Teaching Practices Supporting Students' Awareness of Definition-Related Mathematical Norms

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The mathematics community has a variety of standard norms of practice by which it operates, which are guided by collective values, including in the context of definitions. However, these norms and values are not always shared with students in instruction. Here we examine teaching practices that linear and abstract algebra students perceived as influencing their understanding of norms around mathematical definitions. Teaching practices in linear algebra tended to align more with instructor-centered actions whereas teaching practices in abstract algebra tended to align more with student-centered activities. Implications include the impact of homework and classroom activities on students' perceptions, as well as the long-term use of activities seeming to be more salient in students' perceptions of influential activities.

Keywords: Teaching Practices, Norms, Definitions, Linear Algebra, Abstract Algebra

Mathematical definitions play a central role in both the discipline of mathematics and the teaching of it. They serve as essential tools for naming collections of objects or phenomena, communicating mathematical ideas, and supporting logical derivation (Edwards & Ward, 2008; Ouvrier-Buffet, 2006; Torkildsen et al., 2023). Within the mathematics community of practice (Lave & Wenger, 1991), there are well-established norms for definitions. Broadly, norms are shared expectations about how definitions should function and how they should be crafted. For instance, definitions are expected to clearly identify a property that characterizes a class of mathematical objects, with space for individual agency in how that property is framed, and to be minimal, non-contradictory, unambiguous, and elegant (Edwards & Ward, 2008; Ouvrier-Buffet, 2006; Rupnow & Randazzo, 2023; Van Dormolen & Zaslavsky, 2003; Vinner, 1991).

Understanding definition norms can offer insight into their foundational role in the discipline. However, teaching definitions can be especially challenging since norms in mathematics (generally and specifically to definitions) are often taken-as-if-shared within the community (Yackel & Cobb, 1996; Yackel et al., 2000). That is, they function as tacit expectations that typically go unspoken (Herbst et al., 2011) and thus can remain invisible to students. This could explain findings that identify differences in mathematicians' and students' understandings of the nature of definitions (Edwards & Ward, 2004; Fernández-León et al., 2021; Martín-Molina et al., 2020).

Few studies have directly investigated instructors' teaching of definition norms (Rupnow & Randazzo, 2023; Vroom & Ellis, 2024; Vroom et al., 2024). Rupnow and Randazzo (2023) found that instructors attempt to convey norms aligned with clear communication such as expecting students to use rigorous and precise definitions. Vroom and colleagues (Vroom & Ellis, 2024; Vroom et al., 2024) analyzed classroom video to identify instructional practices aimed at making these norms visible to students as they engaged in revising definitions, for instance, suggesting a particular revision with implicit or explicit reasons related to clear communication. Still, little is known about how students experience and make sense of these efforts. Understanding the student perspective is critical for improving students' opportunities to engage meaningfully with mathematical definitions' norms. Here we address the research questions: (1) What teaching practices did students highlight as impactful to their understanding of definitions in a linear algebra course?, and (2) What teaching practices did students highlight as impactful to their understanding of definitions in an abstract algebra course?

Theoretical Perspective

We adopt Nachlieli and Elbaum-Cohen's (2021) definition of a *teaching practice* as our lens. They describe a teaching practice as comprised of a task and a procedure; the task is an instructor's educational goal while the procedure is an action or actions taken to accomplish that goal. Here, since we are examining students' perceptions of how understandings of norms related to definitions were supported, we infer the instructor's task from each question, which had a specific norm supporting a definition value of the mathematical community embedded in it. We view procedures as supportable by specified instructor-focused actions (e.g., instructor presenting definitions in a certain way) or specified student-focused actions (e.g., students engaging in a certain type of activity in the course at the behest of the instructor). We additionally highlight students' broad claims about the course as a whole or instruction around particular concepts that they viewed as supporting an instructor's task, but that do not fit the grain size of a procedure.

We drew on Rupnow and Randazzo's (2023) work to identify and articulate norms for definitions. In particular, the following norms increase the likelihood of clear communication with definitions: *mathematical definitions are unambiguous*, *definitions are an important part of an axiomatic system*, *definitions are used to support mathematical arguments*, *definitions allow the use of a single phrase to communicate complex ideas*, *names of concepts give insight into what the concept means*, *examples illustrate definitions*, *definitions are based on historical motivations*, *standard formatting choices in mathematics communicate information about definitions*, and *mathematical communication has consistency*. Additionally, the following norms increase the likelihood of freedom of choice in defining: *examples motivate the creation of definitions*, *the relevant features of a concept motivate the creation of definitions*, *abstraction motivates the creation of definitions*, *people make definitions*, *people write definitions purposefully for ease of use*, and *definition conventions vary between mathematicians*.

Methods

A survey was administered to the Fall 2024 sections of one author's Linear Algebra (LA) and Abstract Algebra I (AA) courses. 10 students completed and 3 students partially completed the survey in LA (out of 20), and 12 students completed the survey in AA (out of 27). LA served as the Introduction to Proof, was taught with a mixture of the Inquiry-Oriented (IO) Linear Algebra activities (Andrews-Larson et al., 2017; Wawro et al., 2012; Zandieh et al., 2017) and interactive lectures, and was taken by math majors and minors though most were math minors. AA had LA as a prerequisite, was taught with a mixture of the IO Abstract Algebra activities (Larsen, 2013) and interactive lectures, and was taken by math majors and minors though most were math majors. Five AA survey completers had previously taken the author's section of LA. Lectures in both courses were organized similarly, with LaTeX fill-in-the-blank notes posted before class and filled during class through interactive lectures where the instructor would lead students through the material, with times for students to ask questions and address examples themselves, in line with a chalk-talk lecture (Artemeva & Fox, 2011). IO lessons in LA focused on introducing concepts like span through concrete examples in which students would work in groups. However, students were not expected to write definitions themselves in those activities.

In contrast, lessons in AA introduced various groups in collaborative activities to scaffold students writing their own definition for group and later for group isomorphism.

The survey was available the last two weeks of the semester, and students received extra credit for submitting it. Questions on the survey focused on students' views of definitions and how definitions had been used in the course. The questions had definition norms supporting values of the mathematical community embedded in them, and the norms were ideas the instructor hoped to inculcate in students. In one question, students were asked to drag each stated norm into a "I took these away from the class" or "I did not take these away from the class" box. Norms that were dragged into the "I took these away from the class box" received a follow-up question asking, "What about the course gave you insight into [that norm]?" Students were given a textbox to type answers to these follow-up questions, and we focus on students' responses to the follow-up questions here.

We used Nachlieli and Elbaum-Cohen's definition of a teaching practice to guide analysis. Specifically, we viewed the definition norm from Rupnow & Randazzo (2023) that was embedded in the question as the instructor's task, since the instructor, as an author, did intend to share the norm in the question and students claimed that was an idea communicated in the course. We created procedure codes by having two non-instructor authors independently open code with the definition of a "procedure" in mind. Then these two authors and the instructor author met to consolidate a list of procedure codes. Next, one non-instructor author coded the LA and one non-instructor author coded the AA responses independently. Finally, the same three authors met to discuss and come to consensus on the procedure codes present in each student's response to the provided open-response questions. This process aligns with codebook thematic analysis (Braun et al., 2019), in that researchers had some idea of the types of procedures that might appear in the data but used the data to guide the creation of procedure codes.

Results

We present results by course, first Linear Algebra (LA) student responses, then Abstract Algebra (AA) responses. The bold heading indicates the procedure whereas italics indicate a task (norm goal, referred to as norms below) supported by the procedure.

Table 1. Procedural Codes Highlighted by Participants in Each Course.

Category	Procedural Code	LA Frequency	AA Frequency
Instructor-Focused Teaching Practice	Instructor presents clear and precise articulation of definitions.	6	3
	Instructor clarifies the function of definitions (e.g., utility for proving).	9	4
	Instructor presents motivating example(s) before definition.	2	4
	Instructor presents examples after the definition to illustrate.	5	4
	Instructor verifies examples fit the definition	3	2
	Instructor abstracts properties from the object to define the object.	2	3
	Instructor formats the notes consistently.	5	5
	Instructor emphasizes the importance of notation.	1	2
Student-Focused	Students verify examples fit the definition.	5	2
	Students write definitions (themselves).	1	4

Teaching Practice	Students practice after learning a new definition.	5	6
	Students explore before defining a concept (themselves).	0	8
	Students work individually on proofs for assigned work (tests/quizzes/homework).	7	10
	Students work collaboratively on activities in class.	1	6
	Students discuss readings or proofs for discussion boards.	1	5
	Students compare definitions.	1	2
	Students learn about mathematicians or mathematics history.	2	10
	Students are exposed to different notations outside class.	2	1
Salient Non-Procedure	Particular concepts support greater understanding.	5	9
	The course as a whole supports greater understanding.	9	9

Linear Algebra

Students identified a wide range of instructor and student-centered practices that contributed to their understanding of mathematical definitions, especially those involving the instructor's presentation of definitions, characterization of the utility of definitions, and formatting of notes.

Instructor presents clear and precise articulation of definitions. This procedure supported students' grasp of the *mathematical definitions are unambiguous* norm. This procedure highlights the instructor's direct presentation and elaboration of definitions, which included stating definitions and using them to prove theorems. Students emphasized that the definitions provided were clear. For example, one student noted, "Definitions that I was given were very clear and precise and left little to no room for misunderstanding/miscommunication." Another student observed, "The way definitions and theorems were written in our notes covered all bases, thus left no room for interpretation which is in an important part of any definition, they are supposed to provide clarity." This emphasis on clear and unambiguous presentation helped students understand definitions as foundational elements for logical reasoning, illuminating how "precise definitions act as the building blocks when proving a theorem." Students valued this precision because it "prevents misinterpretation" and allows for "clear results" and the ability to "distinguish similar but distinct ideas".

Instructor clarifies the function of definitions (e.g., utility for proving). This procedure especially supported the norms *definitions are used to support mathematical arguments* and *definitions are an important part of an axiomatic system*. Students cited instances where the instructor explicitly demonstrated how to use definitions in proofs. One student stated, "She showed us how to use definitions to prove theorems". Another student elaborated more, remarking, "I feel like I better understand their [definitions'] purpose and am able to utilize them in the context of proofs and problem solving." Students noted that definitions were purposeful tools that are used "to reach the end result of the theorem" and "in solidifying the validity of our proof and making it more accessible to people who might be unfamiliar with what we are trying to prove." The consistent explanation of how concepts build upon each other using definitions further deepened this appreciation for utility. This pedagogical choice enabled students to see definitions as tools for logical problem-solving, reinforcing the norm of clarity in mathematical communication and their practical utility.

Instructor formats the notes consistently. This procedure especially reinforced the norms *standard formatting choices in mathematics communicate information about definitions* and *mathematical communication has consistency*. One student noted, "Formatting choices, such as uppercase letters for matrices communicate information about definitions, making it easier to identify and interpret mathematical objects.” The consistent use of symbols and notation, such as “the different notations used, like the vector addition and scalar multiplication symbols, uppercase letters for matrices” aided understanding. Furthermore, students felt that the notes were structured; according to one of the students, “The types of theorems, many with corollaries and lemmas stayed consistent throughout the course and those ideas stay consistent throughout other math textbooks, too”. This consistent presentation style, mirroring textbook conventions, illuminated that standardized formatting contributes to mathematical communication.

Students work individually on proofs for assigned work. This procedure directly fostered the norm that *definitions are used to support mathematical arguments.* Students consistently utilized definitions as the foundation for their proofs. As one student put it, “All the proofs I’ve had to do that have been based on definitions.” Another student highlighted this foundational role, stating, “When I wrote proofs I used definitions all the time to support my arguments in each and every case.” The consistent application of definitions in these assignments helped students solidify a relationship between definitions and proofs.

Particular concepts support greater understanding. While we did not code this as a procedure, we noted multiple student responses that focused on learning specific concepts related to the norms *names of concepts give insight into what the concept means* and *definitions allow the use of a single phrase to communicate complex ideas*. With respect to the former, one student wrote, “The names of specific concepts that we’ve learned in this course give insight into what the concept means by hinting at its algebraic or geometric interpretation.” Numerous students noted different concepts for which shorthand was salient. One student noted, “Definitions like span, basis, and linear transformation allow the use of single phrases communicate complex ideas and relationships within vector spaces.” Others noted examples such as the “invertible matrix theorem list” and the concepts of “one-to-one or onto” functions where a single term encapsulates extensive mathematical meaning. Another student said, “Relevant features of concepts like vector spaces or linear transformations, such as the need for operations to satisfy certain properties (e.g., closure, associativity), motivate the creation of definitions that capture these important characteristics for future reference.”

The course as a whole supports greater understanding. Again, while not a procedure, students frequently expressed that the overall structure and approach of the course supported their understanding of definitions as part of a coherent mathematical system. This broad insight was reflected in responses to questions focused on norms of *people write definitions purposefully for ease of use* and *definitions are an important part of an axiomatic system*. This student recognized the utility for their work: “definition[s] are often used to categorize or distinguish concepts, and have been useful in conveying information in a more condensed manner.” Another student noted**:** “definitions are critical to understand proofs.” This interconnectedness was a key takeaway, as one student observed, “I have yet to see any definitions that contradict each other, only those that build on each other.” The consistent use of definitions throughout the course facilitated communication, as students learned to “define terms so that we can talk about more complicated concepts easily.” The course fostered the notion that mathematical definitions are “critical to understand proofs” and essential for consistent communication among mathematicians. This holistic view reflects the instructor’s goal in conveying definitions not just as isolated facts, but as integral components of mathematical practice.

Abstract Algebra

Students identified a wide range of student and instructor-centered practices, especially those involving active engagement, conceptual exploration, and historical or contextual enrichment.

Students work individually on proofs for assigned work (tests/quizzes/homework). Students referenced proof-based assignments especially with regards to the *definitions are used to support mathematical arguments* norm. Students mentioned working individually on tasks such as tests, quizzes, and homework, where they were required to use definitions as the foundation for formal reasoning. These assignments provided repeated opportunities to see how definitions function as essential tools for constructing logical arguments and developing mathematical structure. As one student noted: “I started every proof by recalling the relevant definitions. This demonstrated that definitions are the foundation upon which all mathematical arguments are built.”

Students explore before defining a concept (themselves). Closely tied to this was the theme of student exploration before creating a formal definition, where students engaged with mathematical ideas and patterns before seeing the official class definition. This practice was mentioned in response to several questions, including those focused on the norms *mathematical definitions are unambiguous*, *definitions are an important part of an axiomatic system*, *people write definitions purposefully for ease of use,* and *definition conventions vary between mathematicians*. This practice was valued for helping students build an understanding of a concept before encountering and crafting formal language to describe it. One student remarked: “A lot of the definitions we made in class were motivated by features of the defined object. Usually, these features were discovered first. But ultimately the definition is shaped around these features.”

Students work collaboratively on activities in class. Collaborative classroom work was mentioned by students in relation to multiple norms, including *examples motivate the creation of definitions* and *relevant features of a concept motivate the creation of definitions*, highlighting how group activities helped deepen understanding of definitions through peer explanation and comparison. In these settings, students described learning to refine and revise their definitions as they worked through problems with others. According to one student, “Examples can help people understand what is going on and they make a broader explanation like definitions, like we did in activities.”

Students learn about mathematicians or mathematics history. Historical and contextual information also played a role in shaping students’ understanding of definitions. In connection to several norms, especially *definitions are based on historical motivations*, students referred to learning about the development of mathematical ideas or the mathematicians associated with them. As one student explained: “Historical context was used to explain how definitions evolved to address specific mathematical problems.” Another noted, “Looking at the history of some concepts showed to me that there was motivation at one point to develop these definitions.” This historical framing showed mathematicians engaged in creating definitions or attending to particular definitions in response to their needs at the time.

Instructor formats the notes consistently. In addition to student-driven practices, several instructor-centered strategies were mentioned by students. The most common of these was the instructor’s consistent note formatting, which was highlighted in connection to most of the norms, especially the norms *standard formatting choices in mathematics communicate information about definitions* and *mathematical communication has consistency*. Students reported consistent notes helped them follow the structure of new concepts and clearly identify where definitions were introduced and how they were used. As one student described:

“Definitions were often presented as lists or bullet points, breaking down the properties into manageable components. This modular presentation made complex definitions easier to understand and apply.” This suggests that formatting conventions to aid reading were perceived as useful for students.

Instructor presents motivating example(s) before definition. Students also emphasized the value of motivating examples provided before a formal definition, which helped generate interest and guide intuition prior to engaging with more formal language. Students related this procedure to norms of *examples illustrate definitions*, *examples motivate the creation of definitions*, and *abstraction motivates the creation of definitions*. One student noted: “After noticing a pattern in examples, we would try to generalize what we were seeing.”

Discussion

Students highlighted a variety of course activities that supported their developing understanding of mathematical norms and values supported by work with definitions. While both instructor- and student-centered activities were highlighted by students in each course, instructor-centered activities were referenced more often by linear algebra students and student-centered activities were referenced more often by abstract algebra students. This result aligns with expectations from the course setup. While students in both courses engaged in inquiry-oriented (IO) activities and attended interactive lectures with organized notes, the IO activities in LA focused on creating intellectual need for learning new concepts and a concrete context to anchor students’ developing understandings of span, linear independence, and change of basis (Andrews-Larson et al., 2017; Wawro et al., 2012; Zandieh et al., 2017). However, students were not asked to write new definitions themselves; these were introduced by the instructor in response to the activities. In contrast, students in AA were introduced to examples of groups through concrete activities but then scaffolded through writing a definition for group and group isomorphism themselves (Larsen, 2013). Thus, students in both LA and AA were exposed to ways their instructor formatted definitions and presented definitions in relation to examples, but only AA students were given opportunities to think about definition affordances and conventions as they wrote definitions themselves. This highlights that implementing IO materials is not a monolith with respect to expected student engagement in mathematical practices. Given that many LA courses are not proof-based and serve more non-math than math majors, we recognize that it is potentially beneficial for these materials to center different disciplinary practices.

Moreover, LA also serves as the Introduction to Proof course, meaning the instructor deconstructed how to write proofs and how they relate to definitions often, especially near the beginning of the semester, whereas this was less of a priority in AA, where students were expected to be familiar with this relationship between definitions and proofs already. This aligns with students in LA especially highlighting the instructor’s role in presenting definitions and clarifying their function relative to AA. Nevertheless, students in both courses highlighted long-term activities. This included the utility of the homework, especially proof-based problems, in developing their understanding of definitions and viewing the course as a whole as supporting their understanding of definitions.

While some research at the university level has noted the impact of courses on students’ beliefs about the nature of mathematics as a whole (e.g., Rupnow, 2023; Szydlik, 2013; Ward et al., 2010), here we instead consider the impact of course activities on students’ perception of definitions and associated practices for creating and using them. In so doing, we gain initial insight into how teaching practices can impact students’ perceptions of the values and norms held by the mathematics community.

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