“There’s Nothing Abstract About This Math”: Abstract Algebra Students’ Interpretations of Abstraction

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Abstract Algebra is often cited as being a difficult course for undergraduate students because it is more abstract than computationally based courses such as Calculus. But what does it mean for something to be “abstract”? I conducted a grounded theory analysis of student survey data from an Introduction to Abstract Algebra course to explore how students define abstraction. The data show that students’ personal definitions of abstraction go beyond conceptions of abstraction discussed in the literature, and the concept of abstraction acts as a mathematical artifact that influences the way students view abstract algebra concepts. Preliminary analysis also suggest that students’ definitions of abstraction differed depending on their major area of study, indicating that abstraction is a boundary object whose definitions and uses vary depending on context.

Keywords: abstraction, abstract algebra, artifact, boundary object

As one of the first proof-based courses in the undergraduate math sequence, abstract algebra is often cited as being a difficult course for undergraduate students because of its abstraction level. The purpose of this study is to investigate how students conceptualize mathematical abstraction in an introductory abstract algebra class.

“Abstract” is used as both an adjective and a verb in various contexts in math education. The most well-known treatment of “abstract” as an adjective in Abstract Algebra is found in Hazzan’s (1999) framing of Reducing Abstraction, which identified three interpretations of abstraction. Hazzan writes of “levels” of abstraction of abstract algebra concepts, defining this not as distinct enumerated stages but relative comparisons of how abstract a concept seems to a student.

In the first interpretation, Hazzan states that the level of abstraction of a concept or topic is not a property inherent to that concept, but a “reflection of the relationship between the object of thought and the thinking person” (1999, p. 76). Thus, a person’s opinion of the abstraction level of a concept is influenced by their level of familiarity and prior experience with it.

According to Hazzan, the level of abstraction of a concept can also be described as a reflection of the process-object duality. Students working at a low level of abstraction are often focused on a process involving carrying out an algorithm or procedure, while students at a higher level of abstraction can view a computational process as a generalized object without referring to a specific example.

Hazzan’s third interpretation of abstract is of complexity; “the more compound an entity is, the more abstract it is” (1999, p. 82). For example, a set is more abstract than one element in the set. Finally, Melhuish et al. (2018) expanded the framework and defined a fourth interpretation of abstraction level as the degree of formality or precision of mathematical objects. For example, a formal definition is more abstract than a metaphor or an informal definition.

The interpretations of abstraction identified by both Hazzan (1999) and Melhuish et al. (2018) were based on observations of student work and the strategies students use to make sense of abstract concepts. To date, no studies have directly addressed how students define what it means for a concept to be “abstract” in abstract algebra.

A limitation of Hazzan's study is the student population, which consisted of computer science students taking an abstract algebra class, after having previously taken a linear algebra class. However, some students take abstract algebra as their first introduction to abstract, proof-based mathematics, including students majoring in a variety of subjects. Thus, in this study I investigate how students with a variety of majors (including math, computer science, physics, chemistry, engineering, and economics) and differing levels of mathematical experience interpret abstraction in abstract algebra.

Theoretical Perspective

My theoretical perspective draws from sociocultural theory and the idea of mathematical artifacts as boundary objects. In the sociocultural view, learning occurs through social interaction and is mediated by artifacts, which includes language and symbols as well as physical or conceptual objects, tools, and concepts (Lerman, 2001). Mathematical artifacts are products of history and culture, and their meanings are "long-lasting but potentially changeable" (Lerman, 1996, p. 146).

An often-overlooked artifact encountered in abstract algebra is in the name, "abstract." While the definition of "abstract" is often taken to be self-evident, the concept of abstraction contains multiple meanings, each with their own culture and history. Students from differing backgrounds may have vastly different assumptions of what abstraction is, and these assumptions may influence the way they approach and interpret the subject.

Abstraction as an artifact is a boundary object, an artifact at the boundary between different communities of practice that are used differently by each community (Hoyles & Noss, 2004). Abstraction has long been at the boundary of mathematicians, math educators, and laypeople, who each have different conceptions and uses of abstraction (e.g., Thorndike et al., 1923; Corry, 2003). Students majoring in disciplines such as physics and engineering have different experiences with abstraction compared to math majors. As Hoyles and Noss observe when it comes to boundary objects, "communication of meaning and 'transfer' can be problematic unless and until the different conceptualizations—and the language in which they are expressed—are brought into alignment" (p. 3).

Methodology

Data were collected from students in an Introduction to Abstract Algebra course taught using a rings-first curriculum sequence. The course had an enrollment of 44 students with varied academic majors, including math, computer science, economics, physics, and others.

Halfway through the semester, I administered a written survey to students in the course, to which 33 students responded. For this paper, I analyzed student responses to the following questions:

1. This course is called Abstract Algebra. What does "abstract" mean to you?
2. What concepts in this course so far have seemed the most abstract to you?
3. What concepts in this course so far have seemed the least abstract to you?

I conducted a Grounded Theory analysis (Charmaz, 2014) of all 33 student responses to the survey question. Initial line-by-line coding resulted in 135 codes related to students' experiences with abstraction in this course. After several rounds of coding, I grouped these codes into three major categories representing students' definitions of abstraction, which will be described in the Results section. Following the survey, I conducted one-on-one, semi-structured interviews with 14 students. I used my coding results and memos to select six of the 14 interviewed students representing all major codes. I coded portions of these interview transcripts and selected excerpts to represent each category.

Results

I identified three main categories that students consider in their definitions of abstraction: Structure, Purpose, and Beyond Sensory Experience. Each category was interpreted in two different ways by students. Three of these categories are further split into subcategories (Table 1).

Table 1. Student Interpretations of Abstraction

Category	Interpretation	Subcategory	Examples
Structure	Presence of Structure	Underlying reasons	Abstract algebra "breaks down" math to its fundamentals and explains the underlying reasons behind familiar procedures
		Simplicity	Abstract algebra concepts have simple definitions and axioms which require less justification than contextualized concepts
	Absence of Structure	Arbitrary	Abstract algebra changes familiar algebra rules and definitions without justification
		Indirect	Abstract algebra uses words and "math tricks" instead of numbers
Purpose	Widely Applicable		Abstract begins in applied concepts but moves away from applications Generalizing across dissimilar contexts Able to represent multiple things
	No Applications		Not relevant to student's major or future career

Beyond Sensory Experience	Basis of reality		Abstract math is a "sixth sense" that helps us learn about the world beyond the five senses
	Not Real	Outside the Real World	Things that are beyond perception by the senses are difficult to grasp
		Too strange to believe	Students have difficulty believing concepts are valid, even with proof

Structure

In the first category, Structure, some students interpreted abstraction as the presence of structure. One student, James (all participant names used are pseudonyms), described Underlying Reasons as a "broader view" of algebra:

> We are taking a look at algebra in a broader view. In simple algebra classes you learn the mechanics of algebra and how to apply it. In this class we are looking at things one level higher and proving why the mechanisms we previously used work.

Other students described abstraction as a lack of structure, marked by chaos and uncertainty. Hanna, a math and art double major, described frustration with the seemingly arbitrary nature of rules in abstract algebra, and couldn't see any underlying reasoning behind it. Hanna defined abstraction as "unstructured, random, chaotic… [groups and rings] seem like random 'things' with very specific conditions that someone decided randomly one day."

Purpose

A common theme involved how abstraction related to the purpose or applications of math. Some students said that abstract math is widely applicable to many different areas, seeing it as a generalization of more specific math. Travis described abstraction in terms of the number of applications: "Abstract means abstracting away from applications… groups feel the most abstract to me because they can represent so many things because of how simple the axioms for them are."

On the other hand, Rye, an electrical engineering major, expressed frustration with the lack of practical applications to her major:

> I am a very practical person. And my degree is very practical stuff. And so abstract just feels like, I don't feel like I'm ever going to use this material…it feels like we just chose different conditions, and we are like, 'this is now a ring' and 'this is now a group'. But what is it doing for us? …I actually don't see any importance. Like it just feels like someone's made it up.

The fact that Rye didn't "see any importance" to rings and groups indicates how necessary it is for her to have practical applications when learning a new concept. However, some students enjoyed the fact that there were no practical applications for the course material. Abraham, a biochemistry major, said about the course, "It all seems not very applicable to the real world, which I love. I love that. It's just like exercising the mind for the sake of knowing more things about numbers rather than for, you know, building a skyscraper or something."

Outside the Five Senses

Some students described abstraction in terms of the senses, contrasting abstract math with math that is visual or tangible. Chuck, a math major, suggested that abstraction goes beyond the five senses to learn more than we can normally perceive:

> Maybe math is another human sense. I think that the abstract can go beyond observations and make predictions… actually figuring out what's going on with the universe, sometimes we just can't see or hear or feel it. And I think that's where the abstract and mathematics comes into play and is super useful.

To other students, math that is not tangible and perceivable by the five senses is "not real math," which is how Taylor, a chemistry major, described abstraction. Ian, a physics major, described abstract math as "math that will never be used," "intangible," and "not actually proven." However, he found that a lot of concepts he learned in the course were connected to concepts he had learned in previous courses, leading him to conclude that "there's nothing abstract" about abstract algebra.

Discussion

Much of the literature on abstract algebra assumes that abstraction is inherently difficult for students. However, the students in my study did not consider "abstract" to mean the same as "difficult." There were several students who responded that nothing in the course was difficult for them, but everything in the course was abstract.

The categories of abstraction I found are different from Hazzan (1999) and Melhuish et al. (2018), which were the only previous literature that broadly addressed student perceptions of abstraction in abstract algebra. One possible reason for this is the student population. All of the students in Hazzan's study were computer science majors in an abstract algebra class, while my study included students majoring in math, computer science, and a variety of other STEM fields.

There were notable differences in how students with different academic majors viewed abstraction. A student's major constitutes a community of practice to which they belong, with each community holding different perspectives on and uses for abstraction. The majority of math and computer science majors in this study shared a similar view of abstraction as the presence of structure and purpose. Other majors, the majority holding opposing views, were likely not familiar with the conventions of abstraction in the mathematics community, and instead viewed abstraction through the lens of their primary area of study. In their interviews, several students described how abstraction is used in their major fields like chemistry and economics, and how that influenced their views of abstraction in abstract algebra. Future analyses will explore in depth the relationship between students' major and their understanding of abstraction.

I am in the process of analyzing further data from this same group of students, including a subsequent survey administered at the end of the semester, in which I ask students to rate their agreement with each interpretation of abstraction identified here. These data will reveal whether students' definitions of abstraction changed over the course of the semester. In addition, future analysis of task-based data from this same group of students will shed light on whether students' views on abstraction influenced the way they solved abstract algebra problems and constructed proofs.

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