"I" Find it Difficult Because...: Identifying What Features Students Report as Affecting Difficulty to Understand a Proof

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The student perspective provides valuable information into what students find to be difficult when reading proof. Such information can be used to productively support students as they face these difficulties. In this preliminary report, we investigate what students report as contributing factors of how difficult a proof is to understand. For this report, we focus on data regarding two proofs from the introduction to proof setting. Our findings indicate that the density and width of a proof are key contributors towards the difficulty rating given to the proofs.

Keywords: Reading proof, student perspective, difficulty

Scholars have identified various ways students encounter difficulties with proof (Moore, 1994; Selden & Selden, 2008), typically focusing on students constructing or validating proofs. Meanwhile, understanding a written mathematical proof is also a difficult task. Proofs are full of challenging features, such as the level of abstractness (Rowland, 2001; Weber, 2004) and various writing norms and conventions (Dawkins & Weber, 2014; Lew & Mejia-Ramos, 2019; Melhuish et al., 2025). These norms and conventions include minimizing unnecessary warrants (Dawkins & Weber, 2017, Selden & Selden, 2003), excluding references to people (Dawkins & Weber, 2017, Melhuish et al., 2025), and following the rules of academic language writing (Lew & Mejia-Ramos, 2019). Overall, scholars have identified that students have difficulty with the abstract nature of proof (Alcock, 2010; Weber, 2004), with the complexity of proof (Alcock, 2010), in understanding the mathematical content (Moore, 1994; Selden & Selden, 2008), and having to make implicit warrants and information explicit (Samkoff & Weber, 2015). All of which is taken from an expert's perspective. Though, one may wonder what do students find to be difficult in understanding proof and what impacts this perception of difficulty?

Few existing studies investigate students' perspectives in the reading and understanding of proof (Guajardo & Lew, 2024; Guajardo, Lew, & Melhuish, 2025), particularly regarding what students find difficult in understanding proof (Selden & Selden, 2008). We argue that knowledge of students' perspectives on proof and proving can provide valuable insight into both the teaching and learning of mathematical proof. Namely with students' perspectives, researchers can explore student interactions with proof with greater depth. Additionally, instructors can leverage findings about student perspectives to better understand what students take from class and how to adapt instruction to build on to students' perspectives and existing knowledge base.

In this paper, we offer some preliminary insight into what students find to be difficult in understanding proof. The study presented is part of a larger dissertation study, focused on understanding students' perspectives about various aspects of proof. Here, we present a subset of the data to answer the research question: What do students indicate as factors that contribute towards a higher rating of difficulty to understand a proof?

Theoretical Framing

To situate our study, we conceptualize our thinking of abstraction and complexity of proofs. For abstraction, we take on Hazzan and Zazkis' (2005) three levels of abstraction:

> (a) abstraction level as the quality of the relationships between the object of thought and the thinking person, (b) abstraction level as reflection of the process–object duality, and (c) abstraction level as the degree of complexity of the concept of thought. (p. 103)

In using these levels of abstraction, we consider abstraction to be an individual's perception of the separation between the concept and a concrete model. Separations may be perceived as larger due to the student having less experience with the concept (first level), having a process conception (second level; Dubinsky, 1991), or in dealing with a compounded concept (third degree). With this definition, we can identify how participants view the abstract nature of the concepts throughout the study by considering how they talk about their experiences with the concept, the utility of the concept, and any compounded nature of the concepts.

In terms of complexity, we refer to complexity here differently than Hazzan and Zazkis' third level of abstraction above. For complexity, we use Hanna's (2014) discussion of the width of a proof, "the number of distinct ideas one has to keep in mind" (p. 31). We expand this idea by including the number of moving pieces in a proof one must keep track of (high cognitive load) rather than just the memorization of key ideas or crucial steps to construct a proof.

Methods

Eight undergraduate students, who completed an introduction to proof (ITP) course the semester prior, participated in this study. Table 1 provides participant demographic information.

Table 1. Participant Information

Participant	Race/Ethnicity	Gender	Grade Received in ITP
Adrian	White or Caucasian	Non-binary	B
Claire	White or Caucasian	Female	B
Erik	White or Caucasian	Transman	A
KTRM	Latin/Hispanic	Female	C
Marie	Latin/Hispanic	Female	A
Olivia	Mixed Race	Female	B
Tifanni	Latin/Hispanic	Female	A
Zeus	Latin/Hispanic	Male	B

Participants met with the first author for seven semi-structured interviews. Four of the interviews were task-based interviews where participants read a provided proof (GCD, Partition, Function, and Cardinality) and attempted to understand it to the best of their ability. Participants then answered a series of questions, some of which asked how difficult the proof was. For this analysis, we consider participants' responses to the following questions from the task-based interviews: (a) How well do you feel like you understand this proof?, (b) If you could [rate] this proof in difficulty from 1-10 (ten being very difficult, 1 being not difficult at all), how would you [rate] this proof?, and (c) If given the chance to rewrite the proof, would you change anything about it? We also consider data from the exit interview which included discussion on what participants find to be difficult with understanding proof and reflections on the ratings they gave the proofs throughout the study. For this report, we focus on the two proofs rated most difficult to understand: the Cardinality and the Partition proofs (see Table 2 for the theorem statements).

Table 2. Theorem statements for Cardinality and Partition proofs

Proof	Theorem
Cardinality	The sets $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality.
Partition	Suppose R is an equivalence relation on a set A. Then the set $\{[a] \mid a \in A\}$ of equivalence classes of R forms a partition of A.

To determine the two highest rated proofs, we conducted a points allocation method of ranking based on participants individual rating of each proof: 4 points was awarded to the highest rated proof (most difficult) and 1 point was awarded to the lowest rated proof (least difficult). The Cardinality proof was rated highest by the participants, and the Partition proof was rated second highest. Deductive coding was applied to data from the relevant task-based and exit interviews. The data relevant to the Cardinality proof were coded independently by both authors. Coding was discussed to resolve any discrepancies and reach agreement. Data relevant to the Partition proof was then split equally amongst both authors and were independently coded.

Six codes related to features of proof identified through the larger dissertation study (Guajardo, 2025) were used: abstractness, density, length, width, and writing. Using our definition of abstraction from Hazzan and Zazkis (2005), abstractness was coded when participants explicitly voiced how abstract the proof was or when they discussed visualizing the content in a more concrete way. Density focuses on participants discussing the amount of information hidden from the reader, to which they must deduce on their own. Length was coded when participants referenced the length of the proof as contributing towards the difficulty. For width, we used our definition that was adapted from Hanna's (2014) discussion of width to code when participants were stating difficulty due to having to keep multiple distinct ideas in mind as they were reading. Lastly, the writing code refers to participants indicating the language, notation, or structure of the proof caused difficulties.

Results

Participants considered features of proofs that contributed to the difficulty (or ease) in understanding a given proof. For example, a feature of a proof could be the proof method used. Table 3 shows the features we identified for each participant as contributing towards the difficulty of the proof. When discussing the features of the proofs that affected their difficulty ratings, participants did so independently of other proofs (bolded in Table 3; typically during the task-based interview) and comparatively between proofs (standard font in Table 3; typically during the exit interview). When talking about the features of proof independently of other proofs, participants expressed abstractness, density, length, width, and writing as playing a role in the difficulty rating of the two proofs. While nearly every feature was mentioned for each proof (Table 3), we focus on the density and width features due to length restrictions. Participants primarily expressed density and width as key factors for the difficulty of these two proofs – both in their discussions of the proof individually and their comparisons across proofs.

Table 3. Features discussed by Participants. Bold font indicates feature was discussed individually. Standard font indicates feature was discussed only comparatively. Symbols +, - indicate if that feature was comparatively reported to increase or decrease the difficulty (respectively).

Proof	Features of Proof					
	Abstractness	Density	Length	Line Structure	Width	Writing
Cardinality	**Erik** **Zeus -** + Adrian + Claire +	**Adrian** **Claire** + **Marie** + **Olivia** **Tifanni**		KTRM +	**Adrian** + **Erik +** **KTRM +** **Marie +** **Zeus** Claire +	**Marie** Adrian -
Partition	**Adrian** **Zeus** + Olivia -	**Adrian** **Claire** **Olivia** **Tifanni -** **Zeus** +	Erik -	**Olivia** KTRM -	**Claire** KTRM - Marie - +	**Adrian** **Claire**

For the cardinality proof, most participants indicated density and width played a role in their reasoning for why the proof was difficult to understand. For example, with density participants difficulties with the proof were due to having to unpack a lot of information. Claire explained her rating the proof as having a 4 out of 10 difficulty because the "super specific things of like *why* f, OK, I don't really know [emphasis added]." In this quote, Claire is focusing on the line-by-line arguments made in the proof and unpacking why these arguments are being made. For width, participants expressed having to handle a large quantity of information. Marie expressed the proof was not easy to understand because the proof "gives us like 4 variables in total. The more variables there are, the harder it is to mentally keep track. So even though it's in front of you, that can just be a little harder to keep up with." Here, Marie is having difficulty keeping up with the information presented in the proof. All other features were discussed by at most two participants.

For the partition proof, most participants indicated that density played a role in their reasoning for why the proof was difficult to understand. Similar to their comments regarding the Cardinality proof, participants expressed having to unpack or link information. Olivia explained that she gave the Partition a rating of 9 (and not lower) because she does not "understanding why they're saying what they're saying. Like, at least, when I'm trying to prove an if-then I know what it's supposed to look like. […] with this I'm like, what are we even trying to do here?" Similar to Claire above, Olivia is focusing on having to unpack a lot of information in the proof. As with the cardinality proof, all other features were discussed by at most two participants. This indicates that density and width play a larger role in how participants discuss the difficulty of these proofs.

When looking at how participants compared between proofs, the width of the Cardinality proof was used as justification for a higher difficulty rating than other proofs. In comparing the Cardinality proof to another proof in the study, Adrian explained the Cardinality proof was harder because "it has a lot more stuff […] it's got these different functions and all these different sets that this other one doesn't have." We attribute Adrian's discussion of "a lot more stuff" to having to keep up with more concepts, increasing the width of the proof. Most other features were not mentioned by many of the participants for either proof in the exit interviews. When they were discussed regarding the Cardinality proof, the features were almost exclusively used to explain why the proof was more difficult to understand than others. Participants did not frequently compare the Partition proof to others and when they did, most participants expressed which features made the proof easier to understand than the others. The participants commonly compared the Partition proof to the Cardinality proof as they were typically rated similarly (either hardest to understand or second hardest to understand).

Discussion and Next Steps

In this paper, we discussed the different ways in which undergraduate students rate proofs in difficulty to understand. Participants expressed various features of the proofs as reasons for their ratings. In our data, the features density and width play a critical role in determining how difficult it is to understand these two proofs. This supports the existing literature that suggests that students have difficulty with unpacking information (Samkoff & Weber, 2015) and have trouble with the amount of information that can be present in a proof (Alcock, 2010), but more importantly contributes student voices to the discussion. In particular, students may use these features to compare between proofs: considering which proofs are denser or wider to determine the difficulty of understanding a proof.

Overall, these preliminary findings reveal the complexities of how difficult a proof is to understand. Additionally, it is important to recognize that these features of proof identified by the participants are inherent in proof and cannot easily be hidden or removed. Rather, it is important for researchers and instructors to work toward better supporting their students. One way this may be achieved is by helping students use strategies to overcome these difficulties caused by the features of proof. Guajardo et al. (2025) describe proof comprehension strategies most used by undergraduate students as they attempted to understand proofs. This repertoire of comprehension strategies, paired with our findings on students' perspectives on difficulties, can be used to help students mitigate the difficulties caused by the features and better comprehend the proofs. More research is needed to better understand how to support students in mitigating these difficulties.

Next steps for this study are to continue analysis of the proofs and difficulties identified by participants. Important factors for these participants were not only features as discussed above, but also non-features (aspects of the reader that impact the difficulty rating of the proofs). Two codes were identified through the data that represent non-features; effort to understand and current understanding. With effort to understand, participants reflected on their actions they did or the actions it would take to understand the proof. For their current understanding, participants reflected on how they viewed their own skills to understand (self-efficacy) or their understanding of the content of the proof. From a brief initial analysis, a non-feature that participants find to be a key factor in the ratings give not the proofs is their own understanding of mathematical concepts and arguments in the proofs.

Preliminary Nature

Analysis is still ongoing and theorical framing is still being developed to help situate the discussion of the results. As we discussed above, there are more factors to the difficulty attributed to a proof than just features of the proofs themselves. We welcome any comments and suggestions in moving forward with the analysis of non-features or in any theoretical components that could be productive to consider for our study. In particular, are there any socio-cultural or cognitive theories that could help in analyzing and situating the non-features?

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