Indifference as a Key Aspect of a Global Perspective on Equivalence Relations

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Equivalence is a central concept of mathematics and is a fixture in the K-16 curriculum. Though equivalence has been very well-studied overall, very little is known about how students reason with formal equivalence relations in advanced mathematics. Recent research has demonstrated that one component of productive reasoning about equivalence relations entails the ability to flexibly move between viewing equivalence from both an elementwise (local) perspective and a set-oriented (global) perspective. Building on this work, we report on a pilot study intended to aid the development of a conceptual analysis of equivalence in advanced mathematics - in which we seek to refine the notion of a global perspective on equivalence. In this preliminary report, by showcasing excerpts from task-based clinical interviews with two advanced mathematics students, we propose 'indifference' as a beneficial way of reasoning about equivalence from a global perspective.

Keywords: Equivalence Relations, Indifference, Global Perspective

Equivalence is an idea that transcends many areas of mathematics, from the relatively concrete equality of representations of rational numbers and algebraic expressions to the more abstract notion of decontextualized equivalence relations. Though equivalence has been extensively studied in the mathematics education literature, nearly all of it has focused on students' conceptions and uses of equivalence in K-12 contexts (e.g., Kieran, 1981; Knuth et al., 2006). At the advanced level, the existing research has focused on how students reason with the standard properties that define equivalence relations (e.g., Chin & Tall, 2000; 2001; Asghari, 2004b; Hamdan, 2006; Brandt, 2013). At the moment, the literature is not positioned to address how advanced mathematics students might productively leverage notions of equivalence relations themselves to help solve a problem. Payne & Cook (2025) highlight a distinction between an element-focused local perspective and a set-focused, global perspective on equivalence relations, arguing that the flexibility to transition between these two perspectives is productive for students. These perspectives are broad categories of student reasoning; as defined, their grain-size does not detail how students may identify two elements as equivalent, or the ways in which students might reason about sets of related elements. In the case of a local perspective, this gap has been filled by research that has, for example, examined how students might reason about equivalent fractions (e.g., Smith, 1995) and numerical expressions (e.g., McNeil et al., 2006). Other work has examined general ways of reasoning about equivalence that are relevant across multiple types of objects (e.g., Cook et al., 2022). The same cannot be said for a global perspective, as little has been done to investigate students' reasoning with sets of equivalent elements. Put another way, since a global perspective requires attention to a set of related elements, most equivalence literature has a local focus (e.g., Kieran, 1981) leaving reasoning about equivalence from a global perspective understudied. Building in part upon the work of Payne & Cook (2025) with respect to equivalence relations, we work to answer the question: What aspects of a global perspective on an equivalence relation are productive for students? Based on the equivalence relations literature, and data collected from pilot interviews with advanced undergraduates, we hypothesize that a productive feature of reasoning from a global perspective is the ability to adopt an indifference towards the choice of element within an equivalence class.

Literature and Theory

This is a pilot study to inform a dissertation project aimed at developing a conceptual analysis (Thompson, 2008) that describes ways of knowing equivalence relations that are propitious for students. The literature offers a few insights regarding such understandings. Several authors have previously used the term *global* to describe a perspective that considers sets of equivalent elements, though authors use this term in slightly varying ways. With respect to equivalence relations, Payne & Cook (2025) articulate a *global* perspective on an equivalence relation as being characterized by reasoning about collections of equivalent elements (such as equivalence classes). These authors indicate both that advanced mathematics students reason about equivalence relations in a way that matches this description, and that it is productive. Adopting this definition, a global perspective will refer to reasoning that involves sets of equivalent elements, and a *local* perspective will involve particular equivalent elements. To illustrate, in Arnon et al. (2001) students reasoned about lines with rational slopes as proxies for equivalence classes of fractions (which would require a global perspective) *prior* to learning algorithms to transform fractions into their equivalent forms (which would require a local perspective). However, *most* equivalence literature at the K-12 level has focused (implicitly) on local perspective. Since equivalence classes emerge as a fundamental aspect of equivalence in advanced mathematics, there is a need to study the global perspective as well. Moreover, many studies which focus explicitly on equivalence relations at the undergraduate level concern how students work with their reflexive, symmetric, and transitive properties, and on how these properties relate elements to each other, (e.g., Asghari, 2004a; 2004b; 2005), but do not investigate the specific ways in which students use equivalence relations as a problem-solving tool.

A productive approach that emerged from some earlier work (Payne & Cook, 2025) with students working on equivalence relations tasks involved the students' awareness that when reasoning about equivalence classes, the existence of equivalence classes often made it unnecessary to specify a choice of element from within an equivalence class. This observation warranted further study, though most of the original tasks were not positioned to investigate the strategies that students used when reasoning about an equivalence relation as a global property of a set. In addition to this empirical example, we also find some hints at the notion of indifference in the literature. For example, Piaget et al. (1977, p.155) describe elements of an equivalence class as "indistinguishable", and one of Hamdan's (2006) students describes equivalence relations as unable to discriminate between related elements. Similarly, in the context of teaching homomorphisms one instructor in Rupnow (2021) observed that they could "take any [pre-image of a coset of the kernel] as a representative" (p. 11) due to the induced equivalence relation, though these are comments made in passing, leaving indifference yet to be investigated directly. Since these strategies remain under-investigated, and since we hypothesize that this a productive understanding for students working with equivalence relations tasks, this study was geared towards eliciting more instances of articulated indifference towards particular elements.

Methods

Since the ultimate purpose of this project is to develop a conceptual analysis (e.g. Thompson, 2008) of how students reason about equivalence from a global perspective, the first author conducted task-based clinical interviews (e.g., Clement, 2000) with advanced mathematics students. One goal of conceptual analysis is to build models of student's thinking (Thompson, 2008). Hence, we conducted task-based clinical interviews (e.g., Clement, 2000) intended to explore the mental processes of the participants. Since literature informing a global perspective on equivalence relations remains somewhat sparse, pilot interviews were conducted to be largely exploratory, and to guide the development of a more targeted interview protocol in a future dissertation study. The first author individually interviewed three undergraduate Introduction to Abstract Algebra students three times each in 60-minute sessions and recorded all verbal and written responses on an iPad. We will relate episodes from two of these students, Andrew and Brian, in which they used indifference as a problem-solving tool when reasoning about equivalence relation tasks. The tasks are reproduced below:

Table 1. Two of our tasks, on which we report in the results section.

Task A - Even and Odd Functions - Andrew (pseudonym)	
Task: Suppose f(x) is an even function and g(x) is an odd function. Let $$h(x) = \frac{f\left(\frac{g(x)}{2}\right)}{x^2}$$ If $h(5) = 2.2556$, is $h(-5) = -2.2556$? Answer: No, since $h(x)$ is an even function	Rationale: Function parity partitions the set of all real-valued functions into four subsets: Functions that are even only, odd only, functions that are neither even nor odd, and the function that is both ($f(x) = 0$). The task can be answered by deciding if $h(x)$ is odd, and does not require specification of f(x), g(x), or $h(x)$
Task B - Concentric Circles – Brian (pseudonym)	
Let $(x, y)R(w, z)$ if and only if $x^2 + y^2 = w^2 + z^2$ be a relation that partitions the plane into curves. What is the maximum distance between two points on curves containing $(1,2)$ and $(0, 4)$? Answer: The maximum distance is $4 + \sqrt{5}$	Rationale: If the student observes the equivalence classes are circles, this becomes a matter of finding the distances between points on the opposite sides, but this does not require that students specify the locations of either point.

Tasks were designed to require students to reason either about the partition structure induced by an equivalence relation, or about sets of equivalent elements to observe students' productive use of a global perspective on equivalence relations, and begin to identify useful strategies to approaching these tasks, one of which was hypothesized to be indifference towards the choice of element.

Following Clement (2000), the first author conducted an *interpretive* analysis, which was generally in agreement with Thematic Analysis (e.g., Braun & Clarke, 2006). The notions of a global perspective and indifference both served as initial, a priori codes. Other codes emerged during the analysis that extend beyond the scope of this preliminary report. After each interview was transcribed, the first author coded each transcript for instances of both a *global perspective* and *indifference* towards the choice of element within an equivalence class or a subset of a partition, based on student's observable behaviors. Student's reasoning was inferred to be from a global perspective if they referred to, or made representations of, sets of elements that were related by an equivalence relation, or alternatively if they reasoned about subsets of the corresponding partition. Once a student's reasoning was inferred to involve a global perspective, student's speech was coded as indicating indifference if it contained words or phrases such as "regardless", "independent of", "no matter", or "I didn't specify" in reference to some element of an equivalence class that featured as part of their productive problem solving.

Results

In task A, Andrew quickly recognized that this problem was related to determining whether $\mathrm{h(x)}$ was an even or an odd function, though he could not recall at first precisely what it meant for functions to be even or odd. Citing examples of functions that he knew to be even or odd, he was able to recover a definition that he found useful:

Andrew: for $\mathrm{f(x)}$ to be even, that means that say $\mathrm{f}(1) = \mathrm{f}(-1)$, or in more general terms like $\mathrm{f(x)} = \mathrm{f}(-\mathrm{x})$, like how cosine works.

Interviewer: Okay. So then what about odd functions?

Andrew: So well basically, $\mathrm{g}(1)$ should equal $-\mathrm{g}(-1)$, or in general terms $\mathrm{g(x)}$ should equal $-\mathrm{g}(-\mathrm{x})$. [...] like how sine works [...] that's why I felt like it was saying, "Is it odd?" is because, now I see that's the definition that we're working with there. [...] So going back to the question, 'is this part odd or even', is kind of how I'm looking at that.

Interviewer: How would that information help you answer the question?

Andrew: still going off of this… if $\mathrm{h}(5) = 2.2556$, is $\mathrm{h}(-5)$ negative? That feels like asking if h is odd. So I think I'm kind of trying to work through and see if I can say that h is odd, and then that will definitely include that being true. And if it's even, that would definitely disqualify it from being so.

Here, we claim that Andrew is demonstrating a global perspective about a partition of the set of functions by parity, since his goal has become to assign the function $\mathrm{h(x)}$ to one of the subsets of that partition, and based on the properties of that subset, he will be able to solve the task. However, Andrew goes on to use indifference as a tool in his problem solving, based on an approach to building $\mathrm{h(x)}$ by dividing an even function by x^2, and deciding that it would still be even, independent of a particular choice of $\mathrm{f(x)}$.

Andrew: So I was able to determine [$\mathrm{h(x)}$ was even] using the fact that $\mathrm{f(x)}$ was an even function [...] I really only used that $\mathrm{f(x)}$ was an even function and made a determination that an even function divided by x^2 would still be an even function, and that's what got the answer. [...]

Interviewer: Okay. So even though you didn't have a specific function for $\mathrm{f(x)}$, you were able to determine that $\mathrm{h(x)}$ was an even function?

Andrew: Yes. [...] Based on just the reasoning through the character of a general […] even function [...] Like how I talked about an even polynomial. Reducing it, like dividing by x^2, would always retain its identity as an even function, [...] So regardless of the specific identity of $\mathrm{f(x)}$, the specific function, as long as it's even, which it is, then $\mathrm{h(x)}$ will be even, regardless of the specifics. [...] the reasoning solely ended up relying on $\mathrm{f(x)}$ being even.

We infer that Andrew reasoned from a global perspective because he used a partition of the set of functions. In particular, he used the assignment of elements to certain subsets of that partition to assign other elements to the appropriate subsets. Andrew did not initially recall the formal definitions of even or odd functions, but he used examples to jog his memory and recover the characteristics common to all even and odd functions, which let him proceed without a specific choice of f(x) or g(x). As Andrew noted, his reasoning relied only on the fact that f(x) is even, "regardless of the specific identity", which we argue indicates indifference towards the choice of representative.

Brian adopted a similar indifference in his work on task B, solving the task without determining the exact locations of two points on opposite sides of the circles. After he determined the radii of the circles and added them to find his answer, the interviewer followed up with questions about his reasoning:

Interviewer: Did you ever find the coordinates for, uh, either of those points in your problem-solving to find that distance? So what I'm asking is not, [..] "can you tell me what they are?" What I'm asking is "did you ever find them?"

Brian: I guess I didn't find $(0, 4)$, it's given to me. [...] And then I didn't exactly determine this coordinate. Like I never wrote out, uh…, $\left(0, -\sqrt{5}\right)$. But it's kind of what was going on in my head. Like I knew I was looking for [the opposite] y-value.

Interviewer: Okay. So were you thinking about it in terms of like where that coordinate was or were you thinking about it more in terms of like the diameter and radius in the circle?

Brian: I think more, I was, initially I was like, okay, I need to find this point on the graph, but the properties of the circle will tell me what that exact value is. [...] So once I started thinking about the circle, I kind of left the coordinate idea behind.

Here, we infer that Brian reasoned from a global perspective; he indicated that his reasoning relied on the circles, which are the equivalence classes under this relation. We also consider his reasoning to indicate indifference towards the particular representative, since he notes that in his thinking about the circles, he "left the coordinate idea behind". Here, the symmetry afforded by the equivalence classes enables any choice of point on either circle to be as good as any other, and it is the consideration of the entire equivalence classes that enabled Brian to disregard the coordinates.

Discussion & Questions

We seek to address two gaps in the literature about equivalence relations: the lack of information on the global perspective on equivalence relations, and how equivalence relations are used by students as a problem-solving tool. Our hypothesis addresses both gaps, in the sense that from a global perspective on an equivalence relation, a productive strategy that becomes available is indifference towards the choice of a representative from an equivalence class. In our excerpts, Andrew and Brian productively used indifference to reason about equivalence relations from a global perspective supporting this hypothesis.

Preliminary Nature of the Report

Due to the unexplored nature of indifference as an idea linked to equivalence relations, this pilot study was conducted with the intention of developing a model to be subsequently refined, gaining viability through a cycle of interpretive analysis (Clement, 2000). There remains work to be done to document and articulate other productive forms of reasoning that accompany the use of equivalence relations. To the audience, we pose the following questions: As mathematicians and educators, what are some of the productive strategies that you would consider to be associated with the use of equivalence relations, particularly as global statements about equivalence? and in what mathematical settings do you observe or implement these strategies?

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