An Analysis of Connections Between Epistemological Obstacles Experienced Within the Quantification Unit of a Transition-to-Proof Course

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Students' conceptions play a large role in how they learn math. When students experience challenges, or epistemological obstacles, they may struggle to adapt their ideas that previously served them well. Little research has been done regarding interactions between co-occurring epistemological obstacles. This report utilizes a proposed adjacency matrix methodology for analyzing the co-occurrence of epistemological obstacles. Our results show that students' difficulty noticing hidden quantification had strong thematic connections to two other challenges: quantifying the negation and transforming logical implications into related forms.

Keywords: Epistemological Obstacles, Quantification, Proof and Proving, Logical Implication

Prior research has documented students' persistent challenges when reasoning about logical statements in transition-to-proof courses (TTP; e.g. Dawkins & Roh, 2020; Epp, 2003; Norton et al., 2025). Many of these challenges persist even in response to research-based instruction, and researchers characterize such challenges as necessary in students' development of logical structures, such as logical implication (LI; Norton et al, 2025; Steffe, 1992).

Antonides et al. (2025) observed that existing mathematics education research usually studies students' learning challenges independently, even though students might experience multiple challenges in tandem. Norton et al. (2025) found that students' ways of reasoning impact the profile of challenges that they experience. To support researchers, Antonides et al. (2025) proposed a methodology for mapping connections between co-occurring learning challenges.

This paper aims to add to the literature by investigating challenges experienced concurrently by students during research-based instruction on quantification. Quantifiers, such as "for all," play an essential role in mathematical study as they, when taken together with predicates, allow for reasoning about mathematical objects, their properties, and relationships between objects. Moreover, there is a wealth of literature documenting students' persistent challenges with quantification in logical statements (e.g. Dawkins & Roh, 2020; Epp, 2003; Shipman, 2016).

We asked the following research questions: (a) During research-based instructional interactions on quantification, what challenges do students experience concurrently? and, (b) What ways of reasoning do students demonstrate when challenges are experienced concurrently within the same interaction? In response, we analyzed video of the first six classes of the quantification unit in an undergraduate TTP course. This paper reports on the study's results and shares themes in student reasoning within the two pairs of EOs that co-occurred most frequently.

Theoretical Framework

Following Brousseau (2002), we characterize learning challenges that persist in response to research-based instruction as *epistemological obstacles (EOs)*. They are "the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (p. 82). Students may experience an EO when they become aware of a conflict or perturbation in their thinking, but teachers may notice this tension first.

We situate our investigation of concurrent EOs within Antonides et al.'s (2025) proposed methodology for mapping co-occurrences of EOs experienced within instructional interactions.

This six-step process begins with a literature review of EOs and it culminates in the generation of an adjacency matrix whose rows and columns are these EOs and whose entries indicate the number of times pairs of EOs co-occur in a single instructional interaction. The resulting adjacency matrix can then be visualized as a weighted graph, allowing "researchers to visualize connections between EOs and the strength of each connection" (p. 1321). We elaborate on this six-step process within our Methodology section below.

Methodology

To collect data about EOs, we used the 6-step methodology outlined by Antonides (2025).

Step 1: Literature Review

An extensive literature review was conducted within a larger study, "The Proofs Project" (e.g., Norton et al., 2025). We summarize here the EOs pertinent to our study.

Students consistently struggle with implicit, or "hidden," quantification (coded Qh; Shipman, 2016; Durand-Guerrier, 2003; Epp, 2003). For example, "If $x > 2$, then $x^2 > 4$" implicitly refers to *all* x. When quantification is unhidden, students struggle to quantify a statement's negation (coded Qneg; Dawkins & Roh, 2020). Multiple quantifiers introduce additional struggles (coded Qmult; e.g., Adiredja, 2020; Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000).

Students also experience difficulties when proving quantified statements. For example, they may question why the *Principle of Universal Generalization* (see Copi, 1954) proves a "for all" statement (coded PUG; Dawkins et al., 2022; Norton et al., 2023). Students may also conflate the value of a variable (e.g. $\varepsilon = 1$) with its role (e.g. the distance between two points; coded Qrv; Dawkins & Roh, 2022).

Challenges related to LIs of the form $p \to q$ have also been documented. Students might treat LI as biconditional (coded LIff; Epp, 1999, 2003; Rumain et al., 1983; Wagner-Egger, 2007) or struggle to transform an LI into, for example, its converse or negation (coded LIt; e.g., Antonini, 2004; Hawthorne & Rasmussen, 2015; Hub & Dawkins, 2018). Furthermore, students may conflate the negation of $p \to q$ with its opposite statement, $p \to {\sim}q$ (coded NO; Epp, 2003; Arnold et al., 2024), and they may experience difficulties drawing distinctions between the negation and a disproof of a statement (coded NF; Norton et. al., 2025).

Step 2-4: Data Collection and Analysis

Data consist of a subset of six video-recorded 50-minute classes within the quantification unit of an inquiry-oriented undergraduate TTP course taught by the second author. Table 1 outlines the topics discussed during the classes analyzed within this study (Class 15 was a review day and was not analyzed). Students sat at tables in groups of approximately five students, engaging in group discussion captured on iPads via audio- and screen-recording. We refer to the five groups as the Hexagons, Rectangles, Circles, Squares, and Triangles. We analyzed the recorded videos,

Table 1. The classes analyzed for this report and the respective subjects covered.

Class	Discussion
13	Introduction to the existential quantifier
14	Writing existential proofs, writing nonconstructive existence proofs
16	Comparing constructive vs. nonconstructive existence proofs
17	Introduction to the universal quantifier
18-19	Writing "for all" proofs, constructive vs. non-constructive for all proofs

Table 2. A table with descriptions of EOs that occurred.

EO Code	Description
Qh	Overlooking hidden quantifiers
Qneg	Demonstrating uncertainty about the quantification of the negation
NO	Treating the negation $p \rightarrow q$ as its opposite statement $p \rightarrow \sim q$
PUG	Questioning or neglecting the *principle of universal generalization*
LIt	Difficulty transforming LIs into related forms (converse, negation, etc.)
Qrv	Conflating the role of a variable with its value
NF	Struggling to articulate the difference between negation and disproof
LIff	Interpreting LIs as biconditional (e.g. assuming $p \rightarrow q$ means $q \rightarrow p$)
Qmult	Struggling to make meaning of statements with two or more quantifiers

noting timestamps where EOs were elicited. We marked EOs based on a codebook (see Table 2) developed from our literature review. When students were perturbed, we coded the interaction with an EO that most closely captured their experience. After coding, we demarcated our data into chunks by instructional task. Video from the task launch, group discussions about the task, and subsequent whole-class discussion comprised a single chunk.

Step 5-6: Generating and Interpreting Results

We counted co-occurrences within each chunk as follows. If no EOs were coded, our matrix A remained unchanged. Otherwise, for each distinct pair of EOs (i,j) within a chunk, we updated our matrix A, increasing by 1 the entries in row i column j (A_{ij}) and row j column i (A_{ji}). For example, when Qh, Qneg, and PUG co-occurred within a chunk, regardless of the number of each code's occurrence, a 1 was added to $A_{\text{Qh,Qneg}}$, $A_{\text{Qneg,Qh}}$, $A_{\text{Qh,PUG}}$, $A_{\text{PUG,Qh}}$, $A_{\text{Qneg,PUG}}$, and $A_{\text{PUG,Qneg}}$. All distinct codes experienced by any group were counted as co-occurrences. Using this adjacency matrix, we generated a graph in which each vertex was an EO and weighted edges between EOs represented the pair's number of co-occurrences. Then, we analyzed the strongest connections visible in the graph for themes across chunks in which those connections appeared.

Results

The adjacency matrix and weighted graph, shown in Figure 1, identify which EOs (see Table 2) students experienced concurrently (RQa). We omitted from our graph those EOs not elicited or with no connections. 8 chunks had concurrent EOs, 5 chunks had a single EO, and 2 chunks had zero EOs. Several EOs co-occurred at least once, with Qh and Qneg exhibiting the most connections with other EOs (17 and 11, respectively). The graph highlights two prominent

$$
\begin{array}{c}
\begin{array}{ccccccccc} \text{Qh} & \text{Qneg} & \text{NO} & \text{PUG} & \text{LIt} & \text{Qrv} & \text{NF} & \text{LIff} & \text{Qmult} \end{array}\\
\begin{array}{c} \text{Qh}\\ \text{Qneg}\\ \text{NO}\\ \text{PUG}\\ \text{LIt}\\ \text{Qrv}\\ \text{NF}\\ \text{LIff}\\ \text{Qmult} \end{array}
\begin{pmatrix}
- & 3 & 2 & 2 & 3 & 2 & 2 & 1 & 1\\
3 & - & 2 & 1 & 1 & 2 & 2 & 0 & 0\\
2 & 2 & - & 1 & 1 & 1 & 1 & 0 & 0\\
2 & 1 & 1 & - & 1 & 2 & 0 & 0 & 0\\
3 & 1 & 1 & 1 & - & 0 & 1 & 1 & 0\\
2 & 2 & 1 & 2 & 0 & - & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 1 & 0 & - & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & - & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -
\end{pmatrix}
\end{array}
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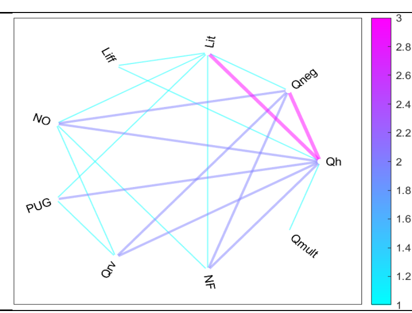

Figure 1. Our study's resulting adjacency matrix (left) and weighted graph of EO connections (right).

Table 3. *All chunks where Qh, Qneg, or LIt co-occur. Gray rows indicate tasks reported on in this paper.*

Class, Chunk	Task Name	Co-Occurring EOs
13, 3	Proof Comparison Task (PCT)	Qh, LIt, Liff
14, 4	Vector Multiplicity Task (VMT)	Qh, Qneg, NO, PUG, Qrv
16, 1	Types of Existence Proofs	Qh, Qneg, NF
16, 2	Proof Evaluation Task (PET)	Qh, Qneg, LIt, NF, NO
17, 1	Proving vs. Assuming Existence Task	Qh, LIt, PUG

connections that appeared within three chunks each: (a) hidden quantification and quantifying the negation (Qh-Qneg), and (b) hidden quantification and transforming an LI (Qh-LIt). The chunks and associated tasks in which Qh, Qneg, or LIt co-occurred are outlined in Table 3. We elaborate on results from the Proof Comparison Task (PCT), Vector Multiplicity Task (VMT), and the Proof Evaluation Task (PET) because they exemplify each prominent connection twice.

Proof Comparison Task (PCT)

In PCT (Figure 2), students considered whether two proofs did or did not prove the given existence statement. *Proof 2* is valid, but *Proof 1* assumes the existence of x: "Suppose that $x^3 - 4x^2 - x + 4 = 0$" implicitly means "Suppose *there exists an x such that* $x^3 - 4x^2 - x + 4 = 0$."

Every group's discussion about PCT was coded as Qh because students experienced difficulty unhiding the quantification of x in *Proof 1*. In the Triangle group, a student said, "If this was in the way of if-then, then *Proof 1* might not be correct because it started with the conclusion. Since it's a 'there exists,' it's fine. You just need to prove the existence." Because *Proof 1* began with the equation, she compared this to beginning a proof of an LI by assuming the conclusion. Whereas she understood this was a flawed approach to proving an LI, she believed it was acceptable when proving an existence statement. Another student said, "Yeah, I think they both work. Just, one of them gives you a single answer and the other gives you three answers." She distinguished the two proofs only in terms of how many "answers" were provided. Following group work, the instructor explained why the first proof assumed the existence of x. This prompted one student from the Rectangle group to ask, "Would it be possible to, like, reword it into, like, 'there exists an x such that if the equation equals 0… x =1 or something? [...] If you were to transform [PCT existence statement] into an implication, then, like how would *Proof 1*…would it still hold?" When the instructor observed that *Proof 1* would be a proof of the converse of the proposed LI, the student was surprised–prompting us to code the interaction as LIff. We also coded LIt because the student experienced difficulty attempting to transform the given statement into an equivalent LI.

Consider the statement:

There exists a positive integer x such that $x^3 - 4x^2 - x + 4 = 0$.

For each proof below, decide whether it does or does not prove this statement.

Proof 1.	*Proof 2.*
Proof 1. Suppose that $x^3 - 4x^2 - x + 4 = 0$. Then, $$\begin{aligned} x^2(x-4) - (x-4) &= 0 \\ (x-4)(x^2-1) &= 0 \end{aligned}$$ So, $x = 4$ or $x = \pm 1$. Since $x = 1$ is a positive integer, the claim is true. $\square$	*Proof 2.* Let $x = 1$. Then, $$\begin{aligned} x^3 - 4x^2 - x + 4 &= (1)^3 - 4(1)^2 - (1) + 4 \\ &= 1 - 4 - 1 + 4 \\ &= 0. \end{aligned}$$ So, the claim is true. $\square$

Figure 2. The PCT from Class 13, chunk 3.

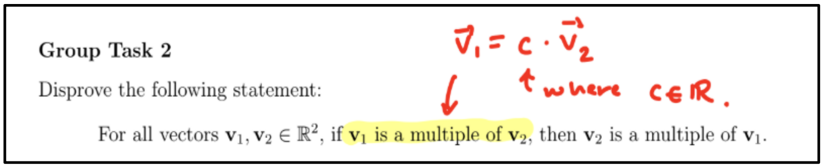

Figure 3. The Vector Multiplication Task (VMT), given during Class 13, Chunk 4.

Vector Multiplicity Task (VMT)

VMT is shown in Figure 3. Before students began work on this task, the instructor provided the definition of "v_1 is a multiple of v_2." Note that both the hypothesis and conclusion hide the quantification of a scalar c within their definitions. Moreover, c need not be the same value in each statement. Four groups arrived at the equation $v_2 = 1/c\ v_1$ and then spent significant time attending to this scalar; the Triangle group focused on recalling linear algebra concepts and never derived the equation. Of the four groups, two groups did not discuss c's quantification (coded Qh) or attempt to write a proof and two groups experienced difficulty attending to quantification of c in the conclusion (coded Qneg) while writing a proof.

The Circle group determined that a disproof entailed providing a counterexample.

Student 1: So since it says for all vectors v_1 and v_2, could we just find a counterexample? [...]

Student 2: Or you can do it specifically. v_1 equals like $\langle 1,1 \rangle$ and you multiply it by 2 …

Student 1: Yeah, or I guess you could say, like, so $v_1 = cv_2$, then that means $v_2 = 1/c\ v_1$ and since 1/c is not a real number… I mean it's a real number… but it's not like an integer.

Student 2: I think it's supposed to be that c is an integer, otherwise it would be true.

While discussing, the group's scribe wrote the statement and its negation symbolically (Figure 4). After solving $v_1 = cv_2$ for $v_2 = 1/c\ v_1$, they questioned whether $1/c$ was a valid multiple. There was no mention of c's quantification in either definition. Their inclusion of "where $c \in \mathbb{Z}$" for both the original statement and their negation (Figure 4) further corroborates that: (a) they overlooked the shift in quantification of c in "v_1 is a multiple of v_2," and (b) c's quantification remained unchanged in the negation. We also hypothesize that their writing, "where $c \in \mathbb{Z}$," seemed only to mimic that of the instructor's (Figure 3), and we conclude that the existential quantification of c remained implicit in the word "where" (coded Qh).

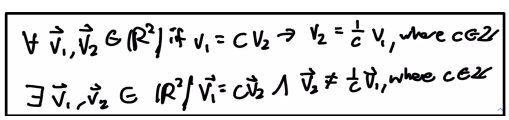

Figure 4. The Circle group's understanding of the VMT's original statement and its negation.

In contrast, the Rectangle group discussed the quantification of the scalar, which they called k. After identifying a counterexample wherein v_1 should be the zero vector, they noted $v_1 = kv_2$ when $k = 0$. However, they struggled with how to show precisely that v_2 was not a multiple of v_1.

Student 1: ...prove that v_2 times $1/k$ does not equal v_1 because… 1 over zero can't exist.

Student 2: Honestly, it doesn't have to be k.

Student 1: Well, it can be any letter.

Student 2: Well, the k has to be different. […]

Student 1: It'd just be $1/k$. Like you'd write, so you'd, like, let $kv_2 = v_1$ and then, like, you show that is $1/k\ v_2$ and let $k = 0$ and you prove that that doesn't exist. If k is 0, $1/k$ can't exist. You don't have to use two separate…

Student 2: But, I'm saying like it doesn't have to be $1/k$. It could be anything. […]

Student 1: Yeah, yeah, but I'm saying, like, the general case.

The group seemed to agree that they must show no scalar existed, but their understandings of "there does not exist" seemed to influence their reasoning differently; Student 2 understood it as "for all," whereas Student 1 seemingly did not. Instead, he viewed $1/k$ as addressing "the general case," perhaps because k was a variable. However, his focus on algebraic manipulation overshadowed his ability to operationalize the meaning of "there does not exist." We characterize this as Qneg, as Student 1's quantification of k in "v_2 is not a multiple of v_1" prevented his noticing that $1/k$ introduced a dependency that removed arbitrariness of the scalar.

Group Task 1

Discuss with your group whether the proof does or does not prove the following statement.

If $x > 0$ then $x + \frac{1}{x} \geq 2$.

Proof. Assume toward a contradiction that there exists an $x \in \mathbb{R}$ such that $x > 0$ but $x + \frac{1}{x} < 2$. Let $x = 1$. Then,

$$x + \frac{1}{x} = (1) + \frac{1}{(1)} = 2 \geq 2.$$

This is a contradiction. Thus, it must be that $x + \frac{1}{x} \geq 2$. □

Figure 5. The Proof Evaluation Task, given during Day 16, chunk 2.

Proof Evaluation Task (PET)

PET (Figure 5), adapted from Shipman (2016), invites students to analyze a flawed proof by contradiction. The task is designed to perturb students' thinking regarding whether they can or cannot choose the value of a variable x when it is assumed to exist. When one assumes an x exists, its value is fixed, yet unknown. This stands in contrast to the logic of proving existence, wherein we may choose a specific x-value. Students experienced a variety of EOs when working on PET, but most notably our strongest connections, Qh-Qneg and Qh-LIt, co-occurred within this task. We note that only one group's discussion was coded as Qh with all others coded Qneg.

A student in the Square group began discussing saying, "'If *P*, then *Q*'… negation is, what, *P* and not *Q*?" The group agreed it was, and one student explained, "Yeah, so it's assuming the negation, then showing the negation is false, so the original statement must be true." In attending to the predicate of the negation (*P* and not *Q*), the group overlooked the quantification of x (coded Qh). With quantification hidden, they concluded the proof was valid.

In the Rectangle group, one student observed, "I feel like if you're assuming toward a contradiction that there exists an x-value then you can't give a constructive example…because there's nothing guaranteeing that the example you give is the assumed existing value." Another student questioned, "Are you allowed to use just one example or do you have to prove for all x-values?" expressing uncertainty about quantification (coded Qneg). Discussion then shifted to what statement the proof *did* prove. One student asserted it was a valid proof of the converse which prompted one student to disagree, and another to say, "That's the negation." They experienced difficulty transforming the LI to fit the logic of the proof (coded LIt).

Discussion

Our results revealed, via number of connections, the centrality of Qh and Qneg in the challenges students experience within instructional interactions on quantification. While we might expect Qh and Qneg to have a strong connection (as negating quantification requires unhiding it first), the equally strong connection between Qh and LIt was surprising. In the discussion that follows, we characterize students' ways of reasoning when the connections Qh-Qneg and Qh-LIt occurred (RQb).

For students to experience the challenge of quantifying the negation, they must first unhide the quantification. Therefore, because challenges are EOs only when they *persist*, students cannot experience both Qh and Qneg simultaneously. So, when Qh and Qneg appeared together, it indicated that, for some groups, quantification remained hidden, and for others, attending to quantification was challenging.

PCT was the only task discussed above in which Qh appeared without Qneg. Because all five groups experienced Qh, we conjecture that students' existing ways of operating on algebraic equations (solving for x), prevented them from noticing implicit quantification. In VMT, only those groups who attempted to write a proof experienced Qneg. Without reaching this stage, quantification of the scalar remained hidden. In PET, only the Squares experienced Qh, as they were focused instead on the logical structure of an LI and its negation. This finding echoes Norton et al.'s (2025) claims that one must conceptualize LI as a single object before they can attend to its quantification. All other groups were perturbed (coded Qneg) by the proof's use of a single example x=1.

Our Qh-Qneg analysis validates existing literature (e.g. Dawkins & Roh, 2020) that students still experience difficulties with quantification after unhiding it. Students' existing ways of reasoning also seem to impact students' abilities to unhide quantification, but the act of proof writing appears to assist students in attending to quantification.

The two tasks, PCT and PET, in which this connection appeared engaged students in determining whether a given (flawed) proof was valid. Each proof contained implicit quantification that perturbed students and, when hidden, resulted in their conclusion that the proof was valid. In the case of PET, those groups that successfully unhid quantification determined the proof was logically flawed. LIt was coded, during whole-class discussion in PCT and group discussion in PET, when students tried to determine a statement that the proof *did* prove.

Our analysis of the Qh-LIt connection suggests tasks that engage students in proof evaluation are effective in eliciting experience of Qh. And, when a proof is invalid, students have opportunities to analyze the logical structure of the proof to determine what, if anything, the proof does prove. This echoes a study by Hub & Dawkins (2018) who used proof evaluation tasks to elicit student reasoning in clinical interviews.

Conclusion

Our implementation of Antonides et al.'s (2025) methodology for mapping connections between EOs supported our identification of key themes in student reasoning. Hidden quantification was highly connected to all challenges experienced by students, and most strongly connected to Qneg and LIt. Analysis of these two connections revealed three key findings. First, students' existing ways of reasoning (e.g. algebraic reasoning and treatment of LI) influenced their ability to unhide quantification. Second, engaging students in proof writing assisted students in noticing quantification. Lastly, proof evaluation tasks in which quantification is implicit are effective in eliciting students' experience of Qh and in offering students opportunities to transform statements to determine the logical structure of a proof.

Studying co-occurring EOs highlighted nuances in students' reasoning and offers perspective on the challenges related to quantification documented in prior literature. Our study showed students' experience of hidden quantification was essential for them to have opportunities to experience other necessary challenges (e.g., Qneg and LIt). This finding suggests the importance of instructors' abilities to notice, elicit, and address head-on EOs during instruction.

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