Expert Perspectives on the Set of Outcomes for a Non-Standard Presentation Counting Problem

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The objective of a counting problem is to determine the cardinality of a set of outcomes, but there is never a singular, unique way to conceive of those outcomes. I report on data of combinatorics experts solving a counting problem whose numerical solution is 2^6, a standard expression to see in combinatorics, but whose set of outcomes is not standard. I demonstrate four distinct models that the participants used to conceive of the outcomes, and how each of those led to different ways of explaining the solution. I argue that these differences among experts reflect the challenging nature of articulating what the set of outcomes for a counting problem is, and how to approach a solution using this set. These data may inform how counting could be presented to learners when adopting a wide view on the multiple possible sets of outcomes for any given problem.

Keywords: combinatorics, mathematicians, expert reasoning

Consider the Roads Problem: How many ways are there to traverse the diagram from point A to point B in Figure 1, if no line segment can be repeated? This counting problem is notable because the outcomes being counted—paths along a particular diagram—are not immediately similar to common archetypal problems. It is possible to create a bijection between the outcomes and length-6 binary strings, which means that there are 2^6 total outcomes, and the Roads Problem can be considered to involve arrangement with repetition. However, as I elaborate later, it is also possible to conceive of the outcomes as involving subsets, or simply as path diagrams independent of auxiliary representations. There is no singular set that can be described uniquely as the set of outcomes, and the model of outcome used can impact how a solution is found, and the form it takes. If a set of outcomes is the foundation of a counting problem, what does it mean for combinatorics if there are multiple contenders for what that set is?

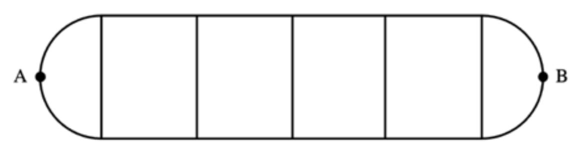


Figure 1. Diagram for the Roads Problem.

This paper reports on data of nine combinatorics experts solving the Roads Problem during task-based interviews. I draw on these data because four distinct models of the set of outcomes emerged from the participants, which demonstrates the multiple faces of a set of outcomes even among experts. These results can provide additional context when working with K-16 students, particularly when conventional solutions might depend on a specific way of conceptualizing or encoding the outcomes.

Literature Review and Theoretical Framework

Student difficulties in combinatorics have been well-documented, including the challenge of distinguishing between problem types. Batanero and colleagues (1997) found that students can have difficulty distinguishing between problem types, and Lockwood (2014) reported that students sometimes use their ‘gut’ when the problem statement does not specifically say order matters. Literature has increasingly applied an approach that values how students create and justify solutions based on the outcomes being counted. For example, systematically listing sets of outcomes (English 1991, 1993; Montenegro et al., 2022), or even thinking about how to do so (Halani, 2012) has been shown to correlate with successful solutions even at the undergraduate level (Lockwood & Gibson, 2016). Some of this work reports on how students conceive of outcomes, and how those conceptions impact the way they find or justify solutions (e.g., Lockwood et al, 2018; Wasserman & Galarza, 2019), showing that certain (sometimes unfamiliar or unintuitive) ways of encoding outcomes are more likely to lead to successful solutions than others.

Lockwood (2014) has advocated for a set-oriented perspective on counting, where attending to the sets of outcomes is a fundamental aspect of a solution. This perspective draws from her (2013) model of student reasoning, which examines the interactivity between *sets of outcomes, formulas/expressions*, and *counting processes.* The set of outcomes for a counting problem is the set of elements being counted, the cardinality of which is the solution to the problem. Formulas and expressions refer to the mathematical formulas and expressions (such as $\binom{n}{k}$ or 2^6) used in the course of the solution, and what is typically used to write the answer. I refer to mathematical expressions that solve a counting problem as *counting expressions.* Counting processes refer to internal or external processes a counter engages with as they solve a problem (such as listing). Wasserman (2019) argued that counting expressions can also represent a particular enumerative strategy or counting process. For example, although $\binom{n}{k}k = n\binom{n-1}{k-1}$ both count headed size k committees chosen from n people, and are thus numerically equal, they represent two distinct ways of counting those committees. The left-hand side first selects the members of a committee before choosing the head, and the right-hand size chooses the heading before selecting the remaining members. I adopt Lockwood's model for student reasoning in this paper, and I discuss how the duality of counting expression can be observed when different conceptions of a set of outcomes are used to solve a counting problem.

Four Models of the Set of Outcomes in the Roads Problem

To illustrate how the duality of counting expressions might arise when solving the Roads Problem, I present four different ways to model the outcomes. These models are my own formalizations of the approaches I observed from experts, and which I expand on in the Results section.

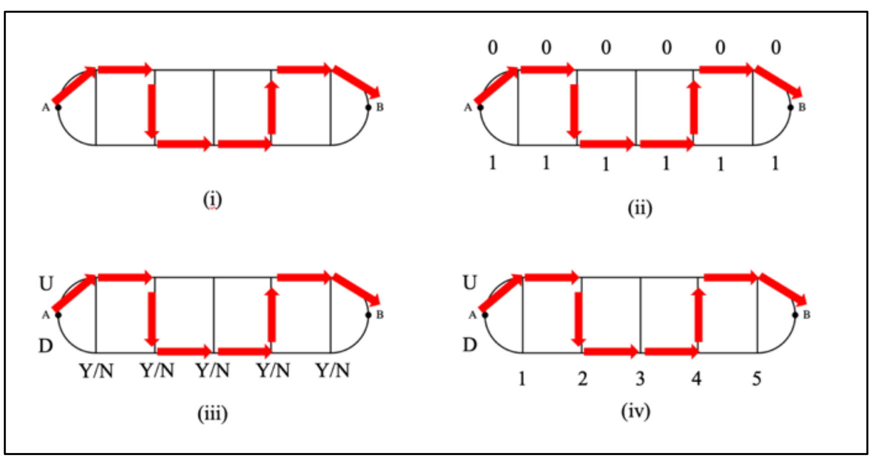


Figure 2. One path shown under four augmented diagrams.

Literal model. The literal model interprets outcomes as visual paths along the diagram. This model does not include an alternate way of encoding or otherwise representing an outcome. I refer to this model as literal because it most closely aligns with the outcomes as described in the problem statement.

Binary model. The binary model encodes paths as a six-character sequence of 0s and 1s, where each 0 or 1 refers to a path segment along the top or bottom of the diagram, respectively. This model does not directly encode vertical path segments, but rather those segments can be deduced from the encoded information. The path in Figure 2.ii represented in the binary model is 001100. This model creates a bijection between all possible paths and six-character binary sequences, and thus there are 2^6 possible outcomes.

Sequential model. The sequential model encodes paths as an initial decision to move up (U) or down (D) from point A, followed by a sequence of choices to cross (Y) or not cross (N) at each vertical path segment. The path in Figure 2.iii represented in the sequential model would be UNYNYN. There are two possibilities for the first decision (U or D), and two possibilities (Y or N) at each subsequent intersection, and so there are $2 \cdot 2^5$ possible outcomes. Note that I distinguish between $2 \cdot 2^5$ and 2^6 because the former illustrates how the first decision (U or D) is a distinct type of decision from the others.

Subset model. The subset model encodes paths as an initial decision to move up (U) or down (D) from point A, followed by a subset of $\{1, 2, 3, 4, 5\}$ to indicate which of the vertical segments occur in the route. The path in Figure 2.iv represented in the subset model would be U$\{2, 4\}$. There are two possibilities for the first decision (U or D), and 2^5 possible subsets of a 5-element set, so thus there are $2 \cdot 2^5$ possible outcomes. I have used 2^5 to represent the ways of counting subsets to reflect the reasoning of the experts reported on in this study, but another counter using the same model might have used a different expression to represent the subsets.

The binary, sequential, and subset models demonstrate distinct models of the outcomes, even if they are used to produce the same or similar mathematical expressions. It is not possible to exchange a representation in one model for a representation in another through a direct character exchange, and each model selects some but not all information about a path to create a representation. For example, despite both binary and sequential models using six-character sequences to encode routes, the outcome 001100 cannot be converted to UNYNUN by a simple exchange of 0s for N and 1s for Ys. Moreover, the binary model selects only information about horizontal segments, and the sequential model selects only information about the vertical segments. In terms of Wasserman's duality of mathematical expression, 2^6 and $2 \cdot 2^5$ represent distinct enumerative processes, and moreover the expression $2 \cdot 2^5$ can even take on more than one interpretation.

Methods

The data presented here were collected as part of a larger study that examined the role of representations in solutions to counting problems for both undergraduate students and combinatorics experts. Both populations were presented with a sequence of tasks meant to elicit the use of representations as part of the solution process. The task sequence for the expert population was comprised of eight problems, the first four of which were selected from the student sequence, and the second four of which were not given to students. I chose expert tasks from the student tasks because they were non-standard presentations of common problem types. Most experts participated in one 90-minute interview, and their work on the tasks comprised most of that interview. However, some experts participated in two shorter interviews due to scheduling constraints, although they were given the same materials. Most of the experts only completed the tasks that were given to both populations, and only one expert completed all eight problems.

I focus here on expert solutions to the Roads Problem, which was the first task given to the experts. I present these data because prior analysis of student solutions on the problem revealed several approaches that reflected differences in how the outcomes were modeled, which seemed to impact how students approached a solution. While the comparison between the populations is not reported here, the expert solutions demonstrate similar impacts between model and solution approach. Moreover, several of the experts commented that the problem was not one they already knew how to solve, and none of them stated the answer prior to reasoning about the outcomes verbally. I thus argue that the work discussed here demonstrates how experts approach an unfamiliar problem that can be solved with their existing combinatorial tools, rather than how experts recall prior solutions.

The experts were recruited for the study because they were either combinatorialists or education researchers who had previously studied combinatorial reasoning. Six combinatorialists and three educators participated in the study. Their pseudonyms and areas of study are presented in Table 1. The research areas of the educators are only summarized to preserve anonymity. Seven of the participants were tenure-track, one was an instructor, and one was a postdoctoral scholar. Three participants had taught graduate-level combinatorics courses, and five had taught undergraduate-level combinatorics courses, or courses that included combinatorics.

Table 1. Participant demographic information.

Pseudonym	Expertise	
	Mathematician (M) or Educator (E)	Research Area
Austin	M	Algebraic and enumerative combinatorics
Craig	E	Student reasoning
David	E	Student reasoning
Erik	M	Algebraic and enumerative combinatorics
Leo	M	Graph theory and discrete geometry
Oscar	M	Logic, computability theory, and graph theory
Seamus	E	Student reasoning
Silvio	M	Group theory
Ted	M	Algebraic and geometric combinatorics

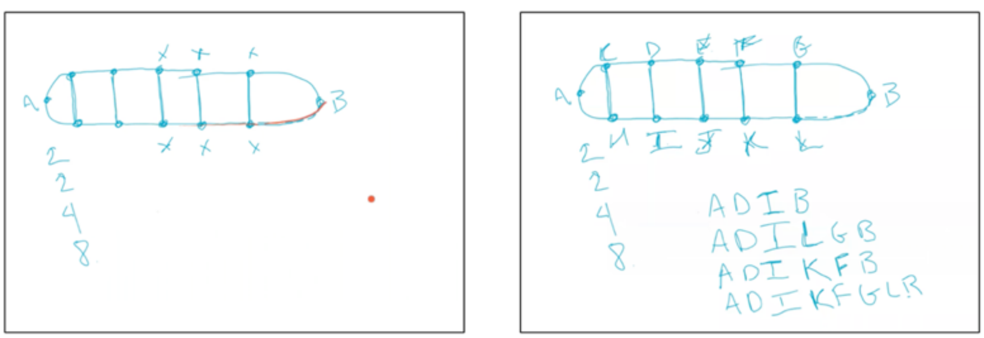
All interviews were conducted over Zoom, and both audio- and video-recordings of the interviews were collected. One participant solved tasks on a blackboard that was visible during the interview, but the remaining participants solved tasks on an electronic tablet whose screen was shared. I created enhanced transcripts of the interviews, where screenshots of the participants' work were used as references alongside the text of the interview. I performed thematic analysis (Braun & Clarke, 2022) on the transcripts to identify the different models of the set of outcomes, problem solving techniques, and how the participants leveraged their conceptualizations of the outcomes to identify a solution.

Results

I present the four models of outcomes I observed, and how these models impacted how the participants sought and justified their solutions. Present in many of these solutions, but not expounded on here, was reasoning about whether a path could move 'backward.' Some participants quickly reasoned that this could not occur, but for Austin this seemed to be the impetus that led to his model.

Literal Model

Craig was the only participant whose work primarily involved reasoning about the literal outcomes, and he spent approximately 37 minutes solving the problem. He began by tracing paths along the diagram with a stylus, which appeared as a red line on the screen that followed his stylus but was not permanent. He organized his tracing by categorizing the outcomes based on the first vertical segment to be part of the path. The first two paths he traced were those that went entirely on the top half or bottom half of the diagram, and the second two paths were those that went on the top or bottom half of the diagram until reaching the last vertical segment. Craig kept track of the number of paths he had traced mentally, and he would record the total number of paths in each category once he had traced all of them (see Figure 3.i). Although it seemed that Craig had developed an internalized way of keeping track of which paths he had or had not traced, he never verbalized it and the red line from his stylus did not appear on screen consistently enough to develop inferences. After tracing out the first four categories, he appeared to have difficulty keeping track of which paths he had already traced, and he labeled each vertex with a letter, as seen in Figure 3.ii. He then used the labels to record paths for a short time, but he commented "This also feels like it's a representational nightmare" before erasing them. It seemed that the primary use of the labeling was to keep track of which outcomes he had traced, and they did not seem to be directly relevant to reasoning about a mathematical expression.



(i) (ii)

Figure 3. Craig's sketching of the literal model, and his use of the model.

After observing that the first five categories included 2, 2, 4, 8, and 16 outcomes, Craig predicted that the last category would include 32 outcomes, and the total number of outcomes would be $2 + 2 + 4 + 8 + 16 + 32 = 64$. When I asked him to describe why there might be powers of two, he responded "Um, having to do with decisions […] each time you reach a point you have a decision. Um, the thing that I don't love about it is that I was trying to think about, uh, for a little bit why I got 2 and 2." By this, I infer that Craig connected the summand of 2^n with the last n vertical paths, but he could not justify why that pattern was inconsistent with the number of paths that used zero vertical segments. I did not push Craig to further justify his solution. Craig appeared to have difficulty using the literal model, perhaps because they were distinct from archetypal outcomes for counting problems, and he had not developed techniques to identify structural patterns within the set.

Binary Model

Austin was the only participant to use the binary model of the outcomes, but he did so only at the end of his solution. The first part of his solution involved decreasing the number of vertical segments in the diagram, and then sketching each of the literal outcomes. By doing so, he observed that there were 2^{n+1} total paths for a diagram with n vertical segments, and he stated "So I'm envisioning a bijection between, um, paths and bit strings." This bijection is illustrated in Figure 4. Austin used this observation to claim that there were 2^6 paths around the original diagram because he knew that there are 2^6 six-character bit strings. This method uses prior knowledge about binary strings to create a solution, and Austin stated "it's unfamiliar […but…] it just absolutely gives up its secrets when you start doing small examples."

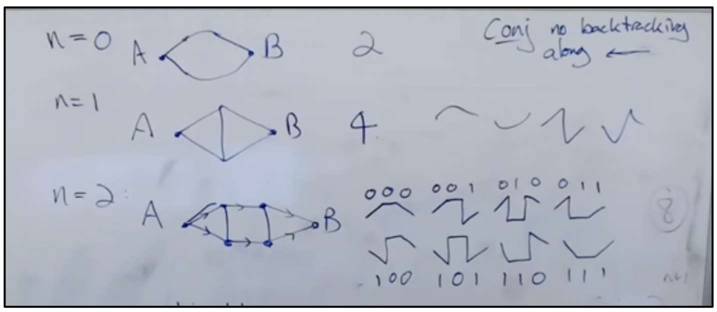


Figure 4. Austin's work using the binary model.

Sequential Model

David, Leo, Oscar, Seamus, and Silvio each used the sequential model. David characterized this method by "we could probably compute the, like, the total number of routes or paths in terms of the number of choices you have to make along the way," which draws an equivalence between a path and a process that would generate that path. Unlike the binary model, where each path was represented by an element of a well-understood set with a known cardinality, the sequential model counted the number of ways a process could be carried out by an agent that traversed the diagram. Similar to David, Oscar and Seamus used language like "if I go down I'd have to go to this point, and again I have two choices there," and "if we went on road two we'd have the same options but in… sort of reverse," indicating they imagined themselves or an agent traversing the diagram, and the path traced out by the agent depended on a sequence of decisions. To further distinguish this model from the binary model, the characterization of the path depended on which vertical components were taken, and each expert differentiated between the first decision (up or down from A) and the subsequent decisions, whereas the binary model depended on which horizontal components were taken and did not distinguish the first path segment from the others. Oscar summarized the method by stating "maybe the right way to think of it is… a route is 'do I take the bridge or not, yes or no?' And then also, but I, realizing that I have two choices initially." This statement seems to indicate that using a sequence of decisions was not merely a process used to count the literal model of outcomes, but instead a complementary way of conceiving of the outcomes themselves.

Subset Model

Erik and Ted used the subset model of the outcomes, which was like the sequential model but included a noticeable difference in terms of how the decisions related to time. Unlike the sequential model, where the experts seemed to think of a path as a sequence of decisions an agent would make while they traversed the diagram, the subset model seemed to make those decisions prior to an imagined traversal. For example, Ted stated "the idea I'm gonna do is say, like, which of these middle paths am I gonna cross?," which indicates selecting the paths prior to traversal, and Erik stated "So I think it really is just any collection of those five vertical roads tells me exactly what I'm going to do […] So, step one is do I go… Do I start up or down. Then, I pick any subset." Erik's statement indicates a distinction between the first decision and subsequent decisions, and both statements indicate choosing the paths prior to diagram traversal. This sentiment seems to be reflected in their final solutions, both of which counted subsets as a single object with cardinality of 2^5. By comparison, although 2^5 appeared in solutions using the sequential model, this expression seemed to indicate one unit repeated five times rather than a single unit. Ted's written work, shown in Figure 5, appears to reinforce that 2^5 stood for the single unit of subset rather than five distinct decisions.

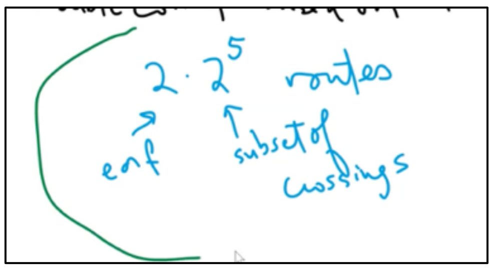


Figure 5. Ted's solution delimitating the first decision from the subset.

Discussion and Conclusion

The expert solutions discussed here shed some light on the nature of sets of outcomes in combinatorics problems. The literal outcomes as described in the statement of the Roads Problem—paths along a diagram—were used as the exclusive set of outcomes by only one participant. All other participants created alternative models of the paths (binary strings, sequences of decisions, and subsets) that seemed to inform how they found a solution and the form of the solution.

It did seem to be the case that the alternative models were strictly artifacts of counting processes the participants employed. The language used by the participants, such as when Oscar said "a route is 'do I take the bridge or not, yes or no?," shows that they were using the literal model to create a secondary model they could hold simultaneously. The second model also demonstrated beliefs about the outcomes that might provide insight on the relationship between outcomes and counters. For instance, the experts who used the sequential model used language that indicated the outcomes would be constructed as an agent traversed the diagram, while the experts who used the subset model used language that indicated the outcomes would be determined prior to an agent traversing the diagram. The binary, sequential, and subset model reflected different temporal relationships between the outcomes and the counter. Whereas the binary model did not include a temporal relationship, the sequential and subset models included language that referenced future or in-the-moment decisions, respectively. The models also showed a difference between counting static complete sets of outcomes (all binary strings), and processes that would generate a single outcome (sequential model).

Acknowledgments

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References

Batanero, C., Navarro-Pelayo, V., & Godino, J. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. Educational Studies in Mathematics, 32, 181-199.

Braun, V., & Clarke, V., (2022). *Thematic Analysis: A Practical Guide.* Sage Publishing.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics, 22,* 451-47.

English, L. D. (1993). Children's strategies for solving two- and three-dimensional combinatorial problems. *Journal of Mathematical Behavior, 24*(3), 255-273.

Halani, A. (2012). Students' ways of thinking about enumerative combinatorics solution sets: The odometer category. In the *Electronic Proceedings for the Fifteenth Special Interest Group of the MAA on Research on Undergraduate Mathematics Education.* (pp. 59-68) Portland, OR: Portland State University.

Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior, 32,* 251–265. doi:10.1016/j.jmathb.2013.02.008

Lockwood, E. (2014). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics, 34*(2), 31-37.

Lockwood, E., & Gibson, B. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics, 91*(2), 247-270. doi:10.1007/s10649-015-9664-5

Lockwood, E., Wasserman, N. H., & McGuffey, W. (2018). Classifying combinations: Do students distinguish between different categories of combination problems? International Journal of Research in Undergraduate Mathematics Education, 4(2), 305–322. https://doi.org/10.1007/s40753-018-0073-x.

Montenegro, J., Rosa Borba, E. de S., & Bittar, M. (2021). Registers of Semiotic Representations Aiding the Learning of Combinatorial Situations. *The Mathematics Enthusiast, 18*(3).

Wasserman, N. H. (2019). Duality in combinatorial notation. For the Learning of Mathematics, 39(3), 16–21.

Wasserman, N., & Galarza, P. (2019). Conceptualizing and justifying sets of outcomes with combination problems, *Investigations in Mathematics Learning,* 11(2), 83-102, DOI: 10.1080/19477503.2017.1392208