Targeting Conceptions of Derivative: Introducing Derivatives in Calculus With Inquiry

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In this study, we leverage the Conceptions of Derivative (CoD) framework (Gerami et al., 2025) to analyze eight U.S. college calculus instructors' instructional tasks for introducing derivatives. During four task-oriented interviews, each instructor proposed up to eight tasks for introducing derivatives with inquiry in their calculus I courses. Based on Zandieh's (2000) process-object model, the CoD framework was used to reveal derivative conceptions that instructors were targeting in their tasks. We identified four components of derivative conceptions targeted in the tasks: contextual framing, epistemological approaches (limit-based, infinitesimal/differential-based, and instrumental/rule-based), mathematical representations, and process-object layers. While pure contextual frames and limit-based approaches dominated early instruction of derivatives for these instructors, some targeted physics and biology contexts as well as infinitesimal/differential-based and rule-based epistemologies. We discuss implications for research on calculus teaching.

Keywords: Derivatives, conception, calculus teaching, instructional tasks

While researchers have explored students' conceptions of derivatives for some time (e.g., Ferrini-Mundy & Graham, 1994; Feudel & Biehler, 2021; Jones & Watson, 2017), little research has examined how college calculus instructors teach derivatives and the conceptions they target during instruction. In this study, we investigate how college calculus instructors introduce derivatives by focusing on the conceptual opportunities they leverage in their instructional tasks. To reveal a broader range of conceptions of derivatives targeted within instructional tasks, we focused on instructors experienced with inquiry-based teaching, assuming they use a broader range of instructional tasks than those from conventional textbooks. We aim to address the following question: Which conceptions of derivative do eight college calculus instructors target in their instructional tasks when introducing the concept for the first time with inquiry?

Literature Review

Our search of the research on teaching of derivatives yielded nine studies. The first four studies were conducted with various motivations, yet they all examine non-intervention based teaching of derivatives by calculus teachers. The remaining studies (Borji et al., 2018; Hähkiöniemi, 2006; Hitt & Dufour, 2021; Kendal & Stacy, 2001) are intervention-based, as they investigate the efficacy of teaching interventions aimed at helping students make connections between multiple representations of the derivative to develop deeper conceptual understandings. Because our question revolves around teachers' natural ways of teaching derivatives, we delve deeper into the non-intervention based studies.

Park (2015) used a discursive approach to investigate how three university calculus instructors defined derivatives at a point, and on an interval (or as a function), and transitioned

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from the former to the latter, given the challenges these concepts pose for students. Although the instructors used secant and tangent lines to graphically illustrate derivative at point, Park identified several disconnections. The instructors did not explicitly connect the graphical and the symbolic representations of derivatives, as they often referred to the slope of secant or tangent lines simply as 'secants' or 'tangents.' While they used graphs to illustrate the derivative at a point (via secants approaching a tangent), they did not use graphical representation to define the derivative at the function layer. Moreover, at the function layer, the instructors often showed derivative as a function with zero, positive, or negative values of derivative "rather than numerical values that change over an interval" (p. 248). Moreover, the transition from the limit to the function layer within the symbolic representation was only shown as changing the symbol for a point (x_0) to a variable (x), without addressing the reasoning behind the change of symbols. Park concluded that by not making these connections explicit, students were left to make some challenging connections on their own.

Within a study about one experienced college instructor's use of a calculus textbook in Norway, Randahl (2016) examined how the instructor introduced derivatives during a sequence of six lectures. The instructor closely followed the textbook's structure and content sequence, defining the following concepts through lecturing: 1) tangent line, 2) slope to the tangent line, 3) the difference quotient, 4) the slope of a curve at a point and the slope of the normal in this order, and 5) The derivative of a function defined using the limit definition. The definitions were presented on the board without further clarification or discussion, after which, the instructor assigned procedural exercises for group work, with some solutions discussed in front of the whole class. Because the instruction mirrored the formal introduction of derivatives in the textbook without providing additional explanations, Randahl concluded that the instruction likely did not enhance students' understanding of the abstract concepts associated with derivatives.

In a study examining teachers' views on differentiation and their knowledge of teaching it, Delos and Thomas (2003) investigated four New Zealand teachers' approaches to teaching derivatives. Although the authors found no common way of teaching derivatives across the teachers, they found that all teachers used graphical representations, particularly the approach of moving secants toward a tangent when finding derivative at a point. Despite their emphasis on graphical representation, the teachers expressed a desire for their students to develop a comprehensive conceptual understanding of derivatives, including understanding it as "a gradient function, as a rate of change, as a limit, and symbolically" (p. 56).

Lastly, in the fourth non-intervention-based study, Hitier and González-Martín (2022) identified the (in)consistencies in the use of derivatives in calculus and mechanics course, needed for solving one-dimensional kinematics tasks. As part of their broader investigation, the authors interviewed four college calculus instructors from Quebec, all of whom used Stewart's (2013) textbook to teach the course. The teachers described their ways of introducing derivatives similar to the textbook's, with interpreting derivative at a point as the slope of the tangent line. They also incorporated the symbolic limit definition both at a point and for a function and made connections with the graphical representations. Although the teachers mentioned using the four representations by Zandieh (2000) in their teaching, they expected students to heavily rely on symbolic calculations to solve derivative tasks about motion (i.e., within the physical representation of derivative).

In summary, these non-intervention-based studies on the teaching of derivatives focus on how teachers often define derivatives. They collectively suggest that instructors often utilize both graphical and symbolic representations, but that they may give less emphasis to physical representations of derivatives. Among the intervention-based studies not reviewed here due to space, there appears to be agreement that using graphical representations for introducing derivatives is effective. However, it remains uncertain whether the physical representations, particularly those related to physics (as opposed to, for instance, population growth or decay), contribute to a better understanding of derivatives.

While these studies shed light on the importance of various representations of derivatives, they have focused exclusively on limit-based understanding of derivatives, reflecting the dominance of this approach in traditional calculus instruction. However, recent literature suggests that instructors may employ alternative epistemological approaches to derivatives that extend beyond the limit framework (Ely, 2021; Keisler, 2011; Samuels, 2017). The Conceptions of Derivatives (CoD; Gerami et al., 2025) framework, which we present next, addresses these gaps by incorporating multiple epistemological approaches and explicit attention to the physical manifestation of derivatives.

Theoretical Framework

The Conceptions of Derivatives (CoD; Gerami et al., 2025) framework extends the Zandieh's (2000) framework (Table 1) to include non-limit-based approaches to thinking about derivatives. Constructed as a cognitive model for students' understanding of the concept of derivative, the Zandieh's framework distinguishes between three derivative "process-object" layers—shown as *ratio*, *limit*, and *function* rows in the table—within four main representations/contexts of derivatives—graphical, verbal, physical, and symbolic. Relying on Sfard's (1991, 1992) notion of process-objects, Zandieh's framework uses "limiting" as the main process for deriving derivatives. The ratio layer is the average rate of change between two points: $\frac{f(x_0+h)-f(x_0)}{h}$. The limit layer is produced by *limiting* the ratio layer: $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, referred to as the derivative of the function at a point. Finally, the function layer is produced by using the same limiting process for all possible points to find derivative as a function: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Table 1. Zandieh's (2000) framework for the conceptions of derivative

		Representations/Contexts				
		Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Difference Quotient)	Other
Process-Object Layer	Ratio					
	Limit					
	Function	.	.	.	.	

Tho CoD framework, shown in Figure 1 is more comprehensive, as it includes two other ways of thinking about derivatives: infinitesimal- or differential-based, and rule- or instrumental-based. The CoD framework has four distinct but interconnected components. *Contextual Framing* is the context through which we frame or interpret derivative problems. Common contexts in calculus are pure mathematics, physics, chemistry, biology, and economics. *Conceptual Epistemology* captures fundamentally different approaches to understanding what derivatives are and how they are derived. The framework identifies three distinct epistemologies: limit-based (traditional approach using limits of difference quotients), infinitesimal/differential-based (using infinitesimal quantities or differential reasoning), and instrumental/rule-based (procedural approach using derivative rules). *Mathematical Representations* encompass the three primary ways derivatives can be expressed mathematically: graphical (visual representations), symbolic (algebraic expressions), and numerical (calculated values). *Process-Object Layering* maintains the hierarchical structure of derivatives from Zandieh's (2000) framework but adapts it to work across different epistemologies: 1) between two points (average derivative/rate of change), 2) at a point (instantaneous derivative), and 3) everywhere (derivative function). Unlike Zandieh's framework where layers are tied to limit processes, these layers can be conceptualized through any of the three epistemological approaches.

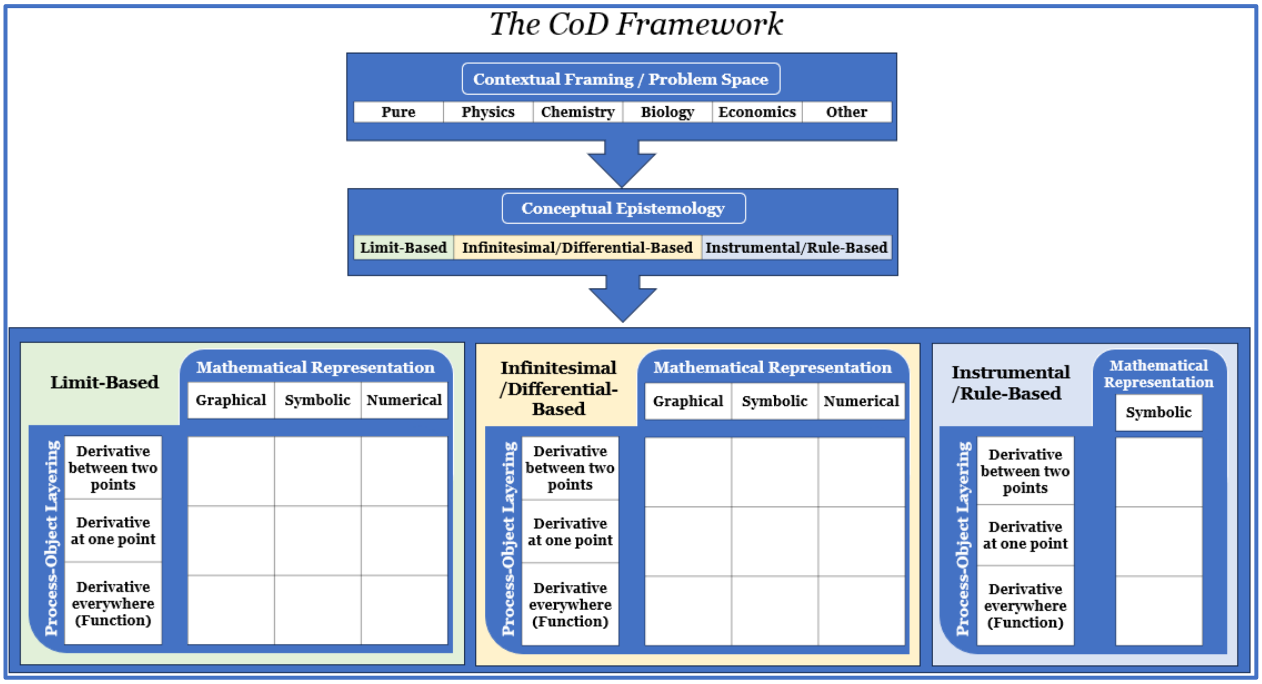

Figure 1. Conceptions of Derivative (CoD) framework

Methods

The data for this study comes from a larger study conducted to investigate calculus instructors' social and content-based framings of instructional tasks for introducing derivatives (Gerami, 2024). Eight instructors were purposefully selected via cluster analysis from a larger subset of experienced college calculus instructors who showed variety in their teaching with inquiry based on a survey (INQUIRE; Shultz, 2020). All using pseudonyms, the instructors were Adrian, Barry, Gopher, Justin, Matthew (he/him pronouns), Monica (she/her pronouns), Alex (any pronouns), and Max (they/them pronouns). The fourth author conducted four 1–2 hours long semi-structured interviews with each instructor. The interviews, organized by Zandieh's (2000) framework, asked instructors to propose up to eight tasks for introducing derivatives graphically, physically, verbally, and symbolically at a point (assuming students already knew about average rate of change), and as a function (assuming students already knew about derivative at a point).

However, the instructors often targeted multiple conceptions in their tasks, expecting students to use multiple representations of derivatives. This motivated us to analyze the conceptions of derivatives, including representations of derivatives, targeted in the tasks. Given that the proposed tasks often had multiple parts, the fourth author identified and separated the smallest calculus-focused parts in each task to streamline the analysis; we call these calculus-specific task units (CTUs). Using the CoD framework and the qualitative software ATLAS.ti, all authors coded the four components of conceptions of derivatives targeted in each task: problem space (pure, physics, chemistry/biology), conceptual epistemologies (limit-, infinitesimal/differential-, or instrumental-based), mathematical representations (graphical, numerical, symbolic), and process-object layers (average derivative, derivative at a point, derivative everywhere/function). Problem space was coded at the task level, while the conceptual epistemologies, representations, and process-object layers were coded at the CTU level. Across the eight instructors, we analyzed 57 tasks containing 252 calculus-specific task units (CTUs). Table 2 shows the distribution of tasks and CTUs by instructor.

Table 2. Number of tasks and CTUs, as well as epistemologies used, by professor

	Instructor							
	Adrian	Alex	Barry	Gopher	Justin	Matthew	Max	Monica
Number of tasks proposed	8	8	8	8	4	6	8	7
Total number of CTUs across tasks	35	35	31	38	10	32	40	31

After coding the tasks independently, we met to compare our codes, discuss differences, and come to consensus. All final codes were agreed upon by all authors. To analyze patterns in conception targeting, we conducted chi-squared tests when possible (five or more CTUs) to determine whether individual instructors showed significant preferences for targeting specific process-object layers or mathematical representations, hypothesizing that instructors would target each layer (or representation) equally.

Findings

Contextual framing varied across instructors, as shown in Figure 2. Of the 57 tasks, 32 used pure mathematical contexts, 22 used physics contexts, and only 2 used biology/chemistry contexts. All instructors employed pure mathematical contexts, with Alex, Barry, Gopher, and Justin using them most frequently. Except for Justin, all instructors used physics contexts at least once, with Matthew, Max, and Monica using them more extensively. Adrian uniquely distributed tasks across pure and non-pure contextual frames, while being the only instructor who used biology/chemistry contexts.

Limit-based approaches dominated instruction across all instructors, appearing in 243 of the 252 CTUs (Figure 3). All eight instructors used limit-based epistemologies extensively. Infinitesimal/differential-based approaches appeared in only four CTUs, used exclusively by Gopher and Justin. Rule-based epistemologies were even less common, appearing in five CTUs across three instructors—Alex, Max, and Monica. While the instructors often targeted a single epistemological approach in their tasks, a few tasks incorporated multiple epistemologies within different CTUs.

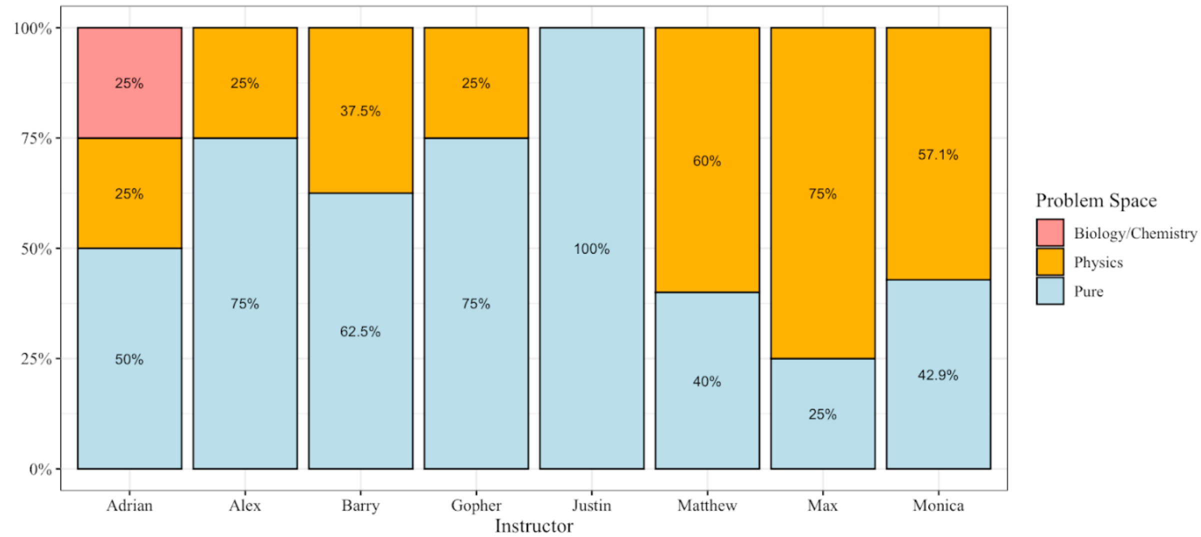

Figure 2. Distribution of contextual framing or problem space across instructors

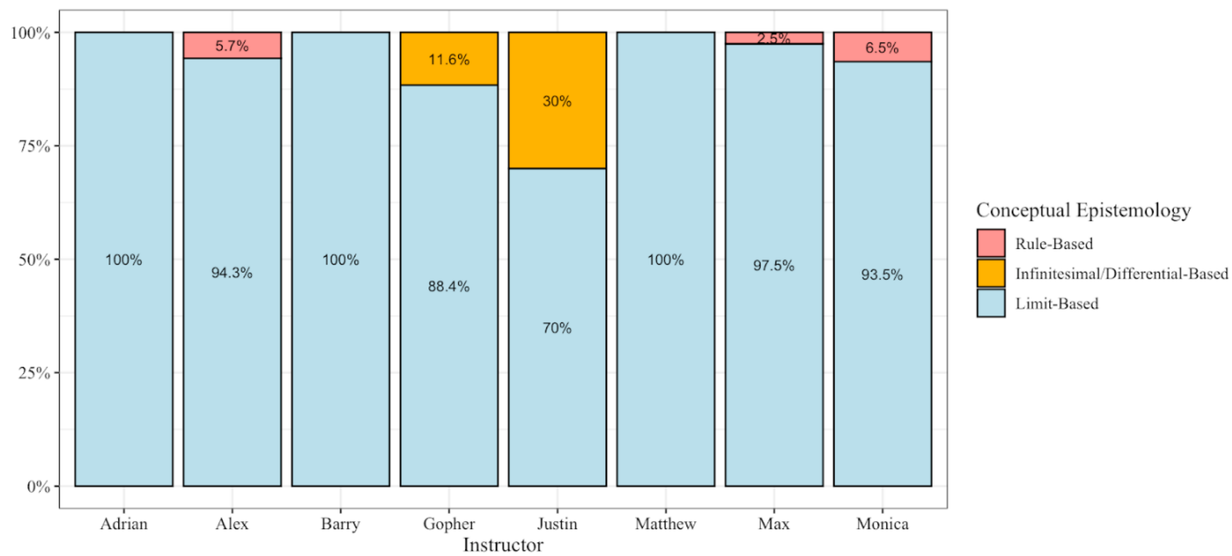

Figure 3. Distribution of conceptual epistemologies across instructors

Within the 243 limit-based CTUs, all instructors except Justin utilized the three mathematical representations and the three process-object layers, though with varying frequencies (Figure 4). Justin did not target the ratio layer or the symbolic representation. Statistical analysis of the rest of the instructors revealed that Gopher and Matthew significantly preferred targeting the limit layer over ratio and function layers ($\chi^2 > 5.99$, $p < 0.05$), while Adrian showed significant preference for symbolic over graphical and numerical representations ($\chi^2 > 5.99$, $p < 0.05$). While Justin's small number of CTUs did not allow χ^2 testing, the remaining instructors did not show significant preferences for targeting specific layers or representations.

The remaining epistemologies—Infinitesimal- or differential-based, and rule- or instrumental-based—were used infrequently. Infinitesimal/differential-based approaches appeared in only four CTUs from Gopher and Justin, exclusively at the limit layer and primarily using graphical representation (i.e., zooming in on the graph of a function to find the instantaneous derivative or the slope of tangent once the function becomes linear). Rule-based epistemology appeared in five CTUs within tasks by Alex, Max, and Monica, all targeting the function layer with symbolic representation, reflecting the procedural nature of applying derivative rules.

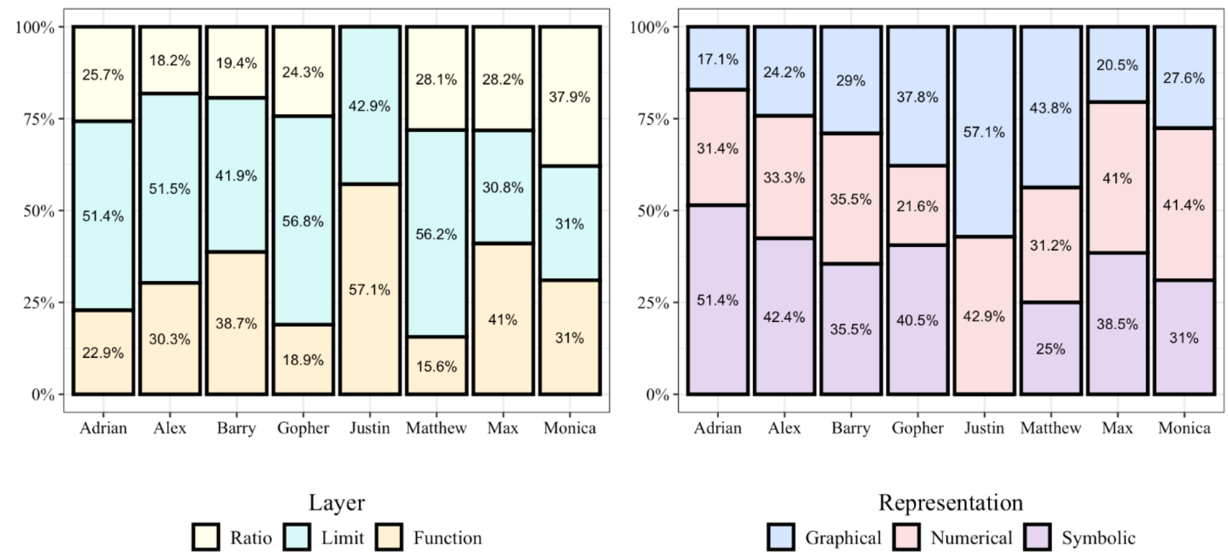

Figure 4. Distribution of process-object layers & representations within limit-based CTUs across instructors

Discussion and Conclusion

Our analysis of introductory derivative instructional tasks from eight experienced college calculus instructors showed that they predominantly targeted pure and physics contextual framing and limit-based epistemology within the conceptions of derivative laid out by the CoD framework (Gerami et al., 2025). However, the prevalence of limit-based epistemology may reflect both our sample selection and the study's original design around Zandieh's (2000) limit-based framework. Additionally, because we specifically asked instructors for tasks introducing derivatives, the focus on limit-based approaches may reflect the instructors' tendency to use this epistemology as the foundational approach when first teaching the concept.

These findings suggest several directions for future research. First, researchers could examine conception targeting across entire instructional materials, rather than just introductory instructional tasks, to capture how targeted conceptions might shift throughout the course. Second, future studies could use the CoD framework to map relationships between instructor targeting of specific conceptions and actual student conception development. Finally, given that colleges increasingly offer calculus courses designed for specific audiences such as biology and business majors, researchers might investigate whether intended student population influences instructors' choices of contextual framing and conceptual epistemologies when teaching derivatives.

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