In Moore's *Advancing Reasoning Project*, his team adapted this problem to be used in a classroom activity for preservice secondary mathematics teachers by incorporating several differently shaped bottles. I adapted this version of the task into two parts. In Part I, students worked together in groups in class to get accustomed to creating graphs/bottles and to introduce the language of covariational reasoning to them. In Part II, students completed the activities outside of class and submitted 3-5 minute video responses of their solutions. Each part had different bottles/graphs, and in Part II, students had the opportunity to make their own bottles/graphs and needed specifically to recognize descriptions of quadratic and exponential relationships. In total, students completed 8 bottle-graph pairs.

GTA Training Task. The *Adapted Bottle Problem* video responses were analyzed according to their correctness and covariational reasoning (see details in Analysis section). For the GTA training material, the goal was to identify two videos for the GTA students to respond to. The learning goals of the GTA activity was to (a) introduce them to procedural vs. conceptual thinking (b) introduce them to covariational reasoning and (c) respond to student thinking by attending to students' covariational reasoning. To do so, the GTAs first read an introduction to procedural and conceptual thinking, including an introduction to Carlson's Precalculus Concept Assessment (Carlson et al., 2010) and potential responses to the *Bottle Problem* (Abell et al., 2024). The GTAs were then given the same *Adapted Bottle Problem* task, including submitting their own 3-5 minute video of their reasoning.

Then, the GTAs watched the two selected videos and received the transcripts of the two precalculus students explaining their reasoning on Part II of the *Adapted Bottle Problem* task. One student exhibited *amounts of change* reasoning considering too large of intervals of change and another student used *thematic associations* with "speed." Neither exhibited entirely correct solutions. Kala's work, though incorrect, used amounts of change (serviceable), constructed expected representations (normative), and carefully color-coded and followed directions (framing/responsiveness). In contrast, Leonhard relied on language around speed to explain his reasoning (non-serviceable), constructed more than one graph for a unique bottle (non-normative), and did not follow instructions to color-code or only explained a subset of his bottles/graphs (framing/responsiveness). The GTAs then submitted a second 3-5 minute video on how they would respond to those students.

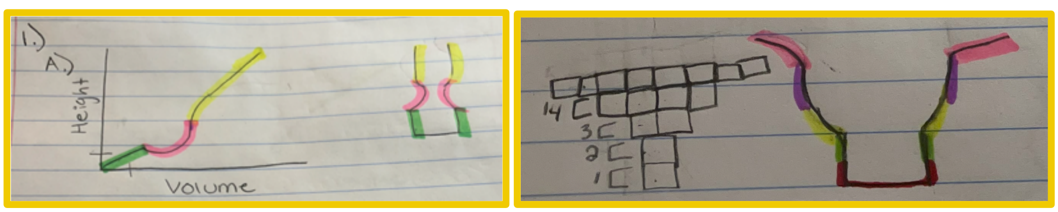

Figure 2. Sample of Kala's work on the Adapted Bottle Problem.

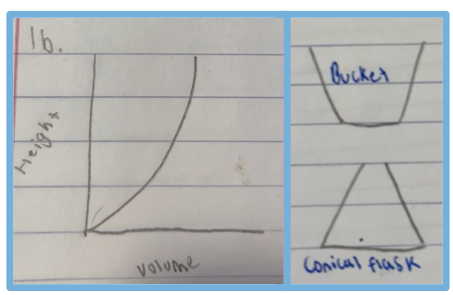

Figure 3. Sample of Leonhard's work on the Adapted Bottle Problem.

Analysis

The goal of the analysis was to attend to the GTA's responses to students' covariational reasoning. To do so, I engaged in a two-part analysis: on GTA responses to the *Adapted Bottle Problem* and on GTA responses to student solutions of the problem. In this manuscript, I focus on the latter part. In this part, I used the SNR framework from Herbst et al. (2023) to code the GTA responses to the student work. I used transcripts of the videos and referenced accompanying student work to code GTA utterances/pointers.

In particular, I created codes for each of the categories of perception. *Correctness* was coded based on the GTA's perception of a student's final graph shape. *Framing* was coded based on the GTA's perception of the prompt. *Normativity* was coded based on GTA's perception of student's production of reasonable responses. *Serviceability* was coded based on GTA's perception of a student's covariational reasoning and/or discussing mathematical relationships, with specific codes pointing to *directional* and *amounts of change* covariational reasoning. However, because these GTAs were new to the construct of covariational reasoning and Kala's amounts of change reasoning was not entirely correct, to code for serviceability, it was only required for the GTAs to attend to quantities students identified and how those quantities were changing together. *Responsiveness* was coded based on GTA's perception of a student's response as a complete response to the problem statement, including color coding.

Moreover, the SNR framework enables the negation of each of these codes (non-normative, for example). Particularly given the open nature of the task, the line between the two can be difficult to define objectively, and so determinations were based on GTA's reactions. For example, one GTA may consider a graph correct as long as the general shape is appropriate (e.g., curved vs. linear segments) whereas another GTA might consider a graph incorrect if the relative steepness of lines in a graph are not considered. Thus, to code for students' responses to student work, an initial code of serviceability, normativity, responsiveness, and correctness was coded. Then, if the GTA's reaction to the work was positive (e.g., praising) the work based on that category, the code remained as is; if the GTA's reaction to the work was negative, the code was negated (e.g., correctness to incorrectness).

Results

In this section, I detail the results of the analysis on GTA responses to Kala and Leonhard's student work. The overall distribution of GTA comments are in Figure 4 below. These results indicate that GTAs largely attended to the serviceable and non-normative aspects of the student work. In the remainder of this section, I detail some of the findings and note what particular aspects of the student work the GTAs perceived.

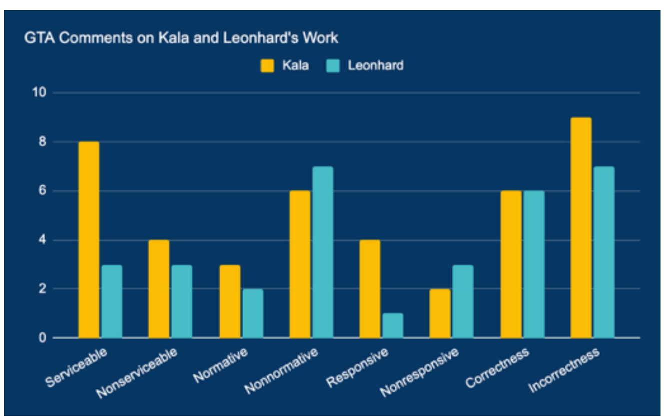

Figure 4. GTA perceptions of Applied Precalculus students' responses to the Adapted Bottle Problem.

GTA Responses to the *Adapted Bottle Problem* and Framing

Although not prompted, many (5) GTAs recognized how difficult the problem itself was, sympathizing with the students. For instance, one GTA remarked that the problems are "not like super intuitive at first glance". In particular, for one of the problems, a GTA remarked they would have preferred the axes to be switched because as is, it "kinda throws off a bit based on conventional [axes]." Two instructors noted that the framing of the problem would require additional instructor support. However, in general, GTAs still had a positive response to the problem, with one stating, "I'm all for it. I think it's good to keep them on their toes."

GTA Responses to Kala's Work

For Kala's work, one of the key perceptions from GTAs were that the work was non-normative. Four GTAs commented on the pink "dip" in Kala's graph in Figure 4 (left). One stated that Kala "dipped the pink part down and said that it decreases but she wasn't fully aware of the fact that she was talking about the decrease in the rate of change." The same GTA called the shape "kinda weird because that wouldn't happen on any of these graphs." These comments attend to the non-normative nature of Kala's solutions, specifically incorporating Kala's covariational reasoning in her description of Kala's contribution.

The second key perception from GTAs was that the work was serviceable. Seven GTAs praised Kala's attempts to use "blocks" (Kala's language) to support her reasoning (see Figure 4 right). One GTA stated "the way she approached it with the squares does demonstrate that she understands the concepts behind the assignment." Another GTA stated that she saw that Kala used this approach "so she can get an idea" of the problem. These comments attend to the serviceability of Kala's approach, specifically using the blocks to support not only constructing the graph but justifying her graph via amounts of change covariational reasoning.

Lastly, and somewhat surprisingly, three GTAs praised Kala's use of color-coding. It remains unclear whether or not the teachers saw color-coding as part of the framing, since Leonhard did not color code and the two GTAs specifically noted a need for him to do so were part of the three that praised Kala's color-coding.

Leonhard's Work

For Leonhard's work, the key perception from GTAs was that the work was non-serviceable and incorrect. One GTA thought Leonhard "did not seem to have as much of an understanding of rate of change" and just "throwing out a few terms." Another GTA noted that Leonhard himself remarked on wishing he had numbers, and stated "since he's given the relationship, you can certainly get numerical values like Kala did." Only three GTA specifically referenced the "faster/slower" speed language (and two praised it). These comments attend to the non-serviceable ways in which Leonhard justified his actions. Overall, GTAs noted some pseudo-analytic actions Leonhard took (even if viewed positively), but not thematic associations.

The second key perception from GTAs was that Leonhard's work was nonnormative. Six students commented that Leonhard produced *two* different bottles for the same graph (Figure 3), though one GTA praised it. As one GTA discussed, "he had a couple options for the bottle. He did say that both of them worked even though both of them were quite different." That is, the two bottles Leonhard produced represent two different rates of change between height and volume, making it impossible for Leonhard to have an appropriate solution. These comments attend to the nonnormative (and also incorrect and nonresponsive) nature of Leonhard's work.

Discussion

Using the SNR and covariational reasoning frameworks, this study showed both that GTA's perceptions span the categories of normativity, serviceability and responsiveness, and that these 11 GTAs primarily focused on the serviceable and non-normative aspects of undergraduate precalculus work on a graphing problem via covariational reasoning. Looking at GTA's comments more closely, it seemed that, even with little introduction to the constructs of covariational reasoning, GTAs were able to perceive and attend to students' covariational reasoning and pseudo-analytic actions in this task. Further research can look at the influence of the task including an instructional goal of covariational reasoning in GTA's perceptions and also the impact additional GTA training in covariational reasoning has on GTA perceptions.

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