A Case for Applying Unit Transformation Graphs to Model Students' Solutions to Transformation Geometry Problems

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In this theoretical report, we argue that unit transformation graphs are an appropriate and useful tool for modeling students' solutions to transformation geometry problems. Developed by mathematics education researchers working from a Piagetian theoretical perspective, unit transformation graphs explicate the mental actions that students use in solving problems, along with the objects (or units) on which they are acting. This theoretical report contributes to an understanding of how unit transformation graphs can be extended, beyond their current use, to model student reasoning in a new mathematical domain. We illustrate the potential efficacy of unit transformations in modeling students' solutions to transformation geometry problems with two researcher-generated solutions to a transformation geometry problem, and we call for future empirical research to model students' solutions to transformation problems with UTGs.

Keywords: Cognition, Learning Theory, Transformation Geometry, Second-Order Models

Transformation geometry—the study of geometry focusing on transformations (such as rigid motions and dilations) and properties that remain invariant under those transformations—is an important component of mathematics education at all levels. For nearly a century, mathematicians and mathematics educators have recommended that transformations play a greater part in geometry curricula (Betz, 1933; King, 2021; Usiskin & Coxford, 1972). The *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 2000) recommend that instructional programs from Grades PK-12 should "enable all students to…apply transformations and use symmetry to analyze mathematical situations" (p. 41). In the Common Core State Standards for Mathematics (CCSSM; Common Core State Standards Initiative, 2010), transformation geometry plays a broad role, elaborated below. At the postsecondary level, transformation geometry together with group theory can be used to classify Euclidean and non-Euclidean geometries—an idea introduced by Felix Klein in his 1872 address to the University of Erlangen, setting forth the influential Erlangen Program. Klein claimed, "The most important notion which is necessary in the following considerations is that of a group of spatial transformations" (cited in Herbst et al., 2017, p. 19).

Within the CCSSM, in Grade 8, students use hands-on activities to verify properties of rigid motions, describe the effects of rigid motions, and understand that two shapes are congruent if (and only if) one can be transformed into the other using a sequence of rigid motions. In high school geometry, students formalize notions of congruence and similarity in terms of transformations, leading to triangle congruence and similarity criteria (Wu, 2013). Transformations are also used in geometric proofs, such as proving that two shapes are congruent by describing a sequence of rigid motions that will map one onto the other. Consequently, middle and high school teachers need to be prepared to conceptualize and explain geometric transformations, including their use in proofs of similarity and congruence; evidence suggests this may pose a challenge (Hegg et al., 2018; Yanik & Flores, 2009).

Given the importance of transformations in students' geometry education, mathematics educators need to better understand how students conceptualize and reason about geometric transformations. From a constructivist perspective, understanding student reasoning in this way involves engaging students in problem situations and, from observations of students' responses, posit answers to the question: "What mental operations must be carried out to see the presented situation in the way one is seeing it?" (von Glasersfeld, 1995, p. 78). Answering this question requires decentering from one's own mathematical knowledge and attuning to the mathematical knowledge of the student. In this sense, the researcher builds a *second-order model*, a model that can explain the researcher's observations of the student's mathematical reasoning (Steffe & Thompson, 2000).

Taking seriously the Piagetian position that students' mathematics is comprised of mental operations (i.e., mental actions), Norton et al. (2023) introduce *unit transformation graphs* (UTGs) as a tool for forefronting students' mental actions and illustrating their coordination in problem solving. UTGs are directed vertex-edge graphs, where vertices represent material being acted on (units) and edges represent mental operations (transformations applied to units). Although UTGs were originally used to model students' numerical reasoning, particularly about fraction multiplication (Norton et al., 2023), we hypothesize that UTGs can be extended beyond numerical contexts, modeling students' coordinations of mental actions generally.

In this theoretical article, we extend the scope of UTGs to the realm of transformation geometry. In doing so, we address the research question: *How, theoretically, can UTGs be used to build models of students' solutions to transformation geometry problems?*

Literature Review

We have research evidence that indicates students across Grades K-16 often experience challenges in making sense of and reasoning about geometric transformations. For example, middle school students, prior to instruction on transformations, may apply "holistic" strategies, rather than measurement-based analytic strategies, to solve transformation problems, with the latter being more likely to lead to correct solutions (Boulter & Kirby, 1994). Similarly, some undergraduate students may hold picture-based meanings of transformations without measurement-based or transformation-based meanings; students who hold either of the latter two meanings are more likely to successfully determine congruence and non-congruence of given shapes (Geotas et al., 2021).

Research has found specific transformations to present challenges for students. Dixon et al. (2022) found that, even after instruction (with traditional curricular materials), many middle school students struggle to solve problems involving rotations. Among 444 middle school students in their study, students answered an average of 59% of questions correctly, with only 46% of students correctly determining whether a given pre-image and image show a rotation and, if so, correctly locating the center of rotation. Hollebrands (2007) found that high school students may struggle to anticipate the results of transformations prior to carrying them out. Yao (2020) found similar results among preservice secondary teachers; moreover, they struggled to determine whether a composition of transformations can be performed as a single transformation. Frazee (2018) found that high school students may struggle to carry out reflections, especially over a slanted line, and to correctly locate a slanted line of reflection for a given pre-image and image, and they may struggle to carry out rotations and locate the center of rotation for a given pre-image and image. However, for both transformations, Frazee found significant conceptual progress after research-based instruction focusing on properties of transformations and developing students' mental processes for conceptualizing them. Research also indicates that preservice elementary school teachers, prior to research-based instruction, may struggle to identify examples from non-examples of reflections, rotations, and translations (Clayton, 2020) and to identify the correct center of rotation for a given pre-image and image

(Edwards & Zazkis, 1993). However, after research-based instruction (similar to that used by Frazee, 2018), Clayton (2020) found preservice teachers to make significant conceptual progress.

Secondary and postsecondary students may struggle to conceptualize transformations as not just motions, but as mappings (i.e., functions) from the plane onto the plane (Hollebrands, 2003; Yanik, 2011; Yanik & Flores, 2009). Hollebrands (2003) found that students' conceptions of domain were critical in understanding geometric transformations as functions. Yanik and Flores (2009) contributed a detailed account of one preservice teacher's trajectory toward developing a mapping-based meaning of translations, with vectors playing an especially important role. Although geometric transformations are formally defined as one-to-one and onto mappings on the plane (Martin, 1982), there are differing opinions about the nature, purpose, and productivity of thinking about transformations as motions versus as mappings (Edwards, 2003).

The challenges that K-16 students experience regarding geometric transformations underscore the importance of mathematics educators better understanding their conceptions and investigating ways in which they can be supported. We posit that to achieve such an understanding, mathematics educators need to better understand students' mental actions for carrying out, and anticipating the results of, reflections, rotations, and translations. This position is supported by research by Frazee (2018) and Clayton (2020), reviewed above, and by Johnson-Gentile et al. (1994), who found significant conceptual gains among middle school students who engaged in computer-based activities using Logo Geometry. They explained, "The need in Logo environments for more complete, precise, and abstract explication [of action commands given to the computer] may account for students' creation of conceptually richer concepts for motions" (p. 137). Although researchers have provided fine-grained analysis of student knowing and learning within transformation geometry contexts, including approaches using APOS theory (e.g., Hollebrands, 2003) and the Pirie-Kieren model (e.g., Yao, 2020), we argue that UTGs offer a uniquely fine-grained lens on students' mental actions that can provide important insight into students' conceptualizations.

Theoretical Framework

We frame our work in terms of Piaget's (1970) genetic epistemology; within this perspective, mathematics is constructed as a coordination of mental actions. A *mental action* is a transformation (a change in the state of some perceived/conceived entity) that may be carried out physically or entirely in one's mind. Within the realm of number, such mental actions include iterating, partitioning, and disembedding; these mental actions are applied to *numerical units*, which are entities conceptualized as 1 (Ulrich, 2015). For example, a student might iterate a unit of 1 to form a composite unit (e.g., 10), or partition a unit of 1 to form smaller equal-size parts (e.g., 1/2). They might disembed one of these parts (removing 1/2 from the whole), treating that part as a unit itself while keeping in mind its relationship to the original whole.

UTGs require researchers to identify the mental actions that students bring to problem situations and to model how they are coordinated. Consider the following example.

> *Cake Task*. Anita is having a birthday party. Her birthday cake is cut into 8 slices for Anita and her 7 friends. After the cake is cut, another friend arrives late to the party. Anita wants to share her slice of cake equally with the newcomer. What portion of the whole cake does Anita get? (adapted from Norton et al., 2023)

To solve this problem, a student might first take the whole cake as a unit. They could then *partition* the cake into eight equal-size pieces, then *disembed* one of those pieces to give to Anita. Then, when the newcomer arrives, this piece is further *partitioned* into two equal-size pieces, and one of them is *disembedded*. To relate the size of this piece to the whole cake, the student might *iterate* it 16 times. Through this coordination of mental actions—partitioning, disembedding, partitioning again, disembedding again, and iterating—the student can conclude that Anita received 1/16 of the whole cake. This coordination of mental actions is illustrated by the UTG in Figure 1, where rectangular nodes represent units and arrows represent the student's mental actions on those units.

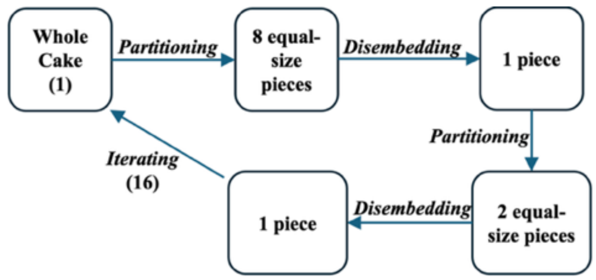

Figure 1. UTG illustrating a coordination of mental actions to solve the Cake Task

Spatial and geometric situations may involve a wide variety of units and mental actions performed on them. Following Antonides et al. (2025), we define a *spatial unit* as a perceived/conceived entity, such as a shape, that is spatially operable—that is, the entity is isolated as an object of attention and acted upon using spatial mental actions. Based on prior research on students' geometric reasoning, we generated a non-exhaustive list of mental actions that we hypothesized may be relevant in students' solutions to transformation geometry problems (see Table 1). We used prior research literature, where available, to suggest definitions of each mental action. We theorize definitions of mental actions not yet represented in the research literature. For example, a student might *unitize* a geometric shape and take it as material for spatial operation (Wheatley & Reynolds, 1996); they might *spatially disembed* part or parts of a geometric shape in problem solving (e.g., isolating one side of a square without losing track of the fact that the side is part of the square) (Barrett & Clements, 2003); or they might *spatially locate* an object in space (e.g., visualizing the location of a cube within a 3-D array, or a point in space corresponding to another point) (Battista, 2004). Understanding precisely how students temporally order mental actions on spatial units can reveal critical insights into their mathematical conceptualizations and reasoning (Antonides & Battista, 2022).

Table 1. Non-exhaustive list of mental actions hypothesized to be relevant in solving transformation geometry problems

Mental Action	Definition
Unitizing	Isolating perceived/conceived material as a distinguishable object through focused and unfocused attention (von Glasersfeld, 1981)
Spatial Disembedding	Isolating (unitizing) part of a spatial object, treating it as a separate spatial entity while maintaining its relationship to its original spatial whole (Barrett & Clements, 2003; Sarama et al., 2003)
Spatial Locating	Identifying the location of a point or other spatial object or region (Antonides et al., 2024; Battista, 2004)
Extending	Imagining a continuation of a spatial object such as a line segment

Directing	Establishing a spatially oriented relationship between two points
Reflecting	Flipping a spatial object, often (formally) over a line conceptualized as the line of reflection (or line of symmetry) (Clements et al., 2001; Edwards, 1991)
Rotating	Turning a spatial object in a particular direction (clockwise or counterclockwise), often (formally) in relation to a fixed point conceptualized as the center of rotation (Clements et al., 2001; Edwards, 1991)
Translating	Sliding a spatial object in a particular direction, often (formally) in relation to or described by a vector (Clements et al., 2001; Edwards, 1991)

Examples of UTGs in Transformation Geometry

To illustrate and support our theoretical argument, we provide two potential solutions (provided by the first two authors) to the K-Activity, described below, including mental actions used in these solutions (indicated by italics). We then illustrate how these two solutions can be modeled using UTGs.

> *K-Activity.* There are two big K's, a black one of the top left, and a blue one on the bottom right [see Figure 2a]. Prove or disprove: The black K can be transformed to the blue K using a translation, a rotation, and a reflection, at most once each.

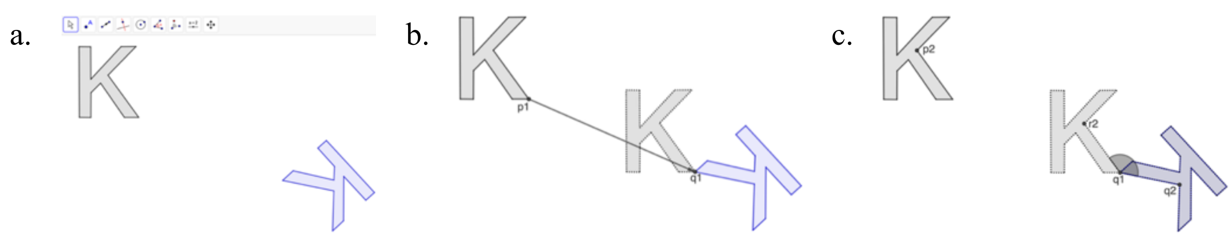

Figure 2. (a) Initial K-activity setup, and potential solution 1, involving (b) translation then (c) rotation

Potential Solution 1. In solving this task, a person might begin by *unitizing* the black K as an object of attention (a spatial unit) and isolating (*spatially disembedding*) a salient point p_1. By attending to the corresponding point q_1 on the blue K, the person *unitizes* the blue K as a spatial unit, then *spatially locates* and *disembeds* the corresponding point, establishing a relational anchor between the two K-shapes. Using these two disembedded points, they mentally construct a vector, $\overrightarrow{p_1 q_1}$, by mentally imagining *directing* p_1 to q_1, and with this they mentally *translate* the black K toward the blue K, producing a new spatial unit (the translated K; see Figure 2b). Then, person may *spatially disembed* another point, p_2, on the black K, then *spatially locate* and *disembed* its translation, r_2, on the translated K, as well as the corresponding point, q_2, on the blue K (Figure 2c). The person may then *connect* points q_1 and r_2, forming a ray, and *connect* points q_1 and q_2, forming another ray; with these, they can imagine *turning* the former ray clockwise toward the latter, conceptualizing the angle $\angle r_1 q_1 q_2$. Then, with this angle, they can *rotate* the translated K, using q_1 as the center of rotation, in a clockwise direction. Through these coordinated mental actions, the person leverages fixed points and constructed angles as invariants to structure the sequence of transformations that map the black K to the blue K.

This person's solution can be modeled using the UTG shown in Figure 3. Rectangular nodes represent spatial units—material isolated as an object of attention and taken as input for spatial mental actions (represented by arrows). The mental act of unitization is not represented by arrows, but rather by the creation of a rectangular node. Dark gray nodes represent the initial figurative material provided in the problem situation (taken by the problem solver as spatial units). Light gray nodes represent spatial units transformed in the process of accomplishing the translation of the black K. White nodes represent spatial units transformed in the process of accomplishing the rotation of the translated black K. The yellow node represents the final state of the black K.

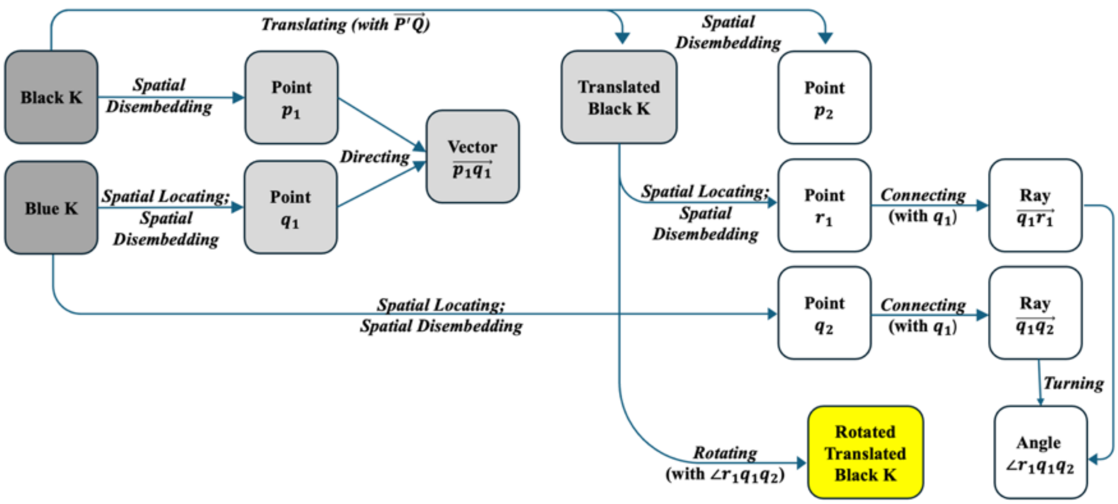

Figure 3. UTG illustrating the coordination of mental actions in Potential Solution 1

Potential Solution 2. As an alternative to the approach in Potential Solution 1, a person could first isolate the blue K as a spatial unit, via *unitization*, and from this *spatially disembed* its long flat side, then *extend* this side to form a line, L_1. They could repeat this process with the black K, *spatially disembedding* its long flat side and *extending* it to form line L_2. In doing so, the person can *spatially locate* a point, z, that they can use as the center to *rotate* the black K clockwise until its long flat side is visually aligned with L_1 (see Figure 4b). The rotated black K is *unitized* as a spatial unit that they will then translate. To carry out and describe this translation, the person could first *spatially disembed* a point p on the rotated black K, then *spatially located* and *disembed* the corresponding point, q, on the blue K. With these, they can conceptualize vector $\overrightarrow{pq}$ by mentally imagining *directing* p in the direction of q. Using this vector, they then *translate* the rotated black K (see Figure 4c), completing the transformation.

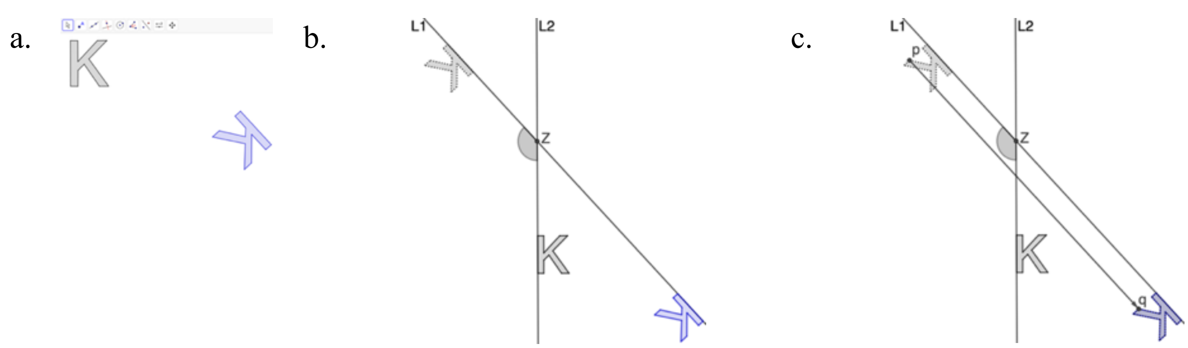

Figure 4. (a) Initial K-activity setup, and potential solution 2, involving (b) rotation then (c) translation

This person's solution can be modeled using the UTG shown in Figure 5. As with the previous UTG, rectangular nodes represent spatial units, and mental acts on those units are represented by arrows. Dark gray nodes represent the blue K and black K (the first two spatial units acted upon in the problem). Light gray nodes represent spatial units transformed in the process of rotating the black K. White nodes represent spatial units transformed in the process of translating the rotated black K. The yellow node represents the final state of the black K.

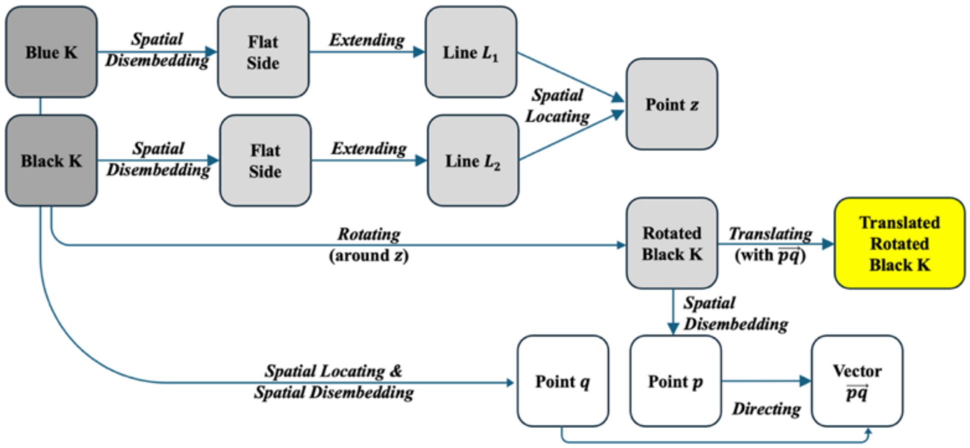

Figure 5. UTG illustrating the coordination of mental actions in Potential Solution 2

Concluding Remarks

UTGs contribute a more explicit lens on the units that students attend to and the mental actions that they coordinate on those units when solving transformation problems. Much prior work has documented the kinds of challenges students face, such as conceiving transformations as motions rather than functions (Hollebrand, 2003; Yanik, 2011), relying on picture-based rather than analytic strategies (Boulter & Kirby, 1994; Geotas et al., 2021), or struggling to anticipate composite transformations (Yao, 2020). UTGs offer a way to model not only the outcomes of these problem-solving attempts but also the evolving sequences of units and mental actions that students bring into coordination.

Additionally, our examples of the K-activity illustrate how different individuals may coordinate similar actions in different sequences: one beginning with translation then rotation, the other reversing that order. By modeling these variations, UTGs make explicit not only the actions themselves, such as unitizing, disembedding, locating, translating, and rotating, but also the ways these actions are sequenced and coordinated across units. This explicit attention to the interplay of units and mental actions offers insight into the diverse strategies students employ when reasoning about geometric transformations.

Finally, we suggest that UTGs can further illustrate the constructive ways that students coordinate units and mental actions, even when their solutions do or do not align with normative or textbook-like solutions. By making visible the variations in how students sequence mental actions, UTGs can reveal the resources that students bring to transformation problems, which in turn help researchers model student reasoning. With this in mind, we call for future research to engage students in solving transformation geometry problems and model their solutions with UTGs. These models can provide opportunities for researchers and teachers to support students in developing more productive strategies for solving transformation geometry problems while also highlighting how students' current reasoning can be a foundation for more productive ways of thinking in geometry contexts.

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