Conceptual Blending as a Problem-Solving Tool in Topology

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The topic of topology is an important and unique course in advanced mathematics that carries two simultaneous complexities: logical proof and spatial reasoning. While proof and problem-solving are two areas that are well documented in mathematics education research, topology itself is a fairly rare topic to appear in the literature of mathematics education written in English. To address the gap in the literature, this paper analyzes a study in which two students attempt a topology question in order to understand their problem-solving techniques and how they leverage their various forms of thinking. We networked the theories of Tall's Three Worlds of Mathematical Thinking and Cognitive Blending to describe the participants' behaviors. The results showed that students leveraged a blend of their thinking to progress through the proof.

Keywords: Topology, problem-solving, proof, Tall's Three Worlds, cognitive blending

Introduction

Topology is an important topic in the world of advanced mathematics, with James (2006) claiming, "Topology, for many years, has been one of the most exciting and influential fields of research in modern mathematics. (p 1)" Often said to be "rubber-sheet geometry", undergraduate courses in topology are primarily concerned with the heavily set-theoretical definitions, theorems, and properties. It is a unique course in that context, as it is a heavily proof-focused topic that is supported by a significant amount of spatial reasoning. Even so, there is a lack of educational research on topology. While the topic was used in developmental psychology (Piaget & Inhelder, 1967) and is a field often brought into studies concerning pre-university math education, research that focuses on RUME in topology is scarce among the literature available in English (Vizcaíno et al., 2023).

Literature Review

Educational research in topology is still in its infancy. While researchers have had some interest in topology, the topic is often used as a setting rather than being central to the study. For example, Hicks et al. (2023) attempted to develop a model for "abstraction through analogical reasoning" (p. 807) and used topology as an example to demonstrate part of their model. Additionally, Clark et al.'s (2017) study focused on the incorporation of "primary source projects" into a topology course. The authors' motivation for doing so was based on the idea that original sources for mathematical concepts framed the issues the mathematicians were facing for the students so that they could better understand why the concepts were developed in the first place. Topology was chosen for this study as it was the course the researcher was teaching at the time. On the other hand, some studies have topology near the research interests. Bardelle (2011) explored students' understanding of the topological concepts of finiteness and boundedness with metric spaces, claiming, "everyday use of mathematical terms combined with a lack of coordination of different representations proves to be a hindrance in the conceptualization of basic properties of subsets of Euclidean spaces" (p. 1). Additionally, O'Brien (2023) establishes the connection between the topology of knot theory and tapestry weaving as she claims that those engaging in tapestry weaving internalize and embody diagrammatic instructions in the same way that a mathematician does with topological diagrams.

For papers that are written in English, we have identified three doctoral dissertations in recent years that are principally concerned with education in topology (e.g., Cheshire, 2017; Gallagher, 2020; Berger, 2021). Cheshire (2017) examined the development and adaptations of concepts that students previously encountered as they make sense of the new version of those ideas in the context of topology, primarily with the topic of continuity. In a similar vein, *Reopening the Case of Schema: A Topological Perspective* is a doctoral thesis that seeks to understand and describe students' adaptations to their bodies of knowledge and approaches to problems as they encounter concepts in topology (Berger, 2021). Both studies found that students may struggle with increasingly abstract topics, but they are generally able to adapt to new problems. Another thesis seeks to understand students' use of examples, gestures, and problem-solving strategies to answer questions in topology (Gallagher, 2020). This work demonstrates and describes the ways that students use verbal, gestural, and visual ideas as tools in topology as well as parts of their own thinking. Gallagher found evidence to suggest that it is useful for students to associate their mathematical concepts with diagrams in order to manipulate them and search for important components of the concepts, concluding that teachers should foster students' intuition. All three of these studies referred to the importance of the nature and mechanisms of students' internalization of topological concepts and how they learn to think about them. These three were all empirical studies that included undergraduate students solving problems in topology, and Berger also included graduate students and a research mathematician.

As is evident by the nature of the topic, proof is a fundamental part of topology. To that end, it is necessary to understand the state of the literature on proof in mathematics education research. The study of proofs, proof writing, and the learning of it is now a relatively well-researched area of mathematics education (Mulhuish et al., 2022; Stylianides et al., 2024). According to Selden and Selden (2013), students in higher level math courses have difficulty producing proofs, partly since proofs outside of an intro-to-proofs course have their own content that students must adapt to while trying to structure their proofs. Selden and Selden categorized a written proof into two parts: the formal-rhetorical part and the problem-centered part. The formal-rhetorical part is that which involves the logical structure of the proof. The problem-centered part consists of the content in the proof oriented around the mathematical constructs and concepts at hand. Thus, to understand how students approach proofs, it is necessary to understand problem-solving itself.

Theoretical Framework

The framework for this study is adapted from Schoenfeld's (1985) problem-solving, Tall's (2013) Three Worlds of Mathematical Thinking, and the theory of Cognitive Blending (Fauconnier & Turner, 2002).

Problem-solving in mathematics is a well-researched topic that has had an influence on the educational practices of teachers for many years (Schoenfeld, 1992), and there are four components of problem-solving outlined by Schoenfeld (1985). The notion of *heuristics* is used to describe the problem-solving strategies that students employ, with the idea of *control* referencing how different heuristics and content *knowledge* resources are used. The subtle concept of *belief system*s also plays a role, as Schoenfeld (1985) claimed, "...their beliefs about mathematics –consciously held or not– establish the psychological context within which they do mathematics" (p. 14). Schoenfeld's problem-solving lens provides a context to ask about students' thought processes in topology.

Tall's Three Worlds of Mathematical Thinking describes the embodied, symbolic, and formal worlds as the three forms of thinking in mathematics (Tall, 2013). The embodied world is that which is influenced by one's perception of the physical world, involving notions such as images, haptics, and motion. The symbolic world involves symbols and operations on them, as often seen in the context of algebra. The formal/axiomatic world describes the thinking dealing with theorems, definitions, and the structured logic-focused concepts of mathematics. This theory enables researchers to describe the different behavior of students as they work through proofs. However, it is worth noting that it is not accurate to the theory to take these worlds as separate, and it is not accurate to suggest that one can move between the worlds of thinking. Instead, the worlds are all descriptions of different manifestations of a given topic as it exists cognitively, and one does not "move" amongst various qualities of a given object. Rather, students can be observed using the different worlds of thinking in conjunction with each other, which demands a theory to describe such a blending of ideas.

The theory of Cognitive (or conceptual) Blending refers to a supposed fundamental cognitive tool of reasoning that involves a person incorporating two worlds of thinking into a single realm to gain further understanding from it (Fauconnier & Turner, 2002). There are two descriptions of a blend: single-scope and double-scope (Bing & Reddish, 2007). Single-scope blending occurs when one world provides framing/information to make sense of the other without any interplay; this is the case when understanding "time flies like an arrow," as one's understanding of arrows is framing a description of time through one's imagination of an arrow in flight. Yet, the phrase does not influence the way one would think of an arrow. Double-scope blending is more advanced, as ideas from both worlds are at play and influencing cognitive creation. It can easily be observed in the context of language that the spoken and written components are distinct yet have great simultaneous influence on each other, creating a double-scope blend when they are used in conjunction.

By networking these theories, we developed a framework that enables a description of how students use different forms of thinking together in their problem-solving by blending them. The research questions to lead this study are:

- What heuristics do students employ as they attempt topology proofs?
- Did blending of the worlds of mathematical thinking occur in students' attempts at solving problems in topology? Was it a useful approach for problem-solving in topology?

Methods

This case study took the form of a single collaborative, structured problem-solving session where two graduate students were presented with three topology problems. The collaborative aspect of the design was intended to encourage the articulation of their thinking as they worked together. The participants of this study were two early graduate students in the researchers' mathematics department. Potential subjects were selected based on their proximity to the beginning of their time in graduate school and their responses to the opportunity. The study involved working on a whiteboard together, and it was audio- and video-recorded. The participants were also asked to construct proofs. The researcher (first author) wrote a problem on the board for the participants to attempt one at a time. Their attempts were examined, though the study was not focused on evaluating the accuracy or validity of their proof or the function of their teamwork abilities; rather, the principal focus was on the nature of their thinking and problem-solving, so they were encouraged to articulate their thoughts. The three topology problems are depicted in Table 1 alongside their motivation for inclusion in the study. These problems were selected to encourage different kinds of reasoning and thinking, thereby gathering a variety of data from the students. This paper focuses on the students' attempt at the second question. The authors are in the process of analyzing the data from the students' attempt at the other questions for future studies.

Table 1. Interview problems and their reason for selection.

Topology Problem	Intention for Selection of the Problem
1. Provide a metric on (-1,1) that induces the usual topology but is complete as a metric space.	The most direct solution involves making a particular connection between spaces that is often associated with a graph, and the notion of a metric is associated with the idea of distance. Thus, the question is heavily influenced by the embodied world of thinking.
2. Determine whether R imbued with the lower-limit topology is path-connected.	Path-connectedness is typically considered in cases of topologies that can be perceived, yet the space in question is not a manifold that can be imagined easily. This might motivate the students to try to work in the embodied world, yet the proof does not directly follow from that perception.
3. Prove that (0,1) with the usual topology is not compact.	Compactness of a topological space is an abstraction of the notion of compactness in metric spaces, which typically benefits from embodied thinking.

After the interview, a transcript of the conversation was created, and images from the video were used to examine the work that the participants did on the board. This transcript of the data was coded with four themes in mind: Embodied, Symbolic, and Formal worlds of mathematical thinking, as well as Blending. The first three are in line with descriptions from Tall's Three Worlds of Mathematical Thinking, while blending was used to identify moments where the participants were considering different worlds of thinking in conjunction to elucidate a concept or make progress in their proof, according to the theory of cognitive blending. This is distinct from the students bringing up different forms of thinking simultaneously without engaging in any detectable interaction between them. Moments of potential blending were identified when different forms of thinking overlapped or followed each other. NVivo qualitative data analysis software was used to examine and code the video along with its transcript. These codes were analyzed further to determine whether they could be considered evidence of a cognitive blend.

Results and Discussion

The two students began this problem with the correct intuition that the space is not path-connected. They started their approach by defining each term they saw, beginning with their understanding of the lower-limit topology. They handled that concept by writing out the form of the basic open sets (intervals of the form [a, b), with a, b real) as seen in Figure 1. Then, to describe path-connectedness, Student 2 stated, "between any two points we can make a path," and Student 1 replied that he thought a path needed to be "between sets." So, Student 2 drew an image of a path in a space by drawing a circle, labeling points x and y, and then connecting those points with a curving line, then verified that path-connected means "between any two points we can make a path?" Student 1 then affirmed this, and Student 2 drew a second circle as he mentioned the notion of the components of the space. This prompted Student 1 to mention "since connected implies path-connected…" which brought them to a discussion about the proper direction of the implication between those two properties. Ultimately, they agreed, mentioning the topologist's sine curve as the example, that path-connectedness implies connectedness, with Student 2 then writing on the board "C → PC," the wrong implication. They then considered whether the space is connected, with Student 1 having believed it is. This motivated students to focus on their definition of path connectedness, as they pondered what the definition of a path involves. After they defined a path, they revisited the correlation between connectedness and path-connectedness, realizing the mistake on the board. After correcting it, they realized that if they can find a separation of the space, then the space must not be path-connected. Student 2 suggested they could assume it is and try to arrive at a contradiction, with Student 1 then arguing that they could just consider what sets are open first to avoid that level of work. They began to ponder what the open sets in this space look like, with Student 1 having quickly decided he does not think that the space has a separation. They did not perform any detectable set theory, as they only wrote down the basic open sets before deciding that there was no open partition.

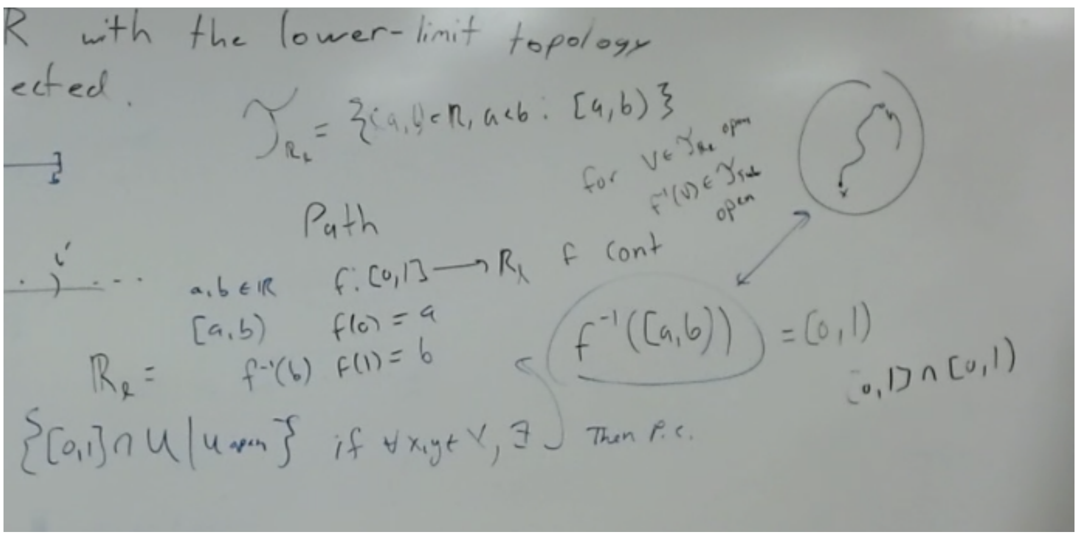

Figure 1. A portion of the students' board work.

The students then engaged in their approach of assuming that the space is path-connected and tried to find a contradiction. Student 2 then stated that he wants to understand what a path would look like between two points "in a basis element" as they decided they need a "continuous function that connects a to b." They began to consider what the preimage under some path of the basic open set [b,a) must "look like", before Student 2 replaced it with the closed set [a,b]. Student 1 then asked the researcher what the definition of a path is, and he asked in response, "How do you think of a path?" Student 1 then gestured a line in the air, then drew the same image that Student 2 drew before, claiming, "it's a continuous sort of connection that exists within the space." The researcher replied that a path is a continuous map from the interval into the space, with Student 2 adding that a space is path-connected if a path exists between "all the points." They then amended their existing definition of a path with quantifiers to extend it into a definition for path connectedness. Student 1 asked what the topology of the interval was, as the researcher affirmed his belief that it has the usual topology, the subspace topology of the real line with the usual topology. The pair thought in silence for a moment, then Student 1 resolved that they wanted to break the continuity of their supposed path. They tried to understand what the preimage of their closed set must look like, with Student 2 having noticed the difficulty in the question coming from the fact that the path may “backtrack” on itself, in other words, the path need not be injective. This was essentially as far as they were able to take the proof on their own, as the rest of the progress was only made with significant interviewer intervention.

Afterwards, the researcher informed the students that they could find a separation of the space. They began to consider this as Student 1 struggled with the set theory, and the researcher guided them to consider explicitly taking the union of particular sets. This helped them realize that they could construct two open sets that are complements of each other, meaning the space is not connected, implying the space cannot be path-connected. This process of the students’ attempt was coded, as Figure 2 depicts the codes as they appear in the transcript, with blue indicating the embodied code, red for symbolic, yellow for formal, and black for identified instances of blending.

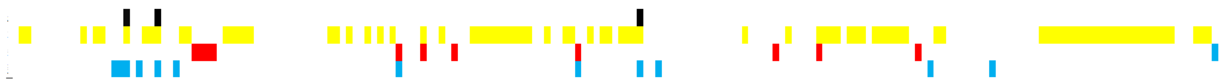

Figure 2. The transcript's coding of embodied (blue), symbolic (red), formal (yellow), and blending (black).

Addressing the research question “What heuristics do students employ as they attempt topology proofs?”, we note firstly that the participants used a significant amount of formal thinking and reasoning, as seen by the frequency of the yellow blocks in the coding of the transcript in Figure 2. They began Question 2 by defining each property and clarifying the relevant topological space by considering the basic open sets. Additionally, they used axiomatic reasoning with relevant theorems to decide which “path” to follow in their solution, recalling that if a space is path-connected, then it is connected, and applying the contrapositive/contradiction to arrive at their result. There was also an intentionality in what activities they considered productive, as at one moment Student 2 said, “Well… let’s just maybe back off from this, since there’s not an immediate separation standing out.” Thus, the heuristics employed include the following: leveraging logical structures to establish and reframe objectives, considering the difficulty of the possible goals, and, as demonstrated in the next question, cognitive blending of various worlds of thinking as strategies to solve their problems.

Next, we consider the second portion of the research questions, “Did blending of the worlds of mathematical thinking occur in students’ attempts at solving problems in topology? Was it a useful approach for problem-solving in topology?” Analyzing this recording yielded various references to Tall’s Three Worlds, and some moments suggest a blending occurring between the worlds of thinking. Firstly, the students handled their concept of a path in such a way that exemplifies a cognitive blend. As they began the problem, the first written “description” they provided for path/path-connectedness was a picture meant to represent the concept of a path. This was also later used to resolve a moment of confusion where one student believed that paths needed to map “between sets.” While the students initially disagreed about the verbal formal description, they immediately agreed on the picture's validity, and Student 1 used that to rectify his misunderstanding of the formal description. This is significant evidence of Student 1 having used single-scope blending as he interpreted this image into a formal understanding, agreeing that the path connects points, not sets. Also, Student 2 intentionally used this image instead of the formal definition to communicate the definition of a path, even though he had just stated that formal definition. Whether he thought the image was more convincing or it would be familiar to Student 1 in such a way that it would help him make that connection, it is clear that Student 2 has internalized the image in great conjunction with the formal definition of a path. It is as if, internally, the image and the definition are communicating the same idea to him, which supports the idea of this cognitive blend between the embodied and formal worlds being a familiar tactic to him.

Blending appeared another time regarding Student 1's description of his concept of a path, as he gestured with his hands while stating that it is a continuous map, before drawing the picture on the board depicted in Figure 1 with the two-directional arrow between the image and the definition. Not only does his explanation describe a blend between his embodied gestures, but he also explicitly connected the ideas on the board with an arrow.

Lastly, while the students used the note "P → C" to reference the theorem that path-connectedness implies connectedness (as well as the converse and its shorthand), this is symbolism that has become common to the students to denote the formal theory. There is no evidence they are handling those icons in any way indicative of a symbolic form of thinking (i.e., there was no action taken on the terms). However, their solution to this confusion, referencing the topologist's sine curve, carried some information that enabled them to reconstruct the theorem without performing any formal tasks. This does suggest some form of blending occurred, though whether it is between any worlds of thinking or merely with their recollection of the topic is unclear.

Conclusions

Ultimately, this data provides evidence that blending the three worlds of thinking is a common and useful cognitive tool for students as they work on problems in topology. Blending was detected and analyzed in the context of successful moments of progress through the problem. It could be suggested that these instances of blending constitute comprehensible connections between mathematicians' different forms of thinking that are not only used in the contexts in which they are developed but may also be internalized into the broader cognitive body of the concepts. Additionally, while no blending was mentioned between the formal and symbolic worlds, this is certainly a possibility if one were to handle a problem by using symbolic logic to restructure the problem (e.g., using DeMorgan's law to "distribute/factor" a negation).

While this study presents compelling evidence that helps address the research questions, there are some limitations. For instance, as this study was task-based, we do not have precise descriptions concerning the students' thinking; we relied on the external evidence of their articulations and board-work to make conclusions about what was happening mentally. This could suggest that there is more blending that is not being detected. Additionally, while these students lack the language of our theoretical framework that would have enabled them to describe their own thought processes, asking them to understand the terms and use them causes an additional cognitive burden that could distract the students and form an unnatural environment for their proof attempt. While there is a larger curiosity regarding how students think about topological concepts in general, the typical class structure may not encourage students to think about their own internal conceptualization of those ideas. Instead, it seems that the majority of the mathematics curriculum focuses on developing students' ability to answer questions, which informed the design of this paper's research question. There are currently plans to develop this project further that involve interviews where the students are asked to describe their thought processes as they work and potentially explain their conceptualizations of topological concepts. As part of the first author's dissertation, we are continuing to refine this theoretical framework to develop a model to further guide our analysis of the data.

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