Investigating the Role of Various Conceptions of Subtraction in Calculus: A Textbook Analysis

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The concept of difference, and its representation as distance within a graphical representation pervades Calculus. Indeed, difference expressions represent change, the very foundation of the subject. Yet, students may often default to a takeaway conception of subtraction, which may be incompatible with viewing subtraction as finding change. Other ways of conceiving of subtraction may support additional ideas in Calculus, related to translating functions, finding areas and volumes, or indexing terms of a sequence. In this preliminary report, we investigate the prevalence of various conceptions of subtraction in the study of Calculus through a textbook analysis of Stewart's Calculus: Early Transcendentals. We report our findings from our analysis and future directions for research.

Keywords: subtraction, difference, Calculus, representations, textbook analysis

The notion of difference, and its graphical representation as a distance between two points, is foundational in the study of Calculus, as it finds the change in values of a variable. Some instances include the limit definition of the derivative ($f(x+h)–f(x)$), the Mean Value Theorem ($f(b)–f(a)$), and the Fundamental Theorem of Calculus ($F(b)–F(a)$). To use a difference expression to model change in a quantity, one must conceive of subtraction as comparison, rather than as takeaway. Yet, the takeaway conception may be at the forefront of students' minds instead, possibly reinforced by concrete activities such as physically taking away objects or "going back" used to introduce subtraction (Selter et al., 2012). Though first teaching subtraction as takeaway at the elementary level is recommended, research has suggested that students need to hold multiple conceptions of subtraction for deep comprehension across the mathematics curricula (Bofferding & Wessman-Enzinger, 2017; Sievert, 2021). However, many undergraduate students may not readily use such a conception of subtraction to represent a horizontal distance between two functions in the Cartesian plane (Sinojia et al., 2023). Other ways of conceiving of subtraction may also be beneficial in Calculus, such as interpreting subtraction as translating a function down some number of units. To better understand the ways in which subtraction conceptions are embedded in the study of Calculus, we decided to conduct a textbook analysis. We chose Stewart's (2021) *Calculus: Early Transcendentals, 9th edition* because of its ubiquity in undergraduate courses. We follow Weinberg (2010) in adopting the lens of the *implied reader of a Calculus textbook* to consider the various conceptions of subtraction needed to meaningfully respond to the text. With this framing, we ask the following questions:

1. How do readers of *Calculus: Early Transcendentals* need to conceive of subtraction to meaningfully and accurately engage with the text?
2. What is the prevalence of various conceptions of subtraction in the text and by topic?

Theoretical Perspective

We adopt the framing of reader-oriented theory for mathematics textbooks, as explicated by Weinberg and Wiesner (2011), in which the text can be viewed as a site of interaction between the author and reader. The authors of a textbook may envision the reader of their text and address them as their target audience, referred to as the *intended reader*. In contrast, the *implied reader* describes the capacities required of the reader to "respond to the text in a way that is both meaningful and accurate" (Weinberg & Wisener, 2011, p. 52). We take up the idea of the implied reader to examine the ways in which subtraction is required to be conceived of in order to meaningfully and accurately engage with Stewart's *Calculus: Early Transcendentals*.

To consider various conceptions of subtraction, we adopt the four models of subtraction identified by Parr et al. (2025) which they observed with undergraduate students on difference tasks designed for Calculus. These four models are: (1) magnitude, (2) takeaway values, (3) takeaway segments, and (4) translation. Table 1 contains the description of each model of subtraction as observed on a number line task or single axis.

Table 1. Four Models of Subtraction adapted from Parr et al. (2025)

	Models $b–a$ as
Magnitude	The distance between positions a and b
Takeaway values	The resulting position starting at position b and moving left a units
Translation	The distance between $b–a$ and 0
Takeaway segments	The resulting segment from removing a segment of length a from a segment of length b

Methods

We performed an initial textbook analysis of James Stewart's *Calculus: Early Transcendentals* in search of each representation of subtraction, led by the first author. We used each representation's description from Table 1. In order to begin to investigate our research questions, the first author identified whether various conceptions of subtraction occurred in each of the 116 subsection of the text (e.g., 4.7). The decision to document the presence of a particular conception, rather than code each individual expression or figure allowed us to avoid double counting repeated properties, definitions, or equations within each chapter. Preliminary coding was only done by the first author, with input as needed from the co-authors. Thus, the current findings remain exclusive to one perspective until others also code more extensively for reliability.

A subchapter would be coded with an "M" if it included any indication of a magnitude representation of subtraction. Mentions of "distance" or "length" were identified as such, as well as any graphical representation containing the length between two points. These concepts require a magnitude representation because this particular form conceives of subtraction as a representation of distance and change between points.

A subchapter would be coded with a "TV" if it contained any instances of a takeaway values form of subtraction. Language including "go back" or "shift down" were considered. The typical visual cue was any subtraction where the result was an independent value similar to the initial point before subtraction. Such instances demonstrate subtraction as a means of producing a new value rather than moving between two existing positions.

Subchapters would be coded with a "T" if they demonstrated a translation form of subtraction. The primary requirement for this representation to be identified was the translation of a segment representing length. If the endpoints of the visual subtraction on display did not match with the minuend and subtrahend, then some form of translation was involved. Otherwise, the subtraction was just considered an instance of the magnitude representation.

Subchapters would be coded with a "TS" if they involved the takeaway segments form at least once. Similar to translation, takeaway segments uses the same minuend and subtrahend as the magnitude form. This fourth model was considered when language involving "pieces" or "segments" was involved, which appeared in visual models as the comparison of smaller and larger lines that overlapped.

As these are visual representations of subtraction, we considered excluding coding to the explicitly graphical applications within the textbook. However, it became apparent early on that not all graphical concepts, such as the representation of the difference between two functions, were given a visual aid (Stewart et al., 2021, p. 40). Because of the space constraint of the textbook, we decided to code all applications of subtraction rather than exclusively graphical representations. Even if subtraction was not written out as an expression in a subchapter, we would still code for representations of subtraction if concepts applying the subtraction forms were referenced to get to these higher-level concepts. The most notable example of this was the expression Δx, which is used to represent the magnitude of the change in x over a particular interval. This symbol is used in later subchapters to represent this change in x while introducing higher concepts like rates of change (p. 146), double integrals in polar coordinates (p. 1063), and surface area (p. 1177). As a subject that builds upon its smaller concepts, we chose to code for references to earlier subchapters requiring subtraction.

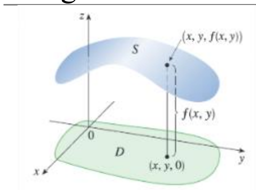
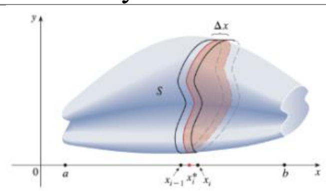
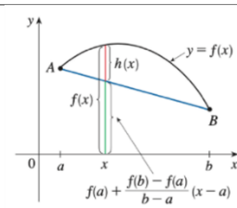
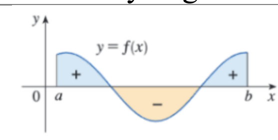
Results

All four representations of subtraction were present within Stewart et al.'s *Calculus* textbook. The magnitude form of subtraction was the most prominent, occurring in 100 of the 116 subchapters (86.96%). Most observed instances can be divided between representations of distance/length and value comparison. Subtraction as length was applied in subchapters exploring rates of change, the precise definition of a limit, polar coordinates, series limits, and vectors. Each of these concepts build off of the basic concept of distance between two points. The distance between an initial and final x value is used in both the creation of derivatives as well as the defining of an integral's interval. These two central Calculus concepts were much of the reason for the magnitude form of subtraction being so prominent across the textbook. Value comparison appeared in additional concepts like the difference of limits, function value differences ($f - g$), rate comparisons, and the difference between two series. Real world examples using this representation of subtraction often involved physical positioning or distance.

Graphing general functions demonstrates how frequently concepts of distance are applied in Calculus. Rather than label the output of a function itself, the textbook would frequently label the length between the x-axis and the output with the function (Table 2). The expression of function outputs as distances from zero puts an emphasis on magnitude and the space between an initial and final value. The epsilon-delta definition of a limit is another interesting representation of distance through the magnitude form of subtraction. The distance between the limit $f(x)$ and $f(x) - \varepsilon$, for some $\varepsilon > 0$, is used as a reference for output value proximity to $f(x)$.

The takeaway values representation was the second most frequent model, and it appeared in 44 of the subchapters (38.26%). There was a greater variety in the applications of the model here, including function transformations, the creation of new functions, sequence notation, value reduction, and repeated subtraction. Function transformations were indicated by the "shifting" of a function around a graph. This became more prominent in later chapters when adding different values of c to an antiderivative. Value reduction occurred in scattered chapters, including the placing of epsilon ball boundaries, subtracting angles to get a third angle, and modeling radioactive decay. Repeated subtraction demonstrated an instance of the operation unique to the takeaway values model, and it was best demonstrated with the concept of an alternating series.

Table 2. Four conceptions of subtraction identified in Stewart's Calculus: Early Transcendentals

Magnitude	Takeaway Values	Translation	Takeaway Segments
			 **FIGURE 4** $\int_a^b f(x)\,dx$ is the net area.
Mapping a function output. The distance from 0 to f(x,y) is labeled f(x,y). Emphasis on distance.	*The (i-1)th term is one term before the ith term. Moving between values performed by counting down.*	*h(x) is the difference between f(x) and the bottom equation. It is then translated onto the x-axis so that Rolle's Theorem can be used to prove the MVT.*	*Subtraction of areas used to find "net area." Introduces integrals with subtraction.*

Sequence notation was the most frequent application of the takeaway values representation of subtraction, even if it is in a more abstract form. Working with the (n – 1)th term requires students to "count back" one term from the nth value. The (n – 1)th term was frequently depicted in graphs as an independent point directly preceding to the nth term. Table 2 presents a visual instance of this more abstract subtraction. When explaining the division of a shape into volume cross-sections, the textbook used the subscript of sequence notation to generalize the visual partitioning beyond any fixed number of segments. Across the chapters, more complicated concepts often employed index notation like this to generalize processes and demonstrate concept application beyond any limiting context. Integral subintervals used subscript subtraction most frequently.

The translation representation of subtraction was the least present within the textbook. There were only 13 subchapters, making up 11.3% of the textbook's sections. The most common occurrence of this representation of subtraction was within settings where the initial point was constant. The greatest example of this is the subtraction of two functions to create a new function. No matter what the result of subtracting two outputs is, the new function's output must be a magnitude relative to zero. The best visual depiction of this was in the proof of the Mean Value Theorem (Table 2). Function subtraction was used to produce a new function $h(x)$, which would have root values for the use of Rolle's Theorem. In a similar manner, subtracting vectors creates a new vector that can share the same initial value as the first two vectors. Problems involving centers of mass also applied this model, visualizing all points with a distance relative to the center. Other real-world problems such as the concept of displacement from a starting point can use this form of subtraction.

The takeaway segments representation of subtraction appeared in 29 of the subchapters (25.22%). Instances of a takeaway segments form being used include the comparison of lengths or quantities as well as area or volume subtraction. Length comparison is helpful when trying to visualize concepts like change over time, differences in rate, or radius subtraction. Displacement and position functions demonstrate subtraction as a tool for finding the remaining or added length when comparing two different points or positions. Area and volume subtraction use the takeaway segments model more consistently because they represent the comparison of established “spaces,” which are closer in concept to a “segment” than an individual point on a graph. Subtracting inner volumes from outer volumes, calculating the area of a portion of a shape, depicting change in area, and representing segments of polar rectangles are all examples of Calculus concepts that applied the takeaway segments form of subtraction in this way. Even integrals were introduced with help from this representation of subtraction, referring to an integral as “net area” (Table 2). Though the positive and negative areas are not overlapping in the same way that the position function’s segments did, their subtraction still represents the comparison of spaces instead of points.

In this study’s current state, the subtraction of two functions to produce a third function was the only Calculus concept where three or more representations of subtraction are applied. If $h(x) = f(x) - g(x)$, then $h(x)$ is the magnitude between the points $f(x)$ and $g(x)$. As a function of its own, all values of h are graphically represented as a distance from zero. This means that the magnitude of $f(x) - g(x)$ is translated to be a distance in relation to the x-axis. This distance can be understood as the difference between the distance from zero to $f(x)$ and the distance from zero to $g(x)$, alluding to the takeaway segments model. The best instance of this is in the proof of the Mean Value Theorem.

After determining the function for the secant line between the two endpoints of the designated interval, function subtraction is used to determine the magnitude of the distance between the secant line and the original function. This new function has two zeros at the endpoints of the interval, as the outputs on the secant line and in the initial function are equivalent. This allows for the use of the previously proven Rolle’s Theorem to complete the remainder of the proof.

Discussion & Future Development

These findings assert that a magnitude representation of subtraction is the most essential for students to understand in Calculus. Graphical interpretations of both derivatives and integrals rely on expressing or representing the distance between points, making the magnitude model of subtraction a necessary resource for Calculus. Yet, this does not mean that the other representations of subtraction should be considered as nonessential. Each of the representations appeared within their own contexts, and thus, students can benefit from each of them to varying degrees. Rather than teach only a magnitude representation of subtraction, it would be beneficial for students to learn how to identify the situations where each form can be helpful.

We pose the following questions for discussion with the audience as we consider our next round of coding of the data:

1. Multiple representations of subtraction can be considered for our data. What is the audience’s perception of the primary (and secondary) representations for the “net area” example?
2. Are there any other examples from the Stewart text that you do not think can be described by one of the models?
3. Are there any limitations of this framework as we consider examples in higher dimensions?

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