Verbalized Transformational Reasoning in Small Group Discourse

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Transformational reasoning is a powerful form of reasoning that can be leveraged during collaborative mathematical activity. However, little research has been done to investigate how transformational reasoning supports collaboration, if at all. Using Brown's (2025) framework for identifying verbalized transformational reasoning and Staats' (2021) poetic structure methodology, the present study sought to investigate how transformational reasoning supported students to collaborate. The results demonstrate that students spent the bulk of their time collaborating on a collective solution by warranting the operation(s) they envisioned transforming the objects.

Keywords: Transformational reasoning, poetic structures, discourse analysis

Inquiry classrooms provide many opportunities for students to engage in collaboration, which has been shown to improve student success. Verbalizing one's reasoning and building on others' reasoning is what drives collaboration. Researchers studying collaborative mathematical activity are interested in understanding how the individual's reasoning, once verbalized, contributes to the collective solution of the mathematical activity. Hence, analytical tools that allow the researcher to track the developing collaborative activity while remaining focused on the students' shared reasoning are needed. Staats (2021) makes the case that common discourse analysis methods "discourse away" the mathematical reasoning of the students (p. 1). To address this concern, she introduced a method of discourse analysis that focuses on student reasoning, called *poetic structures* analysis.

The term poetic structure comes from the work of the linguistic anthropologist Roman Jakobson (1960). Staats comments that in Jakobson's conception, "any time a repetition causes listeners to attend to the form of the statement, or to use the form of a statement to construct meanings, the poetic function of language is in play" (2008, p. 29). A poetic structure is a verbal repetition that involves "similar syntax and at least one word in common" between two phrases (Staats, 2021, p. 5). This definition provides other scholars with a means for identifying meaningful repetition that share mathematical ideas.

Staats (2008; 2021) argued that students communicating their reasoning is the foundation of understanding. Furthermore, "…if communication is a central process for mathematical learning, we must account for the linguistic processes that allow students to construct mathematical ideas" (Staats, 2008, p. 26). Repetition with similar syntax "allows a speaker to recognize an existing idea and modify it in a way that is concise and easy to understand" (Staats, 2008, p. 29). Hence, the use of poetic structures supports the researcher in tracking the ways in which various warrants evolve through conversation between individuals as a collective product is developed (Staats, 2021).

For this study I adopt a socio-cultural perspective of learning. A socio-cultural perspective of learning takes into account that classrooms are socially and culturally situated environments that are not removed from one's prior experience and not separate from one's peers. Coordinating the notions that communication is the foundation of understanding (Staats, 2021), and learning is actively participating in a community of practice (see Lave & Wenger [1991]), communication supports individuals to engage in authentic activity of the community in which they are seeking membership. Furthermore, mathematics classrooms are micro-communities that approximate the mathematics community at large; students learn by becoming active *doers* of mathematics. This suggests that there are two intertwined processes that take place in the individual, a cognitive and a social component of their learning.

I conducted a multiple-case study to understand how transformational reasoning (Simon, 1996) supported students to develop a collective product. I coordinated Brown's (2025) framework for identifying and describing verbalized transformational reasoning with Staats' (2021) poetic structure analysis to understand how verbalized transformational reasoning was used by the students as they developed a collective solution. In this report, I will emphasize the cognitive component of transformational reasoning and demonstrate that the students' verbalized reasoning supported collaboration. As was evident in the ways in which various phrases were repeated and evolved into the collective solution that was produced.

Theoretical Perspective and Relevant Literature

Transformational Reasoning

Simon (1996) coined the term transformational reasoning to describe a type of mathematical reasoning that he noticed students were spontaneously engaging in, that resulted in the students coming away with a sense of how a model worked. Simon defines transformational reasoning in the following way:

> Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. (1996, p. 201)

This definition includes (a) an object (or objects) that is (are) being transformed, (b) the operation(s) that affect the object (or objects), and (c) results of the operation(s) on the object(s). The process of transformational reasoning involves identifying (a) and (b), which allows one to *envision* the transformation taking place, and once (c) is attained one can examine them together to draw conclusions about the entire process to understand why it works. Envisioning the operation(s) on the object(s) is what constitutes transformational reasoning, but the components serve as signals of it.

Scholars have studied transformational reasoning in a variety of math contexts. This scholarship demonstrates that transformational reasoning can support students to: (1) learn about rates of change (Carlson et al., 2002; Johnson, 2012), (2) identify the invariant algebraic properties in related rates problems (Engelke Infante, 2021), (3) ascertain evidence in both proving and proof-related activity (Brown, 2025; Harel, 2001; Harel & Sowder, 1998), (4) continually refine and generalize their arguments (Ellis, 2007), and (5) identify key ideas (Raman, 2003; 2004) for a proof (Zandieh et al., 2008). All of the research here supports the claim that transformational reasoning can foster the development of a deep conceptual understanding.

I studied five pairs of students engaged in different implementations of an inquiry-oriented mathematics task and sought to understand how transformational reasoning contributed to the collective math activity. Specifically, I investigated the following research question: *how does the use of verbalized transformational reasoning facilitate the evolution of the collective product of the students' mathematical activity?*

Methods

Context and Background

The data analyzed in this report come from the NSF-funded project, *ASPIRE in Math* (DUE# 1916490). The *ASPIRE in Math* project sought to support students in their transition to advanced math while also supporting instructors to shift to inquiry-oriented instruction. Among the project deliverables is a curriculum for the guided reinvention of fundamental concepts in real analysis. The curriculum development efforts were guided by the design theory of realistic mathematics education (Gravemeijer, 1994). The curriculum tasks take students step by step through developing a conjecture equivalent to the Intermediate Value Theorem (IVT), developing an approximation method to find roots of continuous functions, and then leveraging the approximation method to introduce concepts like sequences and convergence, among others (Larsen et al., 2022).

All of the data comes from two different public institutions in the Pacific Northwest; these included a research-focused public university and a community college. All of the participants were math majors or minors, and the names in this report are all pseudonyms. The data analyzed consists of videos of each of the pairs of students as they worked on a task, transcripts of the video, and Google Docs of the students' work.

For this report, I analyzed student work on the *fifth-degree polynomial task*, which states: true or false, every fifth-degree polynomial has at least one real root. Students were paired up and asked to justify their answer. The goal of the task is to generate a conjecture that is logically equivalent to the IVT. Often students will observe that the claim is true and justify it using algebra or calculus knowledge. First, the students may recall that odd degree polynomials have end behavior that tends to infinity in one direction and negative infinity in the other, so it must pass through the x-axis. Alternatively, students may use an algebraic approach that leverages factoring to identify potential complex roots in pairs. While both solutions are valid, the class proceeds with the end behavior argument since the goal is to reinvent the fundamentals of analysis.

Once the arguments have been generated, the teacher then works with the students to scrutinize the main components of the argument to tease out the IVT conjecture assumptions. After these activities, the teacher asks the students to write a claim in the form: If [blank], then f has at least one real root. One example of this claim is "if a function f is continuous with both a positive and negative output, then f has at least one real root".

The Cases

The overarching methodology used in this study is a multiple-case study (Yin, 2018). A case consists of a pair of students working on the fifth-degree polynomial task. The case begins when the students share their initial reasoning and ends when the focus shifts to writing a conjecture. See table 1 below for an overview of the cases.

Table 1. Description of the cases

Case	Student Names	Context	Date
1	Chloe & Gabe	Teaching Experiment	Summer 2018
2	Will & Kevin	Whole Class Teaching Experiment	Spring 2020
3	Shay & Justin	Introduction to Proof Class	Winter 2021
4	Andrew & Jovan	Bridge to University Mathematics	Winter 2022
5	Harriett & Josue	Introduction to Proof Class	Fall 2022

Poetic Structures Analysis

Staats (2021) details her methodology for identifying and describing poetic structures in transcript data. A poetic structure is a phrase that "shares syntax and at least one word in common" (Staats, 2021, p. 5). Repeated syntax may take the form of Subject-Verb-Object, or something similar.

In alignment with Staats' (2021) approach, I went through the transcripts line-by-line. In a spreadsheet I recorded (a) the line number, (b) current speaker, (c) phrase that I determined was a repetition, (d) line number of original speaker, (e) original speaker, (f) source of the repeated phrase, and (g) whether or not the repeat was internal or across talk turns. A poetic structure was only recorded if it was on the same topic, so repeats like "3 minus 2" and "4 times 8" would not be considered poetic structures (Staats, 2021). Each poetic structure was the longest stretch of talk that captured the idea the student expressed; this improved the validity in the analysis since it allowed me to capture more of the students' verbalized reasoning (Staats, 2021, p. 6). Staats notes that the method cannot be relied upon to identify all of the poetic structures present in the data, rather the goal of the method is to "value consistency over completeness" (Staats, 2021, p. 8). Lastly, I identified each poetic structure according to Staats' (2021) framework, with the addition of Brown's (forthcoming dissertation) poetic structure of *elaboration* (see table 2).

Table 2. Types of Poetic Structures

Type	Definition
List	Saying a pattern
Echo	Revoicing a short phrase to confirm listening or validate an idea
Comparison	Saying two things are associated
Contrast	Posing two things as alternatives
Interposed List	Two lists are collated
Consolidation	Two or more previous poetic structures are combined
Expansion	A previous phrase has a novel clarifying phrase inserted
Reversal	Two phrases switch places syntactically
General	Repetition mobilizes a prior comment without a distinctive structure
Elaboration	A previous phrase that gets repeated with some words replaced in order to clarify an idea

Transformational Reasoning Analysis

Once the poetic structures analysis was complete, each transcript was coded according to components of Simon's (1996) definition of transformational reasoning. My previous work (Brown, 2025) demonstrated the utility of operationalizing Simon's definition of transformational reasoning. I identified objects, operations, and outcomes. Objects were mainly nouns or pronouns that named or referred to mathematical objects. Operations were adjectives or verb phrases that indicated that the student had modified a mathematical object. Outcomes were most commonly in the form of a claim that included a reference to the object. Each instance of transformational reasoning was recorded in a memo along with a description of my model of the students' envisioning activity. The combination of objects, operations, outcomes, and the description of what a student may be envisioning constitutes (at best) a second order model of student thinking.

Final Steps

Once the poetic structures and transformational reasoning analyses were completed, I overlaid them to analyze how verbalized transformational reasoning was communicated via a poetic structure, if at all. Finally, I conducted a cross-case analysis to extract meaningful patterns in the ways in which the transformational reasoning was used to advance the collective product of students' mathematical activity.

Results

In this section, I will discuss the ways in which students used poetic structures when verbalizing or refining verbalized transformational reasoning as they developed a collective product. I begin by showing how the various components of transformational reasoning were expressed using poetic structures.

Objects and Outcomes

Of the three components of transformational reasoning, objects and outcomes are the most obvious for the observer to witness. Students often make statements about objects and in the claims that they make, one can find the outcome of their transformational reasoning. Objects were not a major component that needed to be negotiated in the fifth-degree polynomial task since the students were largely working with the object as a fifth-degree polynomial as a graph or an equation.

Outcomes were often unchallenged by the students. However, within a students' own explanation the outcomes were commonly rephrased utilizing a poetic structure. For example, consider Shay's (they/them) initial justification for the fifth-degree polynomial claim.

> It'll always have… it will always have at least one real root, because the two… because it's an odd degree. One end will always approach negative infinity and the other will approach positive infinity. So, it'll have to approach… it will have to cross zero at one point on the negative side.

Underlined above is the poetic structure that Shay used to *elaborate* on the meaning of having one real root. The single underline represents a "direct" repeat, and then the double underline demonstrates what was changed from the first sentence to the last. Shay's transformational reasoning likely involved mentally comparing several generic odd-degree polynomial graphs, where they focused on the tails of each graph and observed that one tends to positive infinity and the other to negative infinity. Then it is likely that Shay envisioned tracing a point along the curve. Shay's new outcome "it will have to cross zero…" indicates a shift in their thinking from "always have at least one real root" which likely resulted from envisioning the curve passing through the x-axis.

Similarly, in Will and Kevin's group, Will provided his intuition for the IVT by using a poetic structure to verbalize his transformational reasoning. Will says, "[The IVT] just basically says, if it goes two different places then it has to go to the place in between those two places". In this quote, the double underlined portion is an *expansion* poetic structure. Will expressed his intuition for the IVT as an IF-THEN statement. The transformational reasoning that he verbalized indicates that Will was imagining two points separated on a coordinate-plane. Then the operation "to go" indicates he likely envisioned connecting the two points as one would by drawing a graph on paper. The expansion poetic structure served the purpose of connecting not only his hypothesis to his conclusion, but the objects and operation to the outcome he envisioned.

As was discussed in the two examples here, the operation that one envisions transforming the objects plays a major role in explaining one's reasoning. In what follows, I will discuss how the operation was warranted through poetic structures.

Warranting Operations

Many of the conversations during the fifth-degree polynomial task involved articulating operations that transform objects. Here I will discuss two examples of pairs of students warranting a claim by making the operation they envisioned transforming objects explicit.

Chloe and Gabe were tasked with determining whether or not $p(x) = x^5 - x - 5$ has a real root on $[0,2]$. The pair began by seeking to factor the polynomial. Once that attempt proved fruitless, they shifted gears to graphing the polynomial. The pair proceeded as expected by evaluating the polynomial at integer values zero and one, followed by $x = ½$ and one of the teacher-researchers observed that the pair appeared to be seeking a sign change.

Teacher-researcher: So, you're just trying to see if the output is positive or negative?

Gabe: Right, to see if it crosses the, uh, x-axis.

Teacher-researcher: So, how would that tell you that it crosses the x-axis?

Gabe: For this one it gives insight into seeing what areas, like, if we see that, if we saw that this was positive [pointing with left hand to the y-intercept $(0, -5)$ and with the right hand to a location above the x-axis where $x = ½$] we'd be like, somewhere between here and here [moves hand from the location of $x = 0$ to $x = ½$ on the x-axis], oh it definitely crossed for sure.

Teacher-researcher: Mm, why's that?

Gabe: Cuz it's a continuous function and it's not just going to skip a chunk of the y-axis. So, it would have to cross the x-axis at some point.

There are a number of poetic structures in this short transcript excerpt. The most salient poetic structure to pay attention to is Gabe's elaboration in line 4 above (double underlines). Gabe elaborated on his explanation in line 2, after the teacher-researcher sought to ascertain the goal of Chloe and Gabe's activity. In line 4, Gabe expanded on the operation he envisioned as transforming the objects, but still he glossed over the hidden assumption. In line 5, the teacher-researcher probed Gabe's thinking further. This prompted Gabe to make the operation explicit: he was using continuity. Gabe's statement about continuity is an *elaboration* (dotted underline) given that he replaced the warrant from line 4 but kept the same conclusion. Importantly, the explication of this warrant proved to be a critical point in the students' activity, and it later showed up as a major warrant in their written argument.

Andrew and Jovan. The bulk of Andrew and Jovan's activity involved trying out different operations on fifth-degree polynomials and determining whether or not that operation preserved the fifth-degree polynomial. Consider the following exchange that occurs between Andrew and Jovan.

Andrew: Um, ok so like, if it's a negative 5th degree polynomial, that still has one real root, it's just on the opposite side of the y-axis.

Jovan: Yeah, it would just be going. It would start going down.

Andrew: Mhmm. It would begin going down, do your bumps [as he is speaking, he moves his right hand from left to right, and wiggles his fingers to indicate the 'bumps'], and then continue going in the direction it started, right?

Jovan: Well since it's odd, it would just be, instead of going up from left to right, it would be going… right? [While Jovan is speaking, Andrew draws a curve in the air with his stylus]

Andrew: Right, like the whole function would be coming from the top, squiggly squiggly, and then continue downward through the x-axis [as Andrew is speaking, he sketches the function he is imagining, doing the squiggles as he says squiggles, and showing the overall trend of the curve]. Or if it already was, it would just be downward, but at some point, it will cross…[[Andrew goes on to compare odd and even degree polynomials]]

Jovan: Yeah.

Andrew: So, a negative wouldn't change that…

Throughout this entire exchange, Andrew and Jovan are negotiating Andrew's operation of "negative". Andrew's transformational reasoning likely involved a series of transformations on different objects. For starters, it is likely Andrew appended a negative to the leading coefficient of some polynomial equation he was working with in his mind. Then, given his routine use of DESMOS in class, he likely plotted the equation he had in mind. Once he and Jovan began discussing the claim, it is likely Andrew then shifted to thinking of the graph as emergent, given the way he sketched the graph in the air and described the shape of the graph.

Jovan began by trying to warrant Andrew's operation by describing what he envisioned happened to the graph after operating on it via "negative". Andrew *expands* on Jovan's verbalization in line 3 by describing what he envisioned in terms of the outcome of the transformation. Jovan is still seeking clarification, and he now compares the shapes of fifth-degree polynomials to negative fifth-degree polynomials. Andrew's final *expansion* in line 5 of the excerpt satisfied Jovan's uncertainty, as evidenced by Jovan's "yeah" in line 6. The negotiation of this operation revealed the key point for Andrew and Jovan: that a fifth-degree polynomial has a real root whether its graph emerges up from the left to the right, or down from the left to the right.

Discussion and Conclusion

The findings presented above suggest that the teacher can anticipate that conversations about transformations of objects are likely to be a key part of mathematically productive collaborations. In terms of components of transformational reasoning, these transformations are what I referred to above as the operations. As the students collaborated on the collective solution to the fifth-degree polynomial task, they sought to warrant their belief by verbalizing the operation with which they transformed the objects. Gabe warranted that the polynomial was continuous. This was a pivotal connection with which Gabe and Chloe argued that a root existed. For Andrew and Jovan, the operation that Andrew expanded on allowed the pair to warrant that the fifth-degree polynomial claim was plausible at all.

Additionally, the findings suggest that a signal of successful verbalized transformational reasoning is actively connecting the claim to the observation of the outcome of the transformation that was envisioned. For example, Shay began by stating confidently that a root always exists, but through use of the operation "to approach", they were able to connect the claim to the definition of a root by stating that "it must cross zero". Despite the encouraging findings from this study, more research is needed to ascertain productive ways to support students to engage in transformational reasoning actively.

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References

Brown, D. (2025). Transformational reasoning in an introduction to proofs course. *Proceedings of the Twenty-Fifth Annual Conference for Research on Undergraduate Mathematics Education*, 1068 - 1073. Alexandria, VA.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Charmaz, K. (2006). *Constructing Grounded Theory: A Practical Guide through Qualitative Analysis*. London, UK.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education, 38*(3), 194–229.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for research in Mathematics Education, 25*(5), 443-471.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society, 7*, 234–283.

Jakobson, R. (1960). Closing statement: Linguistics and poetics. In T. Sebeok (Ed.), *Style in language* (pp. 350–377). Cambridge.

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior, 31*(3), 313–330. https://doi.org/10.1016/j.jmathb.2012.01.001

Larsen, S., Alzaga Elizondo, T., & Brown, D. (2022). Following in Cauchy's footsteps: Student inquiry in real analysis. In R. Biehler, M. Liebendörfer, G. Gueudet, C. Rassmussen, C. Winsløw (Eds.), *Practice-oriented Research in Tertiary Mathematics Education*, (pp. 431 – 447). Cham: Springer International Publishing.

Lave, J., & Wenger, É. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge Univ. Press.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*(3), 319-325.

Raman, M. (2004). Key ideas: The link between private and public aspects of proof. *Proceedings of the Twenty-Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 635–638.

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*(2), 197–209.

Staats, S. (2008). Poetic lines in mathematics discourse: A method from linguistic anthropology. For the Learning of Mathematics, 28(2), 26–32.

Staats, S. (2016). From speaking to writing: The role of the reversal poetic structure in problem-solving. *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 1266–1272.

Staats, S. (2018). Coloring conjectures with sound, silence, syntax and gesture. *For the Learning of Mathematics, 38*(2), 2–8.

Staats, S. (2021). Mathematical poetic structures: The sound shape of collaboration. *The Journal of Mathematical Behavior, 62*, 1–15.

Wilmot, D., Schoenfeld, A., Wilson, M., Champney, D., & Zahner, W. (2011). Validating a learning progression in mathematical functions for college readiness. *Mathematical Thinking and Learning, 13*(4), 259–291.

Yin, R. K. (2018). *Case study research and applications: Design and methods* (Sixth edition). SAGE.

Zandieh, M., Larsen, S., & Nunley, D. (2008). Proving starting from informal notions of symmetry and transformations. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection* (1st ed., pp. 125–138). MAA. https://doi.org/10.5948/UPO9780883859759.011