"…it's more of a healthy skepticism…": Descriptions of Undergraduate Students' Mathematical Convictions and Proclivities

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This study used a survey containing two open-ended questions and a ranked choice question. I investigated undergraduate students' mathematical convictions relating to whether they experienced a lack of conviction. Students were asked about their proclivities when an instructor presents a proof to the class. Preliminary findings show that few students experienced a lack of conviction with either theorems or proofs. For theorems, a lack of understanding was often cited. For proofs, students cited doubts about the rigor of those proof types such as picture proofs. Students' proclivities seemed to show that their focus was on understanding a proof's content. Further evidence provided by follow-up interviews corroborated this interpretation and that students may in fact reflect on matters of conviction in mathematical settings.

Keywords: Mathematical conviction, mathematical proof

Mathematical convictions are described as the ways in which one is persuaded about the truth of a theorem given a mathematical argument or statement. Nebulous concepts in mathematics education, like mathematical convictions (Brown, 2018), continue to challenge researchers in the ways we investigate them. This study contributes to the growing field of knowledge around mathematical convictions by investigating undergraduate students' mathematical convictions and proclivities. The purpose of this study was to determine whether and how students consciously operate in a way that foregrounds mathematical convictions. I also investigated the roles that theorems, proofs, and instructors have in matters of students' convictions.

Background Literature

I note that mathematical proofs are often defined as "…valid arguments that establish the truth of a mathematical statement. By an argument, we mean a sequence of statements that [begin with an assumption and] end with a conclusion" (Rosen, 2012, p. 73). Indeed, many textbooks describe mathematical proofs in this way which often do not mention matters of conviction. So, this leads to questions regarding how and why students may be convinced.

Aspects or characteristics that students often cite as contributing to whether they are convinced by a mathematical proof or argument are familiarity with the content and method (Brown, 2018; Chazan, 1993; Healy & Hoyles, 2000), example-use (Healy & Hoyles, 2000; Weber et al., 2022), and whether one understands that proof (Brown, 2018; Selden & Selden, 2003). A contentious aspect reported in the body of research relating to mathematical convictions is that of authority. For undergraduate students, this may be observed by students referencing that the instructor said that the mathematical statement was true or the mathematical proof was valid (Brown, 2014). For mathematicians and educators, a classical perspective is that students are convinced that a theorem is true if and only if they have a mathematical proof. However, prior research described students as being too difficult to convince (Fischbein, 1982) or too easily convinced by mathematical objects that were not proofs (Harel & Sowder, 1998). Interestingly, Weber et al. (2022) described mathematicians' behaviors like students rather than the typical mathematician behavior described in previous research.

Methods

I used a survey intended to capture characteristics of undergraduate students' mathematical convictions and their proclivities as they engage with mathematics. In this preliminary report, I report parts of that survey and the follow-up interviews pertaining to those parts of the survey.

I contacted 231 instructors across the U.S. teaching undergraduate algebra or analysis courses in Fall 2024. Fourteen (6%) agreed to use the survey in their course(s). Two instructors implemented the survey as an assignment, four implemented it as extra credit, and nine forwarded the survey to their students to complete voluntarily. There was a total of 212 students enrolled in these instructors' course sections ($M = 15.14$, $SD = 8.72$). There were 95 students (45%) who responded to the survey. A student's response was removed from analysis if: 1) the duration to complete the survey was less than five minutes, 2) the student was a graduate student, 3) the student did not consent to the study, or 4) the student did not sign the consent form. After this, there were 71 responses (33%) remaining. The average time to complete the survey was 33 minutes while the median time to complete the survey was 11 minutes.

Research Instruments

There were two open-ended questions. The first asked, "was there ever an instance where your professor presented a theorem, and you were not convinced the theorem was true?" If the student responded with "yes," the question continued by asking for further details such as the course in which it occurred, the theorem itself, and why he or she was unconvinced. The second open-ended question asked, "is there a type of proof that you find unconvincing?" If the student responds with "yes," the question continued by asking for further details like type of proof, examples of when it was used, and why he or she was unconvinced by this type of proof.

A ranked choice question had students indicate behaviors they engaged in and their importance during a scenario. These behaviors were generated from assumptions about students' proclivities and prior research regarding students' mathematical epistemologies as it relates to standard practices in the classroom. The prompt was, "when my instructor presents a mathematical proof to me or the class, I personally focus on…." They then rank the following six behaviors on whether they engage in them and how important they are: 1) Trying to understand the content of the proof, 2) Checking whether the proof is valid, 3) Becoming convinced that the theorem is true, 4) Writing what the professor writes or says, 5) How this would be useful for the exam(s) or assignments, and 6) Anticipating the next line of the proof.

Follow-Up Interviews

Students were invited for a follow-up interview if they met two of three criteria: 1) described a theorem that they were not convinced was a mathematical truth, 2) described a proof type that they were not convinced proves, or 3) indicated that they focused on becoming convinced that a theorem is true when the instructor presents a proof. I conducted five follow-up interviews with five different students. The average interview time was 10.8 minutes ($SD = 3.47$ minutes).

I refer to these students as S, K, B, U, and C. Student S was beyond their fourth year with a major in mathematics and completed five proof-oriented courses. Student K was a third-year mathematics major and had completed one proof-oriented course. Student B was a second-year majoring in applied mathematics and had completed one proof-oriented course. Student U was a fourth-year majoring in mathematics and had completed three proof-oriented courses. Student C was a fourth-year majoring in mathematics and had completed three proof-oriented courses.

The questions in the interviews were tied back to responses to the criteria for why they were invited. For example, asking about the role(s) an instructor played or plays in becoming (un)convinced. Also, each student was asked whether they found themselves wondering about whether they're convinced in courses and whether not being convinced is ever an issue.

Results and Preliminary Analyses

Results of the Open-Ended Questions

The results for the open-ended questions are presented below in Table 1. Thirty-eight students (53%) said that they did not experience an instance where their instructor presented a theorem and they were not convinced that theorem was true. Twenty-four students (34%) said that instances of not being convinced by an instructor's presented theorem were due to not understanding. Four students (6%) said that they did not believe that theorem they encountered to be a mathematical truth. These instances included an instructor presenting a theorem relating to converging sequences without justification, the infamous invalid claim that "all horses are the same color," and the reverse triangle inequality. These students described stated their lack of conviction stemmed from the statements not being "obvious" or "fundamentally incorrect."

Most students (83%) said that there was no type of proof that they did not think proved. Four students (6%) said that they were not convinced a proof type proved. They described proofs by picture, proofs by examples, and proofs of uniqueness. For proofs by picture, two students said that pictures can be interpreted differently and drawn in misleading ways. For proofs by examples, the student said having an artificial intelligence program attempt to prove a theorem only used examples which "did not cover all cases." For the student who mentioned uniqueness, they said the proof of "square roots are not rational" and that "when we just name the same variable two different things, it makes sense that it will always come together."

Table 1. Results of the Two Open-Ended Questions

Was there ever an instance where your instructor presented a theorem and you were not convinced the it was true?*				Is there a type of proof that you find unconvincing?*			
	Yes.				Yes.		
No	Did not believe the theorem to be true	Believed it was true, but did not understand why	Other	No	Do not believe that that type of proof proves	Believed it proves but do not understand why	Other
38	4	24	4	59	4	4	2

*There were three instances of a student declining to answer either question.

Results of the Ranked Choice Question

The results of the ranked choice question are presented below in Table 2. I note that a rank of 6 indicated the most importance, a rank of 1 indicated the least importance, and a 0 indicated no importance (i.e., not ranked at all). These rankings were tabulated and then weighted according to the students' rankings. The table is organized from most important to least important. These results show that students indicated understanding the proof's content as frequent behavior and notable importance while checking the proof's validity was infrequent and less important.

Table 2. The results of the Ranked Choice Question

Rank	Students' Focus During Instructor Proof Presentation					
	Trying to understand the proof's content.	Writing what the instructor writes or says.	Becoming convinced that the theorem is true.	Anticipating the next line of the proof.	Usefulness for the exam(s) or assignments.	Checking whether the proof is valid.
6	40	10	3	4	7	3
5	13	14	14	14	6	4
4	5	9	13	13	9	8
3	3	7	9	4	11	12
2	0	5	4	5	11	8
1	1	6	2	5	6	4
0	9	20	26	26	21	32
Weighted	4.72	2.86	2.50	2.44	2.38	1.77

Results from the Five Follow-up Interviews

Four students mentioned that they viewed the instructor as a support mechanism in the process of becoming convinced or resolving issues of certainty. Student C said that instructors "provide tools necessary to be able to understand the proof by giving examples or other theorems." Student S described an instructor's role as engaging with the class by asking whether the theorem or proof makes sense. They emphasized that it was important for the instructor to engage students' thinking with questions relating to specific justifications in a proof. Student K described the instructor's role as a "kind of… comforting thing. I [can] ask them any question, even if it's very complicated." Student U described the instructor's role as "presenting the pieces that you need to follow to be convinced that you or one of the ways you could be convinced a theorem is true." They continued by mentioning that probing questions were especially helpful to "start actually building the theorem."

Few students mentioned strong feelings toward the role of becoming convinced as they engage with mathematics. For example, Student K said, "generally speaking, if it's a complex theorem, I'll break it down into a million pieces and I won't trust it at face value…. I just… can't move on." They continued by saying, "I'll be second guessing myself [otherwise]. Just something I can't help." I asked Student U if being unconvinced is ever an issue, they said that "it's [never] stopped me from trying to engage with a topic…. Perhaps it's more a healthy skepticism that I…probably would benefit from having more than blindly [believing] in the book every time." In other words, "…not being convinced of a theorem just by reading it." Student U believed they developed healthy skepticism by saying, "it's gone in a bit of a weird shape graphic because I started off just blindly believing everything…" and gestured a graph like Figure 1. The reason Student U gestured like this was due to their introduction to proof course instructor who helped develop a sense of skepticism, but as Student K progressed through subsequent mathematics courses, they had rebuilt "blind trust" in textbooks "because…the [questions] we use [in courses] are simply prove this is true or prove this is not true."

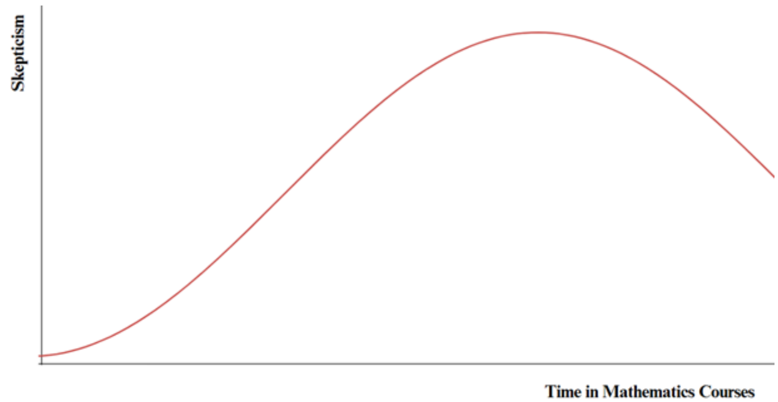

Figure 1. Recreation of Student U's Gestures about Skepticism and Mathematics Courses

Discussion and Conclusion

The open-ended questions revealed that most undergraduate students in this study are convinced that the theorems their instructors present are true and that the proof types they encounter are valid. The instances of students not being convinced that the instructors' theorems were tied to a prevailing desire for more justification and reasoning—a mathematical proof. The instances of students not being convinced by a proof type included picture proofs, proofs by examples, and proofs by uniqueness. Notably, students were articulate about their reasons for why proof types were unconvincing. These reasons relied on skepticism, or doubts, about the rigor of the proof type rather than a misunderstanding.

The ranked choice question revealed that students primarily focused on and find importance in understanding the content of a proof while checking whether the proof is valid was often less important and frequent. This provides some evidence about the role of an instructor. The students in this study seemed not worried about whether an instructor was providing invalid proofs. This is reasonable given the standard structure and practices of mathematics. Interestingly, becoming convinced that the theorem is true was third when the rank was weighted. So, becoming convinced is among the list of proclivities but other mathematically relevant behaviors take priority. In other words, there is evidence that students knowingly think about becoming convinced in their courses. This provides evidence that some students' mathematical epistemologies encompass mathematical convictions given that the scenario was written to exclude specific details such as content, location, and instructor.

The results from the follow-up interviews provide some evidence that students indeed consciously reflect on matters of conviction. For some students, it appeared as ways they come to know and understand mathematics. This reflection was shown to be fostered in the class for one student but not others. Nevertheless, the instructor's role for these students provides understanding which has been observed greatly influencing whether students are convinced (Brown, 2018; Selden & Selden, 2003).

Preliminary Nature of This Report

I do not anticipate having further analysis developed because the analysis is limited by the research instruments' wording. Therefore, I wish to avoid making higher inferences than warranted by the data. However, I wish to develop further research instruments, avenues, and questions pertaining to how reflections on whether one is convinced is taught or enculturated in courses and how these mechanisms influences students' epistemologies as they engage with mathematics (proofs and theorems or otherwise).

References

Brown, S. A. (2014). On skepticism and its role in the development of proof in the classroom. *Educational Studies in Mathematics*, *86*(3), 311–335.

Brown, S. A. (2018). Are indirect proofs less convincing? A study of students' comparative assessments. *The Journal of Mathematical Behavior*, *49*, 1–23.

Chazan, D. (1993). High School Geometry Students' Justification for Their Views of Empirical Evidence and Mathematical Proof. *Educational Studies in Mathematics*, *24*(4), 359–387.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics*, *3*(2), 9–24.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society*, *7*, 234–283.

Healy, L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.

Rosen, K. H. (2012). *Discrete Mathematics and its Applications* (7th ed.). McGraw-Hill Education.

Selden, A., & Selden, J. (2003). Validations of Proofs Considered as Texts: Can Undergraduates Tell Whether an Argument Proves a Theorem? *Journal for Research in Mathematics Education*, *34*(1), 4–36. https://doi.org/10.2307/30034698

Weber, K., Mejía-Ramos, J. P., & Volpe, T. (2022). The Relationship Between Proof and Certainty in Mathematical Practice. *Journal for Research in Mathematics Education*, *53*(1), 65–84. https://doi.org/10.5951/jresematheduc-2020-0034