How Proofs Are Organized: A Systemic Functional Linguistics Theme Analysis

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We present a preliminary linguistic analysis of how proofs are organized. In particular, we performed a Theme analysis—an analysis from systemic functional linguistics (SFL) that identifies the starting points of messages at the level of independent clauses—on proofs from undergraduate textbooks. Theme analysis is one of the ways in which SFL studies the textual metafunction, that is, how texts are organized to relate to their context. Our preliminary results show that the main marked (i.e., unusual) Theme constructions in our dataset are: (a) beginning clauses with Adjuncts, and (b) beginning clause complexes with dependent clauses. Further, we found that textual elements (e.g., thus, so, since, if) were common as part of multiple Theme constructions.

Keywords: discourse, proof, systemic functional linguistics, textual metafunction

We expect that the reader will find the following proof (of the theorem that every finite integral domain R is a field) to be written in a frustratingly opaque manner:

> We need only show that for each $a \neq 0_R$, the equation $ax = 1_R$ has a solution since R is a commutative ring with identity. You let $a_1, a_2, \ldots, a_n$ be the distinct elements of R. Also, suppose $a_t \neq 0_R$. Consider the products $a_ta_1, a_ta_2, a_ta_3, \ldots, a_ta_n$ to show that $a_tx = 1_R$ has a solution. We must have $a_ta_i \neq a_ta_j$ if $a_i \neq a_j$ (because by cancelation $a_ta_i = a_ta_j$ would imply that $a_i = a_j$). $a_ta_1, a_ta_2, \ldots, a_ta_n$ are, therefore, n distinct elements of R. Although, R has exactly n elements all together, and so all the elements of R in some order these must be. In particular, $a_ta_j = 1_R$ for some j. A solution, consequently, the equation $a_tx = 1_R$ has, and a field R is.

This proof is a rewritten version of a proof by Hungerford (2014). We created it by changing the order of clauses within sentences, the order of clause constituents (e.g., subjects, objects), and conjunctions. With this rewritten proof, we wish to emphasize that authors of proofs make choices to ensure that they "relate[] [their] messages to the prior and following text and context" (Schleppegrell, 2012, p. 21). This relating process is what systemic functional linguists refer to as the textual metafunction, and this preliminary report's goal is to study how this textual metafunction is realized in proof texts. Although we eventually intend to perform several of the textual metafunction analyses SFL offers, this report focuses on one textual metafunction analysis: a Theme analysis. We capitalize Theme, which is roughly the first constituent of a clause (more on this below), to distinguish it from its everyday meaning. Thus, we seek to address: How are Themes realized in proofs from undergraduate mathematics textbooks?

Literature Review

Systemic functional linguistics (SFL) originates with linguist Michael Halliday. It studies the grammatical choices humans make within linguistic systems and the functionality of these choices (Schleppegrell, 2012), and for over 50 years, SFL has crossed paths with mathematics education. In 1974, Halliday attended the "Symposium on Interactions Between Linguistics and

Mathematical Education," after which he published a chapter about the mathematics register (Herbel-Eisenmann et al., 2017)—a register being "the constellation of lexical and grammatical features that realizes a particular situational context" (Schleppegrell, 2004, p. 18). Key to understanding registers and SFL are the three metafunctions—ideational, interpersonal, and textual—which are three types of meanings realized in every clause "to talk about the world [...] to interact with other people [...] [and to] organiz[e] language to fit in its context" (Thompson, 2014, p. 30). Below, we describe the metafunction most relevant to our work: the textual.

The Textual Metafunction (in Mathematics)

Two types of features realize the textual metafunction, that is, how messages are organized: structural features (i.e., thematic structure, information structure and focus) and cohesive features (i.e., reference, ellipsis and substitution, conjunction, lexical cohesion) (Halliday, 1994). Whereas the structural features are internal to a clause (or clause complex), the cohesive features focus on how clauses (or clause complexes) relate to one another (Halliday, 1994). (A *clause* is "(potentially) any stretch of language centred around a verbal group," Thompson, 2014, p. 17; a *clause complex* is "a combination of two or more clauses into a larger unit," Thompson, 2014, p. 186). Important for this report is the thematic structure, particularly the concept of *Theme*. Theme, which we discuss in greater depth in our Conceptual Framework, refers to "the element that serves as the point of departure of the message" (Halliday & Matthiessen, 2014, p. 89).

In mathematics education, several scholars have studied aspects of the textual metafunction. For example, O'Halloran (2005) observed that in mathematics texts: (a) marked (i.e., nontypical, not the default) Theme constructions are used to emphasize important content (e.g., "From Equation 3", "When the derivative is small"); (b) dependent clauses as marked Themes are common (e.g., "If $x = 4$"), and (c) conjunctions and conjunctive Adjuncts as textual elements of (multiple) Themes are common (e.g., if, then, therefore). Morgan (1998) postulated that in deductive texts, it is likely that textual elements (e.g., hence, therefore) and reason-providing Adjuncts (e.g., "By Theorem 1") appear in the Theme. (*Adjuncts* are those clause constituents that provide information about the circumstances of a situation, for example, why something happens, Thompson, 2014.) Further, Morgan (1998) suspected that unlike scientific explanations, mathematical deductive texts may favor leading with the cause and then providing the result. On the other end of the textual metafunction spectrum, González and Herbst (2013) and González (2015) have focused on cohesive features by studying conjunctions, and conjunctions and lexical cohesion, respectively.

Last, we wish to acknowledge that there are also studies in mathematics education that focus on the other two metafunctions or combinations of the three metafunctions. For instance, there is work focused on the ideational metafunction (e.g., DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011), the interpersonal metafunction (e.g., DeJarnette, 2018; Herbel-Eisenmann & Wagner, 2010; Mesa & Chang, 2010; Zolkower & Shreyar, 2007), both of these (e.g., Burton & Morgan, 2000), or all three metafunctions (e.g., Herbel-Eisenmann, 2007; Morgan, 1998; O'Halloran, 2005). Finally, not all SFL-related work foregrounds the metafunctions. For example, there is also work that draws on SFL's writing on genre (e.g., Marks & Mousley, 1990; Solomon & O'Neill, 1998).

Conceptual Framework

This report focuses on the realization of the textual metafunction in proofs, and SFL offers multiple analytic tools for analyzing the textual metafunction (e.g., Theme analysis, conjunction analysis). The guiding concept for this report is the concept of *Theme*: "the element that serves as the point of departure of the message; it is that which locates and orients the clause within its context" (Halliday & Matthiessen, 2014, p. 89). In English, placing something at the start signals that it is the Theme. The Theme typically contains given (i.e., already present) information, and the remainder of the clause, the *Rheme*, typically contains new information (Eggins, 2004; Fries, 1994). For each of the three basic grammatical moods in English—which SFL terms *declarative* (e.g., "Danielle watches TV."), *imperative* ("Danielle, watch TV!"), and *interrogative* ("Is Danielle watching TV?")—SFL proposes that there are *unmarked* (i.e., usual) and *marked* (i.e., less usual) Themes. For example, it is natural to start a declarative clause with a Subject (e.g., "I can do without grading") but less common to start with a Complement (e.g., "Grading I can do without"). (A *Complement* is a "nominal or adjectival group that refers to the same entity as the Subject, or describes the Subject," Thompson, 2014, p. 20). So, for declarative clauses, a Subject as Theme is referred to as an unmarked Theme, whereas other thematic choices are referred to as marked Themes. Note that it is SFL tradition to capitalize functional labels like Subject.

In our analysis of Theme, we follow Thompson's (2014) approach, and there are a few key points about his conceptualization of Theme that merit mention:

1. When a clause complex starts with a dependent clause, Thompson (2014) suggests treating the entire dependent clause as the Theme (e.g., "**If it rains**, I'll stay home").
2. Textual and interpersonal elements that precede the first experiential (or topical) element are part of the Theme, which is then called a *multiple Theme* (e.g., "**Thus, surely, you** agree"). Note that a multiple Theme is not multiple Themes, but one Theme consisting of multiple elements that correspond to different metafunctions.
3. Clauses with "there" plus the existential process (usually realized by "be") include the existential process in the Theme (e.g., "**There is/was** a bird outside").

Methods

Data Collection

Thus far, our dataset consists of ten proofs from five popular undergraduate textbooks (i.e., Abbott, 2015; Axler, 2024; Dummit & Foote, 2004; Fraleigh, 2003; Rudin, 1976) that cover typical undergraduate mathematics content: real analysis, abstract algebra, and linear algebra. Within each textbook, two proofs were randomly selected with a random number generator.

Data Analysis

First, the proof texts were separated into T-units: "independent clause[s] together with all the clauses that are dependent on it" (Thompson, 2014, p. 161). T-unit boundaries that occurred within sentences were indicated with a double slash (//). Next, the Theme (and Rheme) of each T-unit were identified, following Thompson (2014). Third, the Theme was studied to: (a) determine whether it was marked, and (b) identify any textual and interpersonal elements part of a multiple Theme. (Textual elements were highlighted in orange and underlined. Interpersonal elements were highlighted in blue and underlined with dots.) For an example of the T-unit separation and coding of a sentence from a proof by Abbott (2015), please see Table 1.

Table 1. Example Theme Analysis of a Sentence

Theme	Rheme	Marked?
If $a = b$,	then $\|a–b\| = 0$ //	marked (dependent clause)
and so certainly $\|a–b\|$	$<\varepsilon$ no matter what $\varepsilon>0$ is chosen.	unmarked (declarative)

The first author coded all ten proofs (i.e., 151 T-units), and, at the time of writing, we are in the process of having each proof coded by at least two authors. We are also currently studying the marked Themes to identify patterns. Our preliminary results are described below.

Preliminary Results

Theme Status

Theme status describes whether a Theme is marked or unmarked, and in our coding, marked constructions appeared roughly a third of the time (49 out of 151 T-units). In particular, we found two marked constructions: (a) beginning a clause complex with a dependent clause instead of the independent clause (33 times), and (b) beginning a declarative or imperative clause with an Adjunct (16 times). We note that among the possible marked options in declarative clauses, Adjuncts fall "somewhere in the middle on the scale of markedness" (Thompson, 2014, p. 150). As this breakdown shows, other marked options, like beginning a declarative clause with a Complement, did not appear at all. Artificial examples thereof (e.g., "To show that x is even is our goal", "A subset of Y the set X is") suggest that such Complement-as-Theme constructions sound strange and may be less functional for the flow and organization of proofs than their unmarked counterparts (i.e., "Our goal is to show that x is even", "X is a subset of Y").

Dependent clauses. For this subsection and the next, let P and Q refer to mathematical statements (e.g., "f is differentiable"). Among clause complexes beginning with dependent clauses, if–then constructions were most common, with "then" sometimes omitted (e.g., "If P, [then] Q"). There were also implicit conditional statements, such as "Assuming that P, [we get] Q", and "pseudo-conditional" statements that described depersonalized mathematical actions in the antecedent (e.g., "Renumbering [...], [we get] Q"). (We use the term "pseudo-conditional" to describe that although these clause complexes can be phrased in if–then terms [e.g., "If we renumber […], then we get Q"], they do not introduce a mathematical assumption.)

Another construction we observed was Thematizing mathematical warrants in dependent clauses (e.g., "Because P, [we get] Q"; "Since P, [we get] Q"). Lastly, we observed constructions that directed the reader toward a goal or direction (e.g., "To prove P, [...]", "Turning to multiplication, [...]"). We note that these uses of dependent clauses as Theme allow authors to adhere to the flow of logic—with data and warrants preceding claims—and, in the last case, orient the reader by framing the argument.

Adjuncts. As aforementioned, Adjuncts provide information about the circumstances of a situation (e.g., when, how, why) (Thompson, 2014). In our dataset, a majority of the Adjuncts appearing as Themes provided justification and typically took the form "By [*warrant*], P" (11 out of 16 times). Initial Adjuncts were also used (5 out of 16 times) to orient the reader and frame the Rheme (e.g., "In the forward direction", "For the [proof of the] first statement", "As regards (a')"). Initial Adjuncts almost exclusively appeared in declarative clauses (15 out of 16 times). The only example of an imperative clause beginning with a (framing-)Adjunct was: "In the general case, let [...]").

Theme Composition

Theme composition describes whether a Theme is simple (i.e., contains only an experiential element) or multiple (i.e., also contains textual and/or interpersonal elements).

Textual elements. There were many textual elements that initiated multiple Themes. To truly do these textual elements justice, a conjunction analysis (Martin & Rose, 2007) would be required. But even without such an analysis, there are several takeaways. Many textual elements introduced (warranted) claims (e.g., “Hence”, “So”, “Therefore”, “Then, “Thus”) or antecedents (e.g., “If”). There were also textual elements that set a (new) stage (e.g., “Now”, “First”), created contrast (e.g., “But”, “However”), and were used to add (e.g., “and”, “also”, “again”). At times, multiple textual elements were combined, mostly to organize the text *and* do mathematical work (e.g., “Now if”, “Again by”, “And hence”, “But then”).

Interpersonal elements. There was only one interpersonal element in a multiple Theme: “certainly” in “**and so certainly |a–b|**<ε”, which adjusts the force of the message (s. SFL’s system of Graduation) (Derewianka, 2011). Since statements in mathematics are either true or false, there may be little need for resources like Graduation that intensify or weaken claims.

Experiential elements. The experiential elements of Themes were: Subjects in unmarked declarative clauses (e.g., “**The *i,j* entry of AB** is [SUM]”) (80 times), dependent clauses in marked clause complex constructions (s. the “Dependent clauses” section) (33 times), Predicators (i.e., the verb that the Subject “does”) in unmarked imperative clauses (e.g., “**Let** n = dim V”) (22 times), and Adjuncts in marked declarative (and, rarely, imperative) clauses (s. the “Adjuncts” section) (16 times). Other than the markedness patterns, we noted that unmarked Subject-Themes often included Pre- and Postmodifiers. For example: “**The very definition of φ_α applied to a constant polynomial $a \in F[x]$, where $a \in F$** gives $\varphi_\alpha(a)=a$”, where the Pre- and Postmodifier are underlined with dashes. Thompson (2014) calls such Subjects *‘heavy’ Subjects*.

Other observations. Linguists disagree about the best way to perform a Theme analysis, particularly around where to draw the Theme boundaries (Thompson, 2014). We experienced this challenge and wish to highlight three structures we observed that were not captured by our Theme analysis but could have been captured if we had drawn different Theme boundaries: (a) leading with projected clauses of the form “We (+ *modal verb*) + *verb* + that + *mathematical statement*” (e.g., “We (must) show that […]”, “We observe that […]”, “We assume that […]”); (b) leading with projected imperative structures (e.g., “Assume that […]”, “Note that […]”, “Observe that”); and (c) leading with depersonalized versions of the previous two structures beginning with anticipatory “it” (e.g., “It follows that […]”, “It needs to be shown that […]”, “It may be noted that […]”). These constructions Thematize comments *on* the subsequent mathematical message instead of the message itself. We posit that this is a functional choice that allows authors to manage the flow of the argument and the links between claims.

Discussion

Although preliminary, our results align with O’Halloran’s (2005) observations and Morgan’s (1998) conjectures about Themes in (deductive) mathematics texts. Our next steps are to increase the number of proofs in our dataset and have all proofs coded by at least two authors. Since we are ultimately interested in what makes a proof flow for (student) readers, we want to also conduct a cohesion analysis. We believe that attending to cohesive features and making observations about how clauses (and clause complexes) are related to one another will complement our current results about how clauses (or clause complexes) are (internally) organized. We have several questions for the audience. First, we have thus far drawn on well-regarded mathematical texts, which appear to be making similar grammatical choices. How could we increase or diversify our dataset to better emphasize that choices are being made and to understand which choices are possibly more functional for communicating proofs in which contexts? Further, since our goal is to eventually understand the effects of these choices on the reading of proof, what methods could we draw on to study how Theme and cohesion choices influence the reading (and comprehending) of proof? Relatedly, how might our current clause-level observations inform the writing of proofs?

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