Undergraduate Students Metacognition While Constructing Mathematical Proofs

Sarah Sparks
University of Northern Colorado

This phenomenological case study examines evidence of undergraduate students' metacognitive reasoning while completing proof construction tasks. Two pairs of undergraduate mathematics majors participated in task-based video recorded interviews where they were presented with contextual scenarios and asked to prove or disprove a related mathematical statement using think-aloud protocols. Participants then took part in stimulated recall interviews where they were shown video clips from their task completion and asked to reflect on their thought processes. This contributed report explores each student's access of metacognitive knowledge and the subsequent actions they took during these proof tasks.

Keywords: Metacognition, Problem Solving, Proof

Mathematical proofs and their underlying logical structures are foundational to the development of mathematical knowledge (Hanna & Barbeau, 2008). The common vocabulary, deductive reasoning, and formatting used to develop and write these proofs are used to communicate new mathematical knowledge (Ernest et al., 2016). Undergraduate mathematics proof-based courses aim to imbue the necessary mathematical knowledge, writing and reasoning skills in students for them to be able to read, construct, and analyze mathematical proofs. The development of these skills is not confined to individual courses, but it is declared to be one of the overarching goals of an undergraduate mathematics program (Committee on the Undergraduate Programs in Mathematics [CUPM], 2015). Although research has explored the cognitive processes related specifically to the proof construction process (e.g., Hanna, 2020; Inglis & Alcock, 2012; Moore, 1994; Tall, 1998), considering that research recognizes metacognition's impact on student success with other mathematical tasks (e.g., Mamona-Downs, 2002), investigating metacognitive aspects of proof construction explicitly may inform how metacognition impacts proof construction and how to leverage this knowledge to improve teaching of proof construction.

For the purposes of this research, I utilize the definition of *metacognition* as "any knowledge or cognitive activity that takes as its' object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about, one's own thinking" (Lerman, 2020, p. 608). Metacognitive processes of *monitoring* and *control* describe the interaction between the *meta-level* (internal metacognitive knowledge and related models of the object-level) and the *object-level* (external to the meta-level, anything external to the individual and their metacognitive knowledge). *Monitoring* is the process where the "meta-level is informed by the object-level" (Nelson & Narens, 1994, p. 11) such that the information taken in during the monitoring process can change the models at the meta-level. *Control* is the process where the "meta-level modifies the object level" (Nelson & Narens, 1994, p. 11). Knowledge at the meta-level can cause action at the object-level, either through starting or ceasing an action. The choice not to take a certain action is part of the control process. In the case of an individual working on constructing a mathematical proof, the theorem statement, their written work, any other related cognitive processes, would all be at the object-level. The meta-level would encompass the individual's judgements about their proof construction process (e.g., how they perceive their own success) and any knowledge of strategies useful to this particular type of proof.

Metacognitive knowledge is partitioned into *declarative*, *procedural*, and *conditional* metacognitive knowledge (Schneider & Artelt, 2010; Shraw & Dennison, 1994). *Declarative* metacognitive knowledge encompasses knowledge a person has about themselves, their cognitive knowledge and abilities. *Procedural* knowledge describes individuals' knowledge of relevant strategies, heuristics, and other tools. (Schneider & Artelt, 2010). *Conditional* metacognitive knowledge is the knowledge of when and why specific cognitive tools and strategies are useful in a given context (Lerman, 2020).

Due to the focus on students' metacognitive processes rather than the validity or rigor of their resulting proof product, I utilize a broad definition of *proof construction* as the process of using a series of logical assertions deduced from a set of premises and previous assertions, and/or a set of instructions that when followed are representative of these deductions, where this collection of assertions and instructions results in an argument that "justifies a claim to oneself and others" (Blanton et al., 2003, p. 113). *Proof* is then defined to be this resulting collection of assertions.

The purpose of this phenomenological case study is to contribute to the field of metacognition research in mathematics education by examining the experiences of undergraduate mathematics students' metacognitive processes during proof construction. Specifically, this research aims to address the questions: What declarative, procedural, and conditional metacognitive knowledge do undergraduate mathematics students demonstrate or claim to have and use during construction of a mathematical proof? What metacognitive processes do undergraduate mathematics students engage in during construction of a mathematical proof? In particular, what monitoring and control processes do undergraduate mathematics students engage in during construction of mathematical proofs? And, in what ways do the cognitive and metacognitive processes of undergraduate students inform each other during construction of mathematical proofs?

Theoretical Perspective and Methodology

Entering this research from a social constructivism theoretical perspective, I considered individual's constructed conceptions, acknowledging that these conceptions are inextricable from the social contexts in which they were constructed (Patton, 2002). Vygotsky claimed the process of learning to include the appropriation and use of tools and signs (Grossman, 1999) that occurs during social interaction. Vygotsky not only considered social interaction with others but interaction within the self, intramental thinking (Ernest, 2010). This sort of internal interaction with the self is enmeshed with metacognition, the individuals reasoning about their own cognition. I specifically considered intramental and intermental thinking, (i.e., the internal and external respectively) (Ernest, 2010). Restructuring one's own conceptions through the process of metacognitive monitoring is analogous to the restructuring of conceptions that occurs with the appropriation of tools and signs inside of social settings. The metacognitive process of control therefore is enmeshed with the appropriation and use of tools and signs itself, as the metacognitive level impacts the individuals' external responses.

This study explores the phenomenon of individual's cognitive and metacognitive processes during the proof construction process. To examine this phenomenon, I conducted a qualitative study with four undergraduate mathematics majors as they completed paired proof-based tasks in an interview setting. Utilizing the participants' verbalizations during the tasks and their self-described experiences and reflections in follow-up interviews, I investigated indications of metacognitive processes for each student and the subsequent actions the students took in their proof construction process. The four undergraduate student participants were from proof-based undergraduate courses at a mid-size university in the United States. Prior to interviews, students were paired based on similar mathematical backgrounds and experience working on mathematics collaboratively together.

Data Collection

The four participants in this study were mathematics majors at a mid-size university in the United States. They each had previously completed several proof-based courses and were currently enrolled in an abstract algebra course designed to specifically focus on proof writing. To address that individuals' conceptions are "formed through their interactions with each other as well as by their individual processes" (Ernest, 2010).

Each pair of participants were given two tasks, and were audio and video recorded and monitored remotely via Zoom© while Livescribe© pens recorded written data. Two to three days after a task, each individual completed a stimulated recall interview, watching selected video clips from their task completion, and asked to reflect on their experience and thinking during those clips. For each task, one pair was given a true claim (Version A), and one pair was given a false claim (Version B) and asked to prove or disprove the given claim. For the second task, each pair was given a different version of the task from the one they were given for the first task. Both tasks are given below

Task 1 Version A prompt. [Adapted from Gilbert and Larson (2012)] A gamekeeper gives player A a positive integer. The players then take turns changing the integer they are given to a smaller integer and then pass that smaller integer back to their opponent, player B. If the integer you received and are changing is even, the two legal moves are: subtracting 1 from the integer *or* dividing the integer by 2. If the integer you received and are changing is odd, the two legal moves are: subtracting 1 from the integer *or* subtracting 1 from the integer, then dividing this result by 2. The game ends when the integer reaches 0, and the player who makes the last move (changing an integer to zero) wins the game.

Prove or Disprove the Claim: Assume the starting integer is given to player A. Any starting integer that when factored is written as $2^k * l$ where k is an even integer and l is an odd integer, will guarantee that player A wins the game.

Task 2 Version A prompt. On a rectangular $n \times m$ grid, there are candies on each square of the grid. The bottom left corner square in the grid contains a bitter candy, all other candies are sweet. The two players take turns picking squares. When a player picks a square, they must eat every candy that is on the square they picked, or on any remaining squares of the grid above and to the right of the square they chose. The players take turns doing this until one of them eats the bitter candy (on the red square). Whichever player eats the bitter candy loses the game. Suppose there are two players, Player 1 and Player 2, where Player 1 makes the first move.

Prove or Disprove the Claim: For a game played on any $n \times m$ rectangular grid, where n and m are both positive integers, Player 1 will always have a winning strategy. (Excluding the trivial case of a 1×1 rectangle)

Data Analysis

Task interview data and stimulated recall interview data were transcribed. All utterances were coded, first deductively to distinguish verbalizations describing internal reasoning versus verbalizations describing something external. The Model of Sign Appropriation and Use (Ernest, 2010, p. 44) was referenced during this deductive coding to help make these distinctions, and code internal reasoning as intramental.

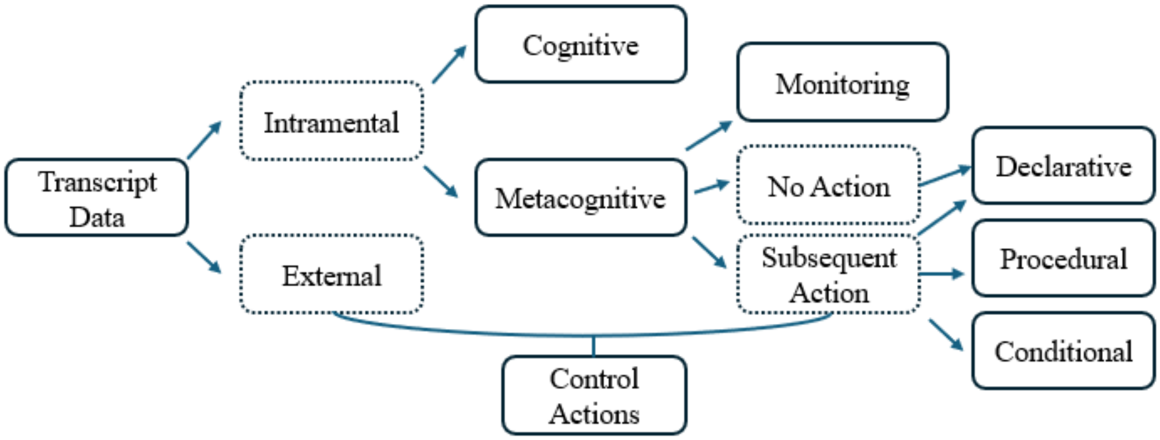

Figure 1. A Flow Chart of the Data Analysis Process

Once verbalizations regarding internal reasoning had been identified, the subject of these utterances was determined. If the subject was something external, that data was coded as cognitive. If the subject of the verbalization was other internal reasoning of the individual, the data was coded as metacognitive.

Data coded as metacognitive was then coded into three categories. Any metacognitive data that was immediately followed by the participant taking an action (or stated intentional inaction) was coded as having a subsequent action. These codes were paired with external coded actions to identify indicated metacognitive control. Metacognitive data where participants stated some intentional reflection about something external was coded as indicating potential metacognitive monitoring. Statements that were declarations of some metacognitive knowledge were coded as stated knowledge, and then deductively coded as declarative. All metacognitive data was deductively coded as indicating declarative, procedural, or conditional metacognitive knowledge. Any data that was coded as metacognitive control, was then connected with external actions and control actions were inductively coded into types of actions.

Consider this example quote from Harper, “I don't know if this general tree thing is ever going to end…Any pattern…has probably already showed up and we just didn't notice it. Let's go back to actual numbers.” This excerpt is related to Harper’s internal reasoning and therefore was coded as intramental. Then, because the subject of these statements is Harper’s own thinking and reflecting on that reasoning and the success of her strategy, this was coded as metacognitive. Harpers initial statement was followed by an action coded as evaluating strategy, and her eventual action was coded as trying examples, as indicated directly in the quote. The relevant metacognitive knowledge was coded as procedural, related to the use of strategies and heuristics.

Results

The control actions, actions participants took immediately following the access of their metacognitive knowledge, included the following: Re-reading the prompt, Reading Aloud, Trying Examples, Conjecturing, Evaluating Argument, Evaluating Strategy, Intentional Inaction, Taking Notes, Math Drawing, Using a Tangible Object, Generalizing, Checking Work, Re-working examples, Organizing Proof, Writing Proof, and Re-reading Proof. Each of the participants engaged in most of these control actions at some point over the course of both tasks. Table 1 offers examples of data for a few of the control action codes, the quoted data, and descriptions of the example control actions.

Table 1. A table of a few demonstrated control actions participants took (Sparks, 2025).

Metacognitive Control Process Actions During Proof Construction		
Control Actions	Description	Example Transcript Data
Re-Working	Re-visiting examples for understanding (not for verification)	“wanting to go back and run through the logic again, remembering what I was trying to prove”
Evaluate Strategy	Actively considering the value or effectiveness of their current approach	“I don't know if this general tree thing is ever going to end…Any pattern…has probably already showed up and we just didn't notice it. Let's go back to actual numbers.”
Organizing Proof	Determining what will be included in the proof or the order	“I don't know, the most efficient way to write this? Let's hit let's write out the tree and then. Yeah, so B can give two…”
Writing Proof	Deciding to move from “scratch work” to writing down a proof or part of a proof	“…I write chaos all over the place on a page just to give myself a sense of what's going on…then I…write it out using more formal language and a series of steps that you can follow…”

Each of the participants had their differences in what types of metacognitive knowledge or processes they demonstrated. Lola often accessed declarative metacognitive knowledge about herself and her proof skills. She also was frequently able to access conditional metacognitive knowledge regarding when and why certain strategy uses might be beneficial in the proof process. Lola also regularly demonstrated evidence of monitoring, reflecting on external outcomes and using that information to inform strategy use or control actions. Harper consistently accessed procedural metacognitive knowledge, using knowledge of strategies and heuristics to initiate control actions.

Riley primarily indicated procedural and conditional metacognitive knowledge and consistently initiated the control actions of evaluating strategies and re-working examples. Spencer indicated declarative metacognitive knowledge, referencing his own perception of his strengths, weaknesses, or affect related to the task completion.

Discussion

There were several noteworthy results that came up during this data analysis. Because mathematical proofs constitute their own genre of writing (Selden & Selden, 2013) one of the insights this research provides is the connection between the access of metacognitive knowledge and the subsequent control actions regarding writing specific components of the proof construction process. The initiation of formal proof writing and the organization of proof structures both were impacted by students’ metacognitive knowledge. In particular, in the majority of instances of these proof writing related control actions, students accessed procedural metacognitive knowledge prior to taking these actions. Participants also made declarative metacognitive knowledge statements that also preceded writing proof and organizing proof related actions, often related to knowing at what point in their proof process they feel has worked well for them to start these components of a task.

In addition to the writing specific aspects of proof construction, another significant result is the control action of re-working examples. Students often accessed metacognitive knowledge before choosing to re-do examples. In this case, not re-doing examples to check for error, but instead to identify reasoning and patterns useful for translating to a more general argument. This is an aspect of these participants' proving process that did does not necessarily make an appearance in other forms of problem solving. While understanding or explaining reasoning is a common aspect of problem solving, the need to structure a generalized argument is not necessarily an expectation or necessity of other forms of problem solving. This data indicates that students are accessing and utilizing metacognitive knowledge to inform the decision to rework examples for this purpose. As this research continues, the types of metacognitive knowledge accessed before, during, and after re-working examples should be explored further.

References

Blanton, M. L., Stylianou, D. A., David, M. M. (2003). The nature of scaffolding in undergraduate students' transition to mathematical proof. N. Pateman, B. Dougherty, J. Zilliox (Eds.), *Proceedings of the 27th Annual Meeting for the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Honolulu: University of Hawaii.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational studies in Mathematics*, *58*(1), 45-75.

Ernest, P. (2016). The problem of certainty in mathematics. *Educational studies in mathematics*, *92*, 379-393.

Ernest, P. (2010). Reflections on theories of learning. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 39–47). Berlin and Heidelberg: Springer.

Grossman, P. L., Smagorinsky, P., & Valencia, S. (1999). Appropriating tools for teaching English: A theoretical framework for research on learning to teach. *American journal of education*, *108*(1), 1-29.

Hamami, Y. (2022). Mathematical rigor and proof. *The Review of Symbolic Logic*, *15*(2), 409-449.

Hanna, G. (2020). Mathematical proof, argumentation, and reasoning. *Encyclopedia of mathematics education*, 561-566.

Hanna, G., & Barbeau, E. (2010). Proofs as bearers of mathematical knowledge. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics* (pp. 85-100). Dordrecht: Springer.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, *43*(4), 358-390.

Krusemeyer, M., Gilbert, G. T., & Larson, L. C. (2012). *A mathematical orchard: Problems and solutions* (Vol. 24). MAA.

Lerman, S. (Ed.). (2020). *Encyclopedia of mathematics education*. Cham: Springer International Publishing.

Mamona-Downs, J. (2002) Accessing Knowledge for Problem Solving. In *Proceedings of the*

International Conference on the Teaching of Mathematics (at the Undergraduate Level)
Mamona-Downs, J., & Downs, M. (2011). Proof: A game for pedants. In *Proceedings of CERME* (Vol. 7, pp. 213-222).
Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in mathematics*, *27*(3), 249-266.
Nelson, T. O., & Narens, L. (1994). Why investigate metacognition? *Metacognition: Knowing about knowing*, *13*, 1-25.
Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof?. *Educational studies in mathematics*, *52*(3), 319-325.
Savic, M. (2015). On similarities and differences between proving and problem solving. *Journal of Humanistic Mathematics*, *5*(2), 60-89.
Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematics Enthusiast*, *10*(1), 303-334.
Schoenfeld, A. H. (1994). What do we know about mathematics curricula?. *The Journal of Mathematical Behavior*, *13*(1), 55-80.
Sparks, S. (2025). *Metacognitive Processes of Undergraduate Students During Mathematical Proof Construction*. In S. Cook, B.P. Katz, & K. Melhuish (Eds.), *Proceedings of the 27th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1095–1101). SIGMAA on RUME. Alexandria, VA.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational studies in mathematics*, 48, 101-119
Vygotsky, L. S. (1987). *The collected works of LS Vygotsky: Volume 1: Problems of general psychology, including the volume Thinking and Speech* (Vol. 1). Springer Science & Business Media.
Zhao, N., Teng, X., Li, W., Li, Y., Wang, S., Wen, H., & Yi, M. (2019). A path model for metacognition and its relation to problem-solving strategies and achievement for different tasks. *ZDM*, *51*, 641-653.
Zorn, P. (Ed.). (2015). *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences*. The Mathematical Association of America.

Theoretical Reports