Analyzing a Student's Observations and Conjectures About Markov Chains: Identifying Scheme Elements and Their Instrumentation of Digital Figures

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This study investigates how a first-year undergraduate mathematics student engaged with a set of digital interactive figures to explore properties of stochastic matrices and Markov chains. Using the frameworks of instrumental genesis and Vergnaud's theory of conceptual fields, we analyzed the student's interaction in terms of their observations and conjectures, two key stages in our Observation, Conjecture, Proof, Theorem (OCPT) pedagogical model. Through task-based interview data, we identified elements of the student's emerging conjecturing scheme, including rules-of-action and theorems-in-action that illustrate their process of sensemaking and experimentation. We argue that this scheme development evidences the student's instrumentation of the figures into a functional instrument for mathematical reasoning.

Keywords: Linear algebra, CAS, eigentheory, conjecture, scheme, instrumental genesis

Linear algebra is a foundational discipline within mathematics, underpinning a wide range of theoretical and applied fields. Its concepts and techniques are essential in areas such as data science, computer graphics, quantum mechanics, machine learning, and systems modeling. Both iterations of the Linear Algebra Curriculum Study Group (Carlson et al., 1993; Stewart et al., 2022), convened by prominent linear algebra educators and education researchers, have proposed agendas and goals for the effective and beneficial teaching of linear algebra. Those recommendations include, among others, a need for the study of linear algebra to be integrated with and responsive to digital technology, as well as to be attentive to the formal theoretical training that has traditionally been reserved for advanced courses in linear algebra.

In keeping with these recommendations, we are interested in developing and advancing a pedagogical model of how students, even at the introductory linear algebra level, can engage in meaningful and authentic formal mathematical practices. We have called this model the Observation Conjecture Proof Theorem (OCPT) model, in contrast to the traditional textbook Definition-Lemma-Proof-Theorem-Proof-Corollary (DLPTPC) mode of content delivery, criticized by Uhlig (2002). OCPT seeks to place the mechanisms of discovery, investigation, and reasoning before the presentation of polished textbook theorems. Lin et al. (2012) define the task of mathematics and the sciences as "first to observe phenomena, then to explain them, and finally to predict." Implicit in this definition is the formalizing process by which the observation of a phenomenon or relationship is formalized, conjectured, into a mathematically coherent, provably true or false statement. This process of conjecturing, and its interplay with observation, is nuanced; example generation and experience are powerful and necessary skill to prompt insightful conjectures, even as examples cannot be accepted as sole proof (Lesseig, 2016; Lynch, Lockwood, & Ellis, 2024), except in the case of counterexamples. The sub elements of the conjecturing activity and the role of observation therein have been described as motivating an initial conjecture, boundary testing on the conditions and assumptions, and encountering structures that may lead to a justification (Lannin et al., 2006).

We are currently primarily interested in facilitating and creating digital environments for students to engage in processes of observation and conjecture, which we call interactive figures.

Our previous collection of interactive figures focused on fundamental concepts of linear combinations, determinants, and eigentheory (Peffer, McDonald, & Stewart, 2025). Markov chains are a critically powerful field in linear algebra and have been recognized as a salient and meaningful application by which to explore concepts in linear algebra and specifically eigentheory (e.g., Harsy & Smith, 2024). However, there is little other instructional or theoretical treatment of the teaching of Markov chains in the literature. In this paper, we will discuss a task-based interview conducted using a new pair of interactive figures targeting the properties of stochastic matrices and Markov chains. We will pay attention to the actions and processes of observation and conjecturing enacted by the student, and the underlying mental schemes involved that enable the student to make use of the interactive figures to explore those properties of Markov chains.

Theoretical Perspectives

With an interest in understanding how the interactive figures facilitate students engaging in the mathematical activities of observation and conjecturing, we first turned to the framework of instrumental genesis to describe this interaction. The theory of instrumental genesis has been adopted in mathematics education literature to describe how students use tools and technology as adaptable problem-solving instruments (Artigue, 2002; Guin et al., 2005). A common, prototype application of instrumental genesis is the analysis of students adapting their algebraic and graphical abilities with a graphing calculator (Trouche, 2020), wherein the calculator (or any other tool) known as the artifact acts to reciprocally influence and be influenced by the learner.

Of particular interest to this study is the process of instrumentation, whereby engagement with the artifact in a goal-oriented, problem-solving scenario influences a student to adapt their cognitive, analytical processes to acknowledge and employ the artifact's utility, thus generating the psychological entity of an instrument. In diagnosing and analyzing the process of instrumentation, whose product is a cognitive reorganization of a student's mental models, it is standard to employ schemas to capture this structural adaptation (Turgut & Drijvers, 2021; Gueudet et al., 2022). By identifying behavioral and analytical features of a schema and distinguishing the ways in which a student has adapted or developed a schema to include and employ an artifact, researchers can investigate and qualify the adaptation of the instrument.

Relatedly, Vergnaud (2009) establishes a theory of conceptual fields, described as a class of situations and problems, together with the schemes and individual constructs and employs to address these situations, and the modes of representation that the individual operates and communicates by. While we might envision the model of OCPT as a networking of conceptual fields that coalesce to constitute a mathematician's formal activity, in this paper we will focus specifically upon Vergnaud's compartmentalization of scheme, which dissects schema into four levels of activity, which are described as (p. 88):

- The intentional aspect of schemes involves a goal or several goals that can be developed in subgoals and anticipations.
- The generative aspect of schemes involves rules to generate activity, namely the sequences of actions, information gathering, and controls.
- The epistemic aspect of schemes involves operational invariants, namely concepts-in-action and theorems-in-action. Their main function is to pick up and select the relevant information and infer from it goals and rules.
- The computational aspect involves possibilities of inference. They are essential to understand that thinking is made up of an intense activity of computation, even in

apparently simple situations; even more in new situations. We need to generate goals, subgoals and rules, also properties and relationships that are not observable.

We wish to employ these components of scheme then to analyze student interactions and their products to describe the forming and adapting schemas at play. As mentioned, the mathematical activity that we are most interested in with these figures is that of observation and making conjectures, and thus we seek to identify the generative rules-of-action that students develop or display in interacting with the figures, and subsequently to interpret the cognitive concepts- and theorems-in-action as they are guiding the student's rules-of-action. In doing so, this scheme documentation should serve to identify the emergent instrumentation of the interactive figures by the student to participate in the described process of observation and conjecture.

As such, our research questions for this study are: How did the student in this study engage with the interactive figures to make sense of conceptual questions regarding Markov chains? What scheme aspects, in the context of observation and conjecture, were displayed?

Method

To collect evidence of scheme relating to observation and conjecture, we designed a qualitative case study consisting of four task-based interviews that were conducted at a public land-grant research university in the USA. Volunteer participants were recruited immediately following their final examination in a first course in linear algebra. These students were all concluding their first year of study, with aspiring majors in mathematics and engineering.

These interviews will constitute data in the first author's PhD thesis, and in this paper, we will discuss one such interview with Alice (pseudonym). At the time of the interview, Alice had just completed her spring semester of her first year of university, intending to study mathematics and having received an A in her first course in linear algebra. The course covered systems of equations, span and linear independence, linear transformations, matrix algebra and invertibility, subspaces of $\mathbb{R}$, determinants, eigentheory, and orthogonality. We chose to present Alice's interview analysis because Alice was entirely unfamiliar with our interactive figures, she had no experience with Markov chains in her linear algebra coursework or other classes, nor had she taken any advanced proof-based mathematics courses.

The materials for the interview included the two digital interactive figures built in Mathematica concerning properties of stochastic matrices and Markov chain iteration. The figures were accompanied by a six-question assignment task mimicking a potential homework. Two questions, Questions 3 and 6, were of particular interest to this study as they asked students to make observations of the properties of stochastic matrices and sequences of Markov chain iterates via the provided interactive figures, though in this paper we will focus exclusively on question 3:

> Question 3: The matrix you have created is called a stochastic matrix: that is, it is a matrix where each column is a probability vector whose entries must be non-negative and sum to 1.0 (probability 100%). Using the first interactive figure, what patterns do you notice about the eigenvalues and eigenvectors of stochastic matrices compared to non-stochastic matrices?

The remaining four questions included conceptual and computational questions involving constructing and analyzing a Markov chain in context of a 2-population dynamic application and were intended to motivate or build from the student's observations. The assignment also provided relevant working definitions such as for probability vectors, stochastic matrices, and Markov chains. Furthermore, the students were provided with a copy of the course textbook and a calculator for their convenience. The student was prompted throughout the interview to proceed as if they were completing this assignment as regular homework and were thus permitted to use any other standard resource they would consult in their own time, including internet sources.

The data collected for this interview included transcribed audio and video recording of the subject, including screen recording of their work with the interactive figures. As well, the written responses of the student on the assignment document or scratch paper were preserved. At the end of the interview, the students were given a brief exit survey asking what else they might have done in the pursuit of this assignment were it a real assignment, how strongly they believed in the validity of their observations made, and for feedback and user sentiment on the interactive figures. A brief analysis of Alice's observation confidence will be included in the Results section.

In order to target a schematic notion of conjecturing, Alice's interview data was individually analyzed via thematic analysis and open coding (Strauss & Corbin, 1998), by identifying novel propositions or formulas that arose as answers to the above question 3, interpreting their testimony and recorded actions for the generative rules-of-action being followed, and subsequently for the implied cognitive concepts-in-action and theorems-in-action informing these decisions, as prescribed by Vergnaud's theory of conceptual fields. Furthermore, these scheme elements were corroboratively identified as being technology-integrated, in actively involving or relating to the use of interactive figures as a digital artifact. In this way, we may show evidence for the student's incipient instrumentation of the interactive figures involving their schemes of observation and conjecture.

Results and Discussion

Alice's Conjectures of Properties of Stochastic Matrices

We begin by looking at Alice's observations regarding the properties of stochastic matrices via Interactive Figure 1. An example of Interactive Figure 1 as it appeared in the interview (Figure 1), as well as Alice's response to question 3 regarding her observation of properties of stochastic matrices and their powers (Figure 2) are shown below. In the interactive figure, a procedurally generated (stochastic or not) matrix A is provided, as well as its power A^k. For both matrices, the figure displays whether the matrix is stochastic, its eigenvalues, and its eigenvectors. The controls of the figure allow the user to vary the size of A, the exponent k, and to select generating a new stochastic or non-stochastic matrix A.

Alice's response to question 3 contains several conjectures. First, Alice has stated that all stochastic matrices have an eigenvalue equal to 1, though makes no qualification on the other eigenvalues. Next, she has conjectured that the eigenvector associated with eigenvalue 1 has entries that sum to 1, and further remarks that this aligns with a probabilistic interpretation, given that question 3 also defines a probability vector as having non-negative entries that sum to 1. It is worth noting that the figure was specifically configured to display an entry sum of 1, since Mathematica otherwise defaults to normalize eigenvectors with respect to the Euclidean norm. However, Alice does not comment on the potential of scalar multiples of an eigenvector to allow different vector sums for otherwise equivalent eigenvectors.

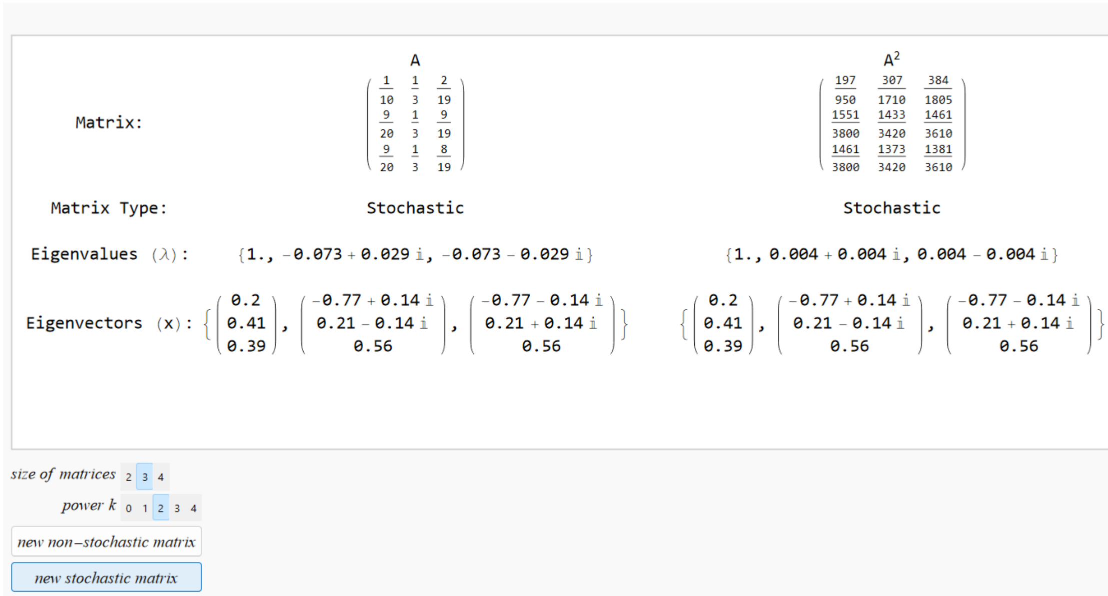

Figure 1. Interactive Figure 1: Properties of stochastic matrices and their powers.

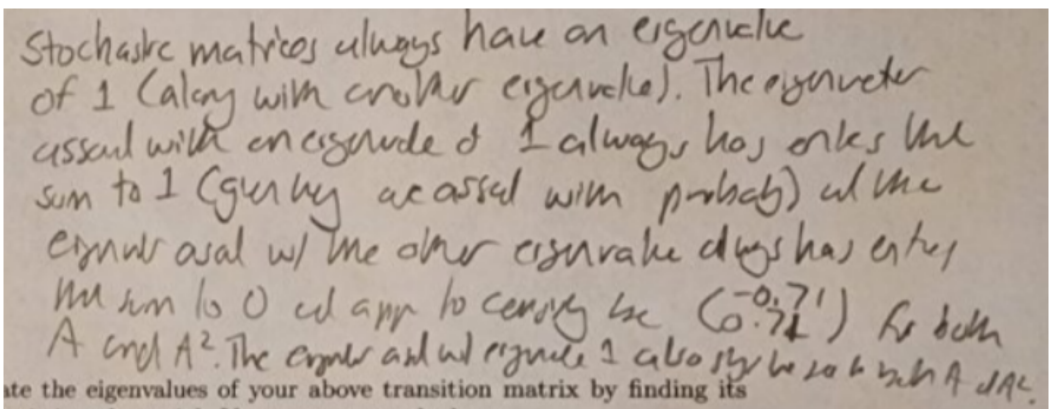

Figure 2. Alice's response to question 3.

While Alice's above response includes other meaningful conjectures, we'll now turn to the interview transcripts to analyze Alice's process of arriving at these ideas.

Alice's Scheme Elements of Observation and Conjecturing

Given free reign over the interview materials, Alice chose to inspect the interactive figures before reading the assignment and the instructions to make mathematical observations via the figures. An excerpt of the interview follows:

Alice: Interesting. These are numbers that don't look very fun to work with. It's probably my first impression. (Alice generates several new stochastic matrices). It's like they all have one. It's my first impression on that.

Interviewer: All have one…?

Alice: As an eigenvalue. (Alice alternates generating several stochastic and non-stochastic matrices). Just interesting for the stochastic matrix.

Immediately from this interaction, we identify several rules-of-action that Alice is employing to inspect the interactive figures to arrive at these mathematical ideas, such as "I look for convenient or otherwise remarkable phenomena" and "I look at several examples of a situation to draw parallels." From these rules of action, Alice was able to draw forth observations such as the eigenvalue 1 property of stochastic matrices. However, she did not make some less convenient observations such the eigenvalue 1 is the largest eigenvalue of a stochastic matrix, which align her comment that the other eigenvalues are "numbers that don't look very fun to work with."

Thus, it is reasonable that Alice has, by way of the interactive figures, constructed mathematically coherent statements about the properties of an otherwise unfamiliar object, stochastic matrices. However, Alice goes on to display other rules-of-action that display a more complex conjecturing activity. After qualifying her statement about the eigenvalue 1 of a stochastic matrix, Alice alternates generating stochastic and non-stochastic matrices, wherein she does view examples of non-stochastic matrices with leading eigenvalues significantly larger or smaller than 1. We might interpret this activity as following several other rules-of-action such as "I look for conditions under which a conjecture does or does not hold" or "I return to the interactive figures to gather additional evidence in support of a conjecture." From these and the preceding rules-of-actions, we interpret several theorems-in-action, invariant logical propositions that Alice holds to be true and to be motivating her actions:

1. A mathematical conjecture should apply to a specific class of objects (e.g. stochastic matrices).
2. A (universally quantified) mathematical statement should be true in all cases; additional evidence may support a statement but does not prove it.
3. The interactive figures are a source of data and examples of mathematical phenomena.

The first theorem-in-action is evidenced by Alice's intentional distinction between the eigenvalues of stochastic and non-stochastic matrices; her alternating matrix generation implied an experimental phase of cross-examination wherein she became increasingly convinced that the presence of eigenvalue 1 was unique to stochastic matrices. This second theorem-in-action, given by Alice's recurrence of viewing multiple examples even after stating a conjecture, is further supported by her exit survey. Alice was asked about her confidence in her conjectures, including specifically the appearance of 1 as an eigenvalue of stochastic matrices. Alice responded:

> I would say I feel somewhat convinced, like I think experimentally, because I pushed the buttons a lot and I was kind of messing with it for a while. I would say that I think I'm like, at least anecdotally, I feel like that's true, but I would want to have like more explanation of what's happening probably before I'd be fully like, this is definitely what's going on. Like I'd want to go to class and have the teacher be like, this is the result you should be getting, or this is like the reason behind why a stochastic matrix has that result.

This excerpt highlights Alice's cognitive relationship between an observed relationship and the justification of that fact. Even though her response indicates a willing appeal to authority, referring wanting the teacher to tell her "the result [she] should be getting," we still gather that Alice recognizes the observations and examples gathered via the interactive figures as providing compelling anecdotal evidence for the truth of her conjecture. This observed cognition about the role of evidence in conjecture and proof further aligns with the literature on the aspects of conjecturing activity and the role of example generation in forming new ideas.

Finally, in the third listed theorem-in-action, we identify Alice's demonstrated relationship to the interactive figures, namely that of reliable example generation and a source of data for her mathematical experimentation. Beyond Alice not seeking additional sources of stochastic matrices with which to make observations, Alice can be seen to make targeted, intentional actions via the interactive figures. These include successively generating examples to search for commonalities and patterns and alternating generating stochastic and non-stochastic matrices to test for differences. By these intentional, mathematically reasoned actions taken within the interactive figures, Alice shows evidence of having begun to instrumentate the interactive figures for engaging in the mathematical activity of making observations and forming conjectures.

Alice's Experimentation with the Definition of Stochastic Matrix. Alice had first looked at and begun to hypothesize about stochastic matrices before she read the definition of stochastic matrix provided in question 3. Her first action after reading the definition of stochastic matrix, as well as the sub-definition of probability vector, was to return to earlier matrix computations and visually check if they satisfied those definitions. Thereafter, seemingly primed to search for probability vectors, Alice returned to the interactive figure as instructed by the problem and was quick to make the following new observations:

> These ones are interesting (gesturing to the second eigenvector of A) because it's well these add up to 0 and these (gesturing to the first eigenvector of A and A2) both add up to 1. Yeah, so adding down this representative eigenvector associated with the eigenvalue of 1 sums to 1. And then for the other eigenvalue, the –0.139, the adding up the entry sums to 0... I'm assuming that they [the eigenvector corresponding to eigenvalue 1] because they're associated with probability like the matrix is.

Armed with a new conceptual tool (in this case, that of probability vectors), we may once again describe several rules-of-action Alice is employing, such as "I test a new definition with examples," "I interpret situations in terms of newly acquired language," and "I ascribe meaning to my observations based on definitions and terminology." These rules-of-action, centered on her activity involving a definition that she accepted, mirror earlier rules-of-action she employed when approaching a conjecture that she formed herself. These common features include testing via new examples, including examples that do not fit her conjecture or definition, and collecting her exploratory data from the interactive figures. This example of exploratory action surrounding the definition of stochastic matrices thus aligns with our previous investigation of Alice's conjecturing scheme aspects.

Concluding Remarks

We have observed, in agreement with literature in student conjecturing, several facets of Alice's conjecturing activity, framed as aspects of her observation and conjecturing scheme. These aspects included both generative rules-of-action by which Alice directed her activity, and the interpreted epistemological theorems-in-action that guided her rule system. Furthermore, we made explicit the role of the interactive figure in Alice's pre-proof activity, in part by referencing her direct actions with the figures in her rules-of-action. In doing so, we have evidenced that Alice was successful in some qualities of instrumentating the interactive figures, specifically for the purpose of engaging in mathematical observation and conjecture.

While we have referenced an Observation Conjecture Proof Theorem pedagogical model of promoting student proof activity, the complete explication of this model and the component scheme descriptions of observing, conjecturing, and proving remain a larger goal of the thesis work and beyond. As such, further analysis of this interview data, as well as the remaining interview data, is necessary and planned. Based on this initial analysis, the first and second authors are confident in the figures' ability to encourage productive activity and plan to incorporate them into their teaching.

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