A Conceptual Framework of Emergent Utility of Advanced Mathematics for Inservice Teachers

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In this theoretical report, we present a conceptual framework that details inservice teachers' emergent utility perspectives towards advanced mathematics and their advanced mathematics coursework. We leverage a river journey metaphor to help illustrate aspects of our conceptual framework. Along the river are utility points that refer to opportunities for teachers to reflect on the usefulness and relevance of advanced mathematics content in relation to its impact on their teaching. Then, we provide examples of both preservice and inservice teachers from extant literature and an initial study conducted in a real analysis course to highlight the four main components of the conceptual framework. We end with implications for designers aligned with the instructional design theory of Realistic Mathematics Education for creating courses for secondary mathematics teachers.

Keywords: Emergent Utility, Inservice Teachers, Emergent Modeling, Teacher Education

Although inservice and preservice teachers have questioned the usefulness or relevance of upper division or advanced mathematics courses for their professional teaching (e.g., Cofer, 2015; Wasserman, 2017; Zazkis & Leikin, 2010), there is growing research that demonstrates how mathematics teacher educators may structure these courses to support teachers in seeing the utility of these courses for their own learning and teaching (e.g., Cruz et al., 2024; Goodchild et al., 2021; Qiu et al., 2025; Wasserman et al., 2023). For example, one common direction is the inclusion of direct content connections between advanced content and content relevant to secondary mathematics. A real analysis connection (Wasserman et al., 2019) could be the concept of continuity as seen in secondary school and a differential equations connection (Apkarian et al., 2023) could be the concept of function. Additionally, there are non-content connections that can be made such as relating experiences in mathematical practices between secondary and university mathematics (Cruz et al., 2024). Collectively, this body of research aims to address the double discontinuity first introduced by Klein (1932, 2016). Our work builds alongside this work, yet, what motivated us is slightly different. Inservice teachers take advanced mathematics courses to pursue graduate degrees, which means they bring experience and knowledge of secondary school mathematics and students to their own graduate learning. More specifically, drawing on Klein, inservice teachers may experience this double discontinuity anew.

The purpose of this theoretical contribution is to call attention to the differences between how inservice and preservice teachers view the relevance of advanced mathematics courses. We propose a conceptual framework that has potential applications for design aligned with Realistic Mathematics Education (RME; Freudenthal, 1973) as an instructional design theory and the concept of emergent modeling. This conceptual framework describes how inservice or preservice teachers (IPSTs) may progress towards seeing how their advanced coursework could have an impact on their teaching practice (i.e., the utility of their courses for their teaching). While both inservice and preservice teachers are important for the conceptual framework, and as such will be discussed, our contribution focuses on inservice teachers. At first, we consider IPSTs together as one entity, because both inservice and preservice teachers have preconceived notions about advanced mathematics. However, the experiences that inservice and preservice teachers bring to their advanced mathematics courses may create fundamental differences in how they engage with the coursework. Thus, we distinguish between each population by discussing both existing literature regarding the support of preservice teachers in making connections and using our preliminary data from inservice teachers.

To illustrate our emergent utility conceptual framework, we draw on the river journey metaphor from Rasmussen & Keene (2019) on waypoints in one's mathematical learning. Rasmussen and Keene (2019) think of waypoints as opportunities for students to engage and connect various ways of reasoning with mathematical concepts (p. 2). These waypoints are content based. We, however, take a different perspective on waypoints. We refer to utility points as opportunities to reflect on the utility of content in relation to its impact on one's teaching. That impact may be different for inservice and preservice teachers. To describe our emergent utility conceptual framework in detail we first situate ourselves in our larger theoretical lens. We then describe the framework using the river journey metaphor and provide examples using an example from existing literature and from empirical data we have collected.

Theoretical Background

We adopt the Emergent Perspective as an interpretive framework to make sense of both students' conceptions of individuals' mathematics ideas and socially embedded classroom practices (Cobb & Yackel, 1996). By adopting the Emergent Perspective, we conceptualize learning as a coordination between students' cognitive development and engagement in mathematical activity. Inservice teachers enrolled in advanced mathematics coursework bring their beliefs about themselves as an instructor, their students, the mathematics they teach, and previously used pedagogical practices. Inservice teachers' previous pedagogical experiences must be considered as a lens through which they consider the relevance of advanced mathematics coursework. Though this framework describes inservice teachers' perspectives on the utility of advanced mathematics coursework, we conjecture it is still viable as a design framework for recommendations about designing courses for preservice teachers. Although preservice teachers may not have experience teaching in a classroom, their perspectives about advanced mathematics coursework might be informed by their beliefs and opinions about their future students (Scherer & Bertram, 2024) or their perception of their mathematical ability (Beisly et al., 2024).

To describe our conceptual framework, we use the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1973). In RME, mathematics is a human activity. Then, the design heuristics of RME support instructors in developing materials that guide students in engaging in mathematical activity. This initial activity can be connected, with help from the instructor as a brokering agent, to formal mathematical logical structures as they are realized by the extant community of mathematicians (Rasmussen et al., 2005). This transition from informal to formal reasoning is directly related to the RME design heuristic of emergent modeling. We leverage this heuristic in our proposed framework. Providing IPSTs with opportunities to develop more formal ways of reasoning may guide them in making instructional moves that will help connect their own students' informal ways of reasoning towards the more formal. From the lens of RME, designers create mathematical tasks intended to elicit students' informal activities. These initial ideas might include sketches, graphs, input-output tables, etc. Students' initial activity forms the foundation for generating ideas and can be folded into classroom discussion by the instructor to become a model of more formal mathematics. As students gain experience working in this experientially real setting, their attention may begin to shift away from the experientially real setting towards mathematical patterns and relationships. Once these mathematical relationships enter the classroom discourse, they can be used by the instructor to connect to the more formal mathematical ideas that they have in mind. Once the students' model(s) become a way to reason about mathematical relationships, as opposed to a way to represent the applied problem, the student uses it as a model for more formal mathematics (Rasmussen et al., 2005; Zandieh & Rasmussen, 2010).

In the next section, we introduce the notions of an IPST's utility of and utility for, advanced mathematics content and later provide examples of the four emergent utility perspectives of our conceptual framework. It should be noted that we view teachers' emergent utility as inextricably linked to the emergent modeling heuristic. As inservice teachers develop new ways of reasoning about advanced mathematics content, it is likely that the ways they perceive the relevance of this content are informed by the lens of their own teaching experiences.

Emergent Utility Conceptual Framework

To help illustrate our framework, we rely on the river journey metaphor (Rasmussen & Keene, 2019). The river metaphor is intended to convey students' non-linear progression through mathematical ideas that may require more or less time based on their previous instructional experiences and current mathematical reasoning. The journey is non-linear in the sense that some students might return to prior waypoints, spend more time in certain areas of the river, or branch off to explore aspects of the river they find interesting. More specifically, we conceptualize the specific ways in which teachers reason about the utility they continue along the river journey as key utility points. We do not comment on the exact nature of the content waypoints, as they are intended to be left general for the sake of encompassing any advanced mathematics course in which IPSTs might enroll. We also want to mention that the amount of time one spends in the river investigating each utility point will vary for each individual and might be impacted by external factors (e.g., shift in the river's features representing a change in the course structure).

When they first enter the river, IPSTs may have preconceived notions about whether or not the course will be helpful for their teaching, as indicated by the starting point in Figure 1. This could be based on the IPST's perception of students' mathematical ability, or their general impression of the advanced mathematics content. At first, IPSTs might perceive the utility of the advanced mathematics coursework as developing their understanding of related course content, but ultimately view the course content as irrelevant for their teaching. Thus, they may choose not to spend as much time at the content waypoint or utility points because they do not view the concepts as relevant for their teaching. Such an IPST develops a utility of a model of perspective towards the advanced mathematics content, as pictured by the top, blue, dotted line in Figure 1.

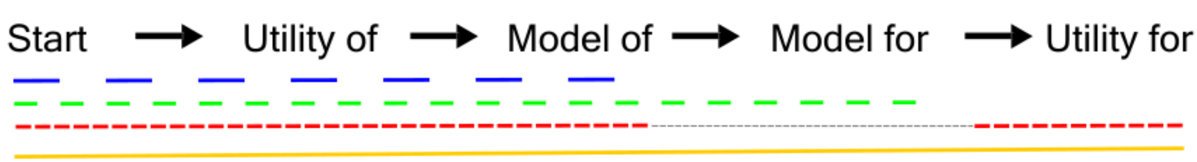

Figure 1. The Emergent Utility Conceptual Framework.

We view advanced mathematics courses as a river filled with opportunities, that some students may perceive as hurdles (e.g., rapids or rocks), to engage in iterative emergent modeling processes where one can spend more or less time building models for the relevant content waypoints and utility points. As IPSTs continue to build models of and for advanced mathematics content, they require opportunities to reflect on their learning in terms of its relevance to teaching secondary mathematics, that is, time to spend at each utility point. Some IPSTs may come to develop models for a majority of the course content, but decide that the time they themselves spent exploring the river would take too long, or be too challenging for their own students. Such a teacher has developed a utility of a model for perspective, as pictured by the second, green, dotted line in Figure 1.

As the IPSTs begin to spend more time at each utility point, they begin to reason about the broader mathematical context and how the content they teach is situated within it. Here, inservice and preservice teachers likely have different knowledge around 'the content they teach' - an inservice teacher is returning to the river and their journey is now informed by previous experiences as a teacher. In other words, they themselves have guided students down a river. It may take more or less time for each individual teacher to reflect on the relevance of the course content to their teacher to the point they are willing to make shifts in their teaching. Eventually, with enough time at each waypoint, the IPST might come to envision changes in their pedagogical decisions about content they will, or have previously taught, that they now realize is directly related to the advanced mathematics content they are learning about. Such an IPST has begun to perceive the utility for advanced mathematics content to impact their teaching practice. In other words, IPSTs are developing an emergent model of/for their own teaching, as it is informed by their new models for advanced mathematics content. In this way, IPSTs are making connections between their formal ways of reasoning about course content and using these connections as a new pedagogical lens to make decisions about their own instruction. However, some IPSTs may not have enough time to spend traversing the river and discovering or reinventing all of the content waypoints but still believe that the course is relevant for their teaching. These IPSTs develop a model of the utility for advanced mathematics content, as pictured by the third, red, dotted line in Figure 1. Such IPSTs may struggle to provide explicit examples of which advanced mathematics content will shift their teaching in specific ways. Despite this, such IPSTs still indicate the impact of the course content on them as educators, which is why the red dotted line is broken by the smaller gray portion in Figure 1.

The IPSTs who develop a model for the utility for advanced mathematics have a new, or renewed, vision of how their formal ways of reasoning about advanced mathematics content will shift their teaching and can provide explicit examples. Such IPSTs have spent enough time at each content waypoint and utility point to reflect on the relationship between their formal ways of reasoning about the advanced mathematics content to determine ways in which this reasoning will shift their teaching practices. This is pictured below in Figure 1 by the unbroken, bottom yellow line. More specifically, an IPST who develops a model for the utility for advanced mathematics content is making new, or different, decisions about course content, pedagogical moves, and curricular innovations in their secondary mathematics classroom.

It should be noted that, for preservice teachers, this may be their first journey down a specific river. For inservice teachers, this may be their second, third, or fourth journey down a specific river. Revisiting the content waypoints additional times, visiting utility points with more teaching experiences, and coordinating this time down additional river journeys may impact the amount of time a specific teacher spends before developing an emergent utility perspective.

Examples of Emergent Utility Perspectives From Literature on Preservice Teachers

First, we consider the results of Liang et al. (2023). The three preservice teachers (William, Maria, and Sterling) were each interviewed twice about their transition to university math positioned as students (first interview) and future teaching practices, using inverse functions as the example topic (second interview). We begin with William because while he had a very positive view about his experiences in advanced tertiary mathematics courses, he continually referred to secondary mathematics as "copycat" math. This was because, according to William, secondary students tend to mimic what the teacher does to solve math problems. This, from William's point of view, is not sufficient when it comes to the learning of mathematical proof at university. William's overall view of tertiary mathematics was that it was fun and interesting for him, but was reserved for individuals who developed a "shield" to survive a transition to proof course to get to the more interesting coursework. This belief about his potential students presupposes an approach to his secondary teaching (forcing students to "use their brains"), but he also appeared to view inverse functions solely as applicable for solving equations. A preservice teacher like William would have a utility of a model for (green line) advanced proof-based content with teaching the concept of inverse functions to his future secondary students. This is because while he has some ideas about how his tertiary mathematics experiences will influence his teaching; he did not appear to leverage his experiences in determining concrete ways to shift his future teacher practice.

Because of her experiences in secondary instructional settings, Maria conceptualized secondary classrooms as competitive (p. 116). She felt pressure from her teachers and instructional settings such as standardized exams, and her impression was that she had to practice problems repeatedly without a strong conceptual understanding in order to perform well (e.g., receive a high grade). However, Maria summarized her experience in tertiary math classrooms as exciting, and recommended to others to "dig deep". Though Maria did not identify herself as a "successful" tertiary mathematics student, she valued the notion of introducing secondary students to proofs that extend beyond the typical geometrically focused examples. A preservice teacher like Maria would have a model of the utility for (red line) logical proof techniques in secondary mathematics classrooms. This is because though Maria was open to the idea of including her experiences in tertiary math classrooms in her secondary teaching, she appeared to require time to "dig deeper" in her math studies and required more opportunities to reflect on ways in which more formal ways of reasoning might shift her future secondary teaching.

When asked to reflect on his experience transitioning from secondary to tertiary mathematics courses, Sterling focused on if and only if, or bidirectional, proofs because he initially struggled to make sense of them. Sterling then provided an example to illustrate how he would support his future secondary students in making sense of these types of statements (p. 119). This example, which involved creating if-then statements about the solutions to the equation x^2=2, illustrates an instance when reversing a true statement does not always produce a new true statement. By providing a non-example of a bidirectional statement to his imagined future students, Sterling was supporting them in making connections he himself struggled with in the transition from secondary to tertiary mathematics. The fact that Sterling was able to reframe an example of a secondary mathematics concept through the lens of his learning in tertiary mathematics contexts, coupled with claims that Sterling possessed a strong background in mathematics at the tertiary level, indicate that Sterling had developed a model for the utility for (yellow line) proof-based content (specifically bidirectional proofs) with teaching the concept of inverse functions.

Examples of Emergent Utility Perspectives From Inservice Teachers

We continue to illustrate our framework by presenting four examples from a study conducted with 10 inservice teachers enrolled in a graduate real analysis course. The following quotes are responses from reflections about the Fundamental Theorem of Calculus (FTC) and continuity and differentiability, with emphasis added.

When responding to a reflection about understanding the fundamental theorem of calculus as related to the concepts of rate of change and accumulation, Demi stated the following:

Demi: Referring to the FTC in this manner would be interesting to a certain level of student. However, to a run of the mill calculus student who is just trying to master the skill of taking the integral I believe would overwhelm them. I would not show this until they were comfortable with the skill or with great explanation beforehand.

This demonstrates that Demi views the relevance of this real analysis content through their own teaching experiences and beliefs about "a certain level" of student. Demi contends that the formal aspects of the FTC would be more difficult than needed in their own instructional context. Demi views the utility of this advanced mathematics through her own teaching experience and appears to believe that because the content was challenging, it means that it would "overwhelm" their students. Such a teacher would possess a utility of a model of (blue line) perspective as her beliefs about her students appeared to mediate her willingness to engage with content waypoints and utility points throughout the course.

When responding to a reflection about comparing the formal, epsilon-delta definition of continuity versus a more informal definition, Eloise stated the following:

Eloise: Honestly, when first completing the homework, I was unsure I really liked/approved of the analysis definition of continuity. However, after doing more problems and reading more about it, it has started to make more sense to me. I find it very interesting since I have never seen continuity defined this way but am starting to appreciate it because I enjoy having a deeper understanding of calculus and it's cool to continue to expand that understanding more still. I will say however that I do not think I will actually change the definition I use to teach continuity … the analysis definition is harder to understand especially for a first-time calculus student.

As Eloise self-describes that she understands the formal analysis definition yet indicates a lack of usefulness in their own teaching due to curricular and testing tensions. Throughout the course, Eloise demonstrated reasoning about continuity that was highly interconnected with other course content but ultimately decided that the reasoning they developed was "harder to understand" for their students and therefore they would not "change the definition" they teach. Whereas Demi's own reasoning about the reciprocal process of differentiation and integration mediated their perception of the level of difficulty of the content and its relevance for their students, Eloise made connections between the course content and their prior teaching practices but still decided to not make changes to the way they teach the concept of continuous functions to secondary students. This teacher possesses a utility of model for (green line) perspective.

When discussing the possibility of teaching derivatives Hannah thought about it "in a variety of ways so the concept can be fully understood." In this way, Hannah appreciates a conceptually focused approach to introducing students to the notion of differentiability. However, Hannah also believes that it would be "less confusing if you did it by practicing finding and taking derivatives." It appears that even though Hannah is open to the impact that advanced mathematics courses could have on conceptual understanding, the connections they made between "finding and taking derivatives" as a process and the relationship between this process and the notion of differentiability are seemingly disparate. Because of this disconnect, such a teacher who is open to changing their teaching practice based on their renewed understanding of the content, but the ways in which these instructional changes can occur is mediated by their own reasoning, would possess a model of the utility for (red line) differentiability as it relates to teaching derivatives to secondary students.

In the same reflection assignment as Eloise, Jocelyn responded:

> *Jocelyn*: Prior to taking this class, I know that I took a Real Analysis class in my undergraduate mathematics classes, but I fell back into the thought that a continuous function is just when you can draw the graph of the function without picking up your pencil. This is the way I have taught my students up until this school year, which is, I believe, on the level of their understanding. But seeing how this idea is used for differentiability makes me understand that this surface level of understanding does not prepare students for what they will see in calculus. I am able to see the connection between continuity and differentiability, which helps me understand both concepts on a deeper level.

It should be noted that Jocelyn was one of the few teachers with calculus teaching experience. A teacher like Jocelyn, who uses formal ways of reasoning to make shifts in their teaching practices has constructed a model for the utility for (yellow line) continuity as it relates to teaching AP Calculus content.

Conclusion

The nature of the transitions between the previously discussed emergent utility perspectives requires further attention and will likely change depending on the individual IPST. However, the purpose of this paper is to set a theoretical basis to inform researcher-designers about the nature of teachers' emergent utility towards advanced mathematics coursework. Inservice and preservice teachers engage with utility points differently. Likely, preservice teachers simultaneously engage with the advanced mathematics content for the first time and have not yet had opportunities to engage with students as a teacher. Inservice teachers have an additional challenge. Their teaching experiences informs the lens through which they view the utility of advanced mathematics coursework, and it could be the case they have a harder time making shifts in their teaching based on formal ways of reasoning about advanced mathematics content.

The ways in which an IPST navigates the river will be dependent on their previous experiences and beliefs. An inservice teacher will likely view the relevance of advanced mathematics content through actual teaching experience (e.g., Demi and Jocelyn) and therefore envision changes to previously implemented curricular materials. A preservice teacher (like William, Maria, and Sterling) can provide imagined examples of activities to include, or problems to address, but these are not considered in light of other issues that emerge from the actual practice of teaching, course creation, implementation, and classroom management. Preservice teachers are leveraging their own formal ways of reasoning and making decisions about whether it applies to imagined students, whereas inservice teachers are levering experiences with actual students and determining whether their own formal ways of reasoning are approachable to their instructional context. Because inservice teachers are likely engaging with the advanced mathematics content in a graduate level course, their new river journey can now be viewed through the lens of both previous instructional experiences as well as their prior teaching. This is what we meant earlier when we said that inservice teachers may be able to spend more time or make it further along the river in their journey. As such, designers must keep this in mind when determining whether certain activities are appropriate for preservice teachers, inservice teachers, or both.

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