A Comparative Case Study of Student Reasoning about Planes and Vertical Change in Multivariable Differential Calculus

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This is part of a larger APOS-grounded study where we investigate students' reasoning about and between multiple components of multivariable differential calculus. We report on the different reasoning of the strongest interview participants from two sections taught by the same instructor. One section used the locally linear approach; the other was taught traditionally. The locally linear approach leverages the geometric understanding of the point-slopes equation of a plane to help students develop key ideas for the differential calculus of two-variable functions. We used semi-structured interviews to investigate students' understanding of multiple components of differential calculus and the relations established between those components. In this report, our analysis focuses on the relations that two high-performing students established between the components of plane and vertical change. Findings suggest that the locally linear approach may help students obtain a deeper understanding of vertical change on a plane than traditional instruction.

Keywords: multivariable calculus, plane, locally linearly approach, APOS, differentials

Introduction

Multivariable calculus is an important tool to model different phenomena in science and engineering. Typically, its study begins with the simplest case, two-variable functions. The study of the didactics of two-variable functions has gained traction in the past twenty years (for a survey, see Martínez-Planell and Trigueros, 2021). In an early paper, Tall (1992) proposed how students may visualize differentials in two and three dimensions. He proposed a model to enable students to think of a two-variable function as locally approximated at a point $(a, b, f(a, b))$ by its tangent plane, with the change in function values modeled by the total differential

$$[df(a,b)](dx, dy) = \frac{dz_x}{dx}dx + \frac{dz_y}{dy}dy$$

where *dx* and *dy* are independent variables representing horizontal change at (a, b) and dz_x, dz_y are the corresponding vertical changes along the tangent plane (Figure 1) for $\frac{dz_x}{dx} = f_x(a, b)$ and $\frac{dz_y}{dy} = f_y(a, b)$. Martínez-Planell et al. (2017) built upon Tall's idea to describe the notion of vertical change on a plane (Figure 2, left) which may be symbolized as

$$\Delta z = \Delta z_x + \Delta z_y = m_x\Delta x + m_y\Delta y.$$

From this idea, you can obtain

$$z - c = m_x(x - a) + m_y(y - b)$$

the point-slopes equation of a plane,

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

the equation of a tangent plane at a point, and directional derivatives (Figure 2, right). While the idea of vertical change appears when interpreting directional derivatives, students find it challenging to construct this notion on a plane. Students do not readily generalize slopes to three dimensions and have difficulty combining rates of change in the *x* and *y* directions into a single rate of change (Martínez-Planell et al., 2015, 2017; Moore-Russo et al., 2011; Weber, 2015; Yerushalmy, 1997). McGee & Moore-Russo (2015) observed that students tend not to think of partial and directional derivatives as slopes of lines on the tangent plane. This led to the design of activities to help students construct a locally linear approach to the differential calculus of two-variable functions.

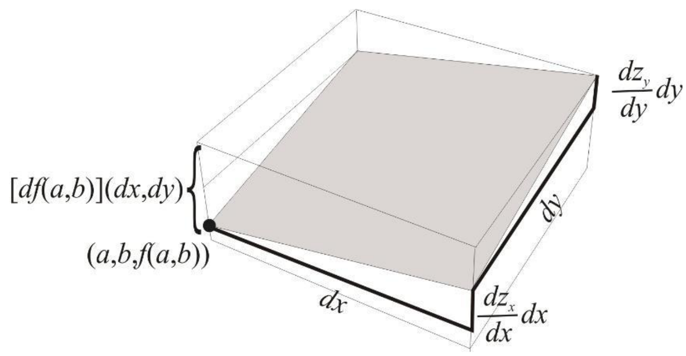

Figure 1. Total differential at a point.

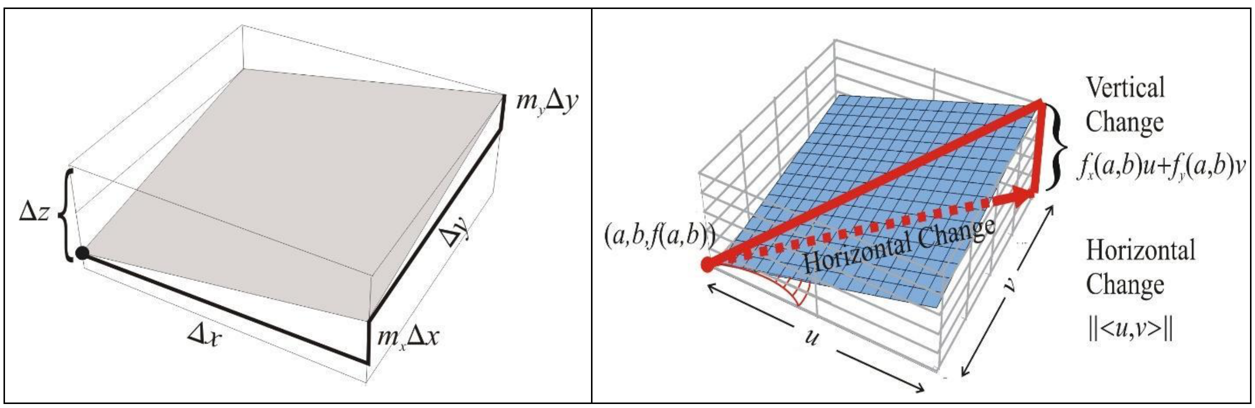

Figure 2. Vertical change on a plane, $\Delta z = \Delta z_x + \Delta z_x = m_x \Delta x + m_y \Delta y$ *and directional derivative,* $D_{\langle \Delta x, \Delta y \rangle} f(a,b) = \frac{f_x(a,b)\Delta x + f_y(a,b)\Delta y}{|\langle \Delta x, \Delta y \rangle|}$.

Borji et al. (2023, 2024) used previously documented activities to observe improved student understanding of partial and directional derivatives. The present study adapted some of these activities for an interview protocol. In this paper, we focus specifically on the relations students established between plane and vertical change. The research question driving this study is: How do two high-performing students, one taught with the locally linear approach and one not, differ in the types of relations they demonstrate between two key components of multivariable differential calculus, plane and vertical change, when solving a series of tasks involving geometric interpretation?

Framework

APOS Theory (Arnon et al., 2014) considers how students carry out Actions (step-by-step manipulations) mechanically until they are able to internalize these as Processes where they can imagine actions without actually performing them. These processes are then encapsulated into a mental Object that in and of itself can be acted on and manipulated. Finally, a Schema of a mathematical concept is a structured collection of Actions, Processes, Objects, and previously constructed Schemas. These components are interconnected so that an individual can recognize when a problem fits within the Schema's scope (coherence) and can activate it to address a task that is presented. Schemas constantly evolve as the students face new problems and learn new mathematics.

In this paper, we use the notion of types of relations between Schema components, introduced by Trigueros (2019), as helpful in describing this evolution. A *correspondence* (C) relation exists between components when, out of habit or repetition and with limited external prompting, the individual correctly relates one component to another without being able to justify the relation. A *transformation* (T) relation between two components exists when a student is able to explain or justify how one component relates to another, possibly through the use of other mediating components. An *equivalence* (E) relation exists between components if properties of one component are conserved or maintained in another such that changes in one component can fluently be interpreted as changes in the other. The result is that a person is able to move easily and efficiently from one component to another, in a consistent manner, across tasks despite the particulars of any given task.

Methodology

One of the researchers taught two multivariable calculus sections for engineering students in Spring 2024. Both sections used the Stewart (2012) textbook and had the same homework exercises with only a few pre-class assignments that differed. Both sections dedicated the same amount of class time to differential multivariable calculus and had common assessments. One section with 19 students was taught traditionally. In particular, the point-normal equation of a plane was underscored and used to develop the differential calculus. This is convenient algebraically but can mask the geometry. With the traditional section, the point-slope equation of a plane was introduced later but was not emphasized. The experimental section with 40 students used activities based on those discussed by Martínez-Planell et al. (2017) and adapted them for use with 3D printed surfaces and the digital application CalcPlot3D (Seeburger, 2016). Within these activities, students developed an approach to compute vertical change on a plane using a 3D-printed surface and later used this to construct the point-slope equation of a plane, prior to exposure to the point-normal equation.

All students were invited to participate in two 45-minute semi-structured interviews at the end of the semester; ten agreed to participate. Most of the 27 questions posed were taken from Borji et al. (2023, 2024). Two of the authors independently coded each participant's response for each question; then, all three discussed and analyzed codes until a consensus was reached. The coding process involved first identifying the components of differential calculus the student was relating in their response, including Function (F), Slope (S), Partial Derivative (PD), Directional Derivative (DD), Gradient (G), Vertical Change (VC), and Plane/Tangent Plane (P). We then decided on the type of relation (i.e., no relation (NR), correspondence (C), transformation (T)) between components, if any, the student was demonstrating in their answer to that particular question. Finally, we looked at each pair of components and identified for which problems that pair had been coded. Looking across the problems and the type of relation coded for each problem, we were able to assign a type of relation for a pair of components. An equivalence (E) relation was assigned to a pair of components only if it was established consistently and fluently across the different problems involving the components.

For brevity, we report only on our coding for the relations established by two students between the components plane or tangent plane (P) and vertical change (VC). Note that the VC code refers to a process conception of vertical change on a plane or tangent plane when there is horizontal change in both the *x* and *y* directions.

Results

In this report, we compare the cases of Jackson and Daphne. They were the strongest performers on the interview tasks in the experimental and traditional sections, and they received course grades of A and B+, respectively. We focus specifically on the relations Jackson and Daphne established between plane/tangent plane and vertical change and the components they used to mediate these relations. Below, we present and discuss transcripts from the three interview questions which provided opportunities for students to relate the plane or tangent plane to vertical change.

For question 1, students were given Figure 3. After finding slope of the line for question 1a, in 1b students were asked: *Starting at any point on the plane, find how much the z coordinate changes if the x coordinate increases by 4 units and the y coordinate increases by 5 units*.

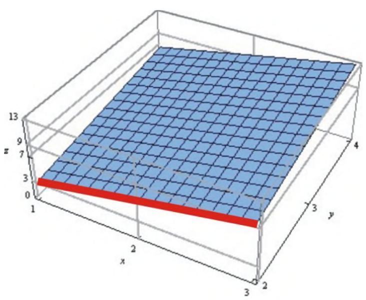

Figure 3. Plane for question 1b.

Jackson: So, I have m_x, to complete the problem I need to find m_y [he correctly finds m_y] so, the change in *x* is equal to 4, change in *y* equals to 5. So, with that in mind … change in *z* is equal to 3 times 4 plus 2 times 5, 12 plus 10, 22

Without prompting or assistance, Jackson used slope and a given amount of horizontal change to determine a resulting change in *z* in one direction. Additionally, he was able to coordinate the change in *z* resulting from movement in two directions to find vertical change. This was coded as a transformation relation because Jackson was able to explain how he moved from plane to vertical change, using slope as a mediator (P-S-VC). We now consider Daphne's response to question 1b.

Daphne: So, $\Delta x = 4$, $\Delta y = 5$ … I'm setting up the equation of a plane… that won't work…Okay, I don't really know how to, like, think about this one.

Interviewer: Let me give you a hint. What would be the change in the *z* coordinate if the *x* coordinate increases by 4 units? Only the *x* coordinate.

Daphne: If the *x* coordinate increases by 4 units, the *z* would increase by 12

Interviewer: Good, and what would happen if the *y* coordinate increases by 5 units?

Daphne: If the *y* coordinate increases by 5 units, then the *z* coordinate will also increase, but I don't know by how much… I guess I could find the … entire equation of the plane.

Interviewer: Just like you found how much z increases when x changes, could you use the same idea to find out how much z changes when y changes?

Daphne: … I guess; I don't know.

Interviewer: Let me give you another hint, can you find a slope in the y direction?

Daphne: [She was able to find the slope in the y direction.] … delta x would equal 3 times 4, and delta y would equal 5 times 2, that's 12 and this is 10, so then, in that case, could you square those two values, add them together? … I just don't know what to do with this information now that I have it.

Interviewer: Okay, so you are standing at a point, you move in the x direction, and you go up 12 units. Then you move in the y direction, and you go up 10 units. So, how much did you go up overall?

Daphne: Oh! It's just 22.

We see that Daphne eventually related the given plane component (P) with the requested vertical change (VC), using the slope component (S) to mediate. However, this happened only with significant scaffolding from the interviewer. We therefore coded "no relation" (NR) for P-VC for Daphne on problem 1b, despite her eventually producing the correct answer of 22. Daphne demonstrated a tendency to rely on habit or repetition, more than once suggesting finding the plane's equation without any indication that she understood its meaning. She first needed to be prompted to compute the slope in the y direction and the changes in z due to moving solely in the x or y direction. Moreover, Daphne showed difficulty coordinating changes in the x and y directions, as Yerushalmy (1997) and Weber (2015) have observed in their work.

Problem 1c asked students to find the equation of the plane in Figure 3. They each did so correctly. Then, they were asked to relate this equation to the previous calculations in part b.

Daphne: I mean, yeah, it includes both of the slopes, and, like, it shows us how x and y, like, change independently of each other …

Daphne produced the point-slope equation of the plane with no assistance. This appears to be a memorized process for Daphne. When asked to relate the equation to prior calculations, she focused on the symbolic calculations in the formulas without any geometric interpretation. Therefore, this was coded as a correspondence relation (P-S-VC). For Jackson:

Jackson: I guess it's, yeah, related in the sense that if I plugged in an (x, y) point, an x point of, x change by four, x equal to 5 [from the initial x equal 1] and a y equal to 7 [from the initial y equal 2], I'd get a change in z of 22. So, depending on where I start, if I were to change x by 4 and change y by 5, my final z value would be z initial plus 22.

For problem 1c, Jackson was coded as establishing a transformation relation between plane and vertical change (P-S-VC). In his explanation, we can see that Jackson assigned geometric meaning to specific algebraic expressions within the plane equation. Specifically, he identified (x-1) and (y-2) as horizontal changes. When Jackson said "depending on where I start," this suggests that he was aware that vertical change is independent of initial point, but depends only on the slopes and horizontal changes. He also went further to show how he used his equation to produce a vertical change of 22 units to correspond with what he calculated in Problem 1b.

Problem 4a (in Figure 4) presented a different plane as the plane tangent to a surface at a particular point and asked, "What can you say about the change in the value of the function if x increases 0.02 units and y increases 0.03 units?"

We note that problem 4a is computationally the same as 1b. The difference is that in 4a, the plane is presented as a plane tangent to a surface and the problem asks about the change in z for the surface, rather than the plane itself. Both Jackson and Daphne first attempted the problem very similarly to one another, using a visual approach.

Jackson: We can expect if x slightly increases in the positive x direction and y slightly increases in the y direction, the z value should increase.

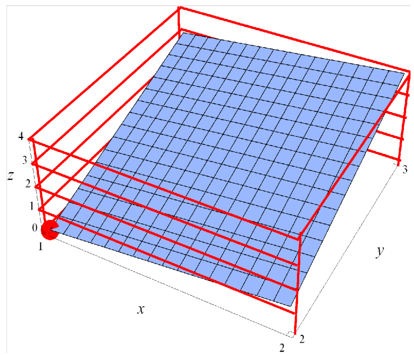

Figure 4. Plane for question 4a.

Daphne, in answering question 4a, first made a connection to problem 1b.

Daphne: This is asking a similar question to, I think, one of the ones on the first or second page. And, then, so if x increases 0.02 and y increases 0.03 units, the change in the value of the function would be positive, cuz, like, this way it goes like that [starting at the base point, pointing along the edge of the plane shown in the figure, corresponding to increasing x], and that way it goes like that [pointing along the edge of the plane, corresponding to increasing y], so the net vector would be positive.

Each student made a claim that the vertical change would be positive. The interviewer then prompted each student to calculate, or explain how to calculate, a value.

Jackson: I'm going to find the equation of the plane, m_x times the change in x plus m_y times the change in y equals z. Next, I was gonna try and find m_x . To find m_x I just fix it at the plane y = 2 and see how z changes when I move from x = 1 to maybe x = 2. And same thing for m_y. I'm in the plane x = 1, and see how z changes when I change y. And my change in x and change in y is gonna look like x - 1 and y - 2 and when I plug in the points, if x increases, okay, my change in x is given to me. They told me 0.02 and my change in y is 0.03, so the main work is just finding m_x and m_y and I should get my final z value.

Here we see consistency between Jackson's approach and a geometric interpretation of algebraic expressions in problem 4 with those in problem 1. For example, he identified "x - 1" as "change in x" and "y - 2" as "change in y." Jackson recognized that .02 could be used instead of x - 1 and .03 instead of y - 2, showing that he could give geometric meaning to the algebraic expressions in the formula. When the interviewer asked Daphne to calculate a value for the vertical change, she did not make any further connections to problem 1b.

Daphne: Yes, I could find, um…I know there is a way to do this. I'm just gonna say that it's around 1 for the z value.

Daphne ultimately abandoned this question with no further explanation or attempt at computation. In contrast to Jackson, Daphne did not identify that she needed to find slopes in the x and y directions, despite having been prompted to do so in Problem 1b. It appears that the process she had used for computing vertical change in problem 1b had not been interiorized; her actions had been guided externally by the interviewer. It was not apparent to Daphne that she could use the equation of a plane to compute vertical change. Therefore, for problem 4a, we coded Daphne as not demonstrating a relation between tangent plane and vertical change.

We see evidence that using the locally linear approach with CalcPlot3D and 3D-printed surfaces enables Jackson to construct a geometric understanding of ideas in differential calculus of two-variable functions in a way that makes sense for him and that he can explain. Jackson, possibly as a result of the classroom activities, was able to give meaning to movements along a plane, slopes, and vertical change. Throughout the interview, Jackson was consistent and fluent in relating the plane, slope, and vertical change components. He considered that properties of the plane component are conserved in properties of the vertical change components, so we considered Jackson to have established an equivalence relation for P-S-VC.

Daphne was quick to produce equations of planes when a problem involved a plane, whether helpful or not. She was not able to relate the algebraic expressions in her plane equations with geometric representations. Overall, we considered that she established the relations P-S-VC as a correspondence relation because she relied on a memorized procedure to produce equations of planes and needed significant scaffolding to relate components in unfamiliar question formats. The relations established by Jackson and Daphne are summarized in Table 1.

Table 1. Comparison of coding of PTP-S-VC relations for Jackson and Daphne

Problem with Description		Coding for Jackson		Coding for Daphne	
		Code	Justification	Code	Justification
1b	Compute VC on plane	T	Computed and explained computation	NR	Computed with significant scaffolding
1c	Create equation of plane and relate to computation of VC	T	Created equation and explained connection between algebraic expressions and geometric features	C	Created equation but could not interpret computation geometrically
4a	Estimate VC on function using tangent plane	T	Made visual estimate and was able to explain how to compute VC	NR	Made visual estimate but was not able to compute VC

Implications and Limitations

This study provides further evidence that constructing vertical change on a plane and geometrically interpreting components of the point-slope equation of a plane are non-trivial for multivariable calculus students. Jackson's understanding of the components of plane and vertical change is more robust than Daphne's, in the sense that he can interrelate the components through slope flexibly across different problems and justify his solutions based on his geometric insight. We hypothesize that the activities using digital visualization and 3D-printed surfaces in Jackson's section supported him in developing these relations. However, we note that greater similarity of classroom activities to the interview tasks in Jackson's case, and the focus on only two participants of all interviewed, limits the conclusions we can draw about the impact of these activities relative to traditional instruction. The study also provides a theoretical contribution by considering the types of relations between Schema components, which is a relatively new idea in APOS theory that can be better understood through further applications to empirical research. Future studies can report on the analysis of all participating students and the relations between all the differential calculus schema components.

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