Parsing Students' Decision-Making During Analogical Reasoning: A Multi-Constraint Approach

Michael D. Hicks
Middle Tennessee State University

General theories of analogy dictate that connections can be made between virtually any two domains, thus suggesting an extremely large number of analogies that one might observe. Unsurprisingly, some analogies are of course more useful than others. How then do students come to observe "useful" analogies? Multi-constraint theory is one such avenue for exploring questions such as these. In this contributed report, I adopt a multi-constraint approach to parsing students' analogical reasoning with the goal of uncovering how certain choices are made when developing ring-theoretic analogies to group-theoretic structures. Analysis revealed that the participants exhibited variations in emphasis of analogical constraints during their reasoning in three ways: (a) adhering to basic structure, (b) adhering to conceptual structure, and (c) flouting constraints to advance exploration. Implications for the teaching and learning of abstract algebra and curriculum development are discussed.

Keywords: Abstract algebra, Analogy, Multi-constraint theory, Similarity-based reasoning

What are all the possible similarities and differences might students observe between a group and a ring? One student might focus on simple aspects of their definition, such as the number of operations each structure possesses. Another might focus on deeper features of the objects, such as recognizing both as being general examples of abstract structures that replace the need to focus on specific objects when writing proofs. Yet, it might be strange to hear a student suggest that groups and rings are similar because they are both spelled with letters from the English alphabet, or that they are different because one describes a collection of things, while the other describes an object you might wear on your finger. While such connections may not be considered all that valuable from an educational perspective, some would still count them as examples of analogies. In fact, highly general theories of analogy dictate that connections can be made between any two domains, thus suggesting an extremely large number of analogies that one might observe between any two situations (e.g., Faulkenhainer, Forbus & Gentner, 1983). A question then arises as to why some connections are more commonly observed over others.

To begin answering questions such as these in the context of mathematical thinking, I adopt a prominent theory of analogical reasoning that delineates types of constraints that occur naturally when reasoning by analogy. In this contributed report, I present an analysis of four students' thinking during their construction of (and reflection upon) ring-theoretic analogues to group-theoretic structures. Unlike previous investigations that focused more-so on *what* students created (e.g., Hejný, 2002; Hicks, 2024; Stehlíková & Jirotková, 2002), this report focuses on exploring underlying reasons for *how* certain choices were made over others when reasoning by analogy (e.g., Lee & Sriraman, 2011)

Theoretical Background: Multi-Constraint Theory

The approach to parsing students' thinking presented in this study relies on a theory of analogical reasoning known as *multi-constraint theory* (Holyoak & Thagard, 1989). This theory posits that the development of analogies is inherently guided by sets of constraints that serve as the foundation for coherent analogy formation. While it can be argued that analogies might be generated between any two given situations, it is rarely the case that a person randomly generates analogies between any two situations as they go about their day-to-day activities. Instead, individuals tend to select particular mappings when generating an analogy. Multi-constraint theory takes as its premise the idea that adherence to a set of constraints leads individuals to develop analogies that satisfy their personal sense of rationality, thus greatly reducing the overall number of analogies that might be created to a smaller subset of "reasonable" analogies as opposed to creating analogies at random. I describe three broad categories of constraints below, all of which serve different purposes in dictating the construction of analogy from an individual's perspective.

Three Types of Constraints

The first constraint is known as *structural consistency*. This constraint suggests that in the course of analogy formation, there is an implicit desire to create, to as great of an extent as possible, a one-to-one mapping between the source and target domains. For example, consider possible analogical comparisons the following statements: (1) *Oliver is taller than Paul*, and (2) *the Sun is more massive than the Earth*. Each of these statements successfully establishes a relation of order between two objects, displaying that there is consistency of a basic structure between the two situations. As such, there is a clear structural mapping between these two domains through a form of proportional reasoning (e.g., Modestou & Gagatsis, 2010).

The second category of constraint is known as *semantic similarity*, which focuses less on the abstract structure that may exist between two domains, and more so on the similarity of the underlying appearance or meanings associated with the two domains. In the previous example comparing the *taller than* relation to the *more massive than* relation, you might notice that the two situations are quite irrelevant to one another despite possessing a common basic structure. Thus, the two statements are low in semantic similarity. In contrast, consider the analogy that underlies the pair of statements (1) *Oliver is taller than Paul* and (2) *William is older than Tim*. In this case, not only is basic structure preserved, but there is also a high semantic similarity. Both situations define relations between people, focusing on properties that are often used to compare and describe people in every-day life. One interesting observation about semantic similarity is that, while it may serve as a constraint to assist with developing analogies in general, it is often argued that a high degree of semantic similarity is less appealing when creating "good" analogies. In fact, preference is often given to analogies that determine similarity between two highly disparate situations that share no surface level similarities (e.g., Gentner, 1983). This concept is ubiquitous to pure mathematical thinking as well, given that isomorphisms only require abstract structure to be similar without any reference to whether the objects themselves within the structures are similar or not (e.g., integers under addition are isomorphic to the free group on one element, yet there is no specific similarity to the elements contained in each group.)

The final constraint is *pragmatic centrality*. This crucial constraint dictates that even when analogies might possess high levels of structural consistency and semantic similarity, an underlying pragmatic purpose must be present for the analogy to be observed. Out of all possible analogies that might be generated, pragmatic centrality serves as the constraint that ensures that analogies are created with purpose. Indeed, pragmatic centrality extends beyond just the analogical creation itself; the constraint of pragmatic centrality is visible when students question the purpose of their choices and reflect upon their overall goal, as well as consider the consequences of their choices. For a summary of the three constraints, see Table 1.

Table 1. Types of Analogical Constraint

Structural Consistency	Adhering to the constraint of purposely generating one-to-one mappings between structures.
Semantic Similarity	Adhering to the constraint of purposely generating analogies between domains with observed similar meanings.
Pragmatic Centrality	Acknowledging underlying purpose of the analogy, either before, during, or after creation.

In this study, I aim to answer the following research question: *In what ways did four students adhere to the three constraints of structural consistency, semantic similarity, and pragmatic centrality while reasoning about ring-theoretic structures by analogy with group-theoretic structures?*

Methods

The data for this study was collected as part of a larger project investigating students' analogical reasoning in abstract algebra. In the Fall of 2019 and the Spring of 2020, rosters were acquired from the abstract algebra courses taught at a large 4-year research university in the southwestern United States. Four students, named Andrew, Brandon, Ellen and Nathan (all pseudonyms) were solicited to be interviewed and participate in a series of 5 clinical task-based interviews (Goldin, 2012). During these interviews, a single camera was positioned toward the student's written work and a lapel mic was given to the student for audio recording. The first interview served to allow students to recall various aspects of group theory and associated concepts and definitions, while the subsequent interviews focused on exploring concepts in ring theory.

The final three interviews all corresponded to the construction of a different ring-theoretic analogy to group-theoretic structures. There were two pertinent goals of these interviews. First was to generate a ring-theoretic analogue to subgroup, homomorphism, and quotient group based on what they knew from group theory by openly responding to the task: *Make a conjecture for a structure in ring theory analogous to _________ in group theory*. Second was to inquire into properties, examples and theorems associated with their newly generated ring-theoretic analogue. During the interviews, the students' understanding of structure was revealed based on what features they attended to while developing their ring-theoretic analogue and was also revealed based on what properties and theorems they decided to investigate following the completion of their analogy.

To analyze the students' interviews, transcripts of the audio were coded alongside the video with respect to the three constraint categories. In particular, the statements made by the students in addition to their written work were scrutinized for any evidence of attention to structural or semantic similarity, as well as any attention to an underlying purpose for their utterances/writing, or considerations of consequences for their construction in the moment (pragmatic centrality). Such utterances were then assigned a code based on the adherence to one or more of the constraints along with descriptions of how the constraint influenced their analogical creation. Following this process, I then identified themes delineating the extent to which each constraint was present across the students' thinking elicited in each interview.

Results

The extent to which each student adhered to the constraints provided insight into their sensitivity to recognizing underlying concepts associated with both group- and ring-theoretic structures, thus reaffirming previous findings related to their sense of analogical structure (Hicks & Flanagan, 2024). But, by attending to the presence of the three constraints in the participants' thinking, a window into their underlying rationale for developing their ring-theoretic analogues was opened, providing further insight into the students' reasons for establishing their analogies. In this section, I present three themes that were identified as possessing particular significance to these constructions: (1) adhering to basic structure, (2) adhering to conceptual structure, and (3) flouting constraints to advance exploration (see Table 2).

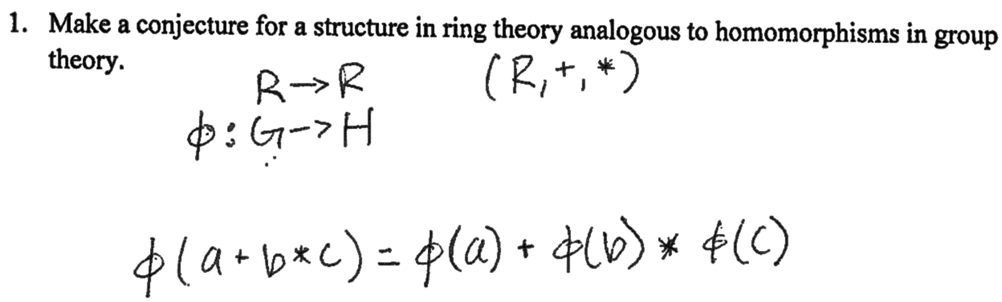
Table 2. Observed Themes

Adhering to Basic Structure	Exhibiting adherence to structural consistency and semantic similarity of surface level concepts, such as naming conventions, written symbols, etc.
Adhering to Conceptual Structure	Exhibiting adherence to structural consistency and semantic similarity of deeper concepts, such as mapping the concept of structure-preserving maps from the domain of group theory to the domain of ring theory.
Flouting Constraints	Ignoring one constraint in an effort to freely investigate properties or aspects of an analogous structure.

Adherence to Basic Structure

Perhaps unsurprisingly, most students adhered to a high level of semantic similarity when generating a name for their ring-theoretic analogy for most structures. For example, each student provided the names "subring" and "ring homomorphism" to describe their newly created analogy in comparison to subgroups and group homomorphisms. Across all structures, the students appeared to be strongly adhering to both constraints of structural consistency and semantic similarity, often with little reference to any sort of underlying purpose to the name. For example, when constructing the ring-theoretic analogue to quotient groups, Nathan introduced the concept of a "normal subring" without any suggestion as to what "normal" might mean in the context of ring theory. I refer to the presence of both structural consistency and semantic similarity of surface level concepts as *adherence to basic structure*.

A notable example of adhering to basic structure occurred when Ellen developed her definition for a ring-theoretic analogue to homomorphism. Despite recognizing differences in the structure of group and ring in general, Ellen made an effort to maintain structural consistency and semantic similarity when developing her analogue for homomorphism by combining the homomorphism property into one statement (see Figure 1.) This allowed Ellen to establish a one-to-one relation between the two homomorphism definitions (i.e., each required only one function and one homomorphism property), while also maintaining a similar appearance for both. However, Ellen did not immediately question the underlying meaning of her newly formed homomorphism property that combined the operations into one expression, thus suggesting that pragmatic centrality was absent from her initial creation.



1. Make a conjecture for a structure in ring theory analogous to homomorphisms in group theory.

$R \to R$ $(R, +, *)$

$\phi: G \to H$

$\phi(a + b * c) = \phi(a) + \phi(b) * \phi(c)$

Figure 1. Ellen's initial written response when developing a "ring homomorphism".

Adherence to Conceptual Structure

The presence of pragmatic centrality at the outset of the generation of an analogical structure influenced the manner by which the structure was initiated. In this data, the presence of pragmatic centrality when initiating construction was often indicative of a strong adherence to both structural and semantic similarity, but with significantly different emphasis on *where* the attention was when developing the structure. For example, Ellen attended heavily to the group homomorphism definition when constructing her ring-theoretic analogue, and thus she relied heavily on written, symbolic similarity. In contrast, Andrew adopted a different approach at the beginning. He first established that there would be a ring R with operations + and *, and asserted that a second ring S should exist. However, he immediately began to question what the operations on S would look like:

> *Andrew*: But these (pointing to ring S) don't have to have the same operations, and I don't know what to use.
>
> *Interviewer*: Can you explain what you mean by that?
>
> *Andrew*: Well, when we have groups, we can something like this: $\phi(a \cdot b) = \phi(a) * \phi(b)$. If this homomorphism is from G to H, then the dot is the operation on G, and the star is the operation on H. So maybe I can use a dot here (*writing the multiplicative operation for S*), and I can just use a minus for… Yeah, I don't know what to use, unless you have a suggestion.

From this, we can see that Andrew's initial construction of the homomorphism property was guided in part by the constraint of pragmatic centrality; even before writing anything, he was already reflecting on the consequences of working with multiple operations in the context of ring theory.

Compared to Ellen's construction, it would appear that Andrew was perhaps less concerned with structural consistency given that Andrew's definition involved two homomorphism properties rather than one. In terms of establishing structural consistency between the two definitions, Ellen successfully mapped one homomorphism property for groups to one homomorphism property for rings. However, I claim that Andrew still maintained a high level of both structural consistency and semantic similarity, but at a different level than Ellen had previously done. This was present in the following interaction:

> *Interviewer*: Could you tell me what your overall thought process was in making this definition?

Andrew: Yeah, so I figured you need to preserve operations. That's basically how a homomorphism is defined, it's a function that preserves operations. So, if that's how it's defined I just have to preserve operations, but with two different operations now.

Thus, while Andrew's adherence to a written, symbolic structural consistency was not as strong as Ellen's, Andrew displayed an adherence to structural consistency and semantic similarity of a more conceptual structure. Andrew successfully encapsulated the concept of *preserving operations* into one object to be mapped between domains, thus achieving a greater degree of a one-to-one mapping despite having to break structural consistency at the basic level. I refer to this phenomenon as *adherence to conceptual structure.*

Flouting Constraints to Advance Exploration

Adherence to one of basic or conceptual structure was not necessarily the norm for all participants and wasn't even standard for any one particular student across all of their created structures. Indeed, a third phenomenon that occurred was that one of structural consistency or semantic similarity would be flouted in favor of the other, whether at the basic or conceptual level. This often occurred when a student began to question their creation in order to begin exploring alternatives. For example, Nathan initially began describing his concept for ring homomorphism by adhering strongly to symbolic semantic similarity. Unlike Andrew, Nathan's construction was tied to ensuring a similar basic appearance to that of group homomorphism with the added caveat that there were two operations defined on rings. As such, Nathan initially flouted structural consistency in order to accommodate the presence of two operations.

However, after developing a definition for a ring-theoretic analogue to homomorphism (see Figure 2), Nathan began to question his choice. He mused:

> Writing it like this makes sense, because that's how the group is written. But, if I wanted to write two functions, I feel like you would have to separate it, like you would be like, psi maps (R,+) to (S,+'), and then zeta or something maps (R,*) to (S,*')… Now there's two of them.

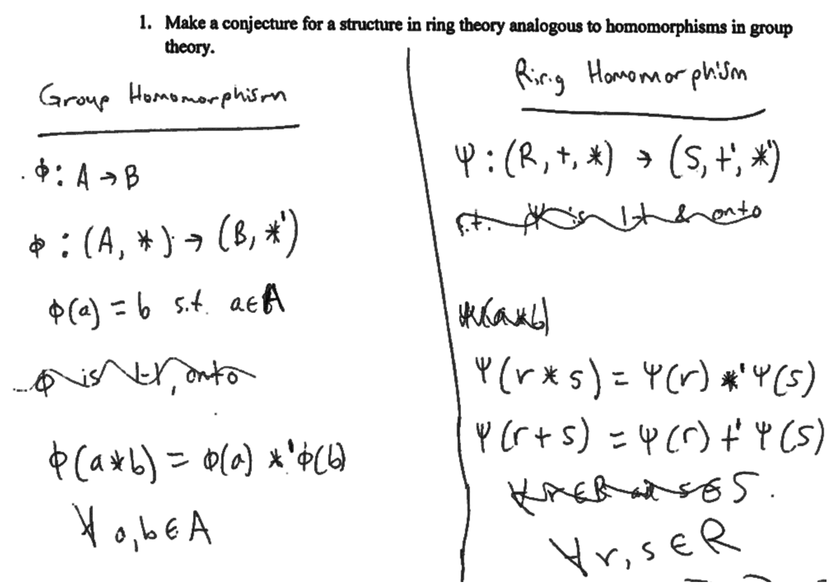


Figure 2. Nathan's written response when developing a "ring homomorphism".

Here, we see that Nathan is now wondering whether there should be two *functions* rather than two homomorphism properties defined on the same function. This investigation led Nathan to ponder whether he could describe both functions in terms of groups (see Figure 3.) In doing so, Nathan purposefully continued to flout structural consistency with the goal of accommodating the presence of two operations for rings.

Unlike Andrew, Nathan never appeared to recognize any one-to-one mapping between the underlying concepts of homomorphism properties for groups and rings during his investigation. Instead, Nathan focused primarily on adhering to a basic semantic similarity, albeit in two forms: (1) establishing similar homomorphism properties for groups and rings, and (2) establishing similar functions defined on the groups and rings where only one operation was mapped at a time.

$$\psi : (R, +) \to (S, +')$$
$$\zeta : (R, *) \to (S, *')$$

Figure 3. Nathan's proposal to define two functions rather than two homomorphism properties.

Discussion

The results of this analysis affirm a subset of previous findings related to parsing students' analogical reasoning. In particular, multi-constraint theory provided another method for observing that students can engage in a creative process of analogical reasoning in order to actively *invent* new (to them) concepts in abstract algebra (see Hicks, 2023). However, the use of multi-constraint theory to parse students' analogical reasoning revealed new perspectives on understanding *why* students may make certain decisions. Of particular significance is the observation that students may flout constraints in an effort to advance their construction or exploration of a new analogy, and that a stronger adherence to pragmatic centrality may develop over the course of an investigation. This supports the view of analogy as a problem-solving process to sometimes be purposefully implemented (e.g., Carbonell, 1983), rather than only a spontaneous process that occurs only by chance.

I end this paper by offering two suggestions stemming from this analysis that may be beneficial for successfully using analogy in mathematics classrooms. First, teachers may benefit from observing the distinction between adhering to basic and conceptual structures and understanding the value of each. Indeed, while it may appear that adhering to conceptual structure is always the ideal goal, adherence to basic structure also resulted in rich initial investigations and allowed the participants in this study a foothold for furthering their constructions. In contrast, many theories of analogical reasoning devalue those analogies that might adhere too strongly to basic structure (e.g., see Gentner's (1983) concept of *literal similarity*), and typical applications of multi-constraint theory investigated how the constraints supported individuals to avoid creating "bad" analogies.

Second, teachers should encourage their students to push on and question their analogical constructions, even when a correct analogy is given from the beginning. As was the case with Nathan's construction of ring homomorphism, the constraint of pragmatic centrality was not initially present. However, once he began to question his construction, he was able to explore alternatives and consider the deeper meaning of what a ring homomorphism is.

References

Carbonell, J. G. (1983). Learning by analogy: Formulating and generalizing plans from past experience. In *Machine Learning* (pp. 137-161).

Falkenhainer, B., Forbus, K. D., & Gentner, D. (1989). The structure-mapping engine: Algorithm and examples. Artificial intelligence, 41(1), 1-63.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, *7*(2), 155-170.

Goldin, G. A. (2012). A scientific perspective on structured, task-based interviews in mathematics education research. In *Handbook of Research Design in Mathematics and Science Education* (pp. 517-545). Routledge.

Hejný, M. (2002). Creating mathematical structure. *European Research in Mathematics Education II, Charles University Prague*, *2002*, 14-24.

Hicks, M. D. (2024). Encouraging mathematical explorations through reasoning by analogy in abstract algebra. *PRIMUS*, *34*(2), 152-167.

Hicks, M. D. (2024). "I'll just try to mimic that": an exploration of students' analogical structure creation in abstract algebra. *Educational Studies in Mathematics*, *117*(2), 303-327.

Hicks, M. D., & Flanagan, K. (2024). Analogical structure sense: A case study of students' analogical reasoning between groups and rings. *The Journal of Mathematical Behavior*, *73*, 101-136.

Holyoak, K. J., & Thagard, P. (1989). Analogical mapping by constraint satisfaction. *Cognitive Science*, *13*(3), 295-355.

Lee, K. H., & Sriraman, B. (2011). Conjecturing via reconceived classical analogy. *Educational Studies in Mathematics*, *76*(2), 123-140.

Modestou, M., & Gagatsis, A. (2010). Cognitive and metacognitive aspects of proportional reasoning. *Mathematical Thinking and Learning*, *12*(1), 36-53.

Stehlíková, N., & Jirotková, D. (2002). Building a finite algebraic structure. *European Research in Mathematics Education–Proceedings of CERME2, Prague, PedF UK*, *1*, 101-111.