Conceptions of Derivatives in Calculus: Merging and Extending Earlier Frameworks

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Researchers studying derivative conceptions often use Zandieh's (2000) framework, which emphasizes graphical, symbolic, verbal, and physical representations within three process-object layers of ratio, limit, and function. While foundational and aligned with traditional calculus instruction, recent literature on derivative teaching reveals additional conceptions. I present the Conceptions of Derivative (CoD) framework, which expands Zandieh's work to offer a more comprehensive perspective on contextual and cognitive processes when thinking about derivatives. The framework provides researchers with a comprehensive lens, while offering instructors awareness and language for intentional targeting of derivative conceptions based on their students' conceptual needs.

Keywords: Derivatives, calculus, conceptions, representations

When considering various conceptions of derivative in calculus, many researchers rely on Zandieh's (2000) framework or its extended versions (e.g., Roundy et al., 2015). As a cognitive framework for learning derivatives, Zandieh's framework has also shaped the research on derivative teaching (e.g., Hitier & González-Martín, 2022) possibly due to its resonance with instructional and curricular approaches to derivatives (Jones & Watson, 2017).

However, in my research on eight experienced college calculus instructors' derivative tasks (Gerami, 2023, 2024), I encountered conceptions beyond those captured by Zandieh's (2000) framework. This motivated the development of an updated framework—*Conceptions of Derivative (CoD)*—to accommodate both my empirical findings and more recent theoretical perspectives on other approaches to thinking about derivatives. This report begins with reviewing Zandieh's (2000) framework and its extended versions, followed by a discussion of its limitations. I then present the extended CoD framework.

Zandieh's (2000) Framework and Its Extensions

In a foundational piece, Zandieh (2000) constructed a theoretical cognitive model for students' understanding of the concept of derivative. For Zandieh, an individual's understanding of derivative is defined "in terms of its overlap or lack of overlap with the mathematical community's notion of that concept" (Zandieh, 2000, p. 104). As shown in

Table 1, the framework is organized by four columns, called contexts/representations, and three rows, called process-object layers. Therefore, each cell in the framework refers to a conception of derivative within a representation or context, and a process-object layer.

Table 1. Zandieh's (2000) framework for the conceptions of derivative

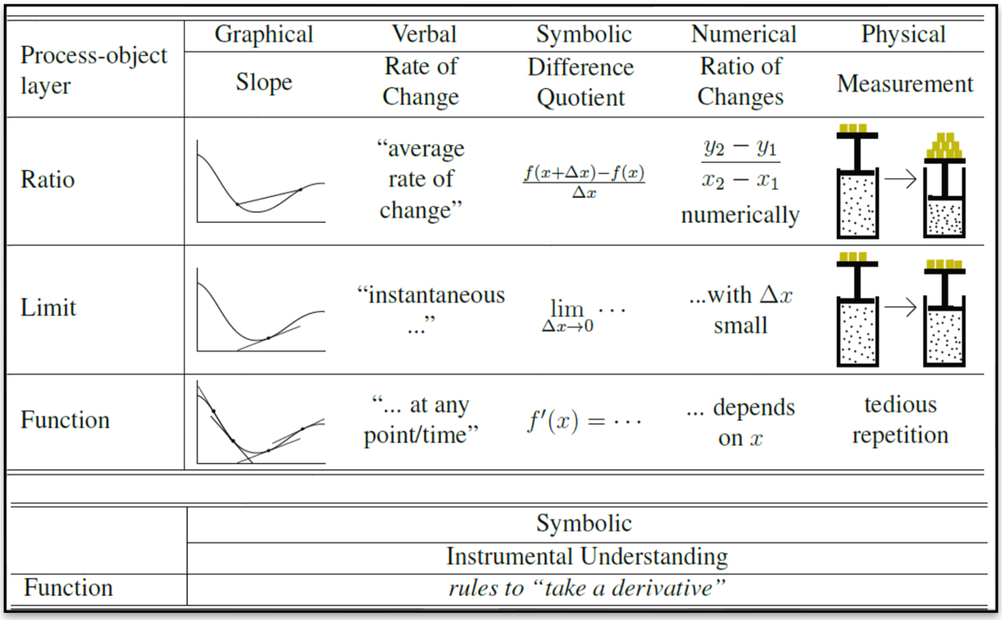
		Representations/Contexts				
		Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Difference Quotient)	Other
Process-Object Layer	Ratio					
	Limit					
	Function					

Defining representation using the notion of a person's different images of the same concept (i.e., concept image; Tall & Vinner, 1981; Vinner & Dreyfus, 1989), she captured four mathematical representations for the concept of derivative: graphical (slope), physical (velocity), verbal (rate of change), and symbolic (difference quotient). Zandieh (2000) also included an "Other" column within the representations/contexts to capture representations that may appear different but are structurally similar. In an example, she noted that rate of change of temperature is another physical manifestation of derivative, like velocity, stating that "Many other physical examples are possible, and there are variations possible in the graphical, verbal, and symbolic descriptions" (p. 105). Nonetheless, for her, these *other* representations within the process-object framework still describe the similar "underlying structure" of the same concept of derivative.

For the rows, Zandieh (2000) used the term "process-object" from Sfard's (1991, 1992) work to capture the duality of some mathematical concepts: processes acting on previously defined objects, and objects as a result of other processes acting on them. To refer to different process-objects within the concept of derivatives in her framework, Zandieh introduced the term "layers" to capture the hierarchical structure of the three process-object rows, as each layer is found by using that layer as a process over the previous layer as an object. The three layers in the framework are ratio, limit, and function. The first layer, ratio, is the object of the average rate of change between two points, which can be represented graphically as a secant slope, average velocity between two times, or symbolically as $\frac{f(x_0+h)-f(x_0)}{h}$ or $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. To find the next layer of derivative—the *limit* layer, we hold the ratio layer as an object and use the *limit* layer as a process on it to decrease h as the horizontal distance between the two points. This gives us the limit of the ratio layer as an object: $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, referred to as $f'(x_0)$ or the derivative of the function at a point. Finally, at the final layer and via the functioning process, we find the instantaneous rates of change at every possible point on the function. This results in the derivative as a function: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Facing challenges to capture all the ways that university students made use of their understanding of the derivative to solve derivative problems, Likwambe and Christiansen (2008) added the *instrumental* representation to capture the "only learning exhibited" by most of their university student interviewees (p. 24). To define this new representation, Likwambe and Christiansen relied on Skemp's (1978) and Lithner's (2003) notion of instrumental understanding as the ability to use a rule or procedure, while specifying it for finding derivatives using *derivative rules*. Likwambe and Christiansen further argued that the instrumental representation is its own category because it does not have any layers (i.e., it does not 'live' in any one specific layer of Zandieh's [2000] framework).

Later, Roundy and colleagues (2015) built on Likwambe and Christiansen's work to further extend Zandieh's (2000) framework by: 1) changing velocity—the physical representation's example—to *measurement* to allow for changes in quantities other than time; 2) adding the *numerical* representation, as calculated values for ratio of changes at the ratio layer and limit layer (for small Δx values), and sequence of numerical ratios of differences at the function layer; and 3) categorizing instrumental representation outside of the rest of the framework as a symbolic representation within the function layer, because using "rules for symbolic derivatives does not require a conceptual understanding of the derivative" (p. 923) and the derivative rules only exist at the function layer. Figure 1 shows this extended framework.



Process-object layer	Graphical Slope	Verbal Rate of Change	Symbolic Difference Quotient	Numerical Ratio of Changes	Physical Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		"instantaneous ..."	$\lim_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \cdots$	... depends on x	tedious repetition

	Symbolic
	Instrumental Understanding
Function	*rules to "take a derivative"*

Figure 1. Roundy and colleagues' (2015) extended framework.

Limitations and Resolutions

While applying Zandieh's (2000) framework to analyze eight experienced college calculus instructors' derivative tasks (Gerami, 2023, 2024), several limitations emerged that motivated the development of an extended framework.

Limitation 1: Placing Non-Limit Approaches

Some instructors in my study conceptualized tangent lines through approaches fundamentally different from the traditional limit-based method. Rather than finding tangents by pushing two points together via limits, they used infinitesimal reasoning—zooming in on a function until it appears flat. This approach represents a distinct epistemological perspective rooted in infinitesimal and differential calculus (Ely, 2021; Keisler, 2011; Samuels, 2017), not merely a different graphical representation of limits.

Similarly, while Roundy et al. (2015) appropriately included instrumental understanding (derivative rules) in their framework, as many students use them to think about derivatives, they positioned it within the limit-based structure. However, students using derivative rules do not conceptualize the rules through process-object layers from ratio to function; it is my understanding as a calculus instructor and mathematics education researcher that they bypass this conceptual development entirely. This suggests instrumental approaches should be placed in a separate epistemological category. Moreover, while Roundy et al. added instrumental understanding within the function layer, I also knew that one could use derivative rules to also find derivative at a point as well as "average derivative" or average rate of change between two points.

These observations revealed three distinct *Conceptual Epistemologies* for understanding derivatives: limit-based, infinitesimal/differential-based, and instrumental/rule-based. Each represents a fundamentally different way of conceptualizing how they are derived.

Limitation 2: Conflating Context and Representation

I also had difficulty with the representation columns *Physical* and *Verbal*. It seemed to me that the physical representation conflates mathematical representation with contextual application. Physics problems are not a way of representing derivatives mathematically; they are contexts where derivatives can be applied. Similarly, while Roundy et al.'s (2015) change to *measurement* addressed some concerns, the distinction between their measurement and numerical columns remained unclear. Moreover, one could use derivatives on problems regarding change within other fields, such as physics, biology, chemistry, and economics.

To resolve this issue, I separated contextual applications into a distinct component called *Contextual Framing*—the context in which derivatives are used. This resolution would also recognize that any epistemological approach can be applied across various contexts.

Limitation 3: Verbal "Representation" as Conceptual Signals

Lastly, my research (Gerami, 2024) suggested that the verbal representations directed instructors toward other derivative conceptions. Rather than being representations themselves, these verbal descriptions serve as signals to other ways for conceptualizing derivatives, which can be different for different people. To address this limitation, I drew on semiotics (de Saussure, 1959) to distinguish between the *signified*, or the mathematical representation itself, and the *signifier*, or the verbal label often used to reference it. This separation clarifies how language can guide conceptual understanding while maintaining mathematical representation as the focus.

The Conceptions of Derivative (CoD) Framework

The Conceptions of Derivative (CoD) framework addresses the limitations discussed above by reorganizing how we conceptualize derivatives using four distinct but interconnected components that capture the multifaceted nature of derivatives (Figure 2).

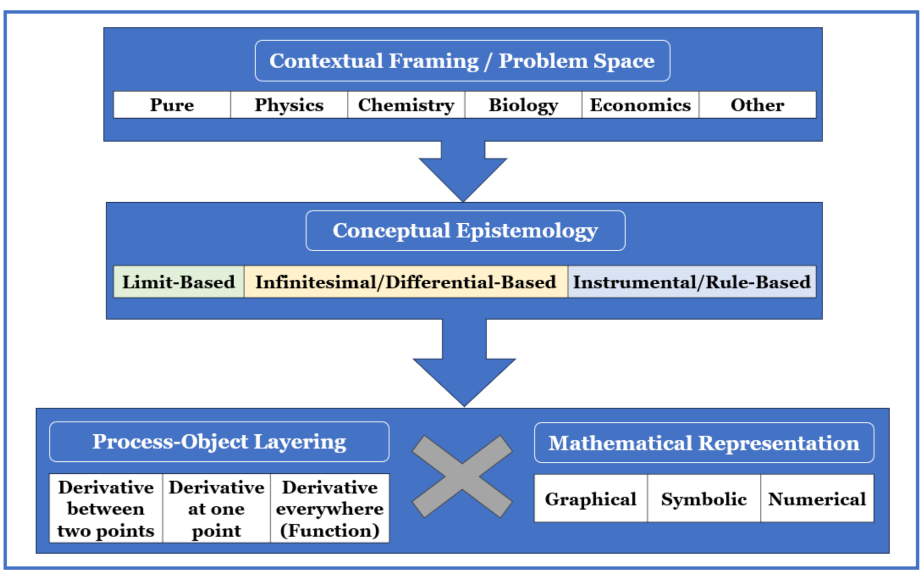


Figure 2. The four main components of the Conceptions of Derivative (CoD) framework

Contextual Framing is the context through which we frame or interpret derivative problems. Common contexts in calculus are pure mathematics, physics, chemistry, biology, and economics.

Conceptual Epistemology captures fundamentally different approaches to understanding what derivatives are and how they are derived. The framework identifies three distinct epistemologies:

limit-based (traditional approach using limits of difference quotients), infinitesimal/differential-based (using infinitesimal quantities or differential reasoning), and instrumental/rule-based (procedural approach using derivative rules).

Mathematical Representations encompass the three primary ways derivatives can be expressed mathematically: graphical (visual representations), symbolic (algebraic expressions), and numerical (calculated values). Following semiotic principles, each representation is understood through both its *signified* (the mathematical object) and *signifier* (the verbal label used to reference it).

Process-Object Layering maintains the hierarchical structure of derivatives from Zandieh's (2000) framework but adapts it to work across different epistemologies: 1) between two points (average derivative/rate of change), 2) at a point (instantaneous derivative), and 3) everywhere (derivative function). Unlike Zandieh's framework where layers are tied to limit processes, these layers can be conceptualized through any of the three epistemological approaches.

The comprehensive framework is shown in Figure 3. As the arrows show, the framework suggests that one encounters a derivative problem within a particular contextual framing. They then approach the derivatives via a specific conceptual epistemology. Within that epistemology, they can work across three layers using various mathematical representations, with verbal signifiers guiding their selection and interpretation of representations. I next explain the layers and representations within each conceptual epistemology.

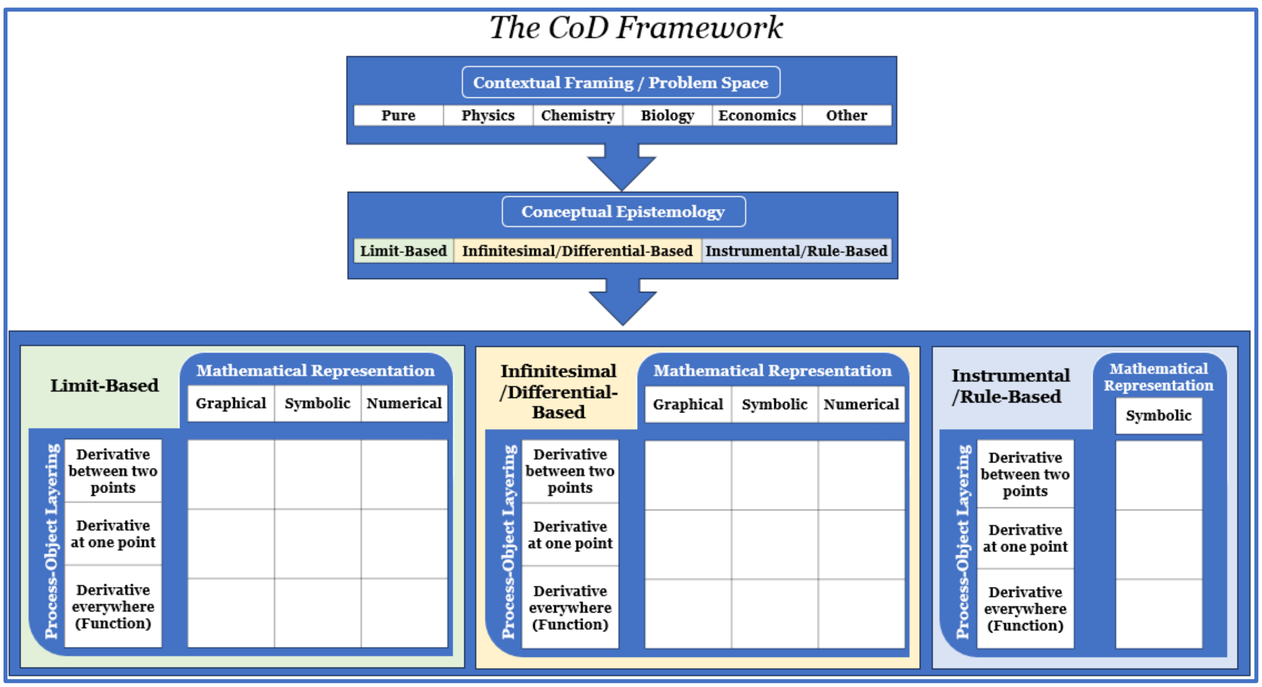


Figure 3. Conceptions of Derivative (CoD) framework

The limit-based conceptions in Figure 4 closely follow Zandieh's (2000) framework, where derivatives emerge through limiting processes. As I have reviewed these above, I do not repeat the details here. Signifiers like *secant/tangent slopes*, *difference quotient*, and *rate of change* are often used to direct attention to specific representations within each layer.

Limit-Based

Process-Object Layers	Representations					
	Graphical		Symbolic		Numerical	
	Signified	Signifier	Signified	Signifier	Signified	Signifier
Ratio		Slope of secant line	$\frac{f(x_0+h)-f(x_0)}{h}$, $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, or $\frac{f(a)-f(b)}{a-b}$	Difference quotient	Secant slope as a number	Slope value
Limit		Slope of Tangent line	$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, or $\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$	Limit of the difference quotient	Tangent slope as a number	Table of tangent slope values
Function		Graph of the derivative function	$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, or $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$	Derivative as a limit	Tangent slopes as numbers	A list of slopes for "all" points

Figure 4. The limit-based conceptions of CoD

Infinitesimal/Differential-Based

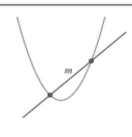
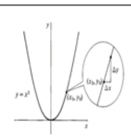
Process-Object Layers	Representations							
	Graphical		Symbolic Infinitesimal		Symbolic Differential		Numerical	
	Signified	Signifier	Signified	Signifier	Signified	Signifier	Signified	Signifier
Ratio		Slope of a line through two points	$\frac{\Delta y}{\Delta x}$	Slope on a line through two points or "real rate"	$\frac{\Delta y}{\Delta x}$	Slope on a line through two points	Slope as a number	Slope value
Zoom		Local slope	$st(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x})$The real number infinitely close to the answer for every non-zero infinitesimal Δx	Infinitesim al rate	$\frac{dy}{dx}$ *at* x_0 *Or* $\frac{f(x_0+dx)-f(x_0)}{dx}$	A quotient or a differential equation $dy = [\blacksquare]dx$ *at* x_0	Local slope as a number	Local slope value
Function		All local slopes	$st(\frac{f(x+\Delta x)-f(x)}{\Delta x})$ for every x and every non-zero infinitesimal Δx	The derivative function	dy/dx *Or* $\frac{f(x+dx)-f(x)}{dx}$	The derivative $f\,'$ or a quotient or a differential equation $dy = [\blacksquare]dx$	Local slopes as numbers	A list of local slopes for "all" points

Figure 5. The infinitesimal/differential-based conceptions of CoD

The process for the infinitesimal- or differential-based conceptions in Figure 5 is 'zooming in.' At the ratio layer, derivatives are real rates of change between two points ($\frac{\Delta y}{\Delta x}$). At the "Zoom" or derivative at a point layer, the rate becomes infinitesimal ($\frac{dy}{dx}$); graphically, this involves zooming in until the function appears linear and one can find the slope. While I cannot provide details here due to space constraints, the infinitesimal approach uses the standard portion of hyperreal numbers, whereas the differential approach uses first-order approximations, so they differ symbolically (see Ely, 2021; Keisler, 2011). The function or everywhere layer extends zooming process across the function's domain. Signifiers like *infinitesimal change* and *differential* are often used to target these conceptions.

Instrumental- or rule-based conceptions in Figure 6 focus on procedural ways of deriving derivatives. Unlike other epistemologies, this approach can be used bottom-up. The everywhere or function layer is found when rules are used to generate derivative functions. The derivative at a point layer can be found by evaluating derivative by 'plugging in' a specific x-value into the derivative function. Although not commonly used, the derivative between two points layer can be found by finding the average of two derivatives from the at a point layer. The symbolic representation and signifiers like *taking derivative* or *using derivative rules* (e.g., power, quotient) are often used for this epistemology.

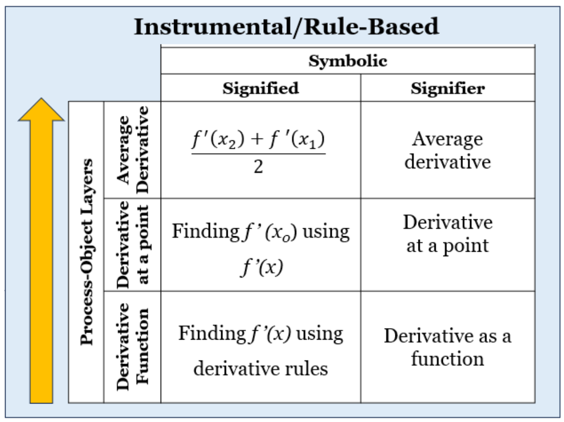


Figure 6. The instrumental/rule-based conceptions of CoD

Conclusion

The Conceptions of Derivative (CoD) framework addresses critical gaps in existing theoretical process-object models by recognizing that derivative understanding encompasses multiple epistemological approaches, not just limit-based reasoning. Rather than privileging a single epistemological perspective, the framework recognizes three distinct conceptual epistemologies—limit-based, infinitesimal/differential-based, and instrumental/rule-based—that can be used within various contextual framings. This offers researchers language to discuss derivative conceptions beyond those in traditional frameworks. Future research can use this framework to investigate how contextual framing influences epistemological choices, how instructors target various conceptions, and how students navigate between different conceptions. Different epistemologies appear to privilege different areas of study. Given that calculus courses are increasingly targeted for specific student audience like biology and business majors, the framework allows instructors and curriculum designers to choose which derivative understanding to emphasize and how.

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References

de Saussure, F. (1959). *Course in general linguistics.* Columbia University Press.

Ely, R. (2021). Teaching calculus with infinitesimals and differentials. *ZDM – Mathematics Education*, *53*(3), 591-604.

Gerami, S. (2023).Calculus I instructors' use of representations in instructional tasks: Introducing derivatives with inquiry. In Cook, S., Katz, B. & Moore-Russo D. (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1185-1191). Omaha, NE.

Gerami, S. (2024). *Instructional tasks for introducing derivatives in college calculus with inquiry* [Unpublished doctoral dissertation].University of Michigan.

Hitier, M., & González-Martín, A. S. (2022). Derivatives and the study of motion at the intersection of calculus and mechanics: A praxeological analysis of practices at the college level. *International Journal of Research in Undergraduate Mathematics Education*, *8*(2), 293-317.

Jones, S. R., & Watson, K. L. (2017). Recommendations for a "target understanding" of the derivative concept for first-semester calculus teaching and learning. *International Journal of Research in Undergraduate Mathematics Education*, *4*(2), 199-227.

Keisler, H. J. (2011). *Elementary calculus: An infinitesimal approach* (2nd ed.). Dover Publications.

Likwambe, B., & Christiansen, I. M. (2008). A case study of the development of in-service teachers' concept images of the derivative. *Pythagoras*, *2008*(68), 22-31.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, *52* (1), 29–55.

Roundy, D., Dray, T., Manogue, C. A., Wagner, J. F., & Weber, E. (2015). An extended theoretical framework for the concept of derivative. In *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 838-843). MAA.

Samuels, J. (2017). A graphical introduction to the derivative. *Mathematics Teacher*, *111*(1), 48-53.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification—The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA notes, no. 25, pp. 59-84). Mathematical Association of America.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *The Arithmetic Teacher*, 9–15.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, *20*(4), 356-366.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS issues in mathematics education*, *8*, 103-127.