Developing Position From Velocity Through Playful Exploration and Feedback

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Kinematics graphs offer potential for developing relationships between derivates and antiderivatives, but students experience challenges with recognizing the meaning of the area under a velocity graph as change in position. We designed a series of playful math tasks offering structured exploration, in which students worked in virtual environments to develop relationships between speed, time, and distance in tabular and graphical formats. Preliminary analysis suggests that students can use rate reasoning to develop accumulated distance under constant velocity graphs, which then can be developed under linear graphs via Riemann sums.

Keywords: Mathematical play, calculus, kinematics, velocity, position

Graphs related to kinematics, such as position, velocity, and acceleration, play an important role in calculus instruction (Barniol et al., 2024; Doorman & Gravemeijer, 2009), offering a rich potential for addressing the relationships between derivatives and antiderivatives (Doorman et al., 2022). An understanding of the kinematics concepts requires students to make sense of the relationships between the graphs of position (the function), velocity (first derivative), and acceleration (second derivative), as well as conceptualizing the antiderivative as the area under the curve between, for instance, the velocity and the change in position (Dominguez et al., 2017). Despite the centrality of these ideas, research indicates student difficulties in making connections between kinematics concepts and their graphical representations (McDermott et al., 1987). In particular, it can be challenging for students to recognize the meaning of the area under the curve for a velocity or acceleration graph (Bajracharya et al., 2012; Beichner, 1994; Martínez-Miraval & García-Cuéllar, 2020), much less to construct meaning for why the area under a velocity curve, for instance, represents change in position.

We became curious about the viability of helping students meaningfully develop the area under a graph (such as a velocity graph) as change in position (i.e., distance traveled) through playful exploration. In working with a group of pre-service secondary mathematics teachers, all who had successfully completed a calculus sequence, we implemented a task to promote an understanding of the relationship between position, velocity, and acceleration in terms of integrals and derivatives (Ely et al., 2025). The task required students to move an astronaut 10 miles in 10 minutes via a constant acceleration (or a series of accelerations) for specified time periods. For instance, if one uses an acceleration of $a(t) = 0.2$ miles / min^2 from $t = 0$ to $t = 10$, the astronaut's velocity is the antiderivative, $v(t) = 0.2t$, $0 \leq t \leq 10$. One could then find the total distance traveled by integration, either by reasoning with the area under the velocity graph or symbolically as $\int_0^{10} v(t)dt = 0.1t^2 \Big|_0^{10} = 10$ miles. We found that our participants struggled to distinguish between acceleration and velocity and did not draw on their understanding of integration to determine exact distance values, findings that mirrored the literature. This led us to wonder whether we could we support students to reason about change in position from velocity graphs. Specifically, could we help students meaningfully consider the area under a velocity curve as a representation of distance traveled by leveraging structured exploration through playful math tasks? Our preliminary findings suggest that students can use their rate reasoning to develop accumulated distance under constant velocity graphs, which then can be developed under linear graphs via Riemann sums with instructor scaffolding.

Background and Theoretical Framing: Area as Distance and Mathematical Play

Areas Under Velocity Curves

We anticipated that students might be able to leverage both geometric reasoning and rate reasoning to develop area as a measure of distance. Regarding the former, the Mean Speed Rule could be relevant to students' reasoning. The rule states that if, over a period of time, Object A has a constant velocity and Object B constantly accelerates (or decelerates) with the same average velocity as A, then the two objects will travel the same distance. Mathematicians at Oxford developed this in the early 1300s, three centuries before coordinate geometry and without referring to area under a velocity curve. One of these scholars, William Heytesbury, reasoned with situational symmetry to justify the rule: If one object constantly accelerates from a velocity of four to eight (no units specified), and another decelerates from four to zero, the latter will offset the former at each moment, so they together must traverse twice the distance of a third object moving at a constant speed of four (Katz, 1998).

Regarding rate reasoning, students may be able to leverage their unit rate understanding to reason about why one can multiply a height by a width to yield an area in the first place (beyond simply relying on the trick of 'canceling units', e.g., ft/sec · sec = ft). Izsák (2025) suggested a general multiplication image that treats velocity as a unit rate. From this perspective we can see a y-value on a velocity graph, such as 20 ft/sec, not just as a one-dimensional height, but as a rate of 20 ft in 1 sec, corresponding to an imagined two-dimensional stack of 20 squares in a 1-sec-width column. With this image, each square is a foot, so one can imagine each second on the x-axis accumulating 20 ft in distance as represented by the area of a two-dimensional column with a width of 1 sec.

Mathematical Play, Exploration, and Feedback

We define mathematical play as entailing exploration, self-selection of goals, immersion, and enjoyment (Ellis et al., 2025). Of particular importance to our focus is exploration, which is a key element of play (Gresalfi et al., 2018) and which can occur in a physical or virtual environment, or even in an imagined realm in which students can run mental simulations. Exploration often starts with free play (Mason, 2019), in which learners familiarize themselves with the task, the environment, and its affordances. It entails asking questions such as "what happens if I try this?" and noticing the results of one's actions. As learners' actions become more goal-oriented, they will attend to the feedback from those actions and modify their subsequent actions in response (Williams-Pierce & Thevenow-Harrison, 2021). Over time, one's actions can then become increasingly anticipatory, with students ultimately able to make mental simulations without having to carry them out in the environment (Simon, 2011).

Researchers have discussed mathematicians' playful activity in terms of exploration and feedback (e.g, Holton et al., 2001; Mason, 2019; Su, 2020). For instance, Su (2020) described an inquiry phase, in which one engages in pattern exploration and reasons from empirical feedback to build conjectures from specific examples. Holton and colleagues (2001) similarly described a process in which mathematicians produce tentative examples without a clear direction to make sense of the problem space before then becoming more systematic and intentional in generating specific cases and conjectures. Mathematicians are adept at engaging in exploration and generating their own feedback, often relying only on paper and pencil or even engaging in purely mental simulations. Students, however, may need more structured environments in which they can try out different actions and observe empirical feedback. For that reason, we designed a series of playful math tasks that encourage exploration while also providing a method for intrinsic feedback. These tasks place students in a virtual environment in which they can explore what happens when they input different speed/time values or acceleration/time values to achieve various goals, such as making an astronaut reach 10 miles in 10 minutes. By providing visual feedback, our intention was to help students structure their exploratory actions so that they could shift to more anticipatory actions over time, specifically by developing their own velocity graphs to make objects travel a specified distance in a specified time.

Methods

We collected data with 7 high-school students (2 9^{th}-grade students, 5 10^{th}-grade students) during an 8-session summer design experiment (Cobb et al., 2003). The sessions took place over a two-week period and each lasted 90 minutes. The students worked in small groups using iPads, and 6 members of the author team were present to observe, engage students in questioning, and run the cameras. We collected video data and screen recordings on each group's iPad as well as all written work. We designed a series of playful math activities to structure students' exploration of constant rate via speed situations, in which students could input various speed and time values to achieve goals such as helping a character reach a specified destination at a particular time. This report is based on the engagement of one trio of students, Adam, John, and Zilar (pseudonyms), during two activities called *Go Karts* and *Describe My Journey*.

Go Karts is a Desmos activity that shows a red go kart that moves at a constant speed of 10 ft/sec, traveling 200 ft in 20 sec. Students can input up to five velocity/start time/end time values in a table to make a blue go kart reach the finish line at the same time. For instance, one could input the following four entries to make the blue go kart tie with the red go kart: 40 ft/sec from 0 sec to 1 sec; 0 ft/sec from 1 sec to 10 sec; 15 ft/sec from 10 sec to 18 sec; and 80 ft/sec from 19.5 sec to 20 sec. Once they have chosen values, students can hit play and watch both go karts run the race to observe the outcome of their actions. We then shifted from tabular inputs to a graphical input, providing students with a blank graph velocity / time graph. Students could then draw graphs to make their go kart tie with the red go kart. Our intention was to support students in creating constant graphs (such as a constant function at 10 ft/sec) and to use their rate reasoning to consider how much distance would be traveled after 1 sec (10 ft), 2 sec (20 ft), and so forth, relating distance traveled to the rectangular area under the constant velocity function. We then anticipated that students might explore linear and piecewise linear graphs, using geometric reasoning similar to the mean speed rule to calculate the areas of triangles under the velocity graph to determine distance traveled. The second activity was *Describe My Journey*, in which groups of students created and traded velocity/time graphs. Each group's task was then to describe the go kart's journey based on the provided graph.

Our coding process is preliminary and ongoing. We began by transcribing the video of the students' activity and enhancing the transcript with relevant images, pauses, and descriptions of the students' gestures. We then completed an initial reading of the data noting where in the tasks students reasoned about velocity and distance. Once we identified those data portions, we examined the supports for students' reasoning, including the use of feedback from the playful math tasks as well as more direct scaffolding from the teacher-researchers (TRs). Our continuing process will next shift to analyzing the interrelated nature between students' exploration, empirical feedback, and emerging conjectures and strategies.

Results

The students explored different graphs with the *Go Kart* activity, developing first a constant graph at 10 ft/sec for 20 sec, and then reasoning geometrically about symmetry above and below the $v(t) = 10$ line to develop piecewise constant velocity functions that yielded a total distance of 200 ft. For instance, they created a graph for the function $v(t) = \begin{cases} 12, & 0 \leq t \leq 10 \\ 8, & 10 < t \leq 20 \end{cases}$ and were able to predict that for the first 10 sec the go kart would travel 120 ft and then it would travel 80 ft for the remaining 10 sec. They also created the piecewise constant function seen in Figure 1a, predicting before running the simulation that their go kart would tie with the red go kart. Adam explained, "The velocity's 30, and that's feet per second, so 30 feet per second and that's 5 seconds, so 30 times five that's 150 feet. So, 150 feet, so that's, and you're trying to go 200. So, you have 150, and, it stops twice so that's not adding anything. And the last one, it's 5 seconds, or, 10 velocity, so five times 10 is 50, and that's 200." The students could also predict that the graph provided by a TR (Figure 1b) would beat the red go kart because it would travel a total of 216 ft in 20 sec (14 ft/sec for 12 sec and 8 ft/sec for 8 sec). However, during the creation of these graphs, the students did not produce linear functions. It was unclear to us whether they would be able to develop area under a linear velocity function to determine total distance traveled.

During the next activity, *Describe My Journey*, Adam, John, and Zilar worked with a graph produced by their classmates (Figure 1c). They could calculate slope values and make sense of the graph in terms of the go kart's journey. They interpreted the constant portions of the graph as constant speeds and they interpreted the portions with negative slope values as periods of deceleration. They could also interpret the portions of the graph below the t-axis. For instance, for the portion of the graph from $t = 5$ to $t = 15$, the slope is -1.1. Adam described the go kart as going backwards for 10 seconds, accelerating at a rate of "1.1 feet per second per second." Similarly, for the portion of the graph from $t = 19$ to $t = 25$, Adam explained, "So you decelerate at 1 feet per second per second for six seconds. This one isn't even that hard!" Describing the final portion of the graph (below the t-axis with a positive slope), John explained, "You slowly go back to zero. So, you're slowing, you're backing up until you're at zero."

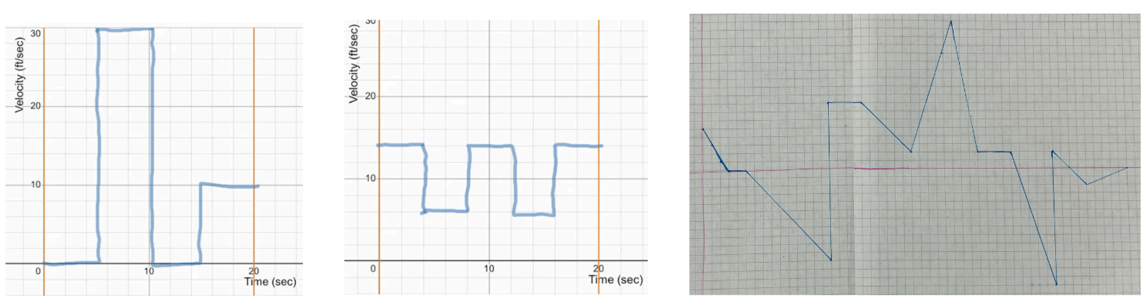

Figure 1. Student-produced velocity/time graph of a go kart's journey (a), instructor-produced graph (b), and student-produced graph for the Describe My Journey activity (c).

At no time during their interpretation of the graph did the students determine distance traveled. A TR therefore asked the students how far the go kart traveled during the constant portion of the graph from $t = 15$ to $t = 19$ where $v(t) = 8$. Adam responded, "You're going 8 feet per second for 4 seconds. So, you're going 32 feet." After the students then determined the distance traveled for the other constant portion of the graph, the TR asked about the linear portion from $t = 19$ to $t = 25$. John reasoned, "You de-accelerate at a rate of one foot per second per second per six seconds" and Adam replied, "So, you go six feet?" John replied, "Well, you start at eight and you end at two for velocity." Adam responded, "You're going 1 feet per second for six seconds," and John objected, saying, "Well, you're *de-accelerating* at a rate of 1 foot per second," and Adam and Zilar both agreed. The TR asked, "Tell me more about the speed. So how is it changing?" John responded, "It's decreasing at one foot per second, per second. From eight feet per second, per second, to two."

The TR then suggested breaking that portion of the graph into a finer picture and asked the students to consider what would happen in the first second. John said, "You go down by one foot per second, per second." The TR asked, "Now, if I ask you how much you travel in that one second, what would you say?" John hesitantly responded, "Seven point five?" The TR asked why that would be the case, and John traced a horizontal line across the portion of the graph from $t = 19$ to $t = 20$ and hesitantly said, "Because if you're going at eight and seven, would it average out to these 7.5 straight line?" We interpret John as imagining a constant line at $y = 7.5$, and the TR suggested that the students pursue this idea further. Adam said, "Like, six point five," indicating the amount traveled in the next second, and John said, "five point five," and Adam continued, "Four point five plus three point five plus two point five…" and John concluded, "Which is thirty."

Preliminary Questions and Discussion

We had anticipated that the students would make a connection to area under piecewise constant velocity graphs by perceiving the area of rectangles as in Figures 1a and 1b. However, there was little mention of area in the students' conversation. Instead, they often appeared to be reasoning with rates but not directly with area. The students understood that x ft/sec meant traveling x ft in 1 sec, $2x$ ft in 2 sec, and so forth by iteration. Our initial observation, both from the data reported here and from our similar work with pre-service teachers, is that students with this reasoning can then approach the problem of finding the distance traveled with a linear velocity with a Riemann sum, leveraging reasoning similar to the mean speed rule; this seems to be a natural concept that emerges from reasoning with rates to determine accumulated distances. This suggests that certain cognitive activities related to calculus are shared by secondary and undergraduate students.

Our initial analysis indicates that the exploration and feedback students engaged in when creating and analyzing graphs for the *Go Kart* activity supported their ability to predict distances using constant and piecewise constant velocity graphs, as over time they began to be able to make those predictions without needing to run the simulations. We would now like to better understand how to leverage structured exploration and feedback to make the area connection more salient for students, or even to perhaps necessitate it (Harel, 2013) through the task structure. We will therefore engage the audience in the following discussion questions: (a) What is the role of exploration and feedback in supporting students' development of area as distance traveled under velocity curves? (b) What adjustments to task structure could we make to support the transition from constant velocity graphs to linear and piecewise linear velocity graphs?

Acknowledgments

This work was supported by the National Science Foundation under ERC-Core #2347820.

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