Two Approaches to Rates and Foundations for Derivatives

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We distinguish 2 approaches to coordinating quantitative meanings for rates, proportional relationships, and variation. The first approach—based on coordinating unit rates, multiple batches, and chunky variation—has dominated existing research on rates and rates of change. The second approach—based on coordinating measurement rates, variable parts, and smooth variation—has not been recognized as distinct from the first. We examine affordances of each approach for reasoning about 2 aspects of rates essential for instantaneous rates of change and derivatives: They are single measures that remain approximately constant as changes in 2 variables tend to 0. We argue that the second approach complements the first in important ways and conclude that it should be recognized as an important focus in future empirical research.

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Past research has recognized that the origins of students' and teachers' challenges with derivatives can often be found in their reasoning about quantities in combination with multiplication and division, variables, rates, and functions (e.g., Boyce et al., 2021; Byerley, 2019; Larsen et al., 2017; Thompson & Harel, 2021). One common challenge is that students and teachers often focus on rates as pairs of amounts rather than as single measures (e.g., Byerley & Thompson, 2017; Frank & Thompson, 2021; Johnson, 2015b; Yu, 2021). In calculus, a view of rates or slopes as pairs of amounts is problematic for developing an understanding of instantaneous rate of change and the derivative of a function. In particular, students who view rates and slopes as pairs of amounts will not be able to coordinate average or instantaneous rates of change of a function with changes in the input variable (e.g., Carlson et al., 2002; Yu, 2024).

In this theoretical report, we distinguish two approaches to mature meanings for rates that we term *unit rate* and *measurement rate*. Although mathematics education research has rarely articulated the distinction between these two meanings, we argue that the distinction matters because each presents different affordances for supporting two aspects essential for understanding how local, linear approximations lead to derivatives. First, a (constant) rate, rate of change[1], or slope is a *single quantity* that measures a quality such as steepness or speed by coordinating changes in two quantities multiplicatively. Second, understanding that m is the instantaneous rate of change of a function at a point relies, in turn, on the idea that *rates can remain constant or approximately constant as* Δx *and* Δy *tend smoothly and continuously to zero*. Our contribution rests on critical examination of ways that past theoretical research on quantitative reasoning has treated multiplication and measurement.

Unit Rates and Measurement Rates

Our delineation of two approaches to rates begins with the distinction between multiplicands and multipliers (e.g., Greer, 1992). If one invokes multiplication when considering a situation where three children each have four cookies, the number of cookies one child has is the multiplicand and the number of children is the multiplier. Sharing or partitive division results

[1] We view rate of change as an application of rate and therefore we use the term rate to include rate of change in the remainder of the article.

from dividing the product by the multiplier to determine the multiplicand. An example would be: “If 12 cookies are distributed evenly among three children, how many cookies does each child get?” Measurement or quotitive division results from dividing the product by the multiplicand to determine the multiplier. An example would be: “If there are 12 cookies, how many children can each get four cookies?”

Given the close connections among division, ratios, and rates, that there are two distinct quantitative meanings for division suggests, in turn, that there should be two distinct quantitative meanings for rates. We introduce these two meanings using red and blue paint mixed in a 4-to-3 ratio. When reasoning with *unit rates*, we consider an initial batch containing 4 gallons of red paint and 3 gallons of blue paint. We imagine variation occurring through partitioning and iterating this initial batch, and we use division based on equal sharing to determine that 4/3 gallons of red paint correspond to each 1 gallon of blue paint. If n is a natural number, then for n gallons of blue paint there will n • 4/3 gallons of red paint. More generally, if r and s are natural numbers, then for r/s gallons of blue paint there will be r/s • 4/3 gallons of red paint. Here, n and r/s function as *multipliers* acting in parallel on two fixed multiplicands, 4/3 gallons of red paint and 1 gallon of blue paint. A similar argument explains a second unit rate, 3/4 blue gallons for every 1 gallon of red paint. Beckmann and Izsák (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have referred to such reasoning as the *multiple-batches* perspective on proportional relationships. The unit rate meaning and multiple-batches perspective have dominated past research on rates (e.g., Lobato & Ellis, 2010; Tallman et al., 2024; Thompson, 1994; Thompson & Thompson, 1994).

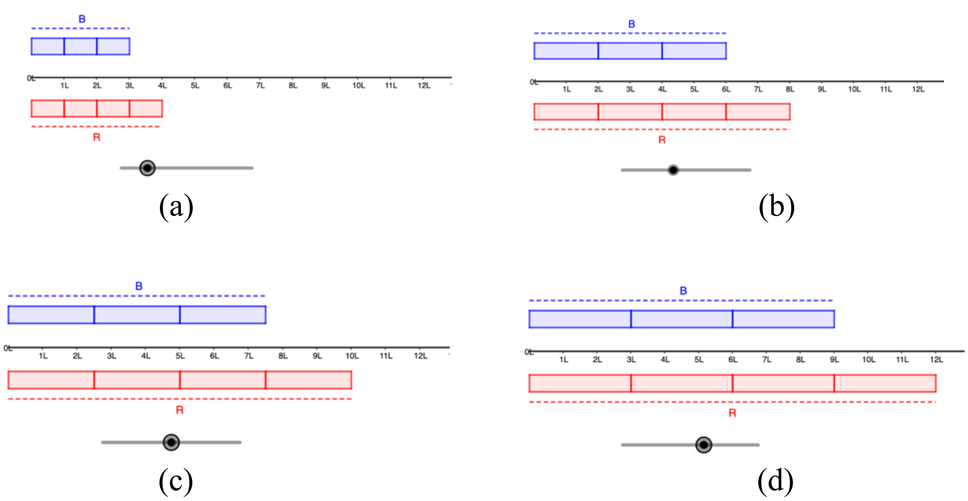

Figure 1. Covarying volumes of paint. All seven parts contain (a) 1 liter. (b) 2 liters. (c) 2.5 liters. (d) 3 liters.

In contrast, when reasoning with *measurement rates*, we organize the same mixture into 4 parts of red paint and 3 parts of blue paint. We imagine variation occurring through increasing or decreasing the volume of each part in such a way that any one part still has the same volume as all remaining six parts. Figure 1 shows screen shots from a dynamic illustration of covarying volumes in which each of seven equal-sized parts increases and decreases in parallel. In contrast to reasoning with unit rates, here we focus on measurement division and ask how much of the blue paint makes the same volume as the red paint. Because the number of parts of blue and red paint are fixed, and because all seven parts contain the same volume, it always takes 4/3 of the blue paint to make the same volume as the red paint, and it always takes 3/4 of the red paint to make the same volume as the blue paint. Notice that if B is any volume of blue paint, the corresponding volume of red paint is $4/3 \cdot B$; and, if R is any volume of red paint, the corresponding volume of blue paint is $¾ \cdot R$. The measures, 4/3 and ¾, are the two measurement rates in this situation. In contrast with unit rates, with measurement rates variation is accomplished by varying a *multiplicand*, B or R, while the corresponding multiplier, 4/3 or ¾, remains fixed. Beckmann and Izsák (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have referred to such reasoning as the *variable-parts* perspective on proportional relationships. The measurement rate meaning and variable-parts perspective has not been recognized in past research on instantaneous rates of change and derivatives as a second, distinct approach.

Research on Rates Grounded in Quantitative Reasoning and Covariation

Research on rates has often been grounded in Thompson's (1994) account of quantitative reasoning. The two features of his account on which we focus are (a) characterizing numerical operations and quantitative operations as two distinct categories of mental operations and (b) characterizing rates as reflectively abstracted constant ratios, or multiplicative comparisons of two quantities varying in constant ratio (pp. 190–192).

Thompson and colleagues discuss multiplication as operating on quantities sometimes and on numbers other times. Thompson and Saldanha (2003) characterized "conceptualized multiplication" as one-to-many relationships: "Envisioning the result of having multiplied is to anticipate a multiplicity" (p. 103). This perspective is consistent with thinking of ratio comparisons where one amount is so many times as, or as much as, another amount. Thompson and Saldanha then discussed multiple proportions that lead to "numerical multiplication." To illustrate, they gave the following analysis for the area of a rectangle that has dimensions 12 inches and 2 centimeters. One side is 12 times as long as 1 inch, one side is 2 times as long as 1 centimeter, and the area is 24 times the area of a rectangle of dimensions 1 inch by 1 centimeter (p. 103). This analysis treats $2 \cdot 12$ as a numerical operation and foregrounds three separate one-to-many relationships where the measures associated with each factor play symmetric roles. Such symmetry contrasts with the conceptually distinct roles played by multipliers and multiplicands discussed above, making it harder to recognize the distinction between unit rates and measurement rates.

How one conceptualizes rates interacts not only with one's conception of multiplicative comparisons but also with one's conception of covariation between two quantities. Existing research has emphasized partitioning and iterating two quantities in parallel (e.g., Lobato & Ellis, 2010; Tallman et al., 2024; Thompson & Thompson, 1994). Thompson and Thompson (1994) provided a particularly succinct statement using speed as the example: "If you go m distance units in s time units at a constant speed, then at this speed you will go $(a/b) \cdot m$ distance units in $(a/b) \cdot s$ time units" (p. 283). This coordination can be envisioned as partitioning both m distance units and s time units into b equal-sized parts and taking a copies of the result. Here multiplication is operating on two quantities, distance and time, in parallel. Other researchers have also examined partitioning and iterating in parallel as a starting point for mature conceptions of rates—for instance, Lobato and colleagues discussed composed units (e.g., Lobato & Ellis, 2010), and Peck (2020) described a "unit rate strategy" as a main outcome in a learning trajectory for slope. Appeals to reflective abstraction have left in a black box how partitioning and iterating two quantities in *parallel* leads to perceiving a fixed multiplicative comparison *between* amounts of each (e.g., between distances and durations of time).

More recently, researchers have examined connections between chunky and smooth variation (e.g., Castillo-Garsow, 2012; Castillo-Garsow et al., 2013) and students' conceptions of ratios and rates (e.g., Johnson, 2015a; Tallman et al., 2024; Yu, 2023, 2024). Castillo-Garsow et al.

explained that a chunky conception of variation focuses on concatenated intervals and endpoints of those intervals. Such reasoning applied in parallel to two quantities is consistent with Thompson and Thompson's (1994) description of constant speed (quoted above) and with Lobato and Ellis' (2010) discussion of composed units. Castillo-Garsow et al. explained further that a smooth image of change focuses on continuous progression through the flow of time. Researchers have recognized that smooth covariational reasoning is critical for building calculus from quantitative and covariational reasoning (e.g., Thompson & Carlson, 2017, pp. 442–443). Still missing from these more recent studies are explanations for how operating on two quantities in *parallel* through partitioning and iterating leads to fix multiplicative relationships *between* the two quantities.

Measurement and Variation

In this section, we connect conceptions of quantity, measurement, variation, and the asymmetric roles of multiplicands and multipliers. Mathematics education research has maintained a distinction between conceiving of a quantity and measuring that quantity. As one example, Thompson (1993) explained: "A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (p. 165). Interestingly, work in mathematics has given a rigorous treatment of such measurement.

1901 Otto Hölder formulated seven axioms for quantity that did *not* include measurement (Michell & Ernst, 1996) but from which he derived measurement. To begin, Hölder's axioms allow one to describe whole and rational number multiples of a quantity: Given any specific quantity Q and any natural number n, Q can be iterated n times to form a quantity $n \bullet Q$ and can be partitioned into n equal-sized parts to form a quantity $(1/n) \bullet Q$ (Michell & Ernst, 1996, sections 3 and 7). It follows that for any positive rational number m/n, where m and n are natural numbers, there is a quantity $(m/n) \bullet Q$, defined as $m \bullet ((1/n) \bullet Q)$, which can be formed by partitioning Q and iterating the result. Note that m/n is a multiplier acting on a multiplicand, Q, and that this action creates chunks. Following Hölder (Michell & Ernst, 1996, sections 10, 11, and 15), one can use completeness of the real numbers to show that for any positive real number t there is a well-defined quantity $t \bullet Q$ whose measure with respect to Q is t. Here, one views the measure, t, as the multiplier that varies as it acts on Q, a fixed multiplicand. We refer to such reasoning as *variation of measure*. The reliance on chunks—in the case of rational multiples—and on completeness—when t varies through irrational numbers—implies that reasoning rigorously about smooth variation in combination with measured quantities is non-trivial.

In contrast, it is also possible to view variation where a quantity increases and decreases without attention to a measure. For example, if a volume of water is poured continuously into a vase, then there is a specific volume in the vase at any given time even if we do not know its measure with respect to any particular unit of volume. This conception of variation aligns with smooth variation. As explained within the paint example, when one applies this perspective to covariation of two quantities, variation occurs in the multiplicand while the multiplier remains fixed. This perspective also resembles that used by Newton: In his history of calculus, Edwards (1979) stated that Newton regarded the speed of a point moving along a straight line as intuitively apparent on physical grounds (p. 192) and took smooth, continuous variation of quantities as given by nature, viewing time as flowing continuously and other quantities as increasing continually in time (pp. 210, 271). We refer to such reasoning as *unmeasured variation*.

Rates, Variation of Quantities, and Derivatives

As stated in the introduction, a quantitative notion of instantaneous rate of change rests on two ideas: Rates or slopes are single quantities and rates of change can remain constant or approximately constant as Δx and Δy tend smoothly and continuously to zero. We agree with Thompson and Carlson (2017) that smooth continuous variation is foundational for building calculus from quantitative and covariational reasoning (pp. 442–443). Below we articulate two approaches to coordinating rates, proportional relationships, and quantitative variation.

Unit rates, multiple batches, and chunky variation

Here we summarize an approach to mature reasoning starting with unit rates. Much of the research on rates has relied proportional covariation in which two quantities are of different types, such as distance and time or height and volume. Such approaches rely on the multiple-batches perspective and on *variation of measure*, at least initially. This follows because, if D and T are two unlike specific quantities such as a distance and a duration of time, then one cannot measure the other directly. The only way to determine new pairs in the same linear relationship is by letting D and T each function as a measurement unit for its own type of quantity (e.g., D measures distances and T measures durations of time). Thus, to explain multiplication as an operation on quantities varying at the same rate, D and T must be iterated and partitioned in parallel, where variation occurs by varying the multiplier, as in varying r/s in $(r/s) \bullet D$ and $(r/s) \bullet T$. As explained above, such variation involves chunks and extension to irrational multiples relies on completeness of the real numbers.

Measurement Rates, Variable Parts, and Unmeasured Variation

Here we summarize an approach to mature reasoning starting with measurement rates. Measurement rates fit most naturally in situations where the two covarying quantities are of the same type—for instance, two lengths or two volumes. In such situations, reasoning about proportional covariation can rely on the variable-parts perspective, *unmeasured variation*, and multiplication as a quantitative operation *between* two quantities. The numbers of same-sized parts are not limited to whole numbers: They can also be positive rational or irrational numbers. For example, if the number of parts is 2.3, then there are 2 whole parts and 3 mini-parts, each 1/10 the size of the whole parts. Covariation occurs by changing the size of one part, and therefore the size of each part or portions of whole parts. Moreover, variation occurs by varying the multiplicand, as in varying B and $4/3 \bullet B$ in the paint example. The parts are free to vary smoothly and continuously.

Entry Points into Derivatives

In this section, we examine the affordances and constraints of unit rates and measurement rates for developing instantaneous rates of change and the derivative. Approaching instantaneous rates of change through unit rates and variation of measure has several advantages. First, considering time as an independent variable may appear self-evident and natural to students—for instance, when thinking about the relationship between speed, distance, and time. Second, a wide range of further applications involve variation of unlike quantities. Examples include the relationships among current, time, and charge; among density, volume, and mass; and among pressure, surface area, and force. Third, past research (e.g., Lobato et al., 2013; Lobato & Siebert, 2002; Tallman et al., 2024; Thompson, 1994) has reported that students of various ages can spontaneously generate sound reasoning in which they partition and iterate in parallel. These results suggest that unit rates and variation of measure provide an accessible entry point for initial reasoning about covariation and proportions in a range of situations where derivatives can be applied. At the same time, our discussion to this point leads to the conclusion that the current practice of engaging students primarily with speed and further situations that use pairs of dissimilar quantities poses an impediment to developing instantaneous rates of change and the derivative because it is difficult to relate unlike quantities through multiplication directly.

Approaching instantaneous rates of change through measurement rates and unmeasured variation of the same type of quantity has a contrasting profile of advantages. Here, it makes sense to interpret the difference quotient $\Delta y/\Delta x$ as a measure that is the result of measurement division: How many or much of Δx it takes to make Δy. This approach makes direct use of the connection between division and measurement, presents rates of change as single numbers that relate Δy and Δx directly, and affords reasoning about smooth and continuous variation.

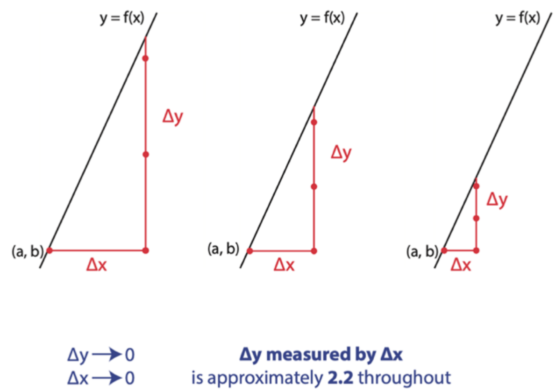

Figure 2. Difference quotient as measure: As Δx tends to zero, the measure $\Delta y/\Delta x$ remains close to 2.2.

To further illustrate a measurement rate interpretation of instantaneous rate of change in the non-linear case, we consider two points on the x- and y-axes that correspond to the x- and y-coordinates of a point moving smoothly and continuously in the coordinate plane. We use the variables x and y for the distances of these two points from the origin. We can regard the distances x and y as unmeasured covarying quantities and the curve that the object travels along as the graph of their relationship. In this situation, if Δx is small enough, the horizontal and vertical distances will change approximately uniformly relative to each other over the interval. Therefore, the measure of Δy in terms of Δx will remain approximately constant. Figure 2 is zoomed in on a small portion of a graph that shows snapshots of Δx (and Δy) decreasing toward zero from a point (a, b) on the graph. On this zoomed-in portion of the graph, the situation is similar to a linear situation, where $\Delta y/\Delta x$ would remain a constant measurement rate, regardless of the size of Δx (and Δy). Because of this approximate linearity, $\Delta y/\Delta x$ tends to a limit as Δx (and Δy) goes to zero, and this value is the instantaneous rate of change of the relationship at the given point, the derivative of y with respect to x. It indicates how y, the vertical distance of the moving object, is changing with respect to changes in x, the horizontal distance, at the given point. In Figure 2, assume that the derivative at the point (a, b) is 2.2. Then, at that point, the change in the vertical distance of the object is 2.2 times as much as the change in its horizontal distance. Therefore, the vertical speed of the point is 2.2 times its horizontal speed at that instant. When the point's horizontal distance changes by a sufficiently small amount from a, its vertical change in distance from b is approximately 2.2 times as much. So, for all changes Δx that are sufficiently small, when the horizontal distance of the object is $a + \Delta x$, the vertical distance of the object is approximately $b + 2.2 \bullet \Delta x$.

Discussion

Our delineation of two distinct approaches to coordinating quantitative meanings for rates, proportional relationships, and variation has implications for both research and instruction. With respect to the research, current quantitative approaches to rates have sought to combine reasoning with unit rates and multiple batches with smooth covariation. As one example, Tallman et al. (2024) reported a teaching experiment with one preservice teacher, Samantha, using a series of constant speed tasks involving a character named Clara. A main goal of the teaching experiment was for Samantha to develop an understanding of constant speed where she would anticipate that "any accumulated distance is some multiple *k* (where $k \in R+$) of the distance Clara travels in one unit of time, and that it therefore takes Clara that same multiple of one time unit to travel that distance" (p. 12). Samantha was able to solve the tasks appropriately by partitioning distances and times in parallel using chunky reasoning. At the same time, the authors argued that Samantha's reliance on chunky variation constrained her reasoning, and they hypothesized that smooth covariational reasoning is essential for constructing rates as reflectively abstracted constant ratios (p. 22). From our point of view, the constraints in Samantha's reasoning may have been an artifact of researchers presenting situations for which it was appropriate to partition and iterate two quantities in *parallel* with an expectation that such reasoning would support a different multiplicative relationship *between* those quantities. We understand this design to be based on a mix of the two approaches to rates and covariation that we have identified and explicated.

With respect to instruction, our analysis suggests that students might benefit from reasoning about rates with pairs of like quantities. We introduced measurement rates using volumes of paint. Further examples in the literature suggest students may be able to generate measurement rates reasoning spontaneously. As one example, Moore (2014) reported a teaching experiment during which Zac developed a preference for using the radius of a circle as a unit of measure and came to reason productively about multiplicative relationships between the length of the radius and arc lengths and sides of right triangles. Moore did not highlight distinct meanings for rates; but, from our point of view, Zac reasoned about measurement rates within a variable-parts perspective. As a second example, Yu (2023) reported a short intervention on linear motion designed "to move students from interpreting a speed of 1.7 m/s as 'for every additional second the person moves an additional 1.7m' into a meaning as 'for any amount of time that passes, *x*, the person moves 1.7*x* cm'" (p. 966). During the unit, college calculus students worked with a dynamic computer environment that allowed them to change horizontal and vertical lengths smoothly and to consider "how many times as large the change in the output is (the length of the vertical segment) with respect to the change in the input (length of the horizontal red segment)" (p. 972). Thus, from our perspective, Yu combined measurement rates with smooth covariation.

We hope to inspire further consideration of new possibilities for coordinating quantitative understandings of rates, proportional relationships, and covariation. Past research has focused on unit rates, multiple batches, and variation of measure, but measurement rates, variable parts, and unmeasured variation have characteristics that scholars have repeatedly described as important for calculus, including derivatives. Thus, we see further investigation into students' reasoning with measurement rates as an important direction for future research.

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