Students' Challenges in Double Integration Over Non-rectangular Regions

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Double integration is challenging for many students. It involves the interrelationship between three variables x, y, z within an object in 3D space expressed in both algebraic and graphical representations. When the integral domain is not a rectangular region, values of x and y along its boundaries may become dependent. However, how students perceive interdependency of variables and what mental actions are involved in double integration have not been fully explored. To answer these questions, multivariational reasoning framework is used in this study to analyze students' cognitive activities. Interview data collected from a university-level calculus enrichment course is analyzed to explore student's reasoning. The findings suggest that students demonstrate several mental actions of multivariational reasoning when performing double integration over different regions. They also suggest some possible directions of extending the theoretical framework of multivariational reasoning to adapt some specific concepts such as range and change of variables in double integration.

Keywords: multivariate calculus, double integral, multivariational reasoning, visualization

Introduction

Double integral is typically taught in multivariate calculus, and it is widely applied across STEM disciplines (Jones & Dorko, 2015). Despite the prolific research in students' understanding of single variable calculus (Thompson & Harel, 2021), relatively less attention is drawn to multivariate calculus (Martínez-Planell & Trigueros, 2021).

Students' thinking towards double integration over rectangular region has been studied as the generalization of definite integration to bivariate function, regarding the two independent input variables (Jones & Dorko, 2015). However, the case of non-rectangular regions has been under-explored until recently (Craig & Pehlivan, 2025). Such problems appear in vector calculus and engineering applications, regarded as a common challenge for students (Khemane et al, 2023).

The difficulty of transitioning from single to double integration may be explained by the complexity of multiple variables and the challenge of visualizing their interrelationship in 3D space. With non-rectangular regions, these two factors are further amplified. The cognitive activities involved when handling multiple variables has been studied by Jones (2022) as multivariational reasoning. Jones categorized types of multivariation structures and induced some mental actions from students' reasoning. This provides a suitable theoretical framework to analyze students' thinking processes in double integration. In this paper, task-based interview data is analyzed to explore students' mental actions of multivariational reasoning involved in double integration over different regions, and to inform potential development of the framework.

Literature Review

The concept of double integration is rooted in bivariate functions and definite integration. A bivariate function is commonly represented in the form of $z = f(x, y)$. With inputs (x, y) as a point in a 2D region, the graph of a bivariate function is a simple surface in 3D space. Trigueros and Martínez-Planell (2010) used APOS theory to study how students transition the concept of single variable functions to bivariate functions in different representations. They found that this process is not straightforward especially when graphical representations are involved.

Definite integrals are associated with a bounded region for the input variable. In the case of positive single variable function, a definite integral can be constructed as the area under the graph over an interval. Jones and Stevens (2022) combined several frameworks into a more general concept of "integral with bounds". Their approach attends to different layers of thinking, involving numeric, graphical, and symbolic representations.

The transition from definite integrals of single variable functions to bivariate functions is indirect. Jones and Dorko (2015) studied how students generalize the concept of multivariate integrals based on single integral. They found that for most students, the generalization is based on the concept of "area under the curve". Martínez-Planell and Trigueros (2020) explored how students relate Riemann sums of bivariate functions to double integrals.

The above studies involve rectangular regions only. Non-rectangular regions are essential to mathematical applications for engineering problems, which makes them research-worthy. Khemane et al (2022, 2023) identified students' common difficulties in distinguishing between rectangular and non-rectangular domains, and swapping the order of integration. Craig & Pehlivan (2025) analyzed students' conceptualization of the bounds of non-rectangular regions. However, these studies did not explicitly delve into students' perception of the interrelationship between variables, and its role in the reasoning process. This study aims to bridge these gaps by exploring students' use of multivariational reasoning (Jones, 2022) in double integration.

Conceptual Analysis

Double integration involves three variables x, y, z being interrelated to each other given a bivariate function, and some constraints on the boundary of the integral domain. To analyze how students perceive and represent this interrelationship, multivariational reasoning by Jones (2022) is adopted in this study. Based on the idea of covariation (Thompson & Carlson, 2017), Jones studied students' cognitive process in coordinating multiple variables. This framework begins with a categorization of multivariation structures into independent, dependent, and nested types, followed by several mental actions involved in students' reasoning processes. Based on this categorization, the conceptual analysis (Thompson, 2008) is described. Double integral over a rectangular region $\int_{y_1}^{y_2} \int_{x_1}^{x_2} z(x, y) dx\, dy$ is associated with independent multivariation since the output variable z varies with two independent input variables x, y. The case of non-rectangular region can be regarded as a special type of dependent multivariation. If the region is vertically simple, the double integral can be written as $\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} z(x, y) dy\, dx$ where the range of the inner variable y depends on the outer variable x. If the region is horizontally simple, the double integral can be captured through $\int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} z(x, y) dx\, dy$, when the range of x depends on y. Figure 1 illustrates these three cases and their associated multivariation structure.

Jones' (2022) framework is further extended to conceptualize students' cognitive activities associated with multivariational reasoning when performing double integration:

1. *Recognition of in/dependence* is involved when students identify the interrelationship between variables given an object before conducting double integration.
2. *Coordination of simultaneous changes* is involved in setting up iterated integral with inner and outer variables, with the range of inner variable dependent to the outer one.
3. *Switching variables/constants* occurs when students perform inner integration, regarding the outer variable as constant. It also occurs when swapping the order.
4. With one variable regarded as a constant, students can regard the bivariate function as a single variable function, hence *reduce into an isolated covariation*.

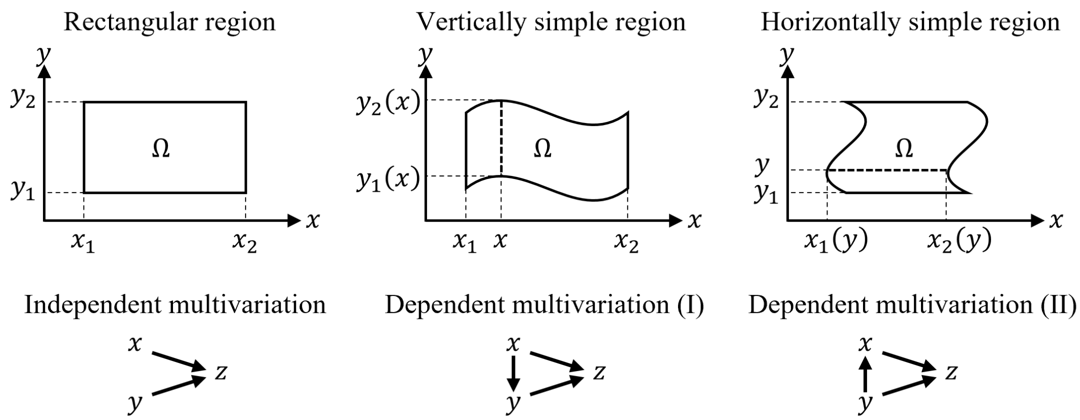

Figure 1. Typology of integral domains and associated multivariation structure.

Method

The data for this study was collected in a university-level enrichment course offered to high-achieving secondary students in Hong Kong. This five-weeks course conducted in summer 2025 by the first author covered single and double integration. The pedagogy involved lectures on function, calculus and visualization, along with interactive activities using dynamic geometry software. After completion of the course, 11 students were recruited for individual task-based interviews, each lasting between 1 to 2 hours. The students were given a bivariate function $z = x + y$ and regions on the xy-plane: square, triangular, quarter-circular (Figure 2). The first two regions correspond to independent and dependent types of multivariation, respectively. The typology of the third region depends on the coordinate system used. Hereby, it can become dependent when x, y are used or, independent for r, θ in polar coordinates.

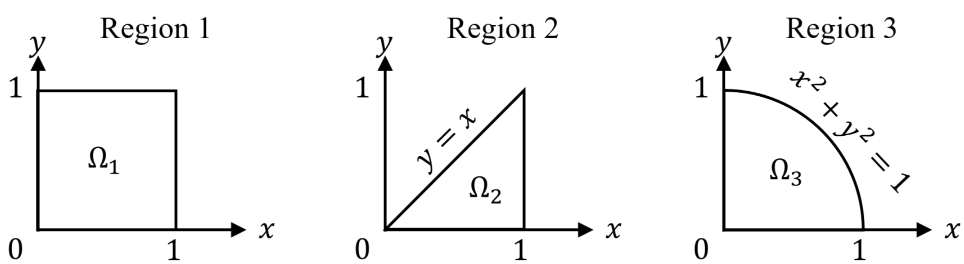

Figure 2. Integral domains used in interview task.

The students were asked to imagine and sketch the graph of the function, express the range of values and dependence of the variables, and eventually evaluate the volume of the solid objects, namely D_1, D_2, D_3, underneath the graph of function and lying on the three regions. These solids and regions were described by the interviewer in words without showing the figures in the beginning. At different stages of the interview, students were provided with dynamic computer graphics illustrating the objects in 3D space virtually to assist their visualization and problem solving. Transcription of the interview recording with students' writing, drawing, hand gestures and usage of computer graphics were analyzed based on Jones' (2022) framework.

Preliminary Findings

The findings below are presented for each task, based on the responses from three students (in pseudonyms), who demonstrated different levels of multivariational reasoning. Due to the preliminary nature of this report, the data from other interviewees has not been included.

Ranges of variables and their interrelationship given a solid object

After provided with the computer graphic of D_1, students were asked to identify the ranges of values of x, y, z within the solid. Henry recognized the base of D_1 as a square and responded that $x = 1, y = 1$ without addressing the ranges of x, y. Consequently, he said z is a line segment going up to a point of 2 and thereby replied $z = 2$. He admitted that z is determined by x, y with the relation $z = x + y$, as a single value. This case indicates a confusion between value and range of variables, which may be rooted in the perception of length when discussing a line segment, rather than the varying positions of a point on it. This can be regarded as the level of "*no variation*" (Thompson & Carlson, 2017) which is before co- or multi-variation take place.

By contrast, Carl claimed that "x is 0 to 1, y is 0 to 1, and z is 0 to 2". After the interviewer asked "if $x = 0.3, y = 0.4$, can z go up to 2?", he realized that the ranges of the variables might not be independent. Then, he stated that the solid is "sandwiched" between two planes $z = 0$ and $z = x + y$, hence modified his answer as z being from 0 to $x + y$ dependently. This graphical interpretation allows a range of z dependent to the position with x, y coordinates within a region. This demonstrates the mental action of *recognizing dependence* of the range of z on x, y, as he realized that values of x, y restrict the allowed values of z. It is notable that Carl did not address this dependence relationship in the beginning, until he applied the graphical representation. This suggests that visualization can enhance students' understanding of interrelationship of variables.

Double integral over triangular region

In the discussion about D_2, Henry thought it would look like a square considering the equation $y = x$, consequently he said that when x increases, y and z also increase. After being asked to consider the solid's projection on the xy-plane, he drew the line of $y = x$ and claimed that the region is a triangle as the line cut through the previous square region into two halves, without drawing the whole triangle with other boundaries. Henry could setup the iterated integral with $dydx$, without explaining why x and y goes from 0. He regarded "starting from 0" as a common habit in integration. When being asked to integrate with a different order $dxdy$, he directly swapped the integral domain $[0, x]$ of y to the outer integral (Figure 3) without rewriting it as an independent interval $[0,1]$. The mental actions of *switching variables/constants* and *coordination of simultaneous changes* seemed to be missing from his reasoning.

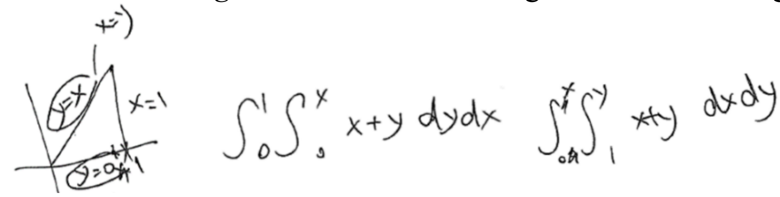

Figure 3. Graph and integrals written by Henry.

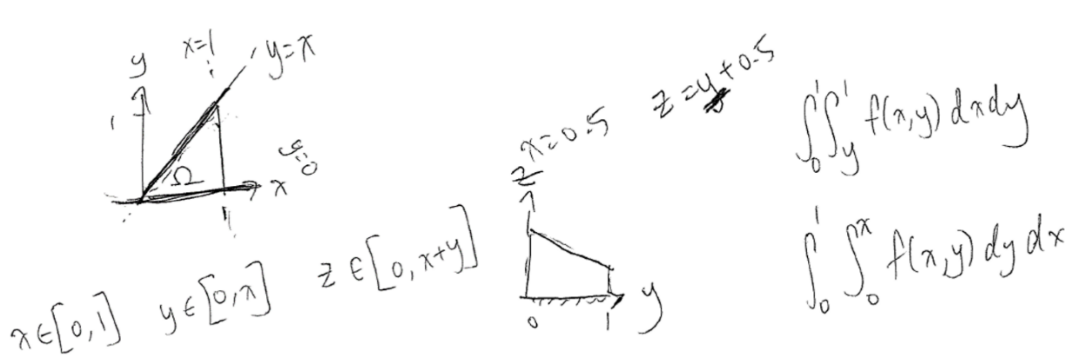

Figure 4. Graph and integrals written by Carl.

Carl, in comparison, could describe the ranges of variables in D_2 using intervals and drew the region accurately. He suggested that y depends on x while z depends on both x, y in this solid. He also rejected the saying that "z depends merely on x" with a visual counterexample: when $x = 0.5$, the solid would have a trapezium cross-section on the yz-plane with $z = y + 0.5$ on top, along which value of z varies with y. Referring to the region, he could write down iterated integrals in both orders of integration correctly (Figure 4). From the framework, Carl could mentally *switch the variable x as a constant* and hence *isolate y, z as a covariation*.

Double integration over quarter-circular region

Dave was asked about D_3. He realized that the base of D_3 is a quarter-circular region with boundary $x^2 + y^2 = 1$ and the axes. When asked about the dependence of variables in D_3, he divided it into two cases with x, y being dependent on the boundary due to the equation, and they are independent inside the region. He then claimed that z is independent to x, y, yet he could clearly describe that z always lie between 0 and $x + y$ within the solid. This response shows a diverse thinking when *recognizing dependence relationship* within non-rectangular region.

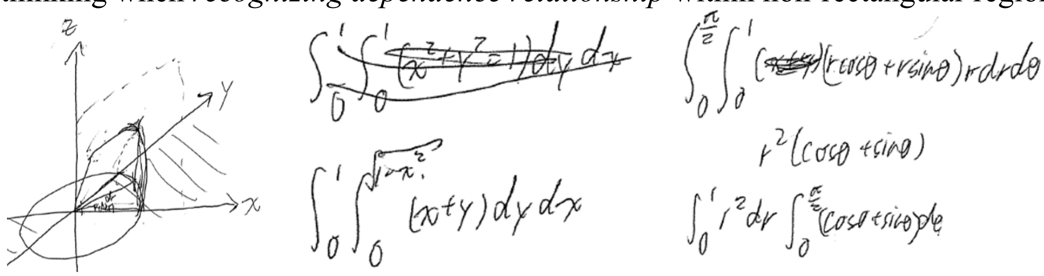

Figure 5. Graph and integrals written by Dave.

When asked to evaluate the volume of D_3, Dave first wrote an iterated integral with both x, y from 0 to 1, and then deleted it. He then wrote another iterated integral with the upper value of y dependent on x, although he agreed that this way is tedious in calculation. Instead, he thought of using polar coordinates. He figured out the ranges of r, θ to set up the iterated integral, then evaluated it with separation method (Figure 5). This case illustrates *recognition of dependence relationship* is involved when students determine their approach of integration.

Concluding Remarks

With the conceptual analysis derived in this study, multivariational reasoning framework (Jones, 2022) is useful for analyzing students' understanding of interrelationship of variables within bounded regions. Students who could complete the tasks with clear explanation of their approaches demonstrated a few mental actions in the framework. However, the conceptualization and expression of double integrals require an understanding of the dependence among the ranges of variables, which is not covered in the framework. The terms "dependent" or "independent" are defined based on values but not ranges of variables. A discussion of range seems necessary for extending the framework to explain students' cognitive process prior to double integration. Compared to rectangular regions, non-rectangular regions with boundary equations impose dependence relationship between ranges of x, y, hence require a higher level of multivariational reasoning associated with visualization. Moreover, the framework can be extended to explain the reasoning processes involved in the transformation of variables and coordinate systems.

In this preliminary report, only three cases have been analyzed due to time and page limits. Although some conceptual analysis has been done, the new extended framework has not been fully developed. Analysis of other interviews in future would validate the new framework.

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