A Preliminary Investigation into Multivariate Calculus Students' Understanding of Trigonometry

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This report describes a preliminary investigation into how multivariate calculus students understand trigonometric functions. Trigonometric functions are an essential component of understanding single-variable calculus which in turn is generalized into multivariable calculus. Therefore, an understanding of trigonometry is not only expected but an important component for success in multivariable calculus courses. Replicating a Weber (2005) study investigating trigonometry students' understanding of trigonometric functions, we surveyed multivariate calculus students to see how they understand trigonometric functions. Similarly, we leveraged the idea of procepts (Gray & Tall, 1994) to aid in interpreting students' descriptions of trigonometric concepts and properties.

Keywords: Multivariate Calculus, Trigonometry, Procepts, Student Understanding

Multivariate calculus is an intense course that often introduces students to both 3-dimensional space and vector notation at the start of term so that by the end of the term they can engage with various fundamental theorems of multivariate calculus. Needless to say, there are a lot of expectations on students' prior knowledge and understanding of various mathematical concepts, one such idea is trigonometry. By the time students are in multivariate calculus instructors assume that students have been working with trigonometric ideas for a number of terms. Furthermore, the fundamental theorems of multivariate calculus are often expressed using vector multiplication which can be defined using trigonometric relationships. However, working with a concept and understanding a concept are often different feats and our study aims to start to investigate how students who are expected to "know" trigonometry actually understand it.

Background and Literature

While multivariate calculus has seen considerably less attention than the teaching and learning of single variable calculus, over the last decade researchers have begun to shift focus to investigating student learning of multivariate topics and offering insights on pedagogical approaches. It comes as no surprise that literature on student understanding in multivariate calculus overwhelmingly focus on the understanding of topics introduced in a multivariate calculus course. A recent meta study that surveyed research studies on the learning of multivariate calculus (Martinez-Planell & Trigueros, 2021) found that much of the research deals with the idea of generalization and how students' knowledge of one-variable calculus influences their understanding of multivariable calculus. This research includes investigations into the student understanding of topics such as generalizing single variable functions to multivariable functions (Kabael, 2011), how student's meaning of function properties generalize to multivariate functions (Dorko, 2017), and students' understandings on definite integral of multivariable functions and how they relate to understandings about integrals of one-variable functions (Jones & Dorko, 2015).

An understanding of trigonometry is an important aspect of calculus readiness, as indicated by the Calculus Concept Readiness (CCR) (Carlson et al., 2015). The CCR is an instrument that assesses foundational understandings and reasoning abilities that have been documented to be essential for learning calculus. Five of the twenty-five assessment items on the CCR specifically address student understanding of trigonometry. However, research also suggests students struggle with trigonometry, specifically with trigonometric comparisons and the recollection of commonly used formulas might be a real hurdle to their understanding of concepts in Calculus II (Ruslimin et al., 2019).

Our work is motivated by one particular study (Weber, 2005) that examined college students' understanding of trigonometric functions informed by Gray and Tall's (1994) notion of procept. Weber used the idea of a procept, described in detail below, to investigate and describe students' conceptual grasp of trigonometric functions, their ability to justify mathematical properties, and their use of multiple representations. Weber found that students did in fact view trigonometric expressions as procepts, often encapsulating processes related to right triangles or the unit circle.

Theoretical Perspective

Building on Weber (2005)'s work, we consider the notion of *procept* as introduced by Gray and Tall (1994). A *procept* is a mental collection of processes along with a concept that represents the result of those processes, summarized using symbolism. For example, Gray and Tall (1994) posit natural numbers as procepts, each symbol, such as 6, denoting the result of a child's process of counting, which later, after additional exposure to counting problems, mask other sophisticated procedures, such as 6 representing variously 4+2, 3+3, 7-1, and so on. Similarly, in calculus, the derivative often denoted $f'(x)$ can be seen as a procept. It both stands symbolically for the result of differentiation of the function $f(x)$, and as shorthand for the procedure itself. In fact, there are two procedures familiarly employed in a first course: the limit definition $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and alternatively the collection of computational rules, which contains the power rule, the chain rule, the product rule, and so on. Gray and Tall (1994) emphasize that the use of a procept, once internalized by a learner, can allow a delayed computation, by "using symbols both as manipulable objects and as triggers to evoke mathematical processes" (p. 138).

The focus of this study is to revisit Weber (2005, p. 93-94)'s research questions with a new population, multivariate calculus students:

(1) *How do students in a [multivariate calculus] course understand trigonometric functions? In particular, do these students understand trigonometric functions as processes and/or procepts? If so, what types of processes do students think of when reasoning about trigonometric functions?*

(2) *Can students in a [multivariate calculus] course use their understanding of trigonometric functions to justify why trigonometric functions have the properties they do?*

In this preliminary report, we report on the first research question due to space limitations but will explore both questions in the RUME presentation.

Research Methods

Participants

The student participants in this study were enrolled in a semester long Multivariate Calculus course at a small public 4-year institution in the Rockies; taught by the first author. The pre-requisite for the course was Calculus II with a minimum grade of C-. Thirty students were enrolled in the course, 28 of whom participated in the study. Most participants were engineering majors (21), including computer engineering (10), engineering (9), and mechanical engineering (2). Mathematics (2) and secondary mathematics education majors comprised just over 10% of the participants. Lastly, there was one physics major (1), one geology major (1), and one participant left the question blank (1). When asked when they were first introduced to the sine function and how long ago, on average participants had known of the sine function for at least 5.5 years. The least amount of time reported was 3 years, however two participants stated 'since high school geometry' which was conservatively estimated to 2 years. The most amount of time reported was 15 years.

Data Collection

Unlike the Weber (2005) study which included a pre-test, post-test, and interviews, our study consisted of a ten-question survey administered in class at the end of the term. Three of the questions were demographic in nature (reported above), one question asked participants to provide known applications of sine, and the remaining six, taken from the original Weber (2005) study, are listed below.

1. Describe sine in your own words.
2. Why is sine a function? What is its domain and range?
3. Approximate; sin 340° and cos 340°
4. What is sin 270° and why?
5. When is sin θ decreasing and why?
6. Why is $\sin^2\theta + \cos^2\theta = 1$?

The first three questions were meant to gain a general sense of how participants think about the sine (and cosine) function and whether they view the functions as operations applied to angles (Weber, 2005). The purpose of the last three questions was to test students' recollection of specific trigonometric properties and elicit a justification for each. This preliminary report will discuss the analysis methods and results of Q1 and Q2 due to space constraints, while the remaining methods and results will be shared in the presentation.

Data Analysis

The first pass of analysis aimed to capture evidence of procepts used by the participants as they described the sine function. The original study exclusively referenced two processes leveraged by trigonometric students. The first approach involved reasoning about trigonometric functions using a unit circle model, and the second described trigonometric functions in term of right triangles. Weber (2005, p.93) described these processes in the context of a student's estimate for sin 20° as:

- Mentally constructing a standard right triangle containing an angle of 20°, and then comparing the length of the side of this triangle opposite to the 20° angle with the length of its hypotenuse. Then, realizing that sine is equal to opposite over hypotenuse, approximating sin 20° to be about 13.
- Mentally constructing a unit circle on the Cartesian plane and then constructing a ray emanating from the origin of the plane that forms a 20° angle with the positive half of the x-axis. Then, focusing on the point of intersection between the ray and the unit circle, noting that the y-value of this intersection will be a small positive number, such as 0.3.

When analyzing responses to Q1 we coded references to such approaches as "unit circle" and "right triangle" respectively.

Two new ways of describing sine were also found multiple times in our data which we named "wave" and "vertical." Participants who described sine in terms of a periodic or sinusoidal function were coded as "wave." An example of which would be, "Sine is the name of a sinusoidal function that oscillates, depending on the given angle" (P2). Responses that used a vertical component devoid of a unit circle, such as “Sine is the trig function that is typically used for y-axis related parts of a function and whatnot” (P13) were coded as “vertical.” All other responses were coded as “other.” Each member of the research team first coded the data individually and came together to discussion any disagreement before reaching complete consensus.

Results

As previously mentioned, in this preliminary report we offer results to the first two questions from our survey. The data set contains responses from 28 participants.

Responses to Question 1: Describe sine in your own words

Most participants described a single strategy (21). Three students offered two approaches, and two mentioned three approaches. Two students did not provide meaningful responses.

Table 1. Student approaches used to describe the sine function

Student descriptions of sine function	
Approach	# of responses
Unit circle (exclusively)	3
Unit circle (total)	6
Right triangle (exclusively)	4
Right triangle (total)	7
Wave (exclusively)	12
Wave (total)	16
Vertical (exclusively)	2
Vertical (total)	3
Other	3

Most participants (16) referenced the “wave” approach, which again was not discussed in Weber (2005). Many of these students mentioned the graph of the function (in Cartesian coordinates), by either illustrating $y = \sin x$ or describing ‘waves,’ ‘periodic functions,’ or ‘sinusoidal functions.’ Twelve of these students only referenced this single strategy. For instance, participant P5 described sine as, “A wave function with period 2π and amplitude 1, starting at the origin.”

Seven students described the sine function as constructing a right triangle and measuring the ‘opposite’ side and the hypotenuse to compute the sine. Five of these students solely mentioned this approach, for example (P16, Figure 1):

Figure 1. P16’s interpretation of sine in terms of a right triangle.

Six participants mentioned the y-coordinate from a point on the unit circle. Two of these students only mentioned this approach. For example, student P1 explained, “Sine is the vertical ‘axis’ so to speak of the unit circle.”

Three students were coded as "vertical" and seemed to reference a y direction or vertical component, devoid of a unit circle. For instance, P8 described sine as: "Sine is a direction in the y or vertical direction."

Five students offered multiple approaches. For example, one student used three approaches by describing sine as a "wave" along with right triangles and the unit circle (P3, Figure 2):

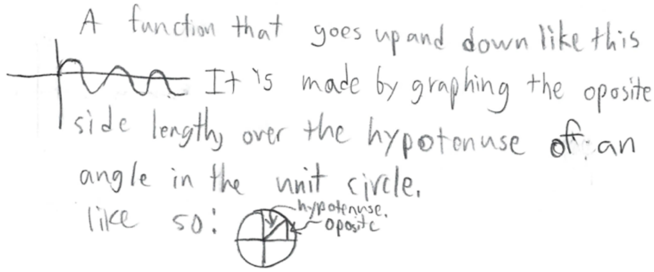

Figure 2. P3's description of sine in terms of wave, right triangle, and unit circle.

Responses to Question 2: Why is sine a function? What is its domain and range?

Many participants (7) ignored the first question, while fourteen wrote responses which could either not be coded or were off topic. Seven responses addressed the question directly. Two of these gave a completely correct answer, emphasizing that a single output was associated with each input. Four more gave a partially correct answer, stating that to each input there is "an" output associated. Lastly, one participant (P27) described a relation, saying "not exactly sure why its [sic] a function, but it relates $\angle A$, X, and H", where they had described sine earlier as relating sides of a right triangle using "the relation $\sin A = \frac{X}{H}$".

When asked about the domain and range of the sine function, a few students (4) gave restricted domains, such as $[0,2\pi]$. Three students skipped answering the question. Two students swapped the domain and range. The slim majority (15) were correct for both domain and range.

Discussion

In this study, we explored multivariate calculus students' understanding of trigonometric functions, in particular the sine function. As expected, this population used more approaches than were found in Weber (2005)'s study. However, we are hesitant to describe the new found approaches ("wave" and "vertical") as procepts since it is unclear to us what process specifically is being encapsulated. We hope further analysis of the remaining interview questions begins to shed light on this and we invite the audience to consider with us.

Consistent with the function literature, most students in our study were unable to, or unwilling to, offer a complete or correct justification of what makes sine a function. We are interested in considering how our current study could serve as a pilot for a follow-up study which would include interviews with participants. We hypothesize that multivariate calculus students have a more robust understanding of function than our survey elicited and if prompted differently, we think the findings would be more sophisticated. We invite feedback from the audience on further ideas for such a follow-up study.

Due to the preliminary nature of this work, we would like suggestions from the audience on the following: (a) What process might lead to a "wave" or "vertical" procept of the sine function? (b) When considering a follow-up study, how might we revise the context of the questions to include multivariate calculus, and would that be advantageous?

References

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. International Journal of Research in Undergraduate Mathematics Education, 1(2), 209-233.

Dorko, A. (2017). Generalising univalence from single to multivariable settings: The case of Kyle. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th annual conference on research in undergraduate mathematics education* (pp. 562-569). San Diego, CA.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of elementary arithmetic. *Journal for Research in Mathematics Education, 26*(2), 114-141.

Jones, S. R., & Dorko, A. (2015). Students' understandings of multivariate integrals and how they may be generalized from single integral conceptions. Journal of Mathematical Behavior, 40, 154-170.

Kabael, T. (2011). Generalizing single variable functions to two-variable functions, function machine and APOS. *Educational Sciences: Theory & Practice*, *11*(1), pp. 484-499.

Martínez-Planell, R., & Trigueros, M. (2013). Graphs of functions of two variables: Results from the design of instruction. *International Journal of Mathematical Education in Science and Technology*, *44*(5), 663-672.

Martínez-Planell, R. & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM- Mathematics Education 3*. DOI 10.1007/s11858-021-01233-6

McGowen, M. A., & Tall, D. O. (2010). Metaphor or Met-Before? The effects of previous experience on practice and theory of learning mathematics. *The Journal of Mathematical Behavior*, *29*(3), 169-179.

Ruslimin A.,Masriyah & Manuharawati. (2019). Difficulties of Undergraduate Students to Understand of 2ndCalculus. International Journal of Trends in Mathematics Education Research, 2(1), 26-30. DOI: http://dx.doi.org/10.33122/ijtmer.v2i1.35

Tall, D. (2004). Introducing three worlds of mathematics. *For the Learning of Mathematics*, *23*(3), 29-33.

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two variable functions. *Educational Studies in Mathematics*, *73*(1), 3-19.

Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, *17*(3), 91-112