Coordinating Point and Rate: A Framework for Students' Reasoning About Slope

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This study examines how precalculus students reason about rate and point as multiplicative objects in the contexts of linear equations and approximation. To capture students' developing conceptions, I introduce a framework distinguishing three levels of abstracted ratio: generalized ratio, unit rate, and interiorized ratio. Findings show that although students progressed toward interiorized reasoning, their understanding was fragile when working with non-integer or symbolic values. They also struggled to coordinate slope with a reference point, which limited their ability to interpret points as records of covariation. Geometric approaches to linear approximation highlighted these conceptual difficulties more clearly than symbolic tangent line equations, underscoring the need for instructional support that fosters covariational reasoning.

Keywords: linear function, linear approximation, rate, point, multiplicative object

Studies have long documented calculus students' struggles with tangent line equations and linear approximations (Amit & Vinner, 1990; Çekmez & Baki, 2016; Moore-Russo & Nagle, 2024; Orton, 1983), yet less attention has been paid to underlying precalculus concepts—especially understandings of rate and point in linear equations. Classroom observations suggest that difficulties may stem not only from gaps in derivative fluency but also from underdeveloped conceptions of linear relationships.

To investigate, I conducted exploratory interviews with three precalculus students, focusing on the research question: How do students conceptualize rate and point as multiplicative objects in the context of linear equations and linear approximation?

Theoretical Framework and Literature

Students who view a function only as a correspondence between input and output (Thompson, 1994b), or who treat $L(x) = f(a) + f'(a)(x - a)$ as a formula to substitute values into, may compute correctly without visualizing the situation or interpreting the slope $f'(a)$. Such disconnections between computation and the graphical tangent have been documented (Code et al., 2014). In contrast, students who conceive x and y as covarying quantities—measurable attributes varying in coordination (Thompson & Carlson, 2017)—are more likely to interpret $L(x)$ meaningfully. A meaningful understanding of $L(x)$, I argue, depends on conceiving both *point* and *rate* as multiplicative objects (Saldanha & Thompson, 1998). A multiplicative object unifies two quantities into a single conceptual entity: a point coordinates horizontal and vertical displacements, and a rate coordinates changes in x and y. Recognizing these as multiplicative objects requires mentally integrating covarying quantities into one structure, rather than viewing them in isolation.

Rate as a Multiplicative Object

The concept of rate connects to the linear function L from two perspectives: calculus and algebra. From a calculus perspective, L embodies $f'(a)$, the instantaneous rate of change $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, interpreted geometrically as the slope of the tangent line. While this definition is central to calculus learning (Orton, 1983; Yu, 2024; Zandieh, 2000), a full treatment lies beyond the scope of this study. Instead, the focus here is on students' conceptions of rate in algebraic contexts, where $f'(a)$ functions as a constant rate of change—the slope of L.

In Thompson's (1994a) framework, a rate is a reflectively abstracted multiplicative structure describing how one quantity changes in relation to another. A ratio may begin as a static comparison but can be abstracted from specific numbers to describe relationships between indeterminate, systematically related quantities. This progression reflects a shift from an *internalized* ratio—reasoning with repeated accumulations of a fixed ratio (e.g., scaling "three pears to four apples")—to an *interiorized* ratio, in which quantities are conceived as continuously varying while maintaining a constant relationship. Interiorized ratio aligns with covariational reasoning, where rate functions as a structural relationship that integrates and governs the coordinated changes of two quantities.

Research shows that students often struggle to construct rate as a multiplicative object. They may treat it procedurally (Lobato et al., 2003), rely on additive reasoning, or focus on isolated values instead of coordinating variation (Clark & Kamii, 1996; Yu, 2024). Even when producing correct values, they may not interpret them as relationships between covarying quantities. Thompson (1994a) emphasized that interiorizing rate requires conceiving it as a structural relationship that remains invariant across magnitudes. Johnson (2015) found that students often viewed rate as an extensive quantity—treating measures as separate totals—rather than as an intensive quantity that unites them into a single multiplicative relationship governing covariation. Together, Thompson and Johnson highlight the difficulty of moving from procedural or context-bound views toward structural conceptions of rate, underscoring the need for analytic categories that capture the partial ways students reason prior to full interiorization.

Point as a Multiplicative Object

A point can be understood not merely as a location—"over this much and up that much"—but as a multiplicative object that records the covariation of two quantities (Thompson et al., 2017). In this view, coordinates represent interdependent values integrated into a single structure. Yet research shows that many learners struggle with this conception. Thompson et al. (2017) found that accuracy in placing the initial point was closely tied to correctly sketching the entire graph. Similar challenges have been documented in contextualized settings where attending to covarying quantities is essential: Ellis et al. (2015) reported students' difficulties in interpreting exponential growth involving height and time, while Frank (2016) documented struggles to reason about points in motion or dynamically changing systems using the same task from Thompson et al. (2017). Collectively, these studies underscore the complexity of conceiving a point not merely as a location but as a representation of a quantitative relationship.

Methodology

Setting

This study took place at a small state university in the southeastern United States. From fifteen students recommended by precalculus instructors for their varied mathematical thinking, three—Tanya, Zada, and Sean (pseudonyms)—agreed to participate for an hourly stipend. Tanya, a dual-enrolled high school student, had completed Algebra 2; Zada, a first-year biology major, had recently finished college algebra; and Sean, a first-year computer science major, had completed precalculus in South Asia.

All precalculus courses used *Algebra and Trigonometry* (Abramson, 2018) as the textbook, which presents slope mainly as the algebraic ratio $\frac{y_2-y_1}{x_2-x_1}$ in slope-intercept form, with point-slope form quickly converted. Graphs focus on parallel and perpendicular lines, per-unit concepts appear only in contextual problems, and nonlinear rate of change is omitted. Multiple representations are included but rely on discrete values, rational slopes, and integer intercepts, reflecting an algebraic–procedural emphasis.

Participants took part in five to seven interview sessions (90 minutes each) near the end of the semester, after instruction on slope, linear equations, and functions. Tanya and Zada completed seven sessions, while Sean completed five due to family circumstances. Using a semi-structured clinical interview format (Ginsburg, 1997) with elements of exploratory teaching (Steffe & Thompson, 2000), early sessions elicited existing understandings through tasks such as constructing equations from graphs or tables, without instructional support. Later sessions introduced targeted prompts, informed by prior responses, to probe reasoning about linear and quadratic functions and linear approximation.

Data Analysis and Development of New Analytic Tool

Data analysis began during data collection, following an iterative, comparative approach (Miles & Huberman, 1994). After each interview, I reviewed recordings, wrote analytic memos summarizing key content, and drafted rough transcripts of notable moments. This early work informed preliminary codes for students' conceptual difficulties with point, slope, linear equations, and approximation, which were refined through constant comparative methods (Glaser & Strauss, 1967) as patterns in reasoning emerged.

As analysis progressed, the coding scheme was refined through distinctions that supported cross-case comparisons of how students coordinated rate and point. Drawing on Thompson's (1994a) framework of internalized and interiorized ratios and related work on covariational reasoning (Johnson, 2015; Thompson & Carlson, 2017), I identified three analytic categories of ratio and rate: generalized ratio, unit rate, and interiorized ratio (Table 1). These categories provided the framework for interpreting students' levels of abstraction of ratio.

Table 1. Types of Abstracted Ratio

Level	Description and Examples of Student Reasoning
Generalized Ratio (Early Abstracted Reasoning)	Treated as a fixed multiplicative value across quantity pairs, applied symbolically without coordinating x and y. Example: "The slope between $(2, a)$ and $(5, b)$ is 3/4 because $(b – a)/(5 – 2) = 3/4$."
Unit Rate (Transition to Interiorized Reasoning)	Conceived as a per-unit change, enabling additive or multiplicative coordination of Δx and Δy, typically for integer Δx values. Example: "If x increases by 1, y increases by 3/4," then uses $3/4\times 2$ to find Δy when $\Delta x=2$.
Interiorized Ratio (Fully Abstracted Reasoning)	Conceived as a structural, covariational relationship, expressed as $\Delta \mathrm{y} = m\Delta \mathrm{x}$, that applies to any Δx, including symbolic or infinitesimal values, supporting flexible and general reasoning. Example: "For any change in x, the change in y is 3/4 times that," applied with h or 0.1.

Figure 1 illustrates how generalized, unit rate, and interiorized ratio conceptions influence students' approaches to linear approximation, highlighting how Δx and Δy are conceived as displacements relative to the reference point $(a, f(a))$ rather than the origin. (Note: I use Δx and Δy instead of dx and dy to reflect students' reasoning in precalculus contexts.)

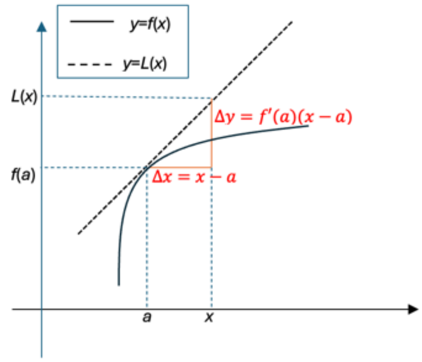


Figure 1. Understanding Linear Approximation from a Covariation Perspective

Results

The results synthesize students' reasoning across tasks on linear equations and linear approximation, organized by a developmental trajectory rather than by participant or chronology. Findings are presented in three parts: (1) pre-abstracted interpretations of rate, (2) emerging abstracted ratios—generalized ratio, unit rate, and interiorized ratio—and (3) conceptions of point as a multiplicative object. Subsections are titled by the transitional strategies observed (e.g., additive, multiplicative) when students' thinking did not fully align with a single level. Tasks were adapted to students' readiness; Tanya contributed the most complete data set, while Zada's more limited mathematical preparation and Sean's reduced participation due to family circumstances yielded partial but informative episodes. Together, these data trace how precalculus students coordinate rate and point and where conceptual fragilities persist.

Slope as a Ratio between Two Points and Construction of Linear Equations

At the outset, all three participants understood slope mainly as the formula $(y_2–y_1)/(x_2–x_1)$ and showed only partial familiarity with "rate of change." When constructing linear equations, Sean applied both slope and point-slope forms, while Tanya and Zada relied solely on slope-intercept form, succeeding only when the y-intercept was obvious. Tanya recognized parallel slopes but struggled to connect tabular data with equations, whereas Zada omitted equation form, misapplied slope in parallel tasks, and added x and y values directly from the table.

In sum, Sean and Tanya displayed stronger reasoning than Zada, yet all three treated slope as an algebraic ratio rather than as an abstracted multiplicative relationship, with little evidence of Cartesian or tabular connections.

Transitioning to Abstracted Ratio

Additive strategies and the emergence of multiplicative reasoning with integer Δx. When asked to find the rate of change between two points $(2, a)$ and $(5, b)$ that lie on the graph of $y = 3x – 7$ (Figure 2A), Zada proposed first calculating a and b and then applying the slope formula. This showed she did not interpret the coefficient 3 as the rate of change for all points on the line but instead viewed slope as something to be computed from particular pairs—an underdeveloped conception of slope as a generalized ratio. To find a and b, she suggested graphing and "counting up by 3" each time x increased by 1.

A. Points (2, a) and (5, b) lie on the graph of $y = 3x - 7$. What is the rate of change between these two points?

B. (3, b) is on the line graph below. What is b?

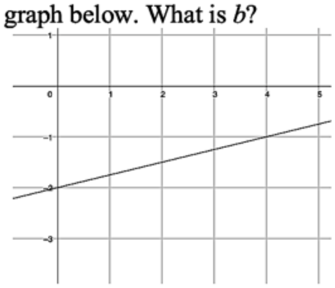


Figure 2. Tasks Requiring a Conception of Slope at an Advanced Level

To move her from counting to multiplication, I asked for the y-value at $x = 10$ on the graph of $y = 3x + 1$. She again counted by 3 until reaching (10, 31) (see Figure 3). For (60, b), however, she hesitated, noting that "it would take forever." With prompts about slope as "a rise of 3 for each run of 1" and the point (60, b) plotted, she initially claimed that b was 6 with a rise of 5, misled by the scale. Only after I clarified the diagram, labeled (60, 1), and reiterated that slope meant a rise of 3 per unit run did she compute the rise as 60 × 3. This episode suggests her multiplicative reasoning with slope was contingent on explicit visual and verbal scaffolds.

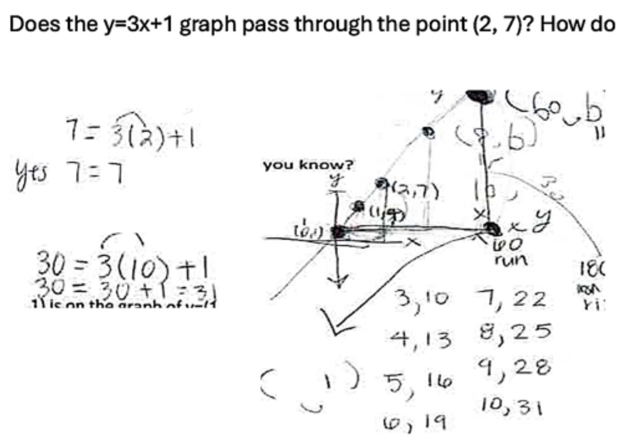


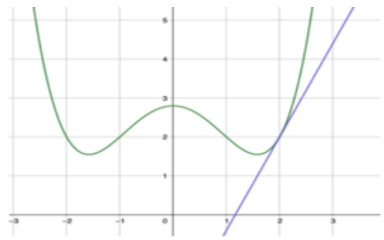
Figure 3. Zada's Transitioning from Counting to Multiplication

From generalized algebraic ratio to generalized geometric ratio. When asked to find b in the point (3, b) (Figure 2B), Sean applied both slope formula and point-slope form, producing $y = \frac{1}{4}x - 2$. He then set up $\frac{1}{4} = \frac{b-(-1)}{3-4}$ and solved for b—indicating that he viewed ¼ as an invariant relationship, a generalized algebraic ratio. Yet he acknowledged knowing no method beyond algebraic manipulation. When I drew segments between (0, –2) and (4, –1) and asked about $y_2 - y_1$ and $x_2 - x_1$, he recalled from high school that they were "differences in the coordinates."

Sean carried this reliance on point-slope form into the linear approximation tasks. To estimate $f(2.1)$ (Figure 4a), he constructed the tangent line $y = \frac{12}{5}x - \frac{14}{5}$ and substituted $x = 2.1$

to obtain 2.24. When pressed to reason directly with slope, he hesitated until I drew displacements and labeled $\Delta x = 0.1$. Then he wrote $\frac{12}{5} = \frac{\Delta y}{\Delta x}$, substituted $\Delta x = 0.1$, and found $\Delta y = 0.24$—explaining, "x is increased by 0.1 and y is increased by 0.24." He also extended this to $\Delta x = 0.2$, showing movement toward conceiving slope as a generalized geometric ratio.

In a later task estimating $\sqrt{1.05}$, Sean again used the tangent line equation but explained that with slope ½ and $\Delta x = 0.05$, "Δy is ½(0.05)." Still, his reasoning reflected manipulation of the equation $½ = \Delta y/0.05$, signaling a generalized ratio rather than an interiorized ratio.



The y=g(x) graph (in blue) is the tangent line of y=f(x) (in green) at the point (2, 2) and the slope of the line g is 12/5.

a) Estimate the value of f(2.1).
b) Find the equation of the line: y=g(x).
c) Estimate f(2+h).

Figure 4. Constructing Tangent Line and Approximating Function Values from Graphs

From a generalized ratio to unit rate: Additive/multiplicative reasoning with integer Δx. Tanya initially treated slope only as a generalized algebraic ratio, writing $\frac{1}{4} = \frac{b-(-1)}{3-4}$ to solve for b in Figure 2B. When asked what a slope of ¼ meant, she replied, "You go up once and go over four," reflecting an action-based geometric ratio. She also noted, "when the run is 1, the rise is ¼," but could not extend this per-unit idea to find b on the graph, repeating instead, "The rise is 1 and the run is 4."

With prompts connecting slope to displacements—"If x increases by 1, how much does y increase?"—she began using ¼ as a per-unit ratio, including decreases: "x is decreased by 1 and y is decreased by ¼." Using (4, –1) as a reference, she reasoned that (3, b) must be "¼ smaller than –1."

Tanya later used unit-rate reasoning more fluently in other tasks. When $\Delta x = -3$, she stated, "Go down 18," interpreting Δy as (–3)(6) without written calculation. In constructing equations, she often used unit-rate language, "Each time I move left by 1, I need to go down by 1/5." In Figure 4b, she subtracted 12/5 twice from the y-coordinate of (2, 2) to find $b = -14/5$. These episodes show her emerging ability to express accumulated change through unit rate reasoning, flexibly switching between addition and multiplication (cf. Thompson, 1994a).

From unit rate to interiorized ratio: Difficulties with non-integer Δx. Although Tanya demonstrated unit-rate reasoning in earlier tasks, she never directly applied the relationship $\Delta y = m\Delta x$ for non-integer values. Instead, she reconstructed the relationship through explicit equations, suggesting slope was not yet interiorized. For instance, in Figure 4b, she used an additive strategy for $\Delta x = -2$ when locating the y-intercept, but for $\Delta x = 0.1$ she set up $12/5 = \Delta y/0.1$, then solved algebraically: $\Delta y = (12/5)(0.1) = 0.24$.

As approximation tasks grew more complex—shifting from numeric to symbolic slopes and from graphs to purely algebraic contexts—she consistently applied slope to estimate values. Yet for non-integer Δx, she reverted to writing $m = \frac{\Delta y}{\Delta x}$ and solving for Δy, reflecting a generalized geometric ratio rather than an interiorized ratio. Her reliance on reconstructing proportions suggests she had not yet conceived slope as an interiorized ratio that operates flexibly on any Δx.

Shortened Treatment of Point as a Multiplicative Object

Although students also struggled with conceiving a point as a multiplicative object, this paper focuses primarily on their reasoning about slope as ratio and rate. All three students had difficulties coordinating quantities—specifically, horizontal and vertical displacements—encoded in both rate and point coordinates. While Sean and Tanya showed somewhat greater facility than Zada, all three struggled at times to use a known point and rate to determine a new point in linear equation and approximation tasks. *Due to space limitations, detailed analyses of students' reasoning about point as a multiplicative object are not included here and will be discussed in a separate paper.*

Conclusion and Discussion

This study examined how students coordinated slope in the context of linear equations and linear approximation tasks, with particular attention to how their reasoning reflected different levels of abstraction of ratio and rate. Building on Thompson's (1994a) distinctions between internalized and interiorized ratios, I proposed three analytic categories—generalized ratio, unit rate, and interiorized ratio—as a framework for analyzing students' reasoning. These categories captured important qualitative differences in how students conceived slope, ranging from applying it as a fixed algebraic relationship, to interpreting it as a per-unit change, to treating it as a structural multiplicative relationship applicable to any Δx.

The framework proved useful in distinguishing students' reasoning even when they produced the same correct answer. For instance, Tanya's use of unit rate reasoning allowed her to flexibly handle integer changes in *x,* yet her reliance on explicit equations for non-integer Δx revealed that she had not fully interiorized slope as an object of thought. Similarly, Sean's progression from algebraic manipulation of slope to its geometric interpretation showed how a generalized ratio could be extended toward more sophisticated reasoning. These distinctions highlight that students' difficulties with tangent lines and linear approximations are not simply procedural gaps in differentiation but reflect deeper issues in how they coordinate quantities through ratio and rate.

By situating students' reasoning within these categories, this study advances prior work on covariational reasoning (e.g., Thompson, 1994a; Johnson, 2015) by offering a finer-grained framework for distinguishing how students engage with slope in algebraic and geometric contexts. The categories of generalized ratio, unit rate, and interiorized ratio clarify not only *where* students encounter difficulties but also *how* those difficulties manifest in relation to symbolic versus graphical reasoning. This contributes to instructional design by highlighting the importance of tasks that explicitly bridge counting strategies, per-unit reasoning, and flexible application of $\Delta y = m\Delta x$. For example, tasks that press students to extend reasoning from integer to non-integer or symbolic Δx may be especially productive in supporting transitions toward interiorized ratio. Future research might investigate how such task sequences influence students' ability to coordinate slope with reference points in linear approximation, thereby deepening their preparation for calculus.

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