Interpreting Components as Coordinates in Two Coordinate Systems

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This study explores how linear algebra students reason with a novel coordinate system, the COSLA system. As part of a larger study, individual clinical interviews were conducted with three students, and their work was analyzed through the ReNaming and ReLocating framework (Lee; 2023, 2024). Findings include various approaches in interpreting n-tuples and vector notations and different ways to reason about a displacement vector. This paper also presents challenges students may face with non-Cartesian systems and suggests future implications for teaching change of basis.

Keywords: student thinking, change of basis, n-tuples, vectors, coordinates

Introduction and Literature

Coordinate systems are widely used in undergraduate mathematics. In precalculus and calculus, students work with various coordinate systems, such as the Cartesian and Polar coordinate systems. They use n-tuples to denote point locations on plane or to express the results of component-wise operations. In linear algebra, students also work with non-traditional linear coordinate systems generated by basis vectors distinct from the Cartesian unit vectors, where vector notation is typically employed.

Several studies have reported on student thinking about vectors in n-tuple form. Students often interpret vectors as a particular string of numbers, independent of any coordinate systems (Hillel, 2000). Similarly, students may conceive the image of a vector under a linear transformation as a string of numbers rather than as a vector representation relative to a basis. (Hillel & Sierpinska; 1994, Sierpinska et al; 1999) More recently, Selby (2016) found that linear algebra students could answer change of bases questions by memorizing formulas, however, often neglected to reason about what is accomplished through the process. To support students in learning change of basis, he proposed a change of basis project emphasizing physical interpretations.

A few other studies have focused on developing instructional materials in the context of change of basis. Wawro et al. (2013) developed a task sequence to support students in reinventing matrix diagonalization, $A = PDP^{-1}$ by working with a non-traditional coordinate system which uses y=x and y=-3x as the new axes. Zandieh, Wawro, & Rasmussen (2017) identified four types (S1-S4) of linear algebra students' symbolizing evolution found in their class the reinvented matrix diagonalization curriculum was applied to. Levenberg (2015) developed a task for pre-service teachers that involved reading values from axes labeled with non-numerical quantities. He found that such activities prompted logical thinking because the relationship among the non-numerical values were directly tied to their associated meanings.

Conceptual Framing

ReNaming and ReLocating

I developed a framework that describes how students reason when identifying locations within a single coordinate system and across multiple coordinate systems. Also, the framework details how they represent numerical values as locations within one or more coordinate planes

(Lee; 2023, 2024). The framework, referred to as the coordinate system framework, includes four distinct components: Naming, Locating, ReNaming, and ReLocating.

Naming describes how locations or shapes in a coordinate plane are assigned names, such as ordered pairs and equations, while Locating describes the reverse process, how ordered pairs or equations are placed into a coordinate plane.

ReNaming refers to when a location or shape, initially measured in one coordinate system, is assigned a new name in a second coordinate system laid atop the first. For example, a circle centered at (1,0) with radius 1 can be named as $(x-1)^2 + y^2 = 1$ in the Cartesian coordinate system and renamed as $r = 2cos\theta$ in the polar system (Figure 1. Left).

ReLocating, on the other hand, occurs when an existing name creates two different locations or shapes in space, depending on the coordinate systems being used. The new one may appear different from the first, but they share the same name. For instance, the polar curve, $r = 2cos\theta,$ can be sketched in the Cartesian coordinate system (Figure 1, Right).

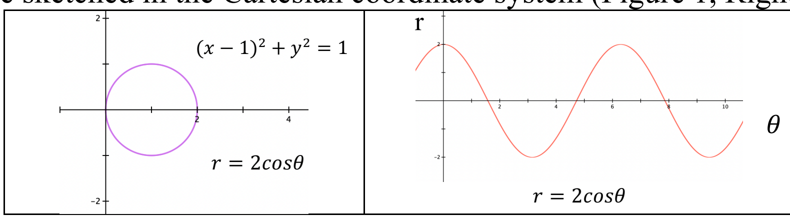

Figure 1. ReNaming Circle with Two Systems (Left), ReLocating $r = 2cos\theta$ into Cartesian System (Right)

Coordinates Not the Same as Components

Coordinates are not always identical to the components in n-tuples (Lee, 2024). Coordinates represent relative sizes with respect to the units employed by a coordinate system, whereas components are the number values listed in an n-tuple or a vector notation. This distinction may not be apparent when considering a point's coordinates within the Cartesian coordinate system, however it becomes salient when reasoning with non-Cartesian coordinate systems. For example, when a student attends to the components of a point's ordered pair (1,2) in the Cartesian coordinate plane, s/he may think of it as "over 1, up 2". His/her interpretation may not have much problem to view (1,2) in terms of the Cartesian basis vectors as such "1 of $\begin{bmatrix}1\\0\end{bmatrix}$ and *2* of $\begin{bmatrix}0\\1\end{bmatrix}$" because the calculation $1\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$ becomes the same as $\begin{bmatrix}1\\2\end{bmatrix}$. However, when thinking of a point viewed in another coordinate system, things get different. For example, there is a point named as $(2,1)$ relative to the basis $\boldsymbol{B} = \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\}$, which would mean "2 of $\begin{bmatrix}1\\-1\end{bmatrix}$ and 1 of $\begin{bmatrix}0\\2\end{bmatrix}$". The location of (2,1) is being represented as a linear combination of the two vectors, $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\2\end{bmatrix}$, that is $2\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}0\\2\end{bmatrix}$ in the new system. Figure 2 illustrates the location of coordinates $(2,1)$ in the new black system.

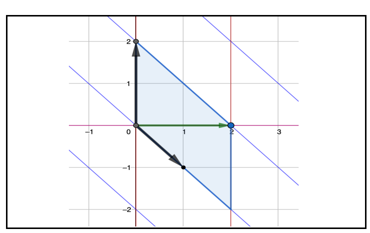

Figure 2. (2,1) in the New Black Coordinate System

While $(2,1)$ is represented as the green vector in the plane, it is also represented as $(2,0)$ relative to the Cartesian basis. In other words, 2 and 1 in (2,1) are the *coordinates* of the location because they are defined relative to the basis $\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\}$, while 2 and 0 in (2,0) are the *components* if it is the calculated result from $2\begin{bmatrix}1\\-1\end{bmatrix}+1\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$ with no attention to the basis.

This proposal reports on student reasoning with a novel coordinate system, the COSLA system, to address the research question: How do students interact with the novel coordinate system and how they interpret components and coordinates- are these interpretations distinct?

Methods

As part of a larger dissertation study, this proposal draws on data from individual face-to-face clinical interviews (Clement, 2000). At the time of interviews, participants were STEM majors who had completed Calculus 1 or 2 as prerequisites and had taken an inquiry-oriented linear algebra course at a large public university in the United States. Both their written work and interview conversations were video recorded. Written work was archived as pdfs, and conversations were transcribed into spreadsheet. Student reasoning was coded line by line on the spreadsheet using the ReNaming and ReLocating framework. All the names used in this report (Hann, Wilson, Jeraldo) are pseudonyms.

Description of the Task

I designed my dissertation task protocols in three packets each called clinical interview, teaching experiment 1, and teaching experiment 2. This proposal focuses on a task administered at the end of clinical interview (Figure 3, Left). The task was designed to explore how students interpret and use a novel coordinate system, referred to as the COSLA system, in which the unit vectors, C_1 and C_2, do not have length 1 relative to the rectangular system, and the origin is not located at (0,0). Prior to this task, students participated in clinical interview tasks themed around Gulliver's Travels, set in an experientially real place, Cocos Island, including a treasure hunt task. Students worked with multiple coordinate systems, such as the Gulliver (slanted) system (Figure 3, Middle) and the Rock (rectangular) system (Figure 3, Right). This series of tasks was developed according to the central idea of Realistic Mathematics Education (RME) informed by Freudenthal (1991). During the clinical interviews, students approximated the location of the Treasure on the Rock's coordinate system as roughly (10, 6.6) or (9.8, 6.6).

Scientists invented an autonomous car, COSLA, to record the environment of Cocos Island. COSLA has its own map system and can only understand location using coordinates relative to the system. COSLA is now at (5,0) in the rock map.

COSLA Map system = $\{C_1, C_2\}$

$$C_1=\frac{\sqrt{3}}{2}\begin{bmatrix}1\\0\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix}$$

$$C_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix}+\frac{\sqrt{3}}{2}\begin{bmatrix}0\\1\end{bmatrix}$$

What is the coordinate command that the scientists call COSLA to come for the Treasure?

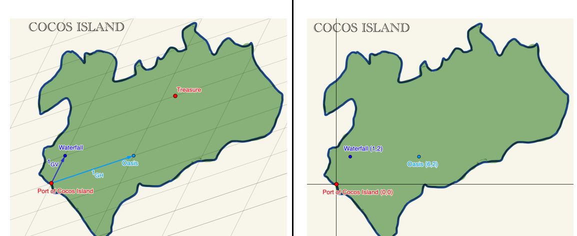

Figure 3. The COSLA Task (Left), Gulliver System (Middle), Rock System (Right)

In the COSLA Task, two coordinate systems are used: the COSLA system, which employs C_1 and C_2 as unit vectors, and the Rock system, which is identical to the rectangular system. Scientists, as inventors of the car, have access to both systems and represent components in C_1 and C_2 in terms of the rectangular coordinate system. That is, $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are the standard unit vectors, and $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$ are relative sizes to the '1'in the standard rectangular system. The car, COSLA, can only interpret commands expressed relative to its own unit vectors, C_1 and C_2.

Results

This section reports how two participants progressed in their search for the coordinate command in the COSLA task. It also examines how components in an ordered pair or a two-dimensional vector came to be understood as coordinates in their reasoning.

Interpreting Components as Coordinates in ReLocating Unit Vector

A participant, Hann, began by interpreting vectors as relative to a coordinate system.

> "We have this definition of our map system[COSLA system]. Okay, so I'm going to interpret these vectors [pointing to the $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ in C_1 and C_2] as being expressed in relation to the rectangular coordinates. So, this one zero, zero one is in rectangular."

He clarified $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ inside C_1 and C_2 as vectors within the rectangular system, and then continued: "What does that [pointing to C_1 and C_2] tell me? …Okay, so we have C_1 which is one zero, C_2 which is zero one (Figure 4)."

He went on to conceptualize C_1 and C_2 as unit vectors relative to the COSLA system and represented them again as $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. The $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ appearing inside C_1 and C_2, and the $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ representing the exact C_1 and C_2, although numerically identical, were distinguished by Hann through coordination with their respective coordinate systems. The two pairs of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ thus came to indicate two distinct locations, which I categorize as ReLocating. To Hann, the components of ones and zeros in each vector were interpreted as coordinates of ones and zeros relative to their respective coordinate systems.

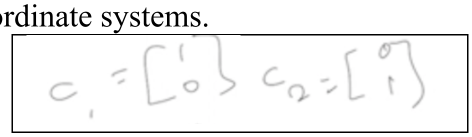

Figure 4. ReLocating Unit Vector to COSLA system

Interpreting Components as Coordinates in ReNaming Unit Vector

Hann next simplified C_1 and C_2 component-wise into vectors and related them to the COSLA unit vectors, saying: " that [COSLA unit vectors] should equal square root three over two, one over square root two in rectangular coordinates…(Figure 5)"

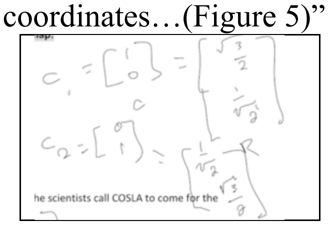

Figure 5. ReNaming COSLA Unit Vector

He placed equal signs even though the entries in the vector notation do not equal their respective components. Here, the equal sign was used to indicate the same physical unit length, despite the unequal components. These are two names for the COSLA unit vector C_1 and C_2, which I categorize as ReNaming. Hann interpreted components in $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{\sqrt{2}}\end{bmatrix}$, $\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{\sqrt{3}}{2}\end{bmatrix}$, as coordinates relative to their respective coordinate systems.

Interpreting Components as Coordinates in ReNaming Displacement Vector

Hann continued his search for the coordinate command by identifying the displacement vector from (5,0) to the Treasure location in the rectangular system. Drawing his earlier work in the clinical interview, he used the Treasure's location (10, 6.6), and again clarified both points, (5,0) and (10, 6.6), as well as the displacement vector from one point to the other, all expressed in rectangular coordinates.

> "What I think I'm looking for is a vector from five zero to the Treasure which is 10 and 6.6. Again, these are rectangular coordinates, and I need this vector to be expressed eventually in terms of the car [COSLA]…We can just do like the head minus the tail, so it's going to be rectangular coordinates. Now, I have the rectangular vector for this command (Figure 6)."

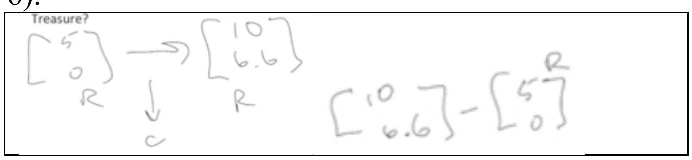

Figure 6. Naming Displacement Vector in Rectangular System

To rename the displacement vector relative to the COSLA system, Hann set up a matrix equation.

> " I need to convert it into the COSLA system, which I think I will need another matrix to do that… You want to multiply it [the matrix with A B C D in it (Figure 7)] by the COSLA coordinates that's one zero, and that will give me my square root numbers [wrote 'C→R' under M (Figure 7) to clarify COSLA to Rectangular]."

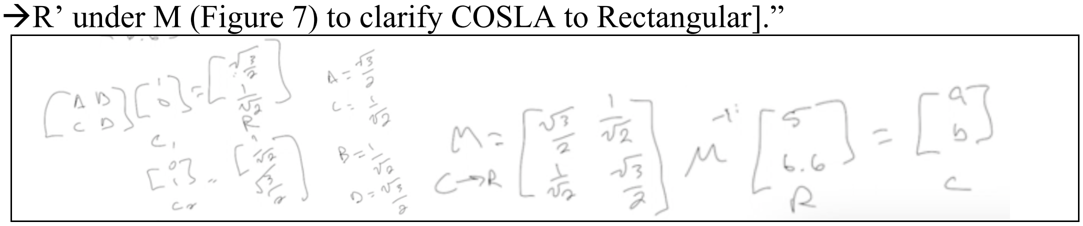

Figure 7. Matrix that Renames Vector

He continued: "to get the coordinate command we're looking for, we need to do M^{-1} times our coordinates we found before…that'll give us our car coordinates. Inverse because we're doing the opposite of what this matrix does. So, M goes from the car to the rectangular, and I want to go from rectangular to the car so taking the inverse."

Hann indicated that the coordinate command is $\begin{bmatrix}a\\b\end{bmatrix}$ (Figure 7) as a result of inverse matrix multiplication, without performing the operation by hand. The vector, $\begin{bmatrix}5\\6.6\end{bmatrix}$, is renamed as $\begin{bmatrix}a\\b\end{bmatrix}$, representing the same displacement vector, which I categorize as ReNaming. Hann interpreted components in these vectors as coordinates relative to their respective coordinate systems.

Interpreting Components as Coordinates in Rectangular System

Wilson, another participant in the interview, approached the same task in a different way. His primary reasoning was to visualize a path that connects the initial location of COSLA to the Treasure as a linear combination of vectors (Figure 8. Left), rather than as a directed displacement vector. In this approach, he did not coordinate between two coordinate systems, but instead interpreted all vector components within the rectangular system. He simplified C_1 and C_2 component-wise into vectors (Figure 8. Middle top), approximated them with decimals (Figure 8. Middle bottom), and used them as a pair of travel vectors. All components were thus expressed relative to the rectangular system. Wilson searched for the COSLA coordinate command as the coefficients a and b attached to each travel vector (Figure 8. Right).

> "The treasure in the rocks map [rectangular map] is 9.8 and 6.6… [he found the Treasure coordinate earlier as (9.8, 6.6)] what I'm trying to find …is how the commands for COSLA would put the treasure to that span… coordinate command… that's what my initial thought is the command for the coordinates to travel this path to the treasure."

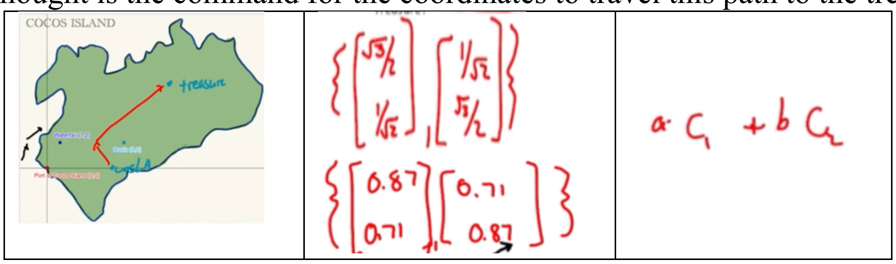

Figure 8. Path from COSLA (5,0) to Treasure (9.8, 6.6)

He continued: "I'm going to expand them [C1 and C2] out… Now that those are C1 and C2 fleshed out (Figure 8. Middle), I guess that's how those set of vectors can get to the point 9.8 6.6."

Wilson then marked travel vectors on the map as multiples of each vector from COSLA to the Treasure (Figure 9, Left) and expressed the path as a vector equation from the initial point (5,0) to (9.8, 6.6) (Figure 9. Right). His drawing and equation reflect an interpretation of the displacement vector as a journey from the initial to the final location. He then spent considerable time plugging whole numbers into a and b in the equation to approximate the Treasure's location as closely as possible.

> "Creating a path from the origin or from a point within a coordinate system to a point that isn't outright just noticeable …How many of these combination of vectors would be used as the coordinate command within this problem."

Rather than ReLocating and ReNaming the units, Wilson expanded every component of the new units and interpreted them relative to the rectangular system. In other words, he treated the components in the vectors C_1 and C_2 as coordinates within the rectangular system.

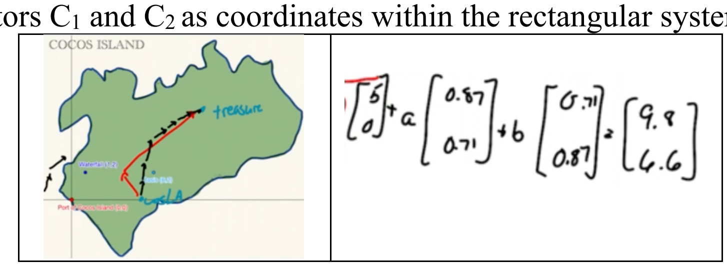

Figure 9. Travel from COSLA to Treasure

Another participant in the interview, Jeraldo, approached the task another way. Like the other two, he began by simplifying C_1 and C_2 component-wise into vectors. He then voiced the need of a transformation matrix to rename coordinates from one system to the COSLA system. Rather than relating the rectangular units to the COSLA units directly, he decided to use the landmarks on the map, the Oasis and the Waterfall, to set up a matrix equation (Figure 10).

$$\begin{bmatrix} 3\sqrt{3} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Figure 10. ReNaming Oasis (Left) and Waterfall (Right)

The Oasis is at (6,2) and the Waterfall at (1,2) relative to the rectangular system. From his matrix equations, Jeraldo appeared to assume that the two systems, rectangular and COSLA, shared the same origin at (0,0). This suggests either that he have not yet had the opportunity to distinguish the two, coordinates and components, or that he did not consider such a distinction necessary, or that he simply did not fully understand the problem.

Discussion

Similar to what Moore et al. (2013) and Zazkis (2008) noted, students express locations or equations within the system in which they have developed thinking- that is, the Cartesian coordinate system. Consequently, when they encounter another system, they tend to think of the new system in terms of an old familiar system. In many curricula and textbooks, components in an n-tuple and vector notation are by default interpreted relative to the Cartesian system, making it compelling for students to view numerical values as corresponding to the Cartesian units. To support students' flexibility in moving and expressing between coordinate systems, it is helpful to include instructions on conversions between non-Cartesian coordinate systems, independent of the Cartesian system.

As noted in the literature section, many of linear algebra studies have emphasized the importance of incorporating more realistic tasks and physical interpretations into instruction. The COSLA task was developed as part of my larger study, in line with these recommendations. I have also created a GeoGebra Applet (**https://www.geogebra.org/classic/vz2mcxqr**) designed to support students' experiences with vectors and coordinate systems, helping move instruction beyond purely abstract and computational approaches.

Lastly, although I was fascinated by Wilson's reasoning, in which he envisioned reaching the Treasure from the initial location as a kind of journey using linear combination, I came across a study reporting that students often employ similar "journey" imagery when adding vectors (Watson et al., 2003). For example, students may justify the equality in $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ by reasoning that it represents moving from A to B and then from B to C. But this imagery can become problematic when interpreting the commutative property for addition such as $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{BC}+\overrightarrow{AB}$ since the latter expression does not give a satisfying meaning to them with the journey reasoning.

Acknowledgements

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