Shore to Shore: Students' Progressive Mathematization of Null Spaces

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This study explores how undergraduate students develop reasoning about the relationship between a matrix and its null space through progressive mathematization within an inquiry-oriented linear algebra context. Using a playful hallway map task, students engaged in a paired teaching experiment to investigate how modifying the number of rooms and hallways affects the dimension of the null space. We drew on a river journey metaphor to examine students' movement along and between shores of contextual or formal symbolic reasoning. Findings reveal a dynamic interplay between contextual reasoning and formal symbolic representations, as students moved fluidly between the realistic hallway context and matrix-based interpretations. Through successive horizontal and vertical mathematization, students gradually developed reasoning that approached a reinvention of the rank-nullity theorem. The results highlight the complexity of this journey and suggest further research into providing supports for instruction.

Keywords: Progressive Mathematization, Student Reasoning, Linear Algebra, Null Space

Inquiry-oriented instruction (IOI) is a student-centered teaching approach where the instructor investigates students' reasoning (Kuster et al., 2017). One source of materials for IOI is materials designed using Realistic Mathematics Education (RME) because it leverages realistic contexts, promotes the instructor acting as a guide, and is designed to take students from reasoning about a context to formal mathematical activity (Gravemeijer, 1999). Sometimes this transition might be seamless, where students might have developed a robust enough understanding within a context to begin reasoning outside of the context to general cases. Other times, students might bring in prior mathematical experiences on the topic and this transition might be messier. In these cases, students go back and forth between contextual and formal symbolic ways of reasoning to consider more general cases (Rasmussen et al., 2005). The result of discussing these different paths is a rich interplay where students can build off each other's knowledge taking them on a complex journey that traverses between developing understandings of realistic contexts and formal symbolic perspectives.

In this paper, we detail one such journey in a playful mathematics task designed to supplement the Inquiry-oriented linear algebra (IOLA) task sequence about subspaces (Andrews-Larson et al., 2023). The goal of the activity was to help students develop ways of reasoning about the relationship between a matrix and its null space by drawing on intuition developed in the context of school hallway maps. We present the results of a paired teaching experiment with students and highlight their complex journey of going between contextual and formal symbolic reasoning which ends with students approaching a reinvention of the rank-nullity theorem.

Theoretical Framework and Related Literature

The task sequence used in this study was developed drawing on RME design heuristics. Of particular interest are heuristics related to students' reinvention of concepts and the purposeful use of realistic contexts (Gravemeijer & Doorman, 1999). In such tasks, students move between a realistic context and formal symbolic reasonings through progressive mathematization. Treffers' (2012) progressive mathematization consists of two types of activity: horizontal mathematization and vertical mathematization. For Treffers, *horizontal mathematization* involves “transforming a problem field into a mathematical problem” (p. 247), where *vertical mathematization* involves making progress within symbolizing more general notions of a concept. In this case, the ultimate goal is to move “from the world of life to the world of symbols” (Freudenthal, 1991, p. 41).

Oftentimes, however, students’ progressive mathematization is more complex than simply developing intuition within a realistic context, transferring that to mathematical symbols, then formalizing concepts through those symbols. Unlike Treffers’ conception, horizontal mathematization can also present itself as students going from formal symbolic ways of reasoning back to reasoning within a realistic context (Rasmussen et al., 2005). For instance, students could learn row reduction and how to interpret its results and then use this method to reconsider a previously examined context. Similarly, vertical mathematization can take place within a realistic context and is not unique to reasoning within formal symbolic mathematics.

Considering horizontal and vertical mathematization in these ways instead highlights how students can move between horizontal and vertical mathematization within and out of a realistic context. Students do not have to solely work within a given context then shift to and stay within formal symbolic reasoning for progressive mathematization to occur. Rasmussen and Keene (2019) used a river metaphor to highlight “places (islets) where students pause to engage various ways of reasoning…as they explore the river and progress in their mathematical journey.” We expand on this metaphor by adding shores to the river which represent the realistic context and formal symbolic mathematics. As students progress down the river, students may choose to work within the context or formal symbols. Perhaps a student is trying to vertically mathematize on the formal symbolic shore but hit an obstacle stalling progress. That student can go back to the other side of the river, or back to reasoning within the realistic context, to anchor what they were trying to formalize within the context. These stopping points along or close to one shore act as opportunities for students to cross the river and horizontally mathematize a concept, where vertical mathematization occurs along one shore. This gradual movement down the river between and along the shores represents students’ progressive mathematization.

In this paper, we focus on students’ progressive mathematization of the relationship between a matrix and its null space. Research regarding null space specifically is limited, especially research focusing on students’ concepts or understandings of null space. Two studies touch on student reasoning about null space, but mostly for other purposes. Wawro (2015) details one student’s interpretation of the Invertible Matrix Theorem. This study is not about null space alone but does involve the student describing relationships between null space and other attributes of matrices or collections of vectors. Caglayan (2019) examines undergraduate students’ classifications of subsets of various vector spaces as either subspaces or not. One of those subsets was the null space of some given vector space. There has been recent work exploring students’ conceptual resources while reasoning about null space (Andrews-Larson et al., 2022) and the ways students are positioned while reasoning about null space in the context of the hallways task (Andrews-Larson et al., In Progress; Smith et al., 2025). Because student reasoning about null space is understudied, we use this topic as the subject of this research study. More specifically, we are interested in the ways students move between reasoning about null space within the realistic hallway map context and formal symbolic reasoning about null space. Our research question is: how do students develop their reasoning about the relationship between a matrix and its null space through progressive mathematization?

Methods and the Hallways Task

Methods of Data Collection

During Summer of 2024, our research team conducted a five-day paired teaching experiment (PTE; Steffe & Thompson, 2000) over Zoom with three participants enrolled in a differential equations course. The third author acted as the teacher-researcher for all five interviews, while the first and second authors took turns acting as the observer-researchers. While the teacher-researcher was engaged in the in-the-moment interactions with all three participants, the observer-researcher was taking field notes, or asking clarifying questions when needed.

When initially designing the study, our research team was curious if participants would position each other differently based on perceived experience with the content (Smith et al., 2025). This was relevant because two of the three participants (Henry and Tyler) previously completed linear algebra before the first interview. We focus our attention on discussions between Henry and Tyler because they represent two different journeys down the river of progressive mathematization. By the start of the fifth day of the PTE, all participants had completed the subspaces unit of the IOLA curriculum, which we refer to as the hallways task.

The Hallways Task

The hallways task has three parts (See Wawro et al., 2013). The first involves the notion of closure under linear combinations of camera data vectors, where 5D vectors represent paths from classrooms in the hallway map pictured in Figure 1. Participants reasoned about the set of camera data vectors that emerge from sending participants from room C back to room C as a set closed under linear combination. Participants referred to these paths as "loops", which became a common contextual term for this set of vectors. In Task 2, participants began to associate camera data vectors with room population change vectors by considering how many people move past certain cameras between each period. This association often led to a discussion of how a system of equations can be represented by a coefficient matrix. The span of the "loop" vectors resulted in a set of vectors that resulted in no change in population because they corresponded to a path that started and ended in the same room. Finding all the "loops" became a way for participants to conceptualize the null space. Finally, in Task 3, participants were provided with a 5 by 7 matrix and asked to create a new one-way hallway map that would correspond to this new coefficient matrix.

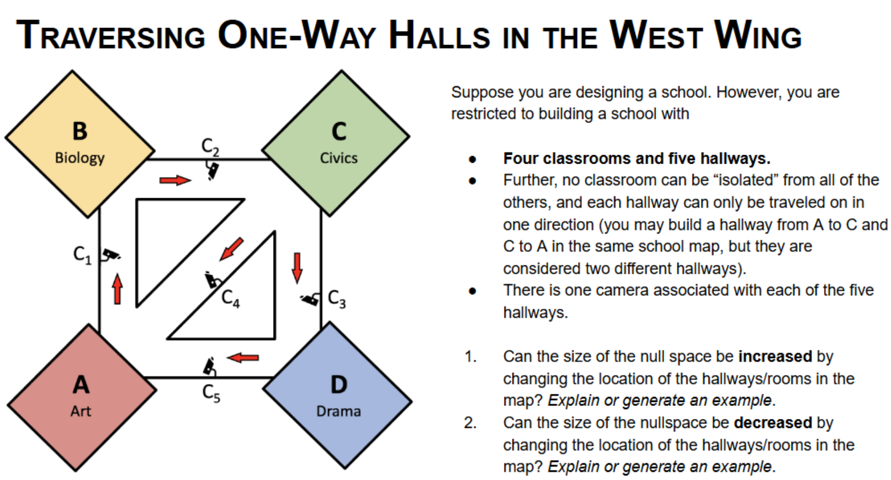

Figure 1. The hallway map from the hallways task sequence with the first Day 5 task statement.

Our team designed an additional task as a follow-up to the hallways task sequence stated at the right of Figure 1. Our initial design was related to the development of a playful math task (Ellis et al., 2024), where participants first attempt to change the dimension of the null space of the original coefficient matrix associated with the hallway map. Then, participants became "structural engineers" tasked with developing a new map where they could change the number of rooms and hallways to determine the impact this had on the dimension of the null space of the associated matrix.

Methods of Data Analysis

Our initial analysis of the data involved reviewing the Zoom recordings as a whole team. We began by noticing instances of collaborative reasoning between the pairs of the three participants. We then highlighted these specific excerpts as examples of collaborative reasoning individually, then coded these excerpts in pairs to develop our collaborative reasoning actions (Smith et al., 2025). We examined the ways participants made space for, acknowledged, or leveraged each other's ideas. For this paper, we reanalyzed the video data from the fifth day, focusing more on participants' mathematical reasoning. We noticed Henry and Tyler were both horizontally mathematizing (i.e., going between reasoning within the realistic hallways context and the more formal matrix representations) and looked to further explore what started this movement and what happened because of this movement. Note that we use the term "realistic context" to represent the context chosen by the designers of the task sequence. We examined two more salient shifts between realistic context and formal mathematical symbols, adding context by noting the catalyst for and results of these shifts. By doing this, we studied Henry's and Tyler's progressive mathematization.

Results

Part 1: Keeping the Number of Rooms and Hallways the Same

At the beginning of Day 5, participants were trying to change the dimension of the null space by generating a different hallway map with four classrooms and five hallways. Tyler was the last to share his hallway maps after some individual working time. Tyler's first map demonstrated his attempt to increase the dimension of the null space by increasing the number of loops:

> Unless I violated a rule I'm unaware of, I think I was able to decrease it by just decreasing loops, but I was not able to increase it. I thought I did, but then I noticed two of the loops just added together to be the third one. So…it's not actually a third independent vector…so I haven't really increased the span at all.

Tyler realized that he had not "increased the span" since the camera data vector that represented the "new loop" was a linear combination of the original two (e.g., $<1, 1, 0, 0, 0> + <0, 0, 1, 1, 1> = <1, 1, 1, 1, 1>$) as pictured in Figure 2. Next, Tyler described his success in decreasing the dimension of the null space by decreasing the number of hallway loops from two to zero:

> There's no loops here on the right. I just want to make sure this didn't violate any things…does there have to be one main loop? Do you have to be able to go throughout the school? Cuz obviously here you kind of get stuck at D, which that's no good. So, I don't know if that's against what we're doing here. If I changed any of them, I would get a loop…. I'm more confident in my inability to increase the null space.

Tyler's continual attention to the "rules" of the task, and his concern with violating them was an indication of his care in contextualizing his mathematical reasoning. It is interesting to note that the task states that students would be able to get from one room to any other room, but as Tyler said, "you're gonna get stuck" in room D. If one were to row reduce the coefficient matrix associated with Tyler’s map, there would be two free variables associated with loops that travel backwards on hallways, which is also “against the rules” of the task.

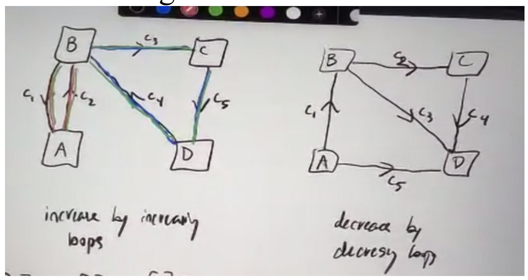

Figure 2. Tyler’s maps with two loops (left - 2D null space) and no loops (right - 0D null space).

Tyler’s contribution led to Henry discussing the number of rows and columns of the coefficient matrix, as represented by the number of classrooms and hallways in the map:

> I think I agree with Tyler, because if you think about the matrix representation then the most amount of linearly independent variables you can have is the number of rows that you have which corresponds to the room distributions. Since you have four rooms, then that means that you can only have four linearly independent variables. Then the amount of free variables would be the number of weights minus the number of independent variables. If we have five weights, then the most linearly independent variables you could have is four, then that would have at least one loop or something like that…I think that assumes though that it has to be onto, in which case, if it doesn’t have to be then you could add more and more zeros.

Henry’s approach of using the greatest number of pivots by comparing the number of rows to columns was accurate. However, when re-presented with Tyler’s hallway map with no loops (see Figure 2), Henry struggled to reconcile the initial statement with his context-based interpretation of the dimension of the null space as a number of hallway loops. Henry went on to say, “I don’t have a solution to my original problem, but the largest possible free space you can have is five, so you would have a 5D null space.” This led to a discussion of a contextual interpretation of a matrix of all zeros, with Henry stating, “I’m not sure how to reconcile the matrix representation with the physical layout.” Tyler goes on to respond:

> I think this is all just connecting to my sort of lingering interest in how we’re actually making these things from matrices. Because it seems sort of convenient whenever there’s like a -1 and a 1, like those pairings to just draw it like that. But I don’t see what’s like absolutely making us do it like that. Like I’m sort of thinking. Based on how we’ve been doing that would be no rooms and no, well not even. It would be like…a bunch of hallways to nothing.

Tyler appeared to be leveraging a zoomed-out perspective of coefficient matrices to question the impact of the restrictions placed on the entries of the coefficient matrix by the task sequence (e.g., the -1 1 pairs within each column). The group continued to consider what a matrix of all zeroes would represent in the context of a hallway map before working on the second task.

During this part of the task, we saw Tyler start his journey by reasoning within the hallways context, then horizontally mathematized by crossing the river to formal symbolic notation. This horizontal movement took the form of Tyler trying to generalize what he was noticing within the hallways context. Henry, however, started on the formal shore but could not make sense of Tyler's findings in relation to his more formal reasoning

Part 2: Changing the Number of Rooms and Hallways

For the second task, participants were positioned as structural engineers and allowed to make edits to the map (e.g., adding rooms or moving and removing hallways) and determine the impact this alteration had on the dimension of the null space. After Henry presented his initial map, he presented two more maps that had five classrooms instead of four:

> Tyler said something about adding rooms and I thought was interesting. So, thinking back to the matrix, I thought that would increase the amount of rows that you have…so then I drew the pentagon because that was the simplest geometry in my head to draw. So, then I was thinking about what would happen if you had no null space. So then according to this, you have five hallways and five rooms so then you have a square matrix, so then you could make something with no null space, so then I made the thing on the right.

Henry appeared to be saying that to have a 0D null space you need to have the same number of rows and columns to ensure there is a pivot in every column, which corresponds to the same number of hallways and classrooms, as indicated in Figure 3. However, it should be noted that Henry's second map emerged from a discussion about a contextual interpretation of the zero matrix. A hallway map with no null space would have no loops, meaning that there is no way to keep the population of each room fixed between class changes. A coefficient matrix with all zeros, would have a 5D null space since it would collapse all camera data vectors to the zero vector. So, even though Henry said the right-hand map in Figure 3 would have no null space, we interpret him to mean this was his attempt at constructing a map that had an associated coefficient matrix of all zeros since, "you get stuck in the room you start in."

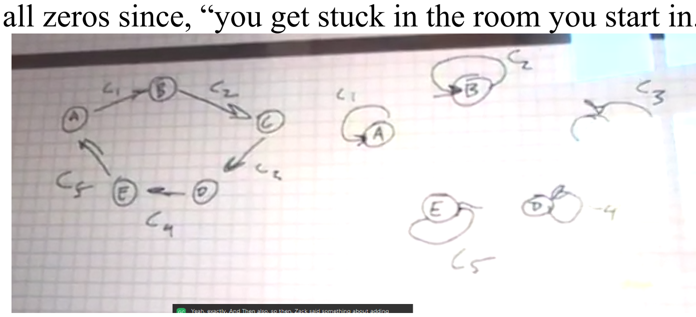

Figure 3. Henry's second set of maps demonstrating 1D null space (left) a no null space map (right).

Similarly to Henry attempting to determine the limits on the possible values of the dimension of the null space, Tyler also attempted to generalize the relationship between the number of rows and columns to determine the dimension of the null space:

> [I] added an additional pathway between C and D going back up. I think that acts as an additional linearly independent vector, so increasing the null space to three. But…I looked at my old one, and sort of realized, I mean yeah if this one has already reduced the null space, then I can just get rid of any of the classrooms or any of the pathways and I can't ever make more loops by reducing, by getting rid of things I already have, so I just said hey I'll start where I'm at and I can remove things to show…you can do either and I've reduced the null space…I'm trying to find out like a rule of thumb for this. But it was sort of confusing because I got there was six columns, and four rows equals two…[I'm] trying to sort of figure out how can we, say we wanted to like minimize how many hallways and things that we're using, how could we do that.

As seen in Figure 4, Tyler was attempting to generalize the relationship between the number of rows and columns and what this meant for the dimension of the null space. However, the difference in the number of rows and columns was not equivalent to the number of loops Tyler generated with his new map (pictured on the left side of Figure 4). The participants ran out of time to completely finalize their thoughts at the end. However, we view Tyler and Henry's progress in investigating the relationship between the number of rooms, hallways, and loops and the number of rows, columns, and dimension of the null space of the associated coefficient matrix as progress towards reinventing the relationship between the dimension of the null space and the number of rows and columns of the that coefficient matrix.

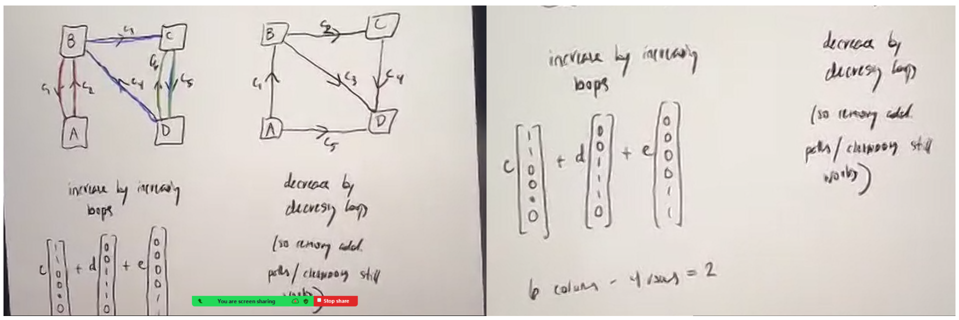

Figure 4. (Left) Tyler's second set of maps with three loops and no loops. (Right) Tyler's attempt at formalizing the relationship between the number of rows, columns, and loops.

Here we see Tyler horizontally mathematize to again cross the river to the formal symbolic side to generalize the relationship between the number of columns, rows, and loops. Henry, on the other hand, horizontally mathematized back to the hallway maps to demonstrate a map that corresponded to his largest value of the null space being five dimensions.

Discussion and Conclusion

In this work, we found that Tyler and Henry had two different approaches to the playful math problem. They were each vertically mathematizing, but on different "shores". The rich discussion between the two resulted in horizontal mathematization, generating connections between shores. For example, Tyler and Henry linked adding columns to adding hallways and adding loops to increasing the dimension of the null space. At the end of the conversation, they were close to understanding how altering the map or matrix impacted the dimension of the null space. For Tyler and Henry, their discussions resulted in successive horizontal mathematization which led to vertical mathematization along both shores of the concept of null spaces.

As both designers and IOI practitioners, we believe that students should be encouraged to revisit and create connections between their formal mathematical experiences and a variety of contextual experiences. This approach paves the way for two new avenues of research. From an IOI instructor's perspective, what are the asset-based approaches to navigate conversations between students who are approaching tasks from opposing shores of the river? From an RME designer's perspective, attention should not only be paid to appropriate entry points but also how frequently the context can be revisited throughout the course and how well these ways of reasoning support the mathematization of later concepts. For example, the hallways context can used to support the guided reinvention of null space as well the rank-nullity theorem.

While this study was limited in number of participants, it charts the start of a journey to a guided reinvention of the rank-nullity theorem. It also highlights the complexity of student reasoning and progressive mathematization down the river. The result of this journey is not only a deep understanding of formal mathematics, but also a rich context from which students can refer to considering new ways to reason about related ideas. Future research will focus on developing this additional activity to supplement the IOLA curriculum.

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