Mapping the Demands of Thinking: Spatial Reasoning and Cognitive Demand in an Undergraduate Discrete Mathematics Textbook

Laurie Short
North Carolina State University

Spatial reasoning and cognitive demand are critical for mathematical learning, yet they are rarely examined together. This study analyzes 178 tasks from the graph theory chapter of Discrete Mathematics: An Open Introduction to explore how these dimensions present themselves and how they interact. Most tasks were of high cognitive demand and required spatial reasoning, especially spatial relations and visualization. Notably, spatial reasoning did not consistently appear as cognitive demand increased in tasks. These findings suggest no clear correlation between the cognitive demand of tasks and the spatial reasoning required.

Keywords: spatial reasoning, cognitive demand, task analysis

Introduction

Strong spatial skills demonstrate advantages for future STEM performance, and notably, there is extensive correlation linking spatial skill to mathematics performance (Gilligan-Lee, Hawes, and Mix, 2022). Spatial reasoning is often overlooked in mathematics curricula at all levels (Gilligan-Lee, Hawes, and Mix, 2022), despite the high demand for STEM skills and the proven malleability of spatial reasoning (Uttal et al., 2013).

Equally important are problem-solving and reasoning skills for STEM success (NRC, 2012). When tasks actively engage students at a high level, they help develop these essential skills. This study examines exercises in the widely used open-source textbook, *Discrete Mathematics: An Open Introduction,* to explore how spatial reasoning and cognitive demand are represented in tasks, aiming to identify patterns in how graph theory problems encourage different types of thinking. This research seeks to determine which spatial skills occur most often and whether their presence aligns with patterns of cognitive demand. This analysis contributes to ongoing discussions about the importance of task design in undergraduate mathematics education and highlights spatial reasoning as a valuable component of mathematical learning.

Literature Review

Spatial Reasoning and STEM Outcomes

Spatial reasoning is the ability to understand, visualize, and manipulate objects and their relationships in space. Spatial reasoning is often thought of in terms of three connected parts: spatial visualization, spatial relations, and spatial orientation (Lohman, 1979). *Spatial visualization* is the ability to mentally construct or interpret a visual representation from a diagram or description. *Spatial relations* refer to understanding how elements are arranged and connected within a visual structure. *Spatial orientation* requires mentally changing your point of view to see an object from a different angle. Research has shown that spatial skills are highly malleable, and small amounts of training can improve spatial reasoning in children and adults (Uttal et al., 2013). Despite its importance in STEM fields (Wheatley & Abshire, 2002; Newcombe, 2010), spatial reasoning remains underemphasized in classroom practice.

Cognitive Demand and Task Complexity

Mathematics education researchers often examine how task design affects the depth of students' thinking (Arbaugh & Brown, 2005). Stein and Lane (1996) emphasize the importance of "starting with high-level, cognitively complex tasks if the ultimate goal is to have students develop the capacity to think, reason, and problem solve" (p.344). Building on this, Stein and Smith (1998) introduced a four-level taxonomy for classifying mathematical tasks based on cognitive demand. In this taxonomy, tasks are categorized as either lower-level demands or higher-level demands. Stein and Smith argue that although choosing a high-level task does not guarantee student engagement at a high level, it is a necessary prerequisite for it (1998).

Intersecting Spatial and Cognitive Demands

Although both cognitive demand and spatial reasoning are recognized as critical, few studies examine their overlap. Cognitive demand of math tasks has often been categorized using Smith and Stein's framework (1998) (Jones & Tarr, 2007; Sapkota and Bondurant, 2024) while spatial reasoning research has focused primarily in K–12 settings (Battista, 2007; Younger, 2018; Hawes, Gilligan-Lee, & Mix, 2023). Few, if any, studies have examined their overlap, and none have systematically addressed this in undergraduate mathematics. Given this gap, undergraduate discrete mathematics provides a useful setting to explore how spatial reasoning and cognitive demand intersect.

The Case of Undergraduate Discrete Mathematics

Discrete mathematics courses often ask students to construct and navigate graphs, networks, and combinatorial structures, activities that are inherently spatial. Graph theory is a core component in discrete math curricula and is a spatially rich domain. Unlike geometric contexts, spatial reasoning in graph theory often involves abstract structures that require students to reason about connectivity, adjacency, and relational structure. An analysis of the graph theory chapter of Levin's *Discrete Mathematics: An Open Introduction* explores which visual-spatial skills appear most frequently, how they align with levels of cognitive challenge.

Research Questions

This study of spatial reasoning and cognitive demand of tasks in *Discrete Mathematics: An Open Introduction* was guided by the following research question: *How are levels of cognitive demand distributed across tasks requiring different types of spatial reasoning and task types in Discrete Mathematics: An Open Introduction?* To address this, the study focused on three sub-questions:

1. How are levels of cognitive demand distributed across task types (e.g., preview activities, in-text exercises, end-of-section problems, etc.)?
2. How does the nature of spatial reasoning vary across these task types?
3. Are there patterns that link levels of cognitive demand with specific types of spatial reasoning required in these tasks?

By investigating these questions, the aim is to understand how spatial reasoning, task type, and cognitive demand interact in this text. These goals shape the study design and analysis outlined in the following methods section.

Methods

This study focused on a chapter introducing one core discrete topic, graph theory, to identify opportunities for students to engage in high cognitive demand tasks and spatial reasoning. *Discrete Mathematics: An Open Introduction* was chosen because it's a free, open-source resource that's "especially well-suited for courses that incorporate inquiry-based learning" (Levin, n.d.) and is widely used as a primary or supplemental text in colleges and universities. The graph theory chapter was selected for this analysis because it appears early in the textbook (Chapter 2), making it easier to determine what topics students know when approaching each task. Additionally, given that spatial processing plays a key role in understanding complex graphs (Trickett & Trafton, 2007), this section provided a fitting context for this study.

All tasks in Chapter 2, except the summary, were analyzed. After examining results holistically, results were disaggregated by the type of problem: investigate, preview activities, in-text examples, reading questions, end-of-text exercises, and additional exercises. Initial coding assigned each task to the type of problem, and tasks were left out if they did not contain a question or ask students to do something. Tasks were the unit of analysis, and overall, n=178 tasks were examined. Grouping by task type, two frameworks were adapted into deductive coding schemes to measure the level of cognitive demand and type of spatial reasoning required for each task.

To identify cognitive demand, each task was coded based on Smith and Stein's Levels of Demand framework (1998). Tasks were identified as lower-level demands or higher-level demands. Lower-level demand tasks were broken into the categories of *memorization* and *procedures without connections*, while higher-level demand tasks were broken into the categories of *procedures with connections* and *doing mathematics*.

Next, a coding scheme was developed to identify the spatial reasoning required for each task. Each task was coded for the type(s) of spatial reasoning required, if any, using Lohman's (1979) framework, resulting in the following codes: *spatial visualization, spatial relations*, *spatial orientation,* and *no spatial reasoning*.

As many tasks contained more than one type of spatial reasoning, codes were treated as mutually exclusive combinations. For example, 'SV' refers to tasks requiring only spatial visualization, while 'SV & SR' includes tasks requiring both spatial visualization and relations. This led to eight mutually exclusive codes, clarifying both which types of spatial reasoning were most frequently used and how often students would need to integrate multiple forms of spatial thinking to complete a task. Figure 1 shows the classic Königsberg bridge problem. This task was coded as *doing mathematics*, since students must explore and reason to solve this task. This task was also coded as *spatial relations and visualization*, as students must understand how the islands and riverbanks (nodes) are connected by bridges (edges) while also mentally tracing and manipulating paths.

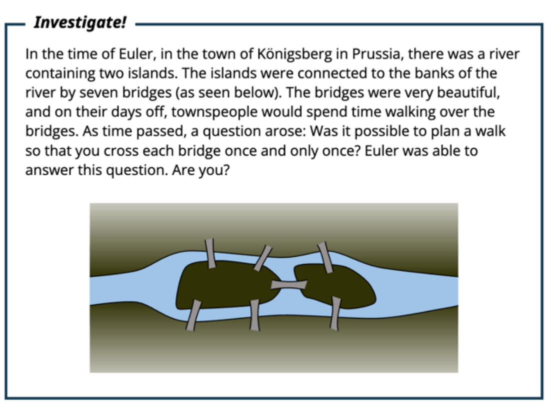

Figure 1. Task coded as doing mathematics and requiring spatial relations and orientation

Findings

Research Question 1: Cognitive Demand of Tasks

This study examined the cognitive demand of 178 tasks in Chapter 2. About 85% were of high cognitive demand (54% *procedures with connections*, 32% *doing mathematics*). The remaining 15% were *procedures without connections*; none were *memorization.*

Disaggregated by task type, most tasks were additional exercises (47%), and nearly all of these were of high cognitive demand, with a roughly even split *of doing mathematics* and *procedures with connections* tasks. End-of-text and reading questions leaned toward *procedures with connections,* while investigate tasks were exclusively high-level, mostly coded as *doing mathematics*.

Research Question 2: Spatial Reasoning Required of Tasks

Of the tasks examined, 67.4% required spatial relations (SR), 64% required spatial visualization (SV), 15.2% required spatial orientation (SO), and 28.7% required no spatial reasoning (NS). Note, that this does not add up to 100%, as some tasks were coded as requiring more than one type of spatial reasoning. Most tasks required some type of spatial reasoning (71.3%), with SR and SV showing up most frequently. Almost half of the tasks that required no spatial reasoning occurred in Section 2.6: Relations and Graphs. Tasks requiring SV and SR together occurred most frequently, with this pair present in 46.6% of all tasks (Table 1). Across task types, spatial reasoning appeared most in investigate and preview tasks (over 85%), somewhat less in additional exercises (76%), and least in end-of-text problems (63%).

Table 1. Mutually exclusive spatial reasoning codes with counts

Spatial Reasoning Only (SR)	Spatial Orientation Only (SO)	Spatial Visualization Only (SV)	SR & SV Only	SR & SO Only	SV & SO Only	SR, SV, & SO	No Spatial Reasoning (NS)
n=13	n=0	n=4	n=83	n=0	n=3	n=24	n=51

Research Question 3: Relationship Between Cognitive Demand and Spatial Reasoning

Spatial reasoning appeared across all cognitive demand levels but was most frequent in *procedures with connections* tasks (77%). *Doing mathematics* tasks required it less (63.2%), and *procedures without connections* the least (46.2%). Again, SV & SR were the most common combination across levels.

By task type, investigate tasks were all high-level, with almost all of these high-level tasks requiring spatial reasoning. Preview tasks also required spatial reasoning, while in-text examples and reading questions had many tasks without spatial reasoning. End-of-text exercises mixed cognitive demand and spatial reasoning. Interestingly, additional exercises showed higher spatial reasoning in *procedures with connections* tasks than in *doing mathematics* tasks.

Discussion and Future Work

Rather than evaluating the quality of tasks or making claims about student learning, this analysis documents how spatial reasoning and cognitive demand co-occur in graph theory tasks, positioning the textbook as a resource for understanding how mathematicians structure opportunities for spatial thinking. Across all task types, a clear pattern emerges in the data: spatial reasoning and high cognitive demand are both emphasized, but they are not always coupled. Further, moving from *procedures with connections* to *doing mathematics* does not indicate that there will be more spatial reasoning required.

Overall, most tasks analyzed were of high cognitive demand, which was expected as this book is inquiry-based. Notably, tasks that appeared at the beginning of each section, investigate and preview activities, were almost all high cognitive demand, aligning with Smith and Stein's recommendation of "beginning with a task that has the potential to engage students at a high level if your goal is to increase students' ability to think and reason" (1998, p. 347). There was a difference in the breakdown of cognitive demand for end-of-text and additional exercises, both of which appear at the end of each section. Additional exercises included a significantly higher proportion of *doing mathematics* tasks, suggesting that assignments drawn from either the initial end-of-text exercises or the additional exercises could vary substantially in cognitive demand.

Spatial reasoning was present in many tasks (71.3% overall), but its distribution among tasks was uneven. Overall, spatial visualization and relations appeared most frequently, and were often grouped, as tasks that required students to think about how aspects of graphs related also required them to visualize the graph. Notably, almost all tasks that appeared at the beginning of each section, investigate and preview activities, required spatial reasoning, and additional exercises showed a slightly higher percentage of tasks that required spatial reasoning than end-of-text exercises.

Regarding the relationship between cognitive demand and spatial reasoning, spatial reasoning did not consistently increase with higher levels of cognitive demand. While many tasks coded as *procedures with connections* involved spatial thinking, spatial reasoning was less frequently seen in tasks categorized as *doing mathematics*. Many of these were focused on proof-based reasoning, where the cognitive demand was of the highest level, but spatial reasoning may not have been necessary. Though there was no clear relationship between cognitive demand and spatial reasoning, the data suggest that there may be a relationship between spatial reasoning and the type of task.

The data show two important findings: (1) spatial reasoning is not automatically embedded in mathematically rich textbook tasks, and (2) spatial relations and visualization are especially prominent forms of spatial reasoning in graph theory tasks. As spatial reasoning is highly malleable (Uttal et al., 2013), these findings can inform pre-college preparation or in-class scaffolding to equip students with the spatial skills necessary for success. These patterns also raise questions about whether selecting high-level tasks alone is sufficient to ensure consistent opportunities for students to engage in spatial reasoning, a hypothesis that warrants further study with student data.

Preliminary Nature of This Report

This report represents an early stage of analysis and is limited to a single textbook chapter with coding conducted by one researcher. Because this study analyzes textbook tasks rather than student work or classroom enactment, claims about how students engage in or develop spatial reasoning are necessarily inferential. Future directions could include expanding the analysis to additional chapters, additional textbooks, additional math content areas, and incorporating interrater reliability. I am especially seeking feedback on two areas still in development: (1) how the coding frameworks for spatial reasoning might be refined to capture nuances in undergraduate mathematics tasks, and (2) how the implications could be framed to best support both pre-college preparation and in-college scaffolding.

References

Arbaugh, F., & Brown, C. A. (2005). Analyzing mathematical tasks: A catalyst for change? *Journal of Mathematics Teacher Education*, 8, 499–536. https://doi.org/10.1007/s10857-006-6585-3

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Information Age Publishing.

Coulanges, L., Bachman, H. J., Libertus, M., & Votruba-Drzal, E. (2024). Examining profiles of U.S. children's screen time and associations with academic skills. *Journal of Children and Media*, 18(2), 235–253. https://doi.org/10.1080/17482798.2024.2327021

Gilligan-Lee, K. A., Hawes, Z. C. K., & Mix, K. S. (2022). Spatial thinking as the missing piece in mathematics curricula. *Science of Learning*, 7, 10. https://doi.org/10.1038/s41539-022-00128-9

Hawes, Z. C. K., Gilligan-Lee, K. A., & Mix, K. S. (2023). Infusing spatial thinking into elementary and middle school mathematics: What, why, and how? In K. M. Robinson, A. K. Dubé, & D. Kotsopoulos (Eds.), *Mathematical cognition and understanding: Perspectives on mathematical minds in the elementary and middle school years* (pp. 13–33). Springer Cham. https://doi.org/10.1007/978-3-031-29195-1_2

Jones, D., & Tarr, J. (2007). An examination of the levels of cognitive demand required by probability tasks in middle grades mathematics textbooks. *Statistics Education Research Journal*, 6(2), 4–27. https://doi.org/10.52041/serj.v6i2.482

Levin, O. (n.d.). Discrete mathematics: An open introduction (4th ed.). *Open Math Books.* Retrieved from https://discrete.openmathbooks.org/dmoi4.html

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. NCTM.

National Research Council. (2006). *Learning to think spatially: GIS as a support system in the K–12 curriculum.* National Academy Press.

National Research Council. (2012). *A framework for K–12 science education: Practices, crosscutting concepts, and core ideas.* National Academies Press.

Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator,* 34(2), 29–43.

Sapkota, B., & Bondurant, L. (2024). Assessing concepts, procedures, and cognitive demand of ChatGPT-generated mathematical tasks. *International Journal of Technology in Education (IJTE)*, 7(2), 218–238. https://doi.org/10.46328/ijte.677

Smith, M. S., & Stein, M. K. (1998, February). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School*, 3, 344–350.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation*, 2(1), 50–80. https://doi.org/10.1080/1380361960020103

Trickett, S. B., & Trafton, J. G. (2007). The use of spatial cognition in graph interpretation. In D. S. McNamara & J. G. Trafton (Eds.), *Proceedings of the 29th Annual Cognitive Science Society* (pp. 2111–2116). Cognitive Science Society.

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin*, 139(2), 352–402. https://doi.org/10.1037/a0028446

Wheatley, G. H., & Abshire, G. (2002). *Developing mathematical fluency*. Mathematics Learning.
Younger, M. (2018). *Spatial skills activities in the middle school mathematics teachers' toolkit: The impact of spatial skill activities on mathematical thinking* (Doctoral dissertation). Ontario Institute for Studies in Education, University of Toronto.