Students’ Mathematical Reasoning About the Period of the Sum of Trigonometric Functions

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This study investigates students’ mathematical reasoning when determining the period of the sum of two cosine functions through exploratory teaching interviews. Two undergraduate students engaged with Desmos applets to explore the sum of two cosine functions. The analysis revealed output-oriented reasoning that focuses on identifying input values where function output repeats, and alignment-oriented reasoning that focuses on where the period of the sum function aligns with the periods of individual trigonometric functions to determine the period of a sum of two trigonometric functions.

Keywords: trigonometric functions, periodic functions, mathematical reasoning

Introduction

The periodic nature of trigonometric functions is often highlighted in mathematics education (Weber, 2005; Ortega, 2017; Brown & Rice, 2011), although relatively few studies examine how students understand the periodic behavior of these functions. This paper presents findings from exploratory teaching interviews (Steffe & Thompson, 2000) aimed at exploring students’ reasoning about the sum of two cosine functions. The study is driven by an interest in how students develop foundational knowledge relevant to Fourier series. Fourier analysis is an interdisciplinary topic connecting mathematics, physics, and engineering education. Although the Fourier series is a significantly advanced mathematical topic, it is often missing from standard undergraduate curricula. When learning Fourier series, the importance of periodic functions is recognized as central (Leith, 2024; Mays and Loverude, 2019; Jia et al., 2011). Fourier series breaks down complex periodic functions into sums of sine and cosine functions, which are used to analyze, design, and optimize models in physics and engineering. The initial phase of the study suggested the need to investigate students’ progression of thinking as they learn about the periodic behavior of sums of sine or cosine functions. Therefore, this paper addresses the research question: *What mathematical reasoning do students use to determine the period of the sum of two cosine functions?*

Literature Review

While many studies (Blackett & Tall, 1991; Doerr, 1996; Brown, 2005; Weber, 2005; Steckroth, 2007; K.A.Thompson, 2007; Moore, 2014) have investigated various aspects of trigonometric functions, their periodic behavior remains underexplored. Educators and practitioners (Miller, 2001; Martinez Ortega, Preciado Babb & Velasco, 2017) emphasize innovative teaching strategies incorporating physical movement to deepen students’ understanding of periodic functions. Stupel (2012) advocated for connecting the periodic properties of trigonometric functions with fundamental number theory concepts, particularly through dynamic visualization tools like GeoGebra that enable students to explore how changes in parameters of a function affect the pattern and periodicity. Studies by Buendía and Cordero (2005) and Shama (1998) explored how students understand the periodic nature of functions and periodicity in general. The findings suggest a disconnect between recognizing periodic behavior in situations and relating this periodic nature to a formal mathematical context. Recent research (Khan, 2025) examined an undergraduate engineering student’s conception of periodic functions, finding that students conceptualize a function's period as the distance between two distinguishable points, such as maxima, where output values repeat, and patterns repeat through the domain. However, despite this saturated work on periodic function understanding, there remains a significant gap in research investigating how students develop mathematical reasoning about the periodic nature of the trigonometric sum, particularly using graphing technology like Desmos.

Theoretical Perspective

Constructivism serves as the foundational learning theory for this research. According to constructivist theory, knowledge is not passively received but actively constructed by an individual. Resnick and Glaser (2016) describe constructivism as a framework in which learners build understanding by integrating new information with their existing knowledge. This paper explores how students develop mathematical reasoning about the periodic nature of the sum of two cosine functions. The exploratory teaching interview (Steffe & Thompson, 2000) rooted in constructivist principles, provided the necessary framework for this investigation. This approach allowed the design of strategic perturbations through researcher prompts and tasks that reveal students' evolving conceptions as they work through the tasks. Khan (2025) reported a student's thinking of periodic functions. The findings enabled the formulation of a hypothetical framework (Table 1) describing students' reasoning about periodic functions and guided the methodological approach of this study.

Table 1. A hypothetical framework of students' thinking of periodic functions.

Setting/Context	Aspects of Concept Image
General conception of periodic functions	A student thinks that a periodic function has repeating and unchanged pattern in a graph.
	A student thinks that a periodic function has same output values after a certain interval.
	A student thinks that a periodic function has repetitive output after a period throughout the domain.
To determine whether a function is periodic	A student thinks that a periodic function has same output values for two reference points and there is an unchanged pattern within these two points. And the pattern keeps repeating throughout the domain.
	A student thinks about an unchanged pattern as a chunk that is repeated throughout the domain.
	A student thinks about a function being periodic in terms of function formula. If there is a horizontal shift P, then $f(x+P)=f(x)$.
Understanding of 'a period' of periodic functions	A student thinks about a period as the distance between two maximum or minimum points of the function.
	A student thinks about a period as the interval for which the function's graph remains unchanged throughout the domain.
	A student thinks about a period as a constant value p. For a horizontal shift p, the function produces same output.

Methodology

This study employs exploratory teaching interviews with two participants to investigate how students use their conception of the period of a trigonometric function in developing their understanding of the period of the sum of two trigonometric functions.

Two volunteer participants, Eloise and EL (pseudonyms), were recruited from a pool of students who participated in another interview that explored their conception of periodic functions. Both participants had prior exposure to trigonometric concepts and functions.

Data were collected through interviews conducted via the Zoom video conferencing platform. The interviews were scheduled as one-on-one sessions and video recorded in full to capture both the participants' verbal explanations and on-screen interactions. During the interviews, participants engaged with a dynamic task using Desmos, a web-based graphing calculator.

In this interview, the participants worked on four tasks that were designed to explore the progression of their thinking about the period of a function that represents a sum of two or more trigonometric functions. In this paper, we report on Task B, where participants were presented with two cosine functions, cos (mx) and cos (nx) where m and n were integer parameters ranging from 1 to 10. In addition, a summation function $h(x) = \cos(mx) + \cos(nx)$ was defined. Using the Desmos applet, the participants were able to graph either the individual functions or the combined function h. The Desmos environment provided interactive sliders for the parameters m and n, allowing the participants to dynamically adjust the values and observe the resulting changes in the graph(s). The following figure is a snapshot of the Desmos applet.

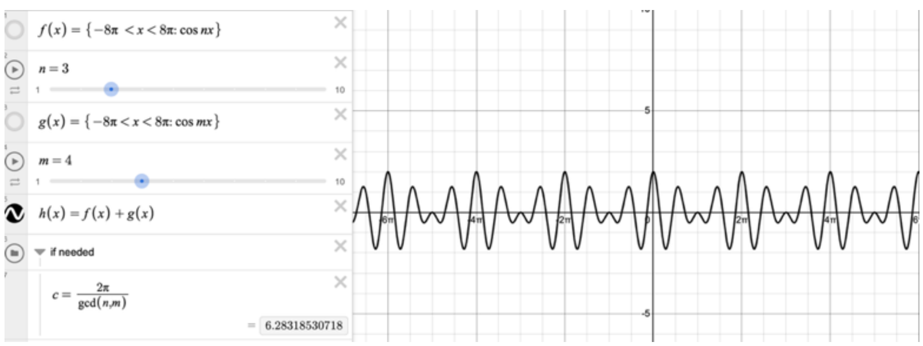

Figure 1. A view of Task B in Desmos

To analyze the data, the qualitative data analysis method (Saldaña, 2021) was implemented. I first transcribed using Adobe Premiere Pro and then manually corrected auto-transcription errors. I read through the data and named codes based on themes. I labeled the data into discrete parts based on the codes I created. Next, I compared similar events in the data and collated excerpts that were labeled with a particular theme. Then, I drew connections between the codes to find how the codes can be grouped into categories to answer the research question.

Results

In this section, I report on Eloise and EL's evolving thinking process in exploring the period of $h(x) = \cos(mx) + \cos(nx)$. Before task B, Eloise and EL were introduced to Task A, where they explored the periods for the functions $y = \sin(3x)$ and $y = \cos(5x)$. First, Eloise determined the distance of a pattern that repeats within a graph of these trigonometric functions with different frequencies, then she tried to choose input values for which the output values repeat. She mentioned the numerical difference between these input values as the period of the functions. EL, on the other hand, approached the same task differently. She used proportional reasoning to rationalize that the period of $\sin(3x)$ would be $2\pi/3$. First, she pointed towards the length of one period of $2\pi/3$, then mentioned that as she knew the period of the sine function is 2π, the period of $\sin(3x)$ would be $\frac{2\pi}{3}$ because if she changes the input for the function within the length of 2π, there are three times as many periods for $\sin(3x)$ compared to $\sin(x)$. The interviewer then asked the participants to think about the period of $y = \sin(3x) + \cos(5x)$. Eloise imagined a graph with some oscillations but was unsure about whether it was periodic. However, after the interviewer presented the graph of this sum function, Eloise again looked for the distance between input values for which the output values are the same. In response to the same question without any probing, EL initially imagined the sum of $\sin(3x)$ and $\cos(5x)$ as a periodic function with a period that would be common for both functions. In her words-

> "If the period of A ($\sin(3x)$) is $2\pi/3$ and the period of B ($\cos(5x)$) is $2\pi/5$, then the period of the combined one will have some integer times $2\pi/3$ and also some possibly different integer times $2\pi/5$. And from what I know $\frac{1}{3}$ and $\frac{1}{5}$, these are not multiples of each other, and their least common multiple, I think, is 1. So, I would say the period would be 2π".

Task B is designed to explore the reasoning students' use in determining the period of the sum of two trigonometric functions. Eloise and EL were presented with the Desmos applet, and they initially explored the applet on their own. After interacting with the applet, Eloise initially perceived that as m and n gets larger, the graph of $h(x) = \cos(mx) + \cos(nx)$ may have a similar pattern to $\cos(x)$ with many peaks in between intervals. While exploring the applet, EL noticed that a regular cosine function with a frequency m makes the period smaller as the value of m gets larger. She also mentioned that the period of the sum of two cosine functions with different frequencies depends on the multiple of the periods of each regular cosine function with frequencies m and n.

Next, in this results section, I will present Eloise and EL's thinking to determine the periods of the function $h(x) = cosmx + cosnx\ where\ 1 \leq m \leq 10, 1 \leq n \leq 10$ for different combinations of m and n.

For Cases $n \neq m$, With no Common Multiple

The interviewer asked Eloise to list the periods of the function h, for different combinations of (n, m) as n, m takes integer values between 1 and 10. Eloise relied on the graph of h to see for which x values the output repeats, focusing on the peak-to-peak distance of the interval. She used this reasoning to find periods of h for combinations like $(1,2), (2,3), (1,10)$. She thought that for (n, m) combinations that $n \neq m$ have no multiple in common, the function repeats every 2π interval. EL, in contrast, for the same situation imagined that the function h will start repeating at 2π, as each $\cos(mx)$ and $\cos(nx)$ need to meet at a place where the period of h is a multiple of both the periods of $\cos(mx)$ and $\cos(nx)$. When (n, m) are different with no multiple in common other than 1, the period of the sum would be 2π. Although Eloise relied on the function formulas, EL conceived the period of h as the distance between points where the individual periods of $\cos(mx)$ and $\cos(nx)$ '*meet*' or coincide.

For Cases Where $n = m$

When $n = m$, EL continued to demonstrate her focus on underlying relationship between the period of h with n and m. However, Eloise initially imagined the period of h would be 2π. To verify her assumption, she utilized the graph of h, and noticed that the period is π. She paused to reconsider and reflected on what the output of h looks like for $n = m$. Eloise realized that if $n = m = 2$, then $h(x) = 2\cos(2x)$, and the period would be half of $2\cos(x)$ which is π. For $n = m = 3$, Eloise focused on the pattern to determine the period of h. She thought $2\cos(x)$ as the base function and compared the period of $2\cos(3x)$ with $2\cos(x)$. She mentioned that the period of $2\cos(3x)$ would be $1/3$ of the period of $2\cos(x)$. At this point, Eloise started recognizing the multiplicative aspect of the pattern of change in the period of h with the changes in n and m. The following figure shows that she wrote a hypothesis. She thinks that when $n = m$, the period of h would be $2\pi/n$.

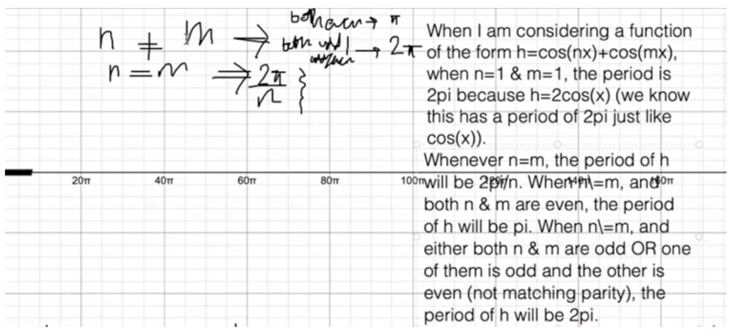

Figure 2. Eloise's initial hypothesis for the period of h, when n=m and $n \neq m$

For Cases $n \neq m$, With Common Multiple

After finding some periods of h for different combinations of (n, m), Eloise also came up with a hypothesis that for cases when $n \neq m$ and n, m are both even, the period of h would be π. When $n \neq m$, and either both n and m are odd, or one of them is odd and the other is even, the period of h would be 2π.The interviewer then offered Eloise to consider periods of h for combinations like $(6,4), (8,4), (3,9)$. Her hypothesis failed when she determined the period $\frac{\pi}{2}$ from the graph of $h(x) = \cos(8x) + \cos(4x)$. Using the graph Eloise demonstrated that the period of a function is the distance of an interval between two repeated outputs of the function for $(n, m) = (8,4)$. She looked for the inputs of h for which she could get the repeated output values. For instance, she thought that if the input for $h(x) = \cos(8x) + \cos(4x)$ is $\pi/2$, then $\cos\left(8 * \frac{\pi}{2}\right)$ and $\cos\left(4 * \frac{\pi}{2}\right)$ both have even integer multiples of π, and the output is 2, the same as the output of $h(0)$. She rechecked the process with an input $\frac{\pi}{4}$, but realized that in this scenario $\cos(4 * \frac{\pi}{4})$ had an odd integer multiple of π and did not return the same output value as $h(0)$. She concluded that as she started with an initial input 0 of $h(x)$, for another input $\pi/2$ of $h(x)$, there exists the smallest interval for which the output of $h(x)$ repeated. Throughout the interview, to determine any period of the sum of trigonometric functions, Eloise strictly focused on getting repeated outputs. She assumed the periods of h are related to (n, m) being the same, even, or odd, and she focused on verifying her assumptions by finding the smallest intervals between two same outputs. Then the interviewer probed her by asking whether the period of h is related to having any common factor between n and m. She quickly comprehended that the period of h would be 2π divided by the greatest common divisor of n and m.

For cases when n and m are multiples of the other, EL continued to compare the periods of $\cos(mx)$ and $\cos(nx)$ individually with the period of $h(x)$. She was thinking about the ratio of 2π and the common multiple of n and m to be the period of h. She thought that if $\cos(2x)$ has a period of π, and $\cos(4x)$ has a period of $\pi/2$; then the period of $h(x) = \cos(2x) + \cos(4x)$ would be the π as there are two of $\pi/2$'s in π. EL also graphed $\cos(2x)$, $\cos(4x)$, and $h(x)$ together and showed that at $x = 0$ and $x = \pi$, all these functions have maximum values and '*they meet at these points.*' Figure 3. below shows that EL marked the outputs for $\cos(2x)$, $\cos(4x)$, and $h(x)$ when the inputs are 0 and π.

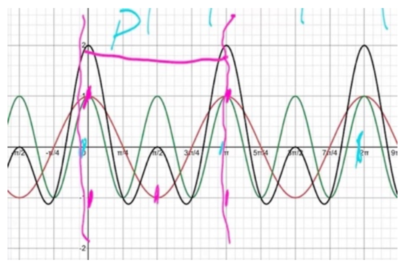

Figure 3. EL shows the periods for the sum function h when n=2, m=4

At this point, EL generalized that the period of $\cos(nx)$ would be $P_1 = \frac{1}{n} * 2\pi$, and the period of $\cos(mx)$ would be $P_2 = \frac{1}{m} * 2\pi$. She thought that the least common multiple of $\frac{1}{n}$ and $\frac{1}{m}$ plays a role in determining the period of h. The interviewer then asked EL to consider $n = 6$ and $n = 9$ to determine the period of h. EL found this combination unique as these are not multiples of each other, but in her words, *'they do combine'*. The following excerpt suggests that EL was referring to the least common multiple in determining the period of h.

Interviewer: What do you mean by they combine?

EL: Do they combine? We have $\frac{1}{6}$ and $\frac{1}{9}$…. So if we have something divided by 6 and we have some other thing divided by 9 (she wrote $\frac{x}{6} = \frac{y}{9}$). Do these ever line up? I think if we do $\frac{3}{9}$ and $\frac{2}{6}$, these are the same. So then, I think this (the period) will be $2\pi/3$

Interviewer: What exactly did you try to get from writing $\frac{2}{6} = \frac{3}{9}$?

EL: I was thinking if we have $\frac{x}{6}$ as some number of periods of the first function, and $\frac{y}{9}$ as some number of periods of the second function. And we want them both to line up at 2π. So if we find two numbers x and y that make this work, then we get the period of the larger function.

Interviewer: Okay.

EL: And then I found 2 and 3 that made this ($\frac{2}{6} = \frac{3}{9} = \frac{1}{3}$) work, and we get $\frac{2\pi}{3}$.

EL was thinking about the least common multiple of n and m as the input within the interval $(0, 2\pi)$ where the output starts repeating for all three functions, $\cos(mx)$, $\cos(nx)$, and h. As she exhibited her thinking of the least common multiple of (n, m) playing a role in determining the period of h for different combinations of n and m, the interviewer asked whether she could generalize her thinking about the period of the sum of two cosine functions. EL spontaneously mentioned that it would be related to the greatest common divisor of n and m. She thinks about the least common multiple of $(\frac{1}{n}, \frac{1}{m})$ as the same as the reciprocal of greatest common divisor of

(n, m). Figure 4. shows EL's formula to determine the period of the sum of two cosine functions.

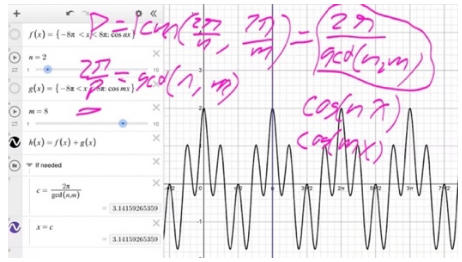

Figure 4. EL's formula to determine the period of h.

Discussion and Limitations

The results of the study suggest two distinct types of reasoning that students use to determine the period of the sum of two cosine functions. The constructs *output-oriented reasoning*, and *alignment-oriented reasoning* represent different ways students utilized their existing mathematical knowledge to make sense of the period of a trigonometric sum.

Eloise's approach of *output-oriented reasoning* focused on interpreting visible and numerical results from the graph. She identified periodic nature of the function by seeking repeating, unchanged patterns and verified the period by finding the smallest input interval producing the same output. Her hypothesis relied on characteristics like evenness or oddness of parameters, verified with graphs rather than focusing on the foundational number theory concepts. This approach is effective in most cases as it aligns with her conceptions of periodic functions but required support to understand the role of the common factor of the parameters n and m in generalization of the period of h.

EL's *alignment-oriented reasoning* centers around the understanding of the multiplicative and proportional relation between frequency and period, recognizing that for $\cos(nx)$ the period is $2\pi/n$. This understanding allowed her to see that for larger frequencies, the trigonometric functions produce smaller periods. She also recognized that the periods of individual functions relate multiplicatively when combined and that their frequencies coordinate to form an aligned period. She thought about how individual cycles 'line up' or 'meet', seeking points where the periods of $\cos(nx)$ and $\cos(mx)$ align with the period of the sum of these functions. EL consistently demonstrated the proportional aspect while comparing the periods. She thinks that if $\sin(x)$ has period of 2π, then $\sin(3x)$ has period $2\pi/3$ because there are "three times as many periods" in the same interval. Using the multiplicative and proportional aspect with the alignment of the periods in her reasoning, she was able to generalize a rule for the sum's period, linking it to the greatest common divisor of the frequencies of the individual cosine functions. This reasoning reflects a robust concept of periodicity, allowing EL to derive a generalizable formula of the sum of two cosine functions.

The distinction between *output-oriented* and *alignment-oriented* reasoning has key pedagogical implications. Both forms of reasoning reflect active engagement in learning; *output-oriented* reasoning emphasizes concrete, visual evidence, whereas *alignment-oriented* reasoning fosters the ability to generalize the foundational mathematical idea. A limitation of this study is that the generalization of the period of h as presented does not hold mathematically when either n or m is an irrational number. Although the interview protocol included tasks designed to explore students' understanding of such cases, the results of these explorations are not included in this paper.

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