Investigating Student Understanding of the Infinitesimal vs. Limit Definition of Integration Through the Curious Placement of dx

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Integrals can be conceptualized with an infinitesimal or limit definition, and holding a certain conceptualization may change how someone frames and solves an integral problem. We present excerpts from two different interview studies where we asked second-semester calculus-based intro physics students integral questions. Student utterances and differing answers to these selected questions led us to define two conceptual resources concerning infinitesimals, which correspond to the different definitions of integration. This work highlights subtle differences in students' interpretation of integration and areas for future study.

Keywords: integration, resource theory, qualitative, interview

Discussions of the infinitesimal vs. limit definition of integration have been on-going in mathematics education for some time (Ely 2020 provides an excellent summary). While many mathematicians prefer a limit definition of integration, applied sciences, such as physics, tend to rely on the infinitesimal definitions of integration (Redish & Kuo 2015). Research also shows that students can hold multiple conceptions of differentials and treat them differently based on these interpretations (Artigue et al. 1990). Students may also favor infinitesimal conceptions of integration despite textbook focus on limits (Nilsen & Knusten 2023). The extent to which this difference in interpreting integrals acts as a barrier for students is unclear. The goal of this paper is not to demonstrate which definition is correct, or which should be taught, but rather to present nuanced findings on this topic and a number of challenges and inconsistencies for these two definitions that we've identified, and the impact they may have on student success in different courses contexts.

The idea that this may be a point of interest came to our attention while conducting interviews using math questions written by a mathematician alongside physics questions written by a physicist. A question we asked, from Jones 2013 which was also written by Jones, was the question $\int_1^2 \; 2/x^3 - x^2 dx$. We observed that this problem does not have parentheses around the integrand. Consistent with a limit definition of integration, integrals have the form of $\int$INTEGRAND *dx*, regardless of whether there are parentheses around the integrand terms or not. The implicit notion is that the *dx* is acting as punctuation, and a marker for the variable over which the integrand will be integrated. Within this model, these parentheses are superfluous. Due to the lack of parentheses, the limit definition of integration does not imply that the *dx* is being multiplied across the terms within the integrand. However, within applied contexts the differential (or infinitesimal) *dx*, or *dm*, or *dt*, represents a small bit of that physical quantity and has units associated with that quantity. The *dx*, for example, is multiplied across every term of the integrand, and therefore parentheses are necessary to ensure that the units work out appropriately. Throughout this paper we refer to the differential, *dx*, as an infinitesimal with the qualities just described. We acknowledge that this is not a rigorous mathematical definition, but we hope it's sufficient to understand our findings. We report preliminary findings about the difference in the interpretation of *dx*, and how it impacts student responses when solving integration problems. The main goal of this paper is to answer the Research Question: "To what extent do students in a second-semester physics course use conventions consistent with infinitesimal definition vs. limit definition of an integral when given a question with a unique placement of *dx* within the integral?"

Theoretical Framework

For this study, we employ the Resources and Framing Theoretical Framework (Hammer et al 2005). Resources and Framing models thinking as consisting of fine-grain "nuggets" that a person has learned and used successfully somewhere previously that are unconsciously brought to bear on a novel task (Hammer, private communication). These resources may have connections to other resources, some of which may be stronger or weaker connections depending on the experience of the person who has used them together in the past (Hammer et al 2005). The accessibility of these resources are influenced by how the individual frames the particular task. Therefore, if someone doesn't articulate a particular resource, this does not mean that they don't possess that resource, rather, the resource was not activated. Within this framework, we as researchers attempt to identify the "good idea" that the individual is attempting to use for a particular problem and if it is successful (i.e., achieves a correct answer), we deem that resource to be productive (Barth-Cohen et al 2023). If the individual is not successful (i.e., gets an incorrect or incomplete answer) we deem that resource to be unproductive. Here, we identify two resources that students may activate in the provided problem, and how these resources connect to either an infinitesimal definition or a limit definition of integration.

Methodology

This study recruited students from the end of a second-semester introductory calculus-based physics class at a large R1 in the upper-Midwest. Students in this course, at minimum, were completing Calculus II in this semester and many were completing Calculus III and/or Differential Equations. Two different sets of interviews were conducted. The first set consisted of three (3 white men) and the second set consisted of four (3 white men, 1 white woman) semi-structured, think-aloud interviews that were conducted and video recorded (Browner 1988). The interviews were transcribed and analyzed through a lens of Resources and Framing (Hammer et al 2005). The protocols featured an array of mathematics and physics questions. The results of one particular question from each interview protocol are presented here. We report on these results combined with textbook observations.

Results & Analysis

Integral without parentheses

We first consider an integral question from Jones 2013. This problem, one among a series of math/physics questions, was given to three individuals. The question was given as was originally written, $\int_1^2 \; 2/x^3 - x^2 dx,$ which did not include parentheses. None of the three students articulated any issue with how this question was written. All three students successfully evaluated the integral. None of the students put parentheses around the integrand; they simply used their knowledge for integrating polynomials and achieved the correct result. We claim this demonstrates students' activation of a "*dx* as punctuation" resource, that is consistent with the limit definition of integration. This may seem relatively light evidence for such a claim, but as stated in the introduction, within a model of infinitesimals, the *dx* is a term that is being multiplied by each of the terms in the integrand. None of the three interviewees commented on the lack of parentheses or drew parentheses before doing the integral. Each of them simply performed the integral. Therefore, we claim that they are treating "*dx* as punctuation."

In discussion with other PER and RUME researchers there was disagreement as to whether or not the lack of parentheses was appropriate, and the extent to which introductory calculus textbooks used parentheses for integrands that had more than one term. An investigation into a number of introductory calculus books showed that all instances of addition and subtraction within an integrand had parentheses around the integrand (Larson & Edwards 2017, Rogawski & Adams 2015, Stewart 2016). It's unclear how prevalent a lack of parentheses is among Calculus instructors and students, and we plan to investigate this further. In discussions with a RUME researcher, they believed the parentheses to be superfluous and were surprised that the introductory Calculus books had parentheses throughout. Additionally, this researcher identified at least one upper-division mathematics textbook that did not use parentheses for integrands that had addition or subtraction within them (Pugh & Pugh 2002). Furthermore, within the same textbook and an additional one there were integrals where the *dx* term was not written within the integral at all (Pugh & Pugh 2002, Abbott 2015). This raises further questions about the practice of experts in mathematics and how that practice is communicated to students.

The curious case of *dx* in the middle of the integral

Within this second interview protocol, four students were given the prompt $\int 2dx(x^2-3x)$. The purpose of this question was to investigate how students would contend with this unique placement of *dx*. The interview protocol prompts "What do you think about this integral?" and then after their explanation, a follow-up of "Could you evaluate this integral?" was asked. The following are the four responses given by our interviewees, with analysis to follow.

Student 3. Student 3 (S3) responds to the initial prompt with "The integral of 2*dx* [and makes a motion with their hands to look like air parentheses] times $x^2 - 3x$, so **I think of this as basically a parentheses being here**, that's how I read that." S3 drew in the parentheses on their page as seen in Figure 1, enclosing the integral symbol, the 2 and the *dx*. And then evaluated the integral of 2*dx* to get $2x + c$, and then wrote their final result as $(2x+c)(x^2 - 3x)$.

$$\left(\int 2dx\right)(x^2-3x)$$

Figure 1. Student 3's annotation of the question prompt. The parentheses indicate the integral of 2 dx as a separate quantity, the result of which they later multiplied by (x^2-3x).

Student 2. Student 2 (S2) states, "It looks out of order. ***dx* should be at the back.**" S2 then moved the *dx* behind the $(x^2 - 3x)$ term, and evaluated this integral correctly. See Figure 2. Note that the answer of S2 is different from S3.

$$\int 2dx(x^2-3x) \Rightarrow \int 2(x^2-3x)dx \Rightarrow \underline{\int 2x^2-6x\,dx}$$

Figure 2. Some of Student 2's work on the given integral problem.

Student 4. Student 4 (S4) states, "Well, **the *dx* could be moved back to the side. So it essentially becomes the integral of two times the quantity x^2 – 3x**." They move the constant 2 out in front of the integral and then write down the expression as seen in Figure 3, moving the *dx* to the right side of the expression, but they do not put parentheses around what is now their integrand. They then evaluate the integral. The interviewer asks, "So I think you kind of already spoke to this. But is there anything wrong about the way that this is written, initially?" Student 4 replies, "I wouldn't say it's wrong. **It's certainly not what I'm used to, but you can put the *dx* anywhere** in that expression and have it still be true."

$$2\int x^2 - 3x \quad dx$$

Figure 3. Student 4's first step on the given integral problem.

Student 1. Student 1 (S1) is significantly more descriptive in their thinking about the integral.

S1: "Um I just say it looks weird, but mathematically I'd say–

INT: "Why does that look weird?"

S1: "Because the d– because the *dx* is in front. Um is it, is it the last term? Because what, I was always taught like in high school– was like your– **your *dx* is like**, that's like **the period of your sentence** like that closes it off. But I know mathematicians get mad at you with this, but like when you like, just multiply differentials like you can't like you can't treat them as fractions, but you– you kinda can… **So I would think that it's perfectly logical to just move it [*dx*] to the end based on, uh, one of the properties of multiplication.**"

After this consideration of the *dx*, and, sounding as if they intend to move the *dx* behind the parenthetical term, S1 is asked to evaluate the integral. S1 immediately changes their mind, draws parentheses around the integral symbol and the *dx* and evaluates the integral the same way as S3. S1 states,

S1: "Ooh. Uh actually um. Now I think I'm gonna go back on myself. I– I actually think that this integral is– is done. Because the *dx* would end the integral."

S1: "So, so I, I– looking at it so now I would, I would integrate the $2dx$ to be $2x$ and then this part is almost treated as in parentheses and I have $2x$ multiplied by $x^2 - 3x$, and then we distribute, $2x^3 - 6x^2$, and that would be your answer."

When the interviewer asks for further explanation of the S1's response, S1 states,

S1: "Yeah. Cause, I don't know if this is the case, but like kind of mathematical like ambiguity. Um, **I'd imagine that you could just multiply the *dx* but I think conventionally the *dx* is at the end**. Like I said, with the period in the sentence. Um yeah, but if I saw this, I guess initially I saw it the other way, but if I– if I saw this, I would imagine that the ***dx* is like the ending of the integral, like the integral starts with the integral sign and ends with the differential.**"

When looking at their final results, two students (S2 and S4) treat *dx* as a quantity (infinitesimal), and move it to the right hand side of the integral before evaluating. The other two students (S3 and S1) ultimately leave *dx* where it is and evaluate the integrand of 2, both being specific and drawing parentheses to make it more clear. These two answers are quite different. Depending on the interpretation of the integral, one arrives at one of two different results.

We identify S3 and S1 as using the "*dx* as punctuation" resource. This resource is very consistent with a limit definition of integration. This is the same kind of resource necessary to interpret $\int_1^2 \; 2/x^3 - x^2 dx$, without parentheses. The *dx* serves as the period at the end of a sentence which begins with the integration symbol, and everything in between is the integrand, the terms to be integrated. S2 and S4 identify the placement of *dx* as "looking weird" and "not what I'm used to," respectively. However, they both move the *dx* term to the end of the integral before evaluating it. We identify this as "*dx* as a quantity (infinitesimal)" resource. This resource is the idea that *dx* within an integral has qualities, such as being subject to multiplicative properties, similar to a variable or a constant and therefore can be moved.

Three of the four students answered this question rather quickly and confidently without a great deal of context or description. S1, however, gave a rather articulate answer that bears further analysis. S1 explicitly states that they were taught in High School that *dx* acts like a period of a sentence, a statement that we would identify as "*dx* as punctuation". They then state that mathematicians don't like it when you multiply differentials like fractions, but that you still can. S1 continues by stating that they can treat *dx* like a term that can be multiplied and that it's logical to move it to the end. These are strong statements that suggest activating a "*dx* as an quantity" resource. However, when asked to evaluate the integral, S1 proceeds with the "*dx* as punctuation" resource and ultimately gets the same result as S3.

A final note on the work of S4 as seen in Figure 3. S4 does not include parentheses around the $x^2 - 3x$, even though there were parentheses around this quantity in the original prompt (See Figure 1.) We connect this lack of parentheses to the "*dx* as punctuation" resource for the "Integral without parentheses" question data. So, we could say that S4 holds and activates "*dx* as a quantity", but then uses notation conventions consistent with "*dx* as punctuation", as they do not deem it necessary to include parentheses on an integral they felt comfortable manipulating. This demonstrates that this individual holds elements of both resources.

Discussion

We have identified two different resources: "*dx* as punctuation" vs. "*dx* as a quantity," and the markers for these resources. We also demonstrated how the activation of one resource, or the other, or both, impacts a student's thinking when solving the two integration problems presented. The activation of one of the two resources can result in different answers to the same problem. While these examples are limited, they demonstrate instances where these two resources could be activated. We highlight a case where activating "*dx* as punctuation" presents no issue for students; there are instances where both interpretations of integration may be necessary, especially in the applied sciences. Although more research in this area is needed, it would seem prudent for both mathematics and physics instructors to be aware of these subtle, yet important, interpretations and distinctions in student thinking.

Future Work

Two questions that we seek to answer are: 1) to what extent is the thinking presented here prevalent among a larger sample of students and 2) to what extent do math and physics instructors think in similar or different ways pertaining to these integral questions. For question 1, we intend to distribute these questions in free-response format to students in Calculus II and Calculus-based Physics II courses in the coming semesters. For question 2, we will be collecting interview data from math and physics instructors to understand not only how they interpret these questions, but also their thinking about the use of parentheses in integrals as was discussed.

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References

Abbott, S. (2015). *Understanding analysis*. Springer.

Artigue, M., Menigaux, J., & Viennot, L. (1990). Some aspects of students' conceptions and difficulties about differentials. *European Journal of physics*, 11(5), 262-267.

Barth-Cohen, L. A., Swanson, H., & Arnell, J. (2023). Methods of research design and analysis for identifying knowledge resources. *Physical Review Physics Education Research*, *19*(2), 020119.

Browner, C. H., De Montellano, B. R. O., Rubel, A. J., Benoist, J., Cerroni-Long, E. L., Charzewska, J., ... & Pollak-Eltz, A. (1988). *A Methodology for cross-cultural ethnomedical research* [and comments and reply]. Current Anthropology, 29(5), 681-702.

Ely, R. (2020). Teaching calculus with infinitesimals and differentials. *ZDM–Mathematics Education*, 53(3), 591-604.

Hammer, D., Elby, A., Scherr, R. E., & Redish, E. F. (2005). Resources, framing, and transfer. *Transfer of learning from a modern multidisciplinary perspective, 89*, 705.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior*, *32*(2), 122-141.f

Larson, R. & Edwards, B (2017) *Calculus* (11th ed.). Cengage Learning.

Nilsen, H. K., & Knutsen, K. H. (2023). First-year engineering students' interpretations of differentials and definite integrals. *International Journal of Research in Undergraduate Mathematics Education*, *9*(1), 173-200.

Pugh, C. C., & Pugh, C. C. (2002). *Real mathematical analysis* (Vol. 2011). New York/Heidelberg/Berlin: Springer.

Redish, E. F., & Kuo, E. (2015). Language of physics, language of math: Disciplinary culture and dynamic epistemology. *Science & Education*, *24*(5), 561-590.

Rogawski, J. & Adams, C. (2015). *Calculus* (3rd ed.). W.H. Freeman.

Stewart, J. (2016). *Calculus: Early Transcendentals* (8th ed.). Cengage Learning