The Persistence of Motion Graph Reasoning Amid Quantitative Conceptions of Coordinate Systems: The Case of Ava

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We report on a qualitative case study of a college algebra student, Ava, who participated in a series of online task-based interviews involving graphs of different dynamic situations. Our analysis focused on the nature of Ava's graph reasoning and coordinate system conceptions within and across these tasks. During Ava's interviews, she primarily demonstrated Variation and Covariation graph reasoning while conceiving of the coordinate system quantitatively. Yet, she also had an expectation that her graphs would represent the motion of the objects in the dynamic situations or the motion of the traces in the graphs themselves. Ava's Motion graph reasoning amid her quantitative conceptions of coordinate systems demonstrates overlap between quantitative and physical conceptions. Her case speaks to the richness of students' conceptions of graphs; even when they conceive of quantitative coordinate systems, they can also expect a graph trace to resemble the motion of an object.

Keywords: undergraduate education, technology, quantitative reasoning, graphing conceptions

Having students engage in tasks where they represent attributes of objects in dynamic situations graphically can be a productive way to foster students' quantitative and covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). However, even when the task is designed in a way that supports their graphical representations of quantitative relationships, students may or may not actually conceive of their graphs in this way. Instead, they may conceptualize them as literal pictures of physical objects (Clement, 1989; Lai et al., 2016; Leinhardt et al., 1990), depictions of motion (Kerslake, 1977), or chronological records of events (Janvier, 1998). Furthermore, students may or may not conceive of a given representational space, such as a Cartesian plane, as a coordinate system that can be used to represent quantitative relationships. Thus, whether or not a particular graphing task is effective depends, in part, on students' graph reasoning (Johnson et al., 2020) and underlying conceptions of coordinate systems (Lee et al., 2020).

Design considerations for digital graphing tasks revealed relationships between students' graph reasoning and their conceptions of coordinate systems (Johnson et al., 2024). We report on a case study combining two theoretical frameworks related to graph reasoning (Johnson et al., 2020) and coordinate system conceptions (Lee et al., 2020) to explain aspects of students' graphing activity that may seem counterintuitive or contradictory. We ask the following research question: What is the nature of students' conceptions of coordinate systems and graph reasoning within and across digital graphing tasks involving different dynamic situations?

Background

Graph Reasoning

Johnson et al. (2020) developed a graph reasoning framework to characterize students' conceptions of graphs, identifying four broad types: Covariation, Variation, Motion, and Iconic. *Covariation* and *Variation* graph reasoning involve students thinking about quantities and their relationships. For example, suppose a student is shown a dynamic situation in which a "Cannon Man" is shot vertically out of a cannon and descends with a parachute (Figure 1) and is tasked with sketching a graph relating his height from the ground and his total distance traveled. A student demonstrating Covariation graph reasoning may sketch a graph and describe how it shows Cannon Man's height increasing and decreasing while simultaneously showing how his total distance traveled continues increasing. If the student exclusively describes their graph in terms of one of these changing attributes, they would be demonstrating Variation graph reasoning (e.g., only describing Cannon Man's height increasing and decreasing).

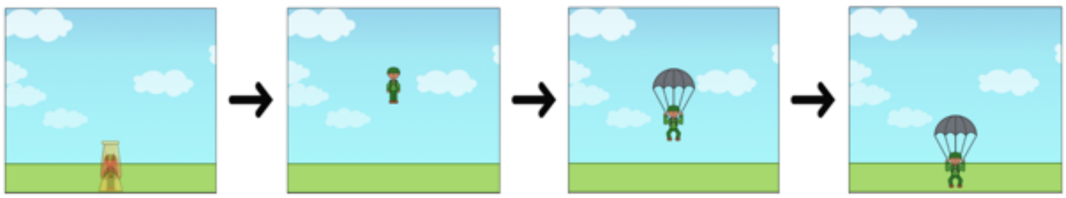

Figure 1. Cannon Man Dynamic Situation

Motion graph reasoning refers to students' reasoning about the physical movement within a dynamic situation. For example, a student demonstrating Motion graph reasoning may consider the Cannon Man dynamic situation (Figure 1) and describe their graph as representing Cannon Man's changing speed. *Iconic* graph reasoning, on the other hand, involves reasoning about physical features of the situation itself; a student demonstrating Iconic graph reasoning may describe their graph as representing Cannon Man's literal path. Johnson et al.'s (2020) framework highlights the diverse ways students may reason about and represent dynamic situations graphically.

Conceptions of Coordinate Systems

Lee et al. (2020) identified two types of coordinate systems students may conceive of when graphically representing dynamic situations: spatial and quantitative. A coordinate system is considered to be *spatial* when it is "mentally overlaid onto the space being represented" (p. 32). When students conceive of spatial coordinate systems, they determine relative locations of objects in the dynamic situations within that space. For example, given the Cannon Man dynamic situation (Figure 1), a student may conceive of a spatial coordinate system and plot points on a graph to represent the literal positions of Cannon Man in the air. Alternatively, students conceive of *quantitative* coordinate systems when they conceive of attributes in a situation as quantities and "project them onto some new space, which is different from the space in which the quantities were originally conceived" (p. 34). For example, in the Cannon Man situation (Figure 1), a student may conceive of a quantitative coordinate system and plot points on a graph to represent the relationship between Cannon Man's height from the ground and his total distance traveled. Notably, even when students are given an implied quantitative coordinate system, they may or may not conceive of the coordinate system quantitatively (e.g., using a coordinate system to represent physical locations even if neither axis entails a distance). Hence a student's meanings for the coordinate system can shape their graphing activity.

Methods

We report on a qualitative case study (Stake, 2005) of a college algebra student who participated in a series of four task-based clinical interviews (Clement, 2000; Goldin, 2000). This case stems from the first author's dissertation (Knurek, 2025a). In a preliminary report, we explored connections between a student's graph reasoning and conceptions of coordinate systems (Knurek et al., 2025b). Ava's case advances the prior work by demonstrating how

Motion graph reasoning can persist amidst a student's quantitative conceptions of coordinate systems. Ava completed interviews online via video conference, during which she interacted with a sequence of dynamic graphing tasks housed in the Desmos classroom (www.desmos.com) known as techtivities (Olson & Johnson, 2022).

During a techtivity, students are shown an animation of a dynamic situation and are asked to consider two specified attributes. Next, students are to explore changes in these attributes by dragging sliders (i.e., "Axis Sliders") along the axes of a Cartesian plane and interacting with a dynamic graph illustrating their slider movements. Then, students are to sketch a graph representing the relationship between attributes. Figure 1 (Cannon Man) and Figure 2 (Dynamic Tent, Toy Car, and 3D Printer) illustrate the four dynamic situations used for the interviews. In the Dynamic Tent techtivity (Figure 2a), students are asked to consider the height and width of a tent. In the Toy Car techtivity (Figure 2b), students are asked to consider a toy car's total distance traveled and distance from a cone. In the 3D Printer techtivity (Figure 2c), students are asked to consider the total amount printed and the height of the printed object.

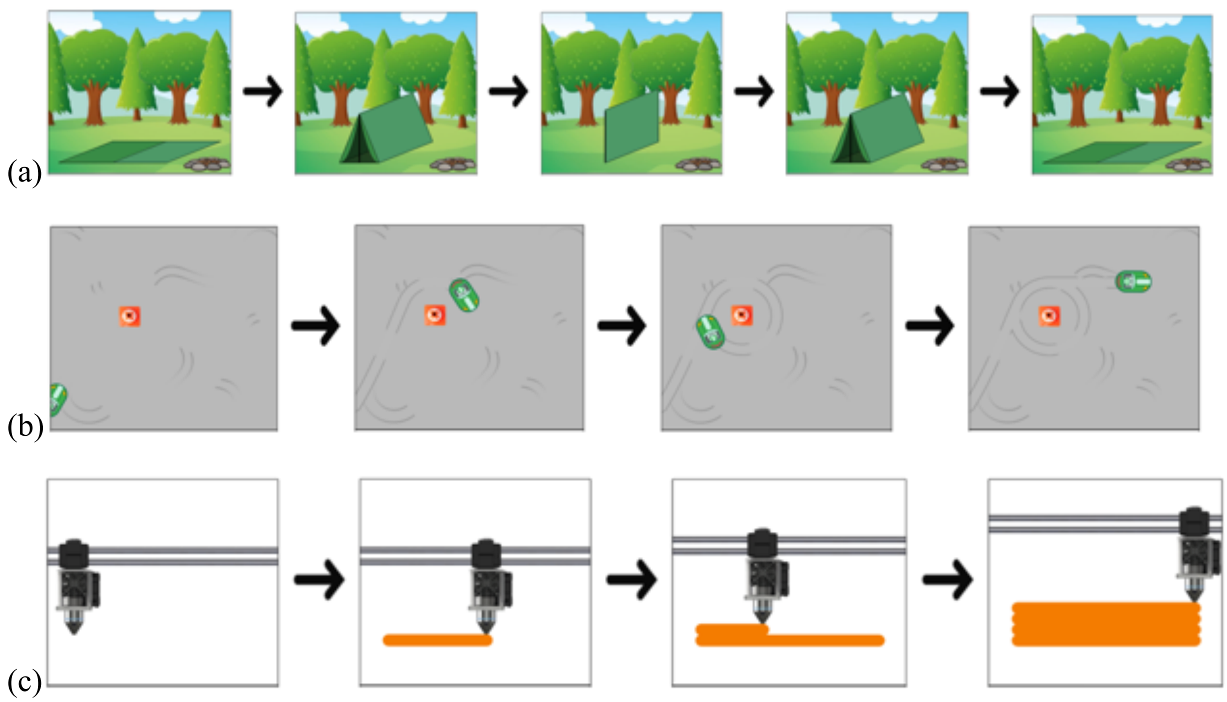

Figure 2. Dynamic Tent, Toy Car, and 3D Printer Dynamic Situations

Data Collection and Analysis

To collect data, the researcher conducted task-based clinical interviews (Clement, 2000; Goldin, 2000) to elicit students' current ways of reasoning by using interview protocols that support flexible and responsive questioning. As such, the four clinical interviews were akin to exploratory teaching (Steffe & Thompson, 2000), which is part of teaching experiment methodology.

Ava's interviews lasted approximately 20-30 minutes and were video-recorded. During the interviews, Ava shared her screen and completed techtivities while the researcher asked questions from the interview protocols, which were designed and refined during an earlier pilot study. The researcher took jot notes (Erickson, 1986) immediately after each interview to document reflections and takeaways to inform minor adjustments to interview questions for subsequent interviews, making the approach iterative and responsive to Ava's emerging reasoning.

To systematically analyze the data, we used Wolcott's (1994) three-part framework: description, analysis, and interpretation. The first part, description, entailed reviewing both the transcripts and screen recordings of Ava's interviews and generating a detailed account of all her words, gestures, and cursor movements. For the second part, analysis, we engaged in conceptual analysis (Steffe & Thompson, 2000) to make inferences about how Ava conceived of quantities, coordinate systems, and graphs based on the descriptions. For the final part, interpretation, we drew on established theories and frameworks to make sense of Ava's thinking and answer the research question. This allowed us to take a step back and consider the broader meaning of our findings, building on the descriptions and analyses we had already conducted to make claims about Ava's reasoning and conceptions within and across techtivities.

Results

Our analysis of Ava's interviews revealed the persistence of her Motion graph reasoning (Johnson et al., 2020) amid her quantitative conceptions of the coordinate system (Lee et al., 2020). Across three interviews (i.e., Cannon Man, Dynamic Tent, and 3D Printer), Ava consistently conceived of the coordinate system quantitatively while demonstrating Variation and Covariation graph reasoning. Yet, she also conceived of some aspects of her graph as representing motion. During her other interview (i.e., Toy Car), Ava conceived of parts of the coordinate system spatially while demonstrating Motion graph reasoning.

In the following sections, we describe Ava's reasoning about each individual attribute while interacting with the Axis Sliders. Then, we present Ava's graph sketches and annotations while highlighting evidence of her graph reasoning and conceptions of coordinate systems.

Variation in a Quantitative Coordinate System

While interacting with the Axis Sliders, Ava consistently conceived of attributes in the dynamic situations as quantities and represented her conceptions of change for each attribute in a new space (i.e., the axes of the given Cartesian plane). For example, during the Dynamic Tent techtivity, Ava dragged the horizontal slider (Width) to the left as the tent's width decreased, then dragged the slider to the right as the width increased. She said, "So it starts at the widest it gets to in this video, and then it slowly like decreases until it gets to zero, and then it comes right back to its initial point." Next, Ava dragged the vertical slider (Height) up as the tent's height increased, then dragged the slider back down as the height decreased. She said, "The height starts at zero and it gradually increases and then like goes back to being zero again." Similarly, during the other three techtivities, Ava used the sliders along the axes to represent her conceptions of change for each attribute. Across all her interviews, Ava demonstrated Variation graph reasoning while conceiving of each axis in the coordinate system quantitatively.

A Coordinate System Involving Spatial and Quantitative Elements

During her Toy Car interview, Ava sketched a graph representing the toy car's proximity to the cone, rather than the relationship between the specified attributes (i.e., the toy car's total distance traveled and the toy car's distance from the cone). She started her graph by plotting a single point on the horizontal axis to represent the maximum distance the toy car traveled (Figure 3). Then, she plotted three points along the vertical axis, labeled "0ft," "5ft," and "10ft." Next, she labeled one of the points on the vertical axis as "cone" and said, "This is where the cone is at. So the cone is five feet from where [the car] started." Finally, Ava plotted points diagonally from left to right in the plane while describing the toy car's proximity to the cone. We numbered each point in Figure 3 to indicate the order in which she plotted these points. Ava described the motion of the toy car while plotting the points:

> When it's further from its journey, it's closer to the cone [plots points labeled 1-4 from left to right on a diagonal line, Figure 3] and then further in its journey, approaching the cone, blah, blah, blah. And then we'll eventually get to the cone at some point in its journey [plots point labeled "5," Figure 3]. And then as it goes down, or as it continues its journey, it's getting further away from the cone [plots points labeled 6-10, Figure 3].

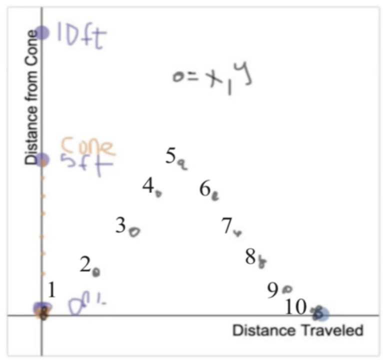

Figure 3. Ava's Graph Sketch during the Toy Car Techtivity

Ava sketched a graph to represent the toy car going to and from the cone. She labeled "cone" on the vertical axis to show the vertical location of the cone in the animation, then plotted points to show how the toy car approached the cone and continued away from the cone. We infer that Ava plotted the points from left to right either because the toy car traveled across the screen from left to right, or because she was focused on the temporal progression of the motion, accounting for experiential time (Thompson & Carlson, 2017). Ava's sketch and response suggests that she conceived of a coordinate plane involving spatial and quantitative elements, and provides evidence of Motion graph reasoning.

Covariation in a Quantitative Coordinate System

During her other three interviews (i.e., Cannon Man, Dynamic Tent, and 3D Printer), Ava sketched graphs representing the relationships between attributes in the dynamic situations. Figure 4 shows Ava's graph sketches for the Cannon Man, Dynamic Tent, and 3D Printer techtivities, respectively. Ava started each graph sketch by plotting points along the axes and in the region bounded by the axes. Then, she connected her points with line segments while describing the relationship between attributes. For example, during the Dynamic Tent techtivity (Figure 4b), Ava said, "When the height is lower at four, we see the width gradually become higher, gets to six, and when the height is even lower, the width is even higher." Ava described the relationships between attributes similarly during the Cannon Man and 3D Printer techtivities. In all three interviews, Ava demonstrated Covariation graph reasoning while conceiving of the coordinate system quantitatively.

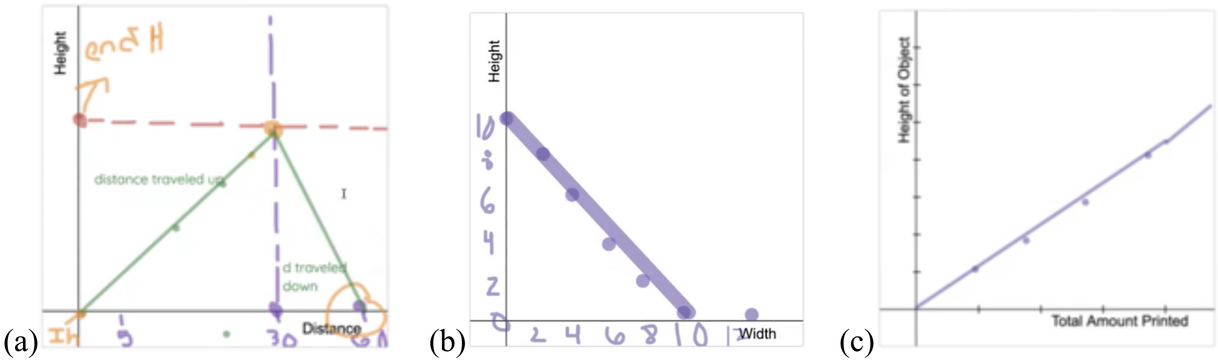

Figure 4. Ava's Initial Graph Sketches

Motion in a Quantitative Coordinate System

After Ava sketched each graph, her subsequent annotations and responses suggested that she conceived of her graphs not only as representing the relationship between attributes, but also as representing visible motion. For example, during the Cannon Man techtivity, the researcher asked Ava, "If Cannon Man just went up at a constant speed and down at a constant speed, would that change anything?" Ava responded, "Yes, because instead of my graph looking like this, it would look more symmetrical like this." She drew a picture in the top-right section of the plane that resembled an upside-down V with a vertical line through it (Figure 5a). She said, "The differences in the slopes would tell you like how fast it's going." Ava described the asymmetrical look of her graph sketch as representing Cannon Man's changing speed, with the different slopes representing different speeds (i.e., how fast it's going). During the Dynamic Tent, after finishing her graph sketch, Ava drew an arrow pointing downwards above her line segment (Figure 5b) and said, "The line goes this way." Then she drew an arrow pointing upwards below the diagonal line, crossed it out, and said, "It doesn't go this way." Similarly, during the 3D Printer techtivity, Ava drew an arrow along her line segment (Figure 5c) and said, "I think it goes in this direction." She proceeded to talk about how her graph showed the motion of the object getting "higher." Although Ava conceived of her graphs in terms of quantities and their relationships across all three interviews, her responses provide evidence that she also had expectations of her graphs representing motion; this included either the motion of the object in the dynamic situation or the motion of the trace in the graph itself. Thus, even though Ava was still conceiving of a quantitative coordinate system and evidencing Variation and Covariation graph reasoning for the majority of her interviews, her Motion graph reasoning persisted. To Ava, a graph should represent more than just the relationship between attributes; it should also incorporate some kind of motion that is reflected in the situation itself.

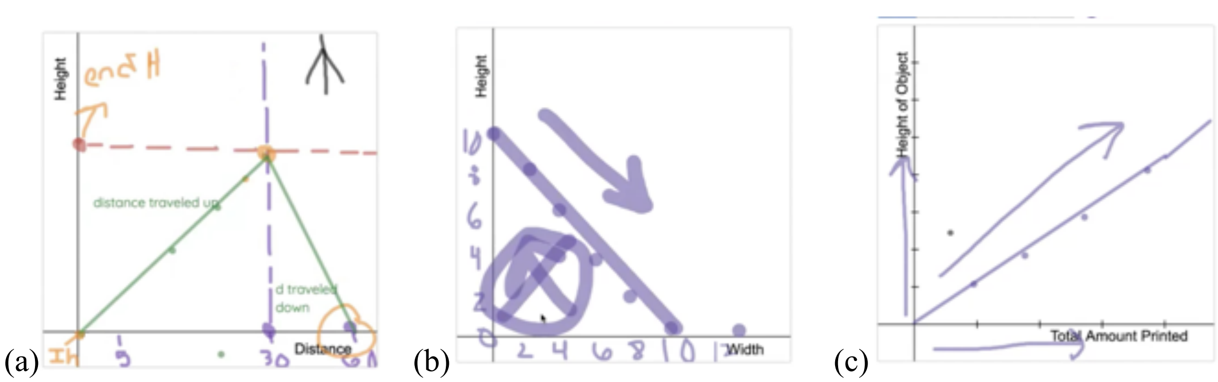

Figure 5. Ava's Graph Sketches with Subsequent Annotations

Discussion and Conclusion

This study provides empirical evidence demonstrating interrelationships between students' graph reasoning (Johnson et al., 2020) and their conceptions of coordinate systems (Lee et al., 2020). Ava's graph reasoning often aligned with her conceptions of coordinate systems in ways to be expected. Across interviews, Ava's Covariation and Variation graph reasoning emerged when she conceived of the coordinate system quantitatively. In conjunction, her Motion graph reasoning persisted amid her quantitative conceptions of coordinate systems. Ava's case shows how seemingly contradictory elements can hold together for students when they have multiple goals for graphing. Ava expected her graphs to represent both the relationships between attributes and the physical motion in the dynamic situations. This speaks to her conception of what a graph *should* represent (Johnson et al., 2020); to Ava, a graph should incorporate both quantitative and physical aspects of whatever situation she is representing.

Our findings have implications for researchers investigating students' graphing activity through tasks involving dynamic situations. Ava's case suggests that even when students incorporate the physical motion of objects into their graphical representations, they also can hold quantitative conceptions of coordinate systems. Thus, there is not a simple one-to-one correspondence between how students reason about graphs and how they conceive of coordinate systems. Researchers can use Johnson et al.'s (2020) and Lee et al.'s (2020) frameworks in tandem in future studies to gain deeper insights into students' graphing activity.

Lastly, we drew on two theoretical frameworks (Johnson et al., 2020; Lee et al., 2020) related to students' conceptions of graphs and coordinate systems, respectively. We acknowledge that other theoretical lenses, such as Moore & Thompson's (2015) constructs of static and emergent graphical shape thinking, could provide additional insight into some of Ava's sketches and responses (e.g., her use of arrows to describe the motion of a graph). However, we intentionally focused our examination of students' graphing activity to Johnson et al.'s (2020) and Lee et al.'s (2020) frameworks to investigate the interconnections of graph reasoning and coordinate system conceptions. In doing so, we were able to uncover some of the nuances involved in these constructs. Future research can continue to investigate students' graph reasoning alongside their conceptions of coordinate systems to gain deeper insights that may support the refinement of these frameworks to better reflect the complexity of students' conceptions and reasoning processes. Studies with larger and more diverse student populations could help validate and extend these findings, providing a clearer picture of how these constructs manifest across different mathematical contexts.

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