A Learning Trajectory for Area Under a Rate Graph

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We made a short sequence of interactive tasks in which a student must estimate the total amount of water in a pool, given the water rate (gals/min) at a handful of moments. Through working on these tasks in a small teaching experiment, an undergraduate who had not seen calculus learned to conceptualize the area under a linearly increasing water rate graph as representing the total accumulated volume of water. From this we developed a learning trajectory, highlighting requisite and developing concepts, actions, and teacher-researcher interactions that were key to his learning. His learning trajectory, which entailed imagery of lower and upper Riemann sums, reveals ways to learn key calculus concepts while maintaining the relationship between rate and amount quantities as a central, rather than an easily jettisoned, part of students' understanding of 'area under a curve'.

Keywords: Calculus, Rate of change, Definite Integral, Accumulation

One of the two main questions of calculus can be summarized as the following: "you know how fast a quantity is changing and you want to know how much of it there is" (Thompson et al., 2013, p. 125). Equipping students to conceptualize the accumulation of an amount of a quantity given its varying rate of change has long been discussed as a significant challenge in the teaching and learning of calculus (e.g., Thompson, 1994). In the U.S., students typically emerge from a standard single-variable calculus class quite able to interpret a definite integral as a gestalt area "under" a curve, but in many cases, they struggle to attribute quantitative meaning to it in a context, and often cannot interpret it as a sum (e.g., Fisher, Samuels, & Wangberg, 2016; Jones, 2015; Sealey & Oehrtman, 2005; Wagner 2016). One recommendation for addressing this issue has been to teach the accumulation of an amount with minimal emphasis on 'area under a curve' (e.g., Thompson et al., 2013). Yet areas under curves are also powerful representations for accumulated amounts, and they can reveal important characteristics about how these amounts relate to their rates of change. Our broader goal is to study how to build a student's understanding of an accumulated amount as an area under a curve *through* leveraging their quantitative reasoning in a context. Our hope is that the relationship between rate quantities and amount quantities can thus become a central, rather than an easily discarded, element of their understanding of 'area under a curve'.

In this paper we develop a short learning trajectory of an important piece of this broader understanding, by conducting a short teaching experiment with a college algebra student who had not taken calculus. We chart out how he learned to interpret the area under a *linear* rate graph (flow of water in gals/min) as representing a total accumulated amount (of water in gallons), by engaging with a sequence of interactive tasks. In so doing he developed a method akin to lower and upper Riemann sums. We believe the linear rate graph case is a pivotal one from which a student could generalize this area-represents-amount idea to more general types of rate curves, while staying grounded in quantitative meanings of accumulation and rate.

Background and Theoretical Framing

Accumulating Amounts From Their Rates of Change

Much of the recent research about the teaching and learning of definite integrals focuses on how students learn to interpret integral notation in quantitative contexts (e.g., Greefrath, et al., 2021; articles in *ZDM* special issue: Ely & Jones, 2023). In some approaches to developing quantitatively-grounded interpretations of integrals, such as multiplicatively-based summation (Jones, 2015) and accumulation-from-rate (Thompson, et al., 2013), the integrand—i.e., the "$f(x)$" in $\int_a^b f(x)dx$—is always interpreted as a *rate*. Specifically, $f(x)$ is the rate at which some amount F changes as x changes, and the definite integral measures how much F accumulates over the interval (a, b). For a student using such an interpretation to imagine an integral in terms of area, the area is always then an amount accumulated under a rate graph (e.g., a graph of $y = f(x)$). Thus it is valuable to study how a learner could develop the idea that the area under a rate graph represents an accumulated amount, even independently of any definite integral notation.

Learning Trajectories

Researchers have defined and theorized learning trajectories in a variety of ways. Simon (1995) introduced the construct of a hypothetical learning trajectory as the learning goal, activities, and thinking in which students engage, and Clements and Sarama (2012) depicted it as a conjectured route through a set of tasks designed to engender the mental actions to support a developmental progression of understanding. For the purposes of this study, we follow Ellis et al.'s (2014) characterization of a learning trajectory as a tool that seeks to explain learning over time, in particular, "an account of changes in students' schemes and operations" (p. 2). This follows with Steffe and Olive's (2010) view that a learning trajectory is ultimately a type of model observers construct of a learner's knowledge. Ellis et al. treat a learning trajectory as including a sequence of *conceptual stages* that can account for a learner's observed activity. Each stage is characterized by the *concepts* or understandings that an epistemic student uses at particular moments of their activity (Ellis, 2022). Also included is the context of the student's activity, such as the *tasks* they performed, the *actions and representations* they reasoned with while doing so, their *interactions* with others, and the *teaching actions* that supported their shifts from one stage to the next. In particular, a learning trajectory seeks to account for how specific contextual elements mediated or facilitated the learner's conceptual changes. The learning trajectory distills the student's whole observed activity and its context into the parts that are key for accounting for how they learned.

Methods

We recruited four undergraduate students at a medium-sized university in the Northwest U.S. to work on a calculus-related activity. They were chosen because they were interested and available, and because they were currently enrolled in college algebra and had not taken calculus. We conducted one 2-hour teaching experiment session with each pair. This paper's results are from one of these pairs. One participant had to leave after 15 minutes, so our results are from the other participant, 'Ian' (pseudonym). Participants worked on a sequence of pool-filling tasks on an iPad. The main teacher-researcher, TR1, asked questions along the way to elicit explanations for their actions and comments, but did not explain concepts or help the participants solve the tasks. A second teacher-researcher, TR2, was present for the first part of both sessions also. Participants also used scratch paper, writing implements, calculators, and a large whiteboard.

The sequence of tasks was programmed using Desmos Classroom. Each task had a similar format and layout [Figure 1], including: (a) an animated video of a swimming pool being filled with water, (b) a table displaying a sample of the water rates (gals/min) at a few moments in time, and (c) a graph of the water rate for the time shown in the animation video. The video had labels showing the rate at which the pool was being filled and the volume in the pool in real time. The number of values shown in the table could be changed.

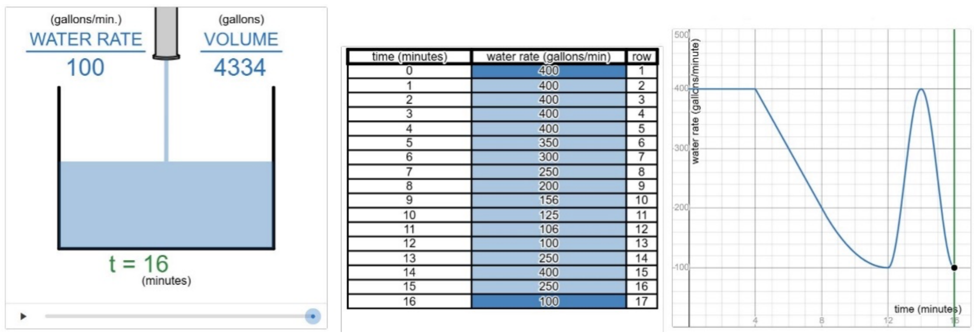

Figure 1. Pool activity, Task 0 (Orientation task): (a) pool animation, (b) rate table, and (c) rate graph

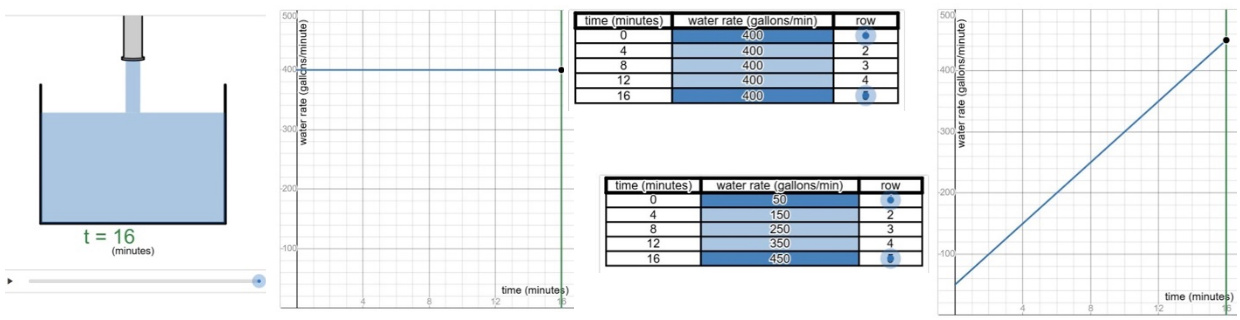

Figure 2. Pool activity, (a) animation, (b) Task 1 table, (c) Task 1 graph, (d) Task 2 table, (e) Task 2 graph

The introductory Task 0 was meant to quickly orient participants to the tools and displays. They were told that the water rate is fluctuating and that the values in the table are as if a picture was being taken of the water rate value at certain moments in time. For Tasks 1 and 2, participants had to extrapolate from just the video and the table the amount of water in the pool at the end of the video ($t = 16$). The water rate and volume readouts, as well as the water rate graph, began as hidden; these were revealed after six guesses or after one relatively close guess. Participants had unlimited guesses and were given general feedback about the accuracy (e.g., "really far below the actual amount"). In Task 1, the water rate was a constant 400 gals/min [Figure 2abc]. In Task 2 the water rate linearly increased from 50 to 450 gals/min [Figure 2ade].

Both sessions were video-recorded, screen-recorded, and transcribed. To analyze the data, we leveraged ongoing and retrospective analysis methods (Steffe & Thompson, 2000) to characterize Ian's changing concepts throughout their sessions. Prior to collecting any data, we formulated a learning goal: the concept that the area under the graph of a rate function measures a total accumulated amount. We developed a hypothetical learning trajectory based on extant research, and modified the tasks. Based on time constraints we narrowed the learning goal to just the area concept for the case of a linearly increasing rate graph, rather than any general rate graph. The hypothetical learning trajectory then served as a source of initial codes for identifying the participants' concepts; we determined as a group whether Ian's task responses provided evidence of a concept from the hypothetical learning trajectory. This initial round of coding also produced emergent codes (Strauss & Corbin, 1990), which we incorporated into a revised learning trajectory for Ian. Given the challenge of understanding and characterizing a student's mental images, we triangulated evidence from multiple sources to infer Ian's thinking, including his written work, utterances, nonverbal gestures, and shifts in his problem-solving approaches. We then examined the interactions between Ian and the TRs, within the context of the task structure, to identify supports for Ian's emerging concepts.

Results

In this section we summarize Ian's activity and his interactions with the teacher-researchers on Tasks 1 and 2. We analyze the concepts, actions, and attention-directing interactions that we found to be key in his learning of the target concept: *The area under the rate graph measures the total accumulation* of, in this case, the volume of water. We identify two of these concepts that Ian already knew and were requisite to his successful learning through the tasks.

Task 1

When Ian began working on Task 1, he saw that the rate 400 appeared in the table five times, so he calculated the total amount of water to be 5×400 = 2000 gals. He was perplexed when the program told him this answer was 'really far below' the actual amount, and he played back and explored the animation. The TRs then directed his attention to the quantities in the table. TR1 said, "the second column gives the water rate," and Ian replied, "I see, okay." TR2 asked, "In what units?" Then Ian responded, "so we just multiply 400 times 16?" When inputted, the program confirmed this to be 'exactly right.' From that moment on, Ian consistently distinguished rates from amounts and attended to which of these he was representing when graphing or calculating. The primary role of Task 1 was not to enable Ian to significantly develop new concepts, but to invoke his understandings of rate (gals/min) in the context at hand. In particular we note a key concept and the intervention that elicited it in the pool context:

- <u>Key (requisite) concept</u>: *A constant rate times a duration of time gives a total amount accumulated over that time interval.*
- A <u>key attention-directing intervention</u>: TRs pointing out the quantity type, and units, of the outputs in the (rate) table.

Task 2

Developing and graphing an underestimate (lower staircase). In Task 2, the water rate linearly increases over the 16 seconds. Initially the table shows the water rate values at every 4 minutes [Figure 2ad], and the graph is not visible. Ian used these values to estimate the total amount of water as 50(4) + 150(4) + 250(4) + 350(4) = 3200 gals. He pointed out to TR1 that he did not need to use the last value on the table (450 gal/min when $t = 16$). The feedback from the program was that this was 'pretty far below' the actual amount of water. Ian said, "I was thinking just like steps from the way it appears," and sketched Figure 3a on paper.

At this point, Ian did several things that, in our analysis, were not crucial to the learning trajectory. He tried several ways to compensate for the lacking amount, trying to incorporate the final rate value 450 (at $t = 16$) somehow in the calculation. He made the program generate more rate values for smaller increments of time and noticed that he could do the same type of calculation. He said that "the more [steps] you have, the more accurate it is"—a valuable insight, but not key in his learning trajectory. He did use these values eventually to find an equation for the rate curve, initially trying a quadratic curve but ending up with the correct linear formula. He did not use the formula however, and after trying another several estimates, the program told him the actual correct amount (4000 gals) and revealed the actual rate graph. This did not seem to surprise him, since he had already figured out the equation of that curve and knew it to be linear.

TR1 decided it would be beneficial to have as a referent on the whiteboard a large graph akin to what Ian sketched on his paper [Figure 3a]. He drew and labelled axes [Figure 3b] and asked Ian to draw a graph of his stairs and asked him how many he wanted to use. Ian said, "there's a number that's quite accurate while still not being massive." He drew his graph with four stairs [Figure 3c], and said, "the staircase I was kind've thinking would look something like that." Ian immediately added, "you just kind've imagine having either a line like that [straight line in Figure 3d] or in between the stairs that averages it [new line in Figure 3e]."

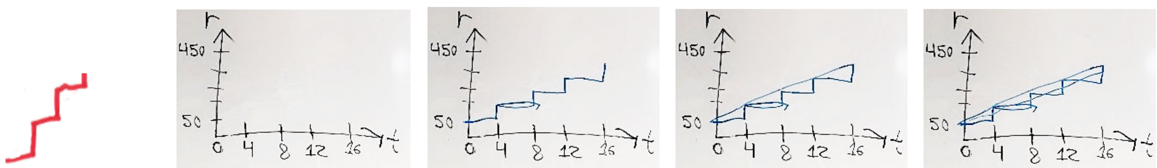

Figure 3a-e. A sequence of rate-vs-time graphs produced by Ian on Task 2.

These actions provided Ian with representations that were crucial to reason with. Ian's ability to represent and interpret his graph indicates that he already had the following complex understanding, and the following key actions enabled him to employ for Task 2:

- <u>Key (requisite) concept</u>: *A rate graph is a continuous record of how a quantity's rate of change varied over time, and each point on it is a multiplicative object* (Saldanha & Thompson, 1998) *representing the rate of flow of water at that moment in time.*
- <u>Key actions</u>: (a) estimating the total amount using constant rates over sub-intervals, and (b) graphing the corresponding (under-)estimating rate function as a 'lower staircase' on the same graph as the actual rate function [Figure 3d].

We briefly note that the other pair of participants we studied were unable to complete Task 2 because they could not consistently employ this concept.

Interpreting the rate graph between under- and over-estimate staircases. Ian noted that his 'lower staircase' underestimates the actual amount of water, and then he drew an 'upper staircase' [Figure 4a] overestimating the actual amount by, as he said, putting the excess "growth" at the beginning rather than at the end of the interval. He added, "This volume [tracing the larger rectangular path we have marked in Figure 4b with a solid line] is bigger than this volume [tracing the smaller rectangular path we have marked in Figure 4b with a dashed line]." Here Ian had begun to use the following new concept:

- <u>Key (new) concept</u>: *The area of a rectangular region under a constant rate graph* (where the rate is in gal/min) *measures a volume* (in gal).

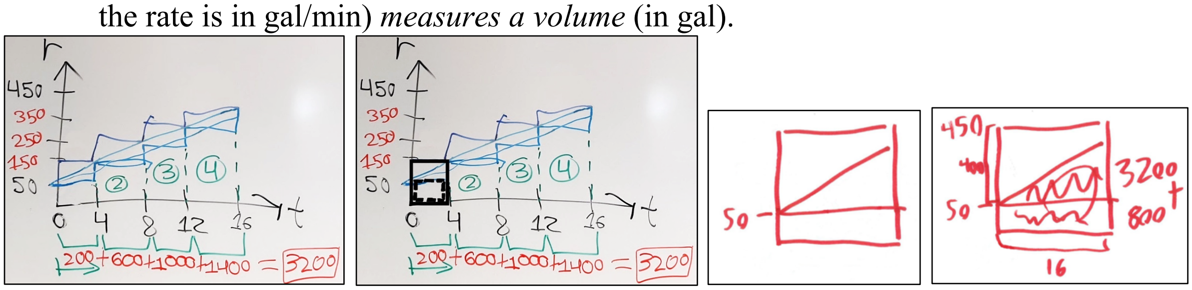

Figure 4a-d. Rate graphs and diagrams Ian used when reasoning with areas on Task 2.

Looking at the graph [Figure 4a], Ian then said there is something between the two staircases that was "more accurate, I guess, to the growth." TR1 asked where in between, and Ian replied, "exactly in between, right?" Although he didn't say it at that moment, later Ian confirmed that this was when he strongly suspected the actual amount would be exactly halfway between his over- and under-estimates, once he saw the graph that displayed all three together (Figure 4a).

Ian then calculated that "exactly in-between" value. He determined the over-estimate amount from the upper staircase (4800), found the difference between that and the lower staircase underestimate to be 1600, took half of that (800), and added this to under-estimate. He got 4000, and since he already knew this to be the exact actual amount, he said, "Okay, so that method does work." His calculation, confirmed that the together with the earlier empirical correct answer feedback from the program, confirmed his geometrical inference from the graph.

When asked why, Ian explained by appealing to the geometry of how the rate graph cut in half the regions between the two staircases: "If you just draw any rectangle and have a triangle in it, it's half. So, I just thought since all of those are squares, it's just half. So, I just tried finding what half would be for that." Here Ian used the following new concept, which was enabled by the following key action:

- Key (new) concept: *The areas between the lower and upper staircases represent the difference in volume between the under- and over-estimates*.
- Key action: Generating the upper staircase overestimate on the same graph as the lower staircase and the actual linearly increasing rate graph.

Area under a linear rate graph. We interpret that Ian's combination of realizations enabled the following type of reasoning. Seeing that the actual volume is exactly halfway between the under- and over-estimates, and also that the region under the actual rate graph exactly splits the difference between the regions under the two staircases, together with his recent concept that the area in a rectangular region represents a volume, Ian was able to infer the following:

- Key (target) concept: *The area under the linear rate graph represents the actual exact water volume*.

This was confirmed by Ian's subsequent activity. TR1 asked if his idea allowed him to use a simpler way to find a volume just by considering the (rate) graph. Ian said, "If it's a straight line then yes," and he drew a simplified geometric diagram with just one large triangle atop a rectangle of width 16 [Figure 4c]. But then he said "No, that's wrong. I'm thinking of it as, that's rate of growth, not that it starts at 50 filled up already. Never mind. Discount that idea." This comment suggests that he was thinking through which aspects of his diagram represented amounts and which represented rates. With TR1's encouragement, Ian continued his geometric calculation [Figure 4d], getting the same answer of 4000 (referred to without units). He said, "that's interesting that it's as simple as that." "That you can just calculate the area and find the… Yeah. Right. Huh. That's pretty neat." This was an empirical confirmation for Ian of his new concept. The fact that he calculated the area of the entire trapezoidal region on its own indicates that he viewed the region as an object whose area measures the water amount, rather than just as a representational tool that suggests how to calculate that amount.

Role of Task 2 elements in Ian's learning trajectory. We have already described the role some key actions and interventions had in Ian's learning trajectory. We also noticed the role that several features of Task 2 had in this trajectory. Task 2 initially provided Ian only with the pool video and the table of linearly changing rate values, and not a rate graph. It required Ian to estimate an amount value, then gave him feedback about that value, and it let him do this six times before revealing the actual volume value and the rate graph. The limited information in the initial table afforded Ian's development of an underestimate of the true amount, since those values naturally appear as 'steps' in the table. The task led Ian to change the number of values in the table, which allowed him to develop more accurate estimates of the rate. The delayed revelation of the actual water amount was important, because it gave him an amount value that he used to empirically confirm his developing concepts about areas as amounts. The delayed revelation of the rate graph may have been less important in Ian's case, since he happened to have already figured out the linear function that fit the table data, and he probably could have graphed it on his own if asked. Finally, we conjecture that Task 2's interactive structure in the first place made his staircases fundamentally situated within his active process of estimating the true amount, rather than simply becoming quantitatively untethered geometric objects.

Discussion

Through these interactive pool-filling tasks, Ian learned that the area under the linear rate graph represents the actual exact water volume. In so doing he used his own method of under- and over-estimating an actual accumulated amount by using lower and upper Riemann sums. Although we did not study what further learning was enabled by Ian's new concepts, here we discuss two promising affordances of his concepts. These affordances help justify why this pool filling task sequence could be more beneficial than another slightly more efficient sequence.

After all, to learn the target concept in just the linear rate case, it would likely be quicker for a learner to average out the increasing water rate over the whole interval, imagining that a pool filling constantly at that rate would end up with the same amount. Graphing could show this averaging geometrically, so they could imagine the areas serving as proxies for the amounts. In comparison with that hypothetical route, Ian's learning trajectory positions him much better to generalize his concept in calculus, because he has developed the key imagery used in calculus to conceptualize an integral as a limit of upper and lower Riemann sums.

A second affordance is that, unlike the students in studies we mentioned earlier, Ian's new area-represents-amount concept is conceptually anchored in the quantitative relationship between rates and amounts, and is not just something he noticed empirically about areas of regions. Yet there was an important part of Ian's learning trajectory that was empirical. In Task 2 he was told the actual amount accumulated in the pool. He noticed this value was halfway between his under- and over-estimates, just as the area under the rate graph is exactly halfway between the areas of the lower and upper staircases. Did he just *associate* the area with the amount, or did he develop any sense of necessity about why the area must measure that amount?

It is difficult to say from our data. Nonetheless we wonder exactly how much necessity is readily available for any learner in the process of constructing the area-represents-amount concept, through any learning trajectory. For instance, another path might allow a learner to more explicitly reason with 'canceling' units: gals/min × min = gals. But does that carry much more sense of necessity for them? Izsák (2025) proposes an approach that grounds quantitative area reasoning with integrals in a particular measurement interpretation of multiplication. This makes the units canceling feel much less like a magic trick, yet in that approach there is still fundamentally an association made between a unit square and a unit of the accumulating quantity (e.g., volume). With this in mind, our study highlights a question for further study: Is some degree of empirical association unavoidable when initially learning the concept that areas under rate graphs represent accumulated amounts? Is the concept's security based on how the learner experiences these areas as increasingly reliable tools for interpreting amounts?

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