Student Perceptions of Norms Supported by Definitions in Proof-Based Courses

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Mathematicians' community values and norms play a vital role in shaping approaches to practice, including with respect to definitions. Mathematicians have suggested that in their instruction they portray norms related to the value of clear communication more than those related to freedom in their use of definitions, but limited work has examined what students take from instruction. Here, we examine survey responses from proof-based linear and abstract algebra courses taught by the same instructor. Results include widespread recognition of definitions' role in supporting arguments and facilitating unambiguous communication as well as some support for mathematical doers' agency in creating and using definitions, suggesting it is possible to portray this freedom in instruction.

Keywords: Values, Norms, Definitions, Linear Algebra, Abstract Algebra

Definitions are central to mathematical practice, as they stipulate the objects coordinated in theorems and algorithms. Extensive work has characterized what makes a good definition, including required aspects like being unambiguous and non-contradictory as well as preferences like elegance and minimality (e.g., Harel et al., 2006; Shir & Zaslavsky, 2001; 2002; van Dormolen & Zaslavsky, 2003; Torkildsen et al., 2023), though recent work suggests that the emphasis on minimality may have received more emphasis in the mathematics education literature than mathematicians place upon it (Torkildsen et al., 2025). Work has also considered the pedagogical value of different definitions for different groups, with teachers often valuing clarity and accessibility for their student audience (Leikin & Winicki-Landman, 2000; Torkildsen et al., 2025; Zazkis & Leikin, 2008).

However, comparatively limited work has examined how mathematicians' work with definitions upholds their broader values and norms of practice as a community. Based on interviews with algebraists, Rupnow and Randazzo (2023) found two such values: 'clarity in and for communication' and 'freedom for the use and creation of definitions'. 'Clarity in and for communication' was upheld by norms like *support arguments/get at truth* and *concept names give insight* while the 'freedom for the use and creation of definitions' value was upheld by norms like *mathematicians write definitions purposefully for utility* and *conventions may vary between mathematicians*. Of note, the interviewed algebraists suggested that their research work supported both of these values, but that their teaching gave limited if any support to 'freedom for the use and creation of definitions' and notions of agency in particular, largely because they were unsure how to do so.

Vroom et al. (2024) examined Introduction to Proof instructors' teaching practices for supporting students in revising their own definitions using the lens of Rupnow and Randazzo's (2023) norms. They noted most of the observed teaching practices focused on unambiguous, precise, and minimal definitions, as well as the fact that definitions are stipulated, may be given insightful names, and generally follow certain conventions of the community. Notably, these norms aligned with the 'clarity in and for communication' value. In contrast, only a few instances aligned with the ideas that people author definitions or contextual conventions supporting the ‘freedom for the use and creation of definitions’ value. However, because the study focused specifically on the revision of definitions rather than their initial formulation, it is possible that opportunities to support the ‘freedom’ value arise more naturally in earlier stages of defining. Together, these works suggest that definition values may appear in mathematicians’ teaching, and potentially the communication value more consistently. As such, more work is needed to understand how students experience such teaching. Here we address the research question: How do students in a linear or abstract algebra course characterize norms and values of definitions communicated in their course?

Theoretical Perspective

We use the values and supporting norms for mathematical definitions characterized by Rupnow and Randazzo (2023) as our lens. This lens is grounded in Laudan (1984)’s Triadic Network of Justification, which characterizes Axiology (Values), Methodology (Norms), and Factual Claims as mutually dependent in a field of study. Rupnow and Randazzo (2023) adapted Laudan’s (1984) framing to the mathematical community’s use and creation of definitions. In so doing, they highlighted two values, ‘clarity in and for communication’ and ‘freedom for the use and creation of definitions’, along with various supporting norms. Here we especially attend to the norms of practice, which served as codes for the open-response questions and were embedded in the closed-response questions, as described below.

Methods

A survey was administered to the Fall 2024 sections of one author’s Linear Algebra (LA) and Abstract Algebra I (AA) courses. 10 students completed and 3 students partially completed the survey in LA (out of 20), and 12 students completed the survey in AA (out of 27). LA and AA were taken by math majors and minors, though in LA most were math minors, whereas in AA, most were math majors. AA required LA as a prerequisite, and five AA survey completers had previously taken the author’s section of LA.

Both courses incorporated inquiry-oriented instruction, with linear algebra using the Inquiry-Oriented Linear Algebra tasks focused on span and linear independence (Wawro et al., 2012), linear transformations (Andrews-Larson et al., 2017), and change of basis and eigentheory (Zandieh et al., 2017). These sets of tasks were used to introduce the respective concepts, but more formal treatments and formal definitions of the respective concepts were given on subsequent days in fill-in-the-blank notes during interactive chalk-talk lectures (Artemeva & Fox, 2011). Abstract Algebra tasks focused on motivating and defining a group and group isomorphism (Larsen, 2013). Students were given the opportunity to work with examples of a group to abstract the concept of a group and define it themselves before fill-in-the-blank notes with an interactive chalk-talk lecture formalized what had been discussed in the previous weeks of tasks. Similarly, students worked with Cayley tables to consider examples and non-examples of isomorphisms before defining the concept themselves and subsequently having fill-in-the-blank notes during chalk-talk lectures. As Vroom et al. (2024) note, this inquiry setting opened up the opportunity for meaningful engagement with norms (from the instructor’s perspective), particularly *definitions arise from/relate to examples*, *mathematicians make definitions* and *mathematicians write definitions purposefully for utility*.

The survey was available the last two weeks of the semester, and students received extra credit for submitting it. Questions on the survey focused on students’ views of definitions and how definitions had been used in the course. Open response questions analyzed here focused on how mathematical definitions are similar and different from dictionary definitions, the purpose of definitions, ways definitions were used in the course, how their understanding of definitions had changed during the course, and whether the instructor did or could do something more illuminating about definitions. Closed response questions provided simplified versions of the norms from Rupnow and Randazzo (2023) (e.g., 'Definitions allow the use of a single phrase to communicate complex ideas' was provided to ask about the *Definitions as shorthand* norm). Students were given a textbox in which to type a response to each open response question and were asked to drag each stated norm into a "I took these away from the class" or "I did not take these away from the class" box for the closed response question.

We used the existing definition norms from Rupnow and Randazzo (2023) as a starting point for analysis of the open responses but made space to add additional codes if needed, in alignment with codebook thematic analysis (Braun et al., 2019). One non-instructor researcher coded the LA open-ended items independently and another non-instructor researcher coded the AA open-ended items independently. Based on this, three additional norm codes were created: *mathematical definitions are subject to verification/justification*, *mathematical definitions are expressed through symbolic language and notation*, and *mathematical definitions can evolve over time.* We perceived the first two norms to support the 'clarity in and for communication' value, and the third norm to support the 'freedom of choice for the use and creation of definitions' value. Then these two researchers plus the instructor researcher discussed and came to consensus on codes present in each student's response to each question in both courses.

Results

Here we highlight students' responses with respect to the two values separately. Table 1 notes the number of students whose responses received a norm code (Open) or who signified that "I took these away from the class" by dragging the prompted norm into a box (Closed). Throughout we signify norms with italics.

Table 1. Student norm counts by course and question

Value	Norm	LA Open (n=13)	LA Closed (n=13)	AA Open (n=12)	AA Closed (n=12)
Clarity in and for communication	Rigor/precision	4	0*	4	0*
	Clarity/no ambiguity in communication	10	8	10	6
	Axiomatic system/work with definitions	6	10	4	12
	Support arguments/get at truth	11	12	12	12
	Norm of precision perpetuated through teaching	0	0*	4	0*
	Norm of precision irrelevant to laypeople	0	0*	2	0*
	Definitions as shorthand	6	11	2	10
	Concept names give insight	0	8	0	5
	Examples illustrate definitions	2	11	2	10

	Definitions are based on historical motivations	0	2	0	4
	Need more than definitions to understand concepts	4	0*	1	0*
	Definitions scaffolded for undergrads	2	0*	1	0*
	Norms of mathematical communication through formatting choices	1	6	0	7
	Mathematical communication has consistency	2	11	2	9
	Mathematical definitions are subject to verification/justification	0	0*	1	0*
	Mathematical definitions are expressed through symbolic language and notation	4	0*	2	0*
Freedom of choice for the use and creation of definitions	Definitions arise from/relate to examples	0	8	2	9
	Definitions permit inspection of relevant features	0	10	1	8
	Concepts are made through abstraction	0	5**	0	9
	Mathematicians make definitions	1	7	4	8
	Mathematicians write definitions purposefully for utility	7	7	4	7
	Conventions may vary between mathematicians	0	7	0	9
	Norm of agency perpetuated through teaching	1	0*	4	0*
	Mathematical definitions can evolve over time.	0	0*	1	0*

*No question asked on this topic
**Count reduced by one because a student noted they no longer meant to choose this in a later open-ended question

Clarity in and for Communication

A predominant norm for LA and AA students was the role of definitions to *Support arguments/get at truth*. This norm was articulated and selected across both open and closed responses, indicating a strong perception of definitions as foundational for logical reasoning and mathematical communication. Students noted how their instructor clarified this function of definitions, particularly their utility in proofs. One LA student stated, “She showed us how to use definitions to prove theorems.” Another LA student elaborated, remarking, “I feel like I better understand their purpose and am able to utilize them in the context of proofs and problem solving.” Many students emphasized how definitions provide a necessary foundation for constructing valid arguments, arriving at mathematical truths, and developing deeper results. As one AA student described the purpose of mathematical definitions: “To serve as the building blocks for the development of theorems, proofs, and mathematical structures.” This highlights students' ability to see definitions not as isolated facts, but as interconnected components within a logical framework.

Students also showed awareness of the norm *mathematical communication has consistency*. This norm highlights the mathematical community's expectation for a standard and consistent way of conveying mathematical ideas to promote clarity and understanding. Responses reflecting this norm were noted in open responses but were affirmed more often in the prompted closed responses. One LA student reflected, "I've seen definitions in practice and developed a stronger sense of what they do and don't contain," demonstrating how exposure to definitions helped them notice patterns in how definitions are framed. An AA student echoed this norm, stating, "To establish a consistent rule or mathematical relationship, to describe a proven mathematical principle." Another AA student emphasized its broader significance: "They ensure clarity and consistency which allows us universal understanding and the ability to construct complex mathematical frameworks." These comments underscore that students viewed definitions not merely as isolated facts, but as critical tools for constructing precise, shared meaning.

The norm *Axiomatic system/work with definitions* was highlighted by students in both LA and AA. In open responses, this norm was mentioned by 6 LA students and 4 AA students, while in closed responses, it was selected by 10 LA students and 12 AA students. This pattern suggests that when explicitly prompted, students across both courses largely agreed with this norm as an expectation of practice, though fewer highlighted it unprompted. In the following LA student's response, we interpret species to mean examples of concepts: "This entire course relies on definitions building on each other. Knowing these different definitions allow[s] me to make assumptions as to how certain species will behave and more importantly why." Another LA student recognized the structure of mathematical definitions in describing a word or process: "During this class the way mathematical definitions were presented to me has helped me understand how topics relate to one another better. The use of theorems, lemmas, and corollaries, specifically was very helpful." Similarly, an AA student emphasized the foundational role of definitions: "It is clearer now that definitions play a huge part in the understanding and development of mathematics." This recognition reveals that students understood definitions as foundational, logical building blocks in mathematics.

Beyond their role in supporting arguments and axiomatic systems, students also recognized definitions for their *clarity/ no ambiguity in communication.* This norm was mentioned in both open and closed responses, suggesting that students valued definitions as tools for precise expression. One LA student characterized mathematical definitions as **"**more precise and straightforward**"** than dictionary definitions. They are designed to be **"**unambiguous**"** and **"**to distinguish similar but distinct ideas." Another LA student observed that the wording in mathematical definitions is specific, taking **"**into account every nuance." Nevertheless, this precision can sometimes make definitions long or hard to understand, as one LA student noted, "I personally think mathematical definitions are overly complex when they could be simpler. I understand that they are trying to not be ambiguous, but they are just complex." An AA student also focused on avoiding ambiguity, stating: "The purpose of mathematical definitions is to provide precise and unambiguous descriptions of ideas, forming the foundation for rigorous reasoning and logical deductions. They ensure clarity and consistency which allows us universal understanding and the ability to construct complex mathematical frameworks." These observations reveal that students recognized the communicative function of definitions, understanding their role in ensuring clarity, avoiding ambiguity, and supporting shared understanding across mathematical discourse.

Student responses, particularly in closed questions, revealed an awareness of the role of definitions as shorthand in mathematics. The norm *Definitions as shorthand* was characterized in open responses by 6 LA and 2 AA students and in closed responses by 11 LA and 10 AA students. This indicates that students, particularly when prompted, perceived definitions as concise ways to encapsulate complex ideas. This LA student noted, "Definitions package up entire concepts into concise words; This not only helps with communicating and understanding, it also helps with drawing out theorems." An AA student also reflected that seeing definitions as shorthand was a way "I used to keep the definition in the back of my head and then every now and then used it, but this course has made me study the definition and completely understand it. Made me realize how important the[y] are." Students' responses demonstrate the practical and communicative power of definitions as shorthand in mathematical practice.

Student responses revealed an awareness of certain aspects of the role of examples in mathematical understanding. The norm *Examples illustrate definitions* was not as frequently mentioned in open responses (2 LA students and 2 AA students), but it was more frequently chosen in closed responses (11 LA students and 10 AA students). One LA student noted: "The use of theorems and definitions paired with relevant examples or problems was helpful." Additionally, one Abstract Algebra student illustrated in their open response: "She gave examples of the definitions and how they truly were related to certain problems." Students in both courses indicated that examples help solidify understanding, associate concepts with processes, and demonstrate how definitions operate in practice. This shows that students viewed definitions not only as abstract entities but as tied to concrete examples and intuitive reasoning, though they especially attended to this when prompted.

Freedom of Choice for the Use and Creation of Definitions

Student responses, particularly in closed questions, showed an awareness of the norm *Definitions arise from/relate to examples*, which highlights that mathematical definitions are tied to concrete examples and intuitive reasoning. As one AA student illustrated: "They [definitions] provide a framework for constructing examples that satisfy certain properties or counterexamples that fail to meet specific criteria."

The norm *Mathematicians write definitions purposefully for utility* was consistently mentioned by students, with 7 LA students and 4 AA students in open responses and 7 LA students and 7 AA students in closed responses. Students both spontaneously articulated and recognized with prompting that definitions are not arbitrary but are designed for practical application. For instance, one LA student noted, "I feel like I better understand their [definitions'] purpose in the context of proofs and problem solving." Demonstrating an appreciation for the pragmatic use of definitions in mathematics, one AA student elaborated:

> A mathematical definition defines a mathematical structure or process and how to apply that process / utilize the structure. A definition in the dictionary is more to give meaning to a word. Both give meaning to some abstract object, but a mathematical definition has more nuance than just giving meaning.

Another AA student stated:

> Writing proofs has been extremely beneficial in my understanding of proofs. Before I became more familiar with proofs, I did not think definitions in math had much purpose other than just making rules in math that we have to follow. It is clearer now that definitions play a huge part in the understanding and development of mathematics.

These reflections show students had developed a sophisticated view of definitions not as static rules, but as essential tools in both reasoning and application.

The idea that *Mathematicians make definitions* appeared in open responses from 1 LA student and 4 AA students, and more frequently in the closed responses (7 LA students and 8 AA students), highlighting students' awareness that mathematical definitions do not merely exist but are created by individuals. This LA student expressed, "Mathematical definitions allow us to not only understand the world of math but also create it in a way that is universal," emphasizing mathematicians' active role in constructing tools for the discipline. An AA student further noted:

> My understanding of mathematical definitions is, in general, more developed than it was before because I was exposed to a lot of definitions in this class. I have more appreciation for the process of writing a definition and I know how to read mathematical definitions more accurately.

These reflections point to students' recognition of definitions as thoughtfully constructed tools for shaping and communicating mathematics.

The norm *Conventions may vary between mathematicians* was not spontaneously mentioned in any open responses, but it was selected in closed responses by 7 LA students and 9 AA students. While students did not highlight that mathematical definitions and conventions reflect choices made by mathematicians depending on context or purpose in response to the purpose of definitions, they did agree with this sentiment when prompted more generally.

Discussion

Students in both courses largely highlighted the same norms with fairly similar frequencies. Considering that the same instructor taught both courses and that both courses are proof-based, this perhaps should not be surprising. Across the open and closed responses in both courses, students commonly highlighted *support arguments/get at truth* as a purpose of definitions as well as that definitions support unambiguous rigorous communication (*rigor/precision* and *clarity/no ambiguity in communication*). Students also highlighted *axiomatic system/work with definitions*, *definitions as shorthand,* and *mathematical communication has consistency*, though these arose more in the closed than open responses. This suggests that students agreed with these norms when prompted to consider them but did not immediately highlight them from prompts focused on differences between mathematical and dictionary definitions, the purpose of definitions, or ways definitions were used in the course.

Of note, the norms *mathematicians make definitions* and *mathematicians write definitions purposefully for utility* also arose in the data, suggesting a recognition of mathematicians' (and potentially their own) agency in writing and using definitions. Relatedly, though less commonly, some students did suggest that agency in using and creating definitions was supported through instruction. This contrasts with the algebraists' claims about their teaching practice (Rupnow & Randazzo, 2023) and most of the Introduction to Proof instructors' teaching moves supporting students' definition revisions, though revising definitions comprises a limited portion of instructors' overall work in a course (Vroom et al., 2024). Thus, while supports for the 'freedom for the use and creation of definitions' value may not yet be commonly documented in the literature, it is possible to support this value.

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