Proceedings of the 28th Annual Conference on Research in Undergraduate Mathematics Education

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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-eighth annual hybrid Conference on Research in Undergraduate Mathematics Education in Alexandria, Virginia from February 26 – 28, 2026 hosted by Virginia Tech Mathematics Department. The hybrid modality enabled presenters and attendees to participate fully online such that travel was not a requirement. Of the 479 conference attendees, approximately 18% of them attended virtually.

The program included plenary addresses by Dr. Jennifer A. Czocher, Dr. Felix Ho, and Dr. Daniel Reinholz and 151 paper reports, including contributed, preliminary, and theoretical research reports, and 124 posters. Each paper report underwent a rigorous peer review process by three or more reviewers, including at least one member of the conference Program Committee. Each poster report was reviewed by two members of the Program Committee. The following are the acceptance rates for each type of reports for this year's conference:

Contributed Report: 89/114 (78%)
Preliminary Report: 42/58 (72%)
Theoretical Report: 20/28 (71%)
Poster Report: 126/138 (91%).

Two accepted poster reports were withdrawn prior to the conference. A new set of primary subject areas for submissions were introduced for this year's conference. The table below offers an approximation for the areas of research that were represented during the paper sessions. It is sorted by the most represented.

Primary Subject Area	Number of papers
Student Thinking	49 (32%)
Equity (access, achievement, identity, and power)	23 (15%)
Classroom Pedagogies & Teaching Practices (including tutoring)	18 (12%)
Affect (including attitudes, beliefs, anxiety)	17 (11%)
Curriculum & Curriculum Design	13 (9%)
Teachers & Instructors Professional Learning (including GTA Training, PD)	11(7%)
Assessment	6 (4%)
Classroom Dynamics & Discourse	5 (3%)
Institutional & Departmental Practices	5 (3%)
Research Methods	4 (3%)

The proceedings are organized by types of papers that represent current work in undergraduate mathematics education:

- Contributed Reports share findings from a completed research study that make meaningful contribution to existing literature.
- Preliminary Reports share preliminary findings from an ongoing research project in early stages of analysis.
- Theoretical Reports share scholarly works whose primary aim is to contribute to the development of theories or theoretical perspectives.
- Poster Reports are 1-page documents sharing a description of a research study at any stage of completion.

These proceedings, along with previous years' proceedings are available on the sigmaarume.org website. As announced at the conference, the searchable database of all RUME Proceedings has become available for use where readers can search by keywords, title, and authors. We thank Dr. Shandy Hauk for spearheading this effort.

Many members of the RUME community volunteered for the Program Committee where they reviewed many submissions such that every submission was reviewed by at least one member of the Program Committee. The Program Committee aided us in putting together this program and their hard work is greatly appreciated. The Local Organizing Committee were responsible for the smooth running of the presentations and on-site activities, which would not have been possible without the help of many volunteers, and we thank them for their tireless efforts to host a conference that runs smoothly.

Thank you to all the researchers who submitted such strong proposals and ultimately made the conference a fun and joyous event.

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Logic in Introduction-to-Proof Students: Mathematical Truth Table Tasks

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Mathematics uses standard logic, in which the truth value of the conditional 'if A then B' is completely determined by the truth values of A and B. Specifically, when A is false, 'if A then B' is, by definition, true. This makes the mathematical conditional out of line with natural language, where research using truth table tasks shows that conditionals are often considered irrelevant in false-antecedent cases. What does this mean for mathematics undergraduates? Do they interpret conditionals according to standard logic, or do their responses align with everyday interpretations? Do they make consistent interpretations, or do these vary by content? This report contributes a first investigation of this issue.

Keywords: conditionals, if-then, logic, reasoning, truth tables

Introduction

In mathematics, conditionals – *if-then* statements – are ubiquitous. Many theorems and conjectures are naturally expressed as *universal* or *generalized* conditionals of the form '$\forall x$, if $A(x)$ then $B(x)$' (Durand-Guerrier, 2003). This is so well understood that the quantification is often left implicit (Shipman, 2016), as in these example theorems:

- If $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a, then f is continuous at a;
- If G is a finite group and H is a subgroup of G, then $|G| = |H||G:H|$;
- If p is a prime number and $p \nmid a$, then $a^{p-1} \equiv 1 (\text{mod } p)$.

The interpretation of universal conditionals is underpinned by the standard mathematical interpretation of *if-then* as the *material conditional*, a truth-functional logical connective (Edgington, 2020). For propositions A and B, the truth value of 'if A, then B' is defined for every possible combination of truth values of A and B, and is completely determined by those truth values per the standard truth table – see Table 1.

Table 1. Defining truth table for if-then as a truth functional logical connective.

A	B	if A then B
T	T	T
T	F	F
F	T	T
F	F	T

Truth functionality is necessary for *if-then* to interact with the other connectives *and*, *or* and *not* in a way that enables unambiguous evaluation of logical equivalence and argument validity (Smith, 2020). And the standard truth table is the only viable option (Edgington, 2020). We want a universal conditional 'if $A(x)$ then $B(x)$' to be false only when there exist cases corresponding to the TF row, with $A(x)$ true but $B(x)$ false. We want it to be true if there are no such cases, compelling us to define it as true for TT, FT and FF rows. This interpretation confirms validity for standard argument forms. For instance, *modus tollens* inferences from *if A then B* and *not-B* to *not-A* are valid because the premises are true together only in the FF row of the table, where A is false (Velleman, 2006). However, the truth-functional conditional is

psychologically unnatural. It defines a conditional as true whenever its antecedent is false, where natural-language conditionals are often considered irrelevant in such cases (Nickerson, 2015). This issue is the focus of this report, which describes the development and use of mathematical *truth table tasks*.

Theoretical Background

Conditionals and their interpretation have been studied in mathematics education (Dawkins & Norton, 2022; Inglis & Attridge, 2016) and more generally in psychology (Nickerson, 2015), philosophy (Edgington, 2020) and linguistics (Douven et al., 2020). Reasoning *from* conditionals is often investigated using *conditional inference tasks*, which have been used to study mathematics students' reasoning about both abstract content (Attridge & Inglis, 2013) and everyday and mathematical content (Alcock et al., under review; Alcock & Inglis, under review). Interpretation *of* conditionals is often studied using *truth table tasks*, which have not so far been used with mathematics students or with mathematical content.

Truth table tasks come in various forms. One simple type presents a conditional and asks participants to generate examples that are true and examples that are false. People commonly generate TT cases for 'true' and TF cases for 'false'; a minority of participants additionally generate false-antecedent cases for 'true' (Nickerson, 2015).

A more standard type presents participants with a conditional and with cases corresponding to each truth table row; the task is to judge the truth value of the conditional in each case. Tasks vary in their response options: participants might be forced to choose between 'true' and 'false' or might be allowed a third option of 'irrelevant', 'indeterminate' or similar (Nickerson, 2015). When offered this third option, people typically use it for one or both false-antecedent cases. For instance, Newstead et al. (1997) asked participants to state whether each presented case 'supports', 'contradicts' or 'tells us nothing about' the relevant conditional. Across three studies, they found that people almost always judged that TT cases support the conditional and TF cases contradict it. For FT cases, responses were split almost equally between contradicts and tells us nothing; for FF cases, responses were split almost equally between supports and tells us nothing. This effectively gives the 'truth table' in Table 2.

Table 2. Common responses in truth table tasks, Newstead et al. (1997).

A	*B*	if *A* then *B*
T	T	T
T	F	F
F	T	F or I
F	F	T or I

Tasks also vary in semantic content. Newstead et al. systematically varied their content using conditional promises, threats, tips, warnings, and conditionals that they called temporal, causal, and universal. Universals (e.g., 'If the student is doing economics, then he is a socialist') are arguably closest to mathematical conditionals because they are effectively about set inclusion, though of course such claims would be treated as debatable – the bivalent logic of mathematics does not usually apply to everyday content (Oaksford & Chater, 2020). For universal conditionals, Newstead et al. reported higher proportions of tells-us-nothing responses, at 64% for FT and 63% for FF cases respectively; this meant that 46% of responses followed a TFII pattern sometimes referred to as a *defective implication* interpretation (following Wason &

Johnson-Laird, 1969). The full set of patterns considered by Newstead et al. were termed *implication* (TFTT, as in the truth-functional conditional), *equivalence* (TFFT, treating a conditional as a biconditional), *defective implication* (TFII), and two patterns based on their data, *defective biconditional* (TFFI) and *Pattern X* (TFIT). The implication pattern was extremely rare: interpretations of everyday conditionals do not align with standard logic, as is well documented across tasks of various types (e.g., Oaksford & Chater, 2020).

What does this mean for students learning mathematics? When faced with mathematical content, do they import everyday interpretations, or does their reasoning align with standard logic? This study was designed to find out.

Method

Designing a truth table task with 'realistic' mathematical content is nontrivial. Mathematical conditionals are easy to come by – as noted in the Introduction, many theorems can be expressed in this form. But theorems, of course, are true, whereas a truth table task requires cases for all four truth-value combinations, TT, TF, FT and FF. This means that both the conditional and its converse must permit counterexamples. I therefore used simple mathematical content to construct conditionals that sound sensible because a) the semantic contents of their antecedents and consequents involved related concepts, as is pragmatically expected in everyday conditionals (Grice, 1989), and b) the first case to come to mind would likely be a TT case. To keep the tasks short while providing at least some mathematical variety, I used three conditionals:

- If *n* is a positive odd number, then *n* is prime.
- If rectangle *R* has area 12 square units, then its perimeter is 14 units.
- If *x* is a whole number, then $x^2 > x$.

I used all three conditionals in each of three truth table tasks (using the word 'claim' rather than 'conditional' in these tasks to make the language more accessible).

- In the *generate* task, participants were asked to give one or more examples that would make each conditional true and one or more than would make it false;
- In the *true-false* task, participants were shown the conditional with TT, TF, FT and FF cases, and asked to judge the conditional true or false for each one;
- In the *true-false-irrelevant* task, participants were shown the conditional with TT, TF, FT and FF cases, and asked to judge the conditional true, false or irrelevant for each one.

For the true-false and true-false-irrelevant tasks, the provided cases are shown in Table 3. Rectangles were shown diagrammatically – see the example true-false-irrelevant task in Figure 1.

Table 3. TT, TF, FT and FF cases provided for each conditional in the true-false and true-false-irrelevant tasks; for task presentation format see the appendix.

Conditional	TT	TF	FT	FF
If *n* is a positive odd number, then *n* is prime.	3	9	2	8
If rectangle *R* has area 12 square units, then its perimeter is 14 units.	3×4	2×6	1×6	2×3
If *x* is a whole number, then $x^2 > x$.	3	1	3/2	1/2

Consider this claim:

If n is a positive odd number, then n is prime.

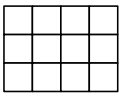
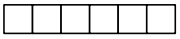
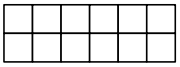
For each n-value listed below, would you say the claim is true, false, or irrelevant?

1. $n = 3$ A
2. $n = 8$ D
3. $n = 2$ C
4. $n = 9$ B

Consider this claim:

If rectangle R has area 12 square units, then its perimeter is 14 units.

For each rectangle shown below, would you say the claim is true, false, or irrelevant?

1. D
2.  A
3.  C
4.  B

Consider this claim:

If x is a whole number, then $x^2 > x$.

For each x-value listed below, would you say the claim is true, false, or irrelevant?

1. $x = \frac{3}{2}$ C
2. $x = 3$ A
3. $x = 1$ B
4. $x = \frac{1}{2}$ D

Figure 1. Example true-false-irrelevant task page (small letters to the right track randomization).

For each task, the order in which the conditionals appeared was counterbalanced. For the true-false and true-false-irrelevant tasks, the order in which the cases appeared for each conditional was randomized. Participants were first-year UK mathematics undergraduates in the third week of an introduction-to-proof course (the researcher had no involvement in the course). Each participant received one of the three tasks on paper, and papers were arranged so that students in adjacent seats would receive different tasks. The task took five minutes. For the generate task, 29 participants consented to have their data used. For the true-false task, 30 consented but one clearly misunderstood the task and two did not respond for all three conditionals so, for simplicity, data from 27 is presented. For the true-false-irrelevant task, 26 participants consented but one did not respond for all conditionals, so data from 25 is presented.

Results

The headline result is that in judging truth values for mathematical conditionals, introduction-to-proof students adhere much more closely to the defective TFII interpretation than to the truth-functional TFTT interpretation of standard logic. That is, they interpret mathematical conditionals as true in TT cases, false in TF cases, and – where permitted and indeed sometimes where not permitted – as irrelevant in FT and FF cases. Moreover, content appears to affect interpretations, and a substantial minority of students do not implement a consistent interpretation across contents. I report results for the generate task, then break down the results for the true-false and true-false-irrelevant tasks by conditionals then by participants.

Generate Task

Results from the generate task were clear and in line with common responses to tasks with everyday content (e.g., Nickerson, 2015): for examples that would make the conditional true, participants gave TT cases; for examples that would make it false, they gave TF cases. A small number made interpretation or calculation errors (e.g. failing to notice the whole number requirement for x and providing a range such as $0 < x < 1$). But most responded predictably: Table 4 gives examples that were most commonly listed first.

Table 4. Commonest first responses for examples that make each conditional true/false, with percentage of participants giving that response.

Conditional	True	False
If *n* is a positive odd number, then *n* is prime.	3 (66%)	9 (66%)
If rectangle *R* has area 12 square units, then its perimeter is 14 units.	3×4 (86%)	2×6 (69%)
If *x* is a whole number, then $x^2 > x$.	2 (55%)	1 (66%)

True-False Task

Results for the true-false task – see Table 5 – showed that for all three conditionals, small minorities of participants aligned with the material interpretation, with response pattern TFTT. But most went against it, either choosing 'false' for both false-antecedent cases (TFFF) or subverting the task by introducing 'irrelevant' as an extra option and applying it to those cases (TFII). No participant gave a biconditional (TFFT) response. For the conditional 'If x is a whole number, then $x^2 > x$', there were fewer TFFF responses but several TFTF responses, judging the conditional true for the case $x = 3/2$. This differs from the common everyday responses summarized in Table 2, but the task required written one-word responses only, so I have no qualitative data on why this might be. I hope this might generate discussion at the conference.

Table 5. Numbers giving the commonest response patterns for each conditional (26 participants).

Conditional	TFFF	TFII	TFTF	TFTT
If n is a positive odd number, then n is prime.	17	3	1	4
If rectangle R has area 12 square units, then its perimeter is 14 units.	17	4	1	1
If x is a whole number, then $x^2 > x$.	11	3	8	3

Analysing by participants reflected these results in that:

- 8 responded TFFF for all three conditionals;
- 5 responded TFFF for the n and R conditionals but TFTF for the x conditional;
- 3 responded TFII for all three conditionals;
- 1 responded TFTT for all three conditionals.

Of the remaining ten participants, nine were inconsistent – some gave different response patterns for all three conditionals. As in Dawkins and Cook's (2017) work on disjunctions, this shows that students do not necessarily recognize logical structures or apply these in the same way across different mathematical content. One participant responded IFII for all three conditionals, apparently unwilling to assign a truth value in any but the clearly false case. Although unusual in these data, this is in line with interpretations in which conditionals are not considered true unless there is an inferential link between antecedent and consequent (e.g., Douven et al., 2020).

True-False-Irrelevant Task

Results for the true-false-irrelevant task – see Table 6 – showed that for all three conditionals, participants who were offered the 'irrelevant' option commonly used it, leading to response pattern TFII. No participant offered this option responded according to the material conditional (TFTT), and again no participant gave a biconditional (TFFT) response. As in the true-false task, there were a few TFTF responses for the x conditional.

Table 6. Numbers giving the commonest response patterns for each conditional (25 participants).

Conditional	TFFF	TFII	TFTF	TFTT
If n is a positive odd number, then n is prime.	2	20	0	0
If rectangle R has area 12 square units, then its perimeter is 14 units.	4	17	0	0
If x is a whole number, then $x^2 > x$.	1	16	5	0

Analysing by participants reflected these results in that:

- 14 responded TFII for all three conditionals;
- 1 responded TFII for the n and R conditionals but TFTF for the x conditional.

Of the remaining ten participants, all were inconsistent across the conditionals – again, some gave different response patterns for all three.

Discussion

This is the first study to design and use a mathematical version of a truth table task to investigate interpretations of mathematical conditionals. Its main result is that for simple mathematical content, the interpretations made by introduction-to-proof students align much more closely with typical interpretations of everyday conditionals than with standard mathematical logic. That is, students most commonly think of conditionals as true in TT cases,

false in TF cases, and irrelevant in FT and FF cases. When forced to choose between true and false for false-antecedent cases, most pick 'false', giving a TFFF rather than a TFTT response. Many students are not consistent in their interpretations across content (cf. Dawkins & Cook, 2017). Finally, content seems to have an effect, with the conditional 'If x is a whole number, then $x^2 > x$' prompting some TFTF responses.

One question is how much this matters. Although the truth-functional material conditional underlies mathematical interpretations of logical equivalence and validity, 'real' mathematics rarely involves propositional conditionals. Consequently, it is possible to provide an introduction to proof that does not address propositional logic, instead focusing on the distinction between a (universal) conditional and its converse – several commonly recommended books do this (Alcock & Sa, 2025). Moreover, a defective interpretation does not appear fatal to strong mathematical reasoning. Although a material interpretation is technically necessary for validity of modus tollens inferences (Durand-Guerrier, 2003), these also submit to a longer but valid workaround ('if A were true, then B would be true, but B is not true, so A cannot be true'), and evidence from other tasks too indicates that this interpretation is common among successful undergraduates (Inglis & Simpson, 2009).

One way to approach an informed view on this question would be to find out how professional mathematicians respond to truth table tasks. It could be that all give TFTT responses, in which case an argument about developing expertise suggests that teaching the material conditional is worth the effort. But mathematicians, too, have little cause to consider propositional conditionals in their day-to-day work, so it is possible that they have little practice in thinking about these and that some would be influenced by everyday language and respond according to the defective TFII interpretation. In that case, the question becomes more nuanced – if the truth-functional interpretation is mathematically 'correct' but not necessary for a successful mathematical career, perhaps teaching it should not be a priority. By the time of the conference, I will have evidence on mathematicians' responses to these tasks.

In the meantime, my current view is that mathematics undergraduates should encounter propositional logic as part of exposure to the foundations of their subject (Alcock, 2025). However, we should expect that they will find it unnatural, and we should think carefully about whether and how our explanations leverage everyday content. In some cases, such content might support students in reaching appropriate interpretations (Epp, 2003). In others, it might encourage the reverse, or at least it might fail to raise the right questions. For instance, many textbooks that do cover the material conditional do so by presenting an everyday conditional (e.g. 'If it stops raining by Saturday, then I will go to the football game', Lay, 2014, pp.4-5); they invite students to consider when they would consider it false, then argue that it should be considered true in all other cases. As Alcock and Sa (2025) point out, this fails to engage with the irrelevance problem. To remedy this, the psychology literature suggests that an intermediate step might be useful: although people consider conditionals irrelevant for false-antecedent cases, they are often willing to concede that conditionals are *compatible* with those cases (Nickerson, 2015). The present research with mathematical content did not test this, but a follow-up is underway that will replicate the true-false and true-false-irrelevant tasks and add one testing judgements of compatibility. Again, I will hope to share the data at the conference.

Acknowledgments

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Coordinating Point and Rate: A Framework for Students' Reasoning About Slope

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This study examines how precalculus students reason about rate and point as multiplicative objects in the contexts of linear equations and approximation. To capture students' developing conceptions, I introduce a framework distinguishing three levels of abstracted ratio: generalized ratio, unit rate, and interiorized ratio. Findings show that although students progressed toward interiorized reasoning, their understanding was fragile when working with non-integer or symbolic values. They also struggled to coordinate slope with a reference point, which limited their ability to interpret points as records of covariation. Geometric approaches to linear approximation highlighted these conceptual difficulties more clearly than symbolic tangent line equations, underscoring the need for instructional support that fosters covariational reasoning.

Keywords: linear function, linear approximation, rate, point, multiplicative object

Studies have long documented calculus students' struggles with tangent line equations and linear approximations (Amit & Vinner, 1990; Çekmez & Baki, 2016; Moore-Russo & Nagle, 2024; Orton, 1983), yet less attention has been paid to underlying precalculus concepts—especially understandings of rate and point in linear equations. Classroom observations suggest that difficulties may stem not only from gaps in derivative fluency but also from underdeveloped conceptions of linear relationships.

To investigate, I conducted exploratory interviews with three precalculus students, focusing on the research question: How do students conceptualize rate and point as multiplicative objects in the context of linear equations and linear approximation?

Theoretical Framework and Literature

Students who view a function only as a correspondence between input and output (Thompson, 1994b), or who treat $L(x) = f(a) + f'(a)(x - a)$ as a formula to substitute values into, may compute correctly without visualizing the situation or interpreting the slope $f'(a)$. Such disconnections between computation and the graphical tangent have been documented (Code et al., 2014). In contrast, students who conceive *x* and *y* as covarying quantities—measurable attributes varying in coordination (Thompson & Carlson, 2017)—are more likely to interpret $L(x)$ meaningfully. A meaningful understanding of $L(x)$, I argue, depends on conceiving both *point* and *rate* as multiplicative objects (Saldanha & Thompson, 1998). A multiplicative object unifies two quantities into a single conceptual entity: a point coordinates horizontal and vertical displacements, and a rate coordinates changes in *x* and *y*. Recognizing these as multiplicative objects requires mentally integrating covarying quantities into one structure, rather than viewing them in isolation.

Rate as a Multiplicative Object

The concept of rate connects to the linear function *L* from two perspectives: calculus and algebra. From a calculus perspective, *L* embodies $f'(a)$, the instantaneous rate of change $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, interpreted geometrically as the slope of the tangent line. While this definition is central to calculus learning (Orton, 1983; Yu, 2024; Zandieh, 2000), a full treatment lies beyond

the scope of this study. Instead, the focus here is on students' conceptions of rate in algebraic contexts, where $f'(a)$ functions as a constant rate of change—the slope of L.

In Thompson's (1994a) framework, a rate is a reflectively abstracted multiplicative structure describing how one quantity changes in relation to another. A ratio may begin as a static comparison but can be abstracted from specific numbers to describe relationships between indeterminate, systematically related quantities. This progression reflects a shift from an *internalized* ratio—reasoning with repeated accumulations of a fixed ratio (e.g., scaling "three pears to four apples")—to an *interiorized* ratio, in which quantities are conceived as continuously varying while maintaining a constant relationship. Interiorized ratio aligns with covariational reasoning, where rate functions as a structural relationship that integrates and governs the coordinated changes of two quantities.

Research shows that students often struggle to construct rate as a multiplicative object. They may treat it procedurally (Lobato et al., 2003), rely on additive reasoning, or focus on isolated values instead of coordinating variation (Clark & Kamii, 1996; Yu, 2024). Even when producing correct values, they may not interpret them as relationships between covarying quantities. Thompson (1994a) emphasized that interiorizing rate requires conceiving it as a structural relationship that remains invariant across magnitudes. Johnson (2015) found that students often viewed rate as an extensive quantity—treating measures as separate totals—rather than as an intensive quantity that unites them into a single multiplicative relationship governing covariation. Together, Thompson and Johnson highlight the difficulty of moving from procedural or context-bound views toward structural conceptions of rate, underscoring the need for analytic categories that capture the partial ways students reason prior to full interiorization.

Point as a Multiplicative Object

A point can be understood not merely as a location—"over this much and up that much"—but as a multiplicative object that records the covariation of two quantities (Thompson et al., 2017). In this view, coordinates represent interdependent values integrated into a single structure. Yet research shows that many learners struggle with this conception. Thompson et al. (2017) found that accuracy in placing the initial point was closely tied to correctly sketching the entire graph. Similar challenges have been documented in contextualized settings where attending to covarying quantities is essential: Ellis et al. (2015) reported students' difficulties in interpreting exponential growth involving height and time, while Frank (2016) documented struggles to reason about points in motion or dynamically changing systems using the same task from Thompson et al. (2017). Collectively, these studies underscore the complexity of conceiving a point not merely as a location but as a representation of a quantitative relationship.

Methodology

Setting

This study took place at a small state university in the southeastern United States. From fifteen students recommended by precalculus instructors for their varied mathematical thinking, three—Tanya, Zada, and Sean (pseudonyms)—agreed to participate for an hourly stipend. Tanya, a dual-enrolled high school student, had completed Algebra 2; Zada, a first-year biology major, had recently finished college algebra; and Sean, a first-year computer science major, had completed precalculus in South Asia.

All precalculus courses used *Algebra and Trigonometry* (Abramson, 2018) as the textbook, which presents slope mainly as the algebraic ratio $\frac{y_2-y_1}{x_2-x_1}$ in slope-intercept form, with point-slope

form quickly converted. Graphs focus on parallel and perpendicular lines, per-unit concepts appear only in contextual problems, and nonlinear rate of change is omitted. Multiple representations are included but rely on discrete values, rational slopes, and integer intercepts, reflecting an algebraic–procedural emphasis.

Participants took part in five to seven interview sessions (90 minutes each) near the end of the semester, after instruction on slope, linear equations, and functions. Tanya and Zada completed seven sessions, while Sean completed five due to family circumstances. Using a semi-structured clinical interview format (Ginsburg, 1997) with elements of exploratory teaching (Steffe & Thompson, 2000), early sessions elicited existing understandings through tasks such as constructing equations from graphs or tables, without instructional support. Later sessions introduced targeted prompts, informed by prior responses, to probe reasoning about linear and quadratic functions and linear approximation.

Data Analysis and Development of New Analytic Tool

Data analysis began during data collection, following an iterative, comparative approach (Miles & Huberman, 1994). After each interview, I reviewed recordings, wrote analytic memos summarizing key content, and drafted rough transcripts of notable moments. This early work informed preliminary codes for students' conceptual difficulties with point, slope, linear equations, and approximation, which were refined through constant comparative methods (Glaser & Strauss, 1967) as patterns in reasoning emerged.

As analysis progressed, the coding scheme was refined through distinctions that supported cross-case comparisons of how students coordinated rate and point. Drawing on Thompson's (1994a) framework of internalized and interiorized ratios and related work on covariational reasoning (Johnson, 2015; Thompson & Carlson, 2017), I identified three analytic categories of ratio and rate: generalized ratio, unit rate, and interiorized ratio (Table 1). These categories provided the framework for interpreting students' levels of abstraction of ratio.

Table 1. Types of Abstracted Ratio

Level	Description and Examples of Student Reasoning
Generalized Ratio (Early Abstracted Reasoning)	Treated as a fixed multiplicative value across quantity pairs, applied symbolically without coordinating x and y. Example: "The slope between $(2, a)$ and $(5, b)$ is 3/4 because $(b - a)/(5 - 2) = 3/4$."
Unit Rate (Transition to Interiorized Reasoning)	Conceived as a per-unit change, enabling additive or multiplicative coordination of Δx and Δy, typically for integer Δx values. Example: "If x increases by 1, y increases by 3/4," then uses $3/4 \times 2$ to find Δy when $\Delta x = 2$.
Interiorized Ratio (Fully Abstracted Reasoning)	Conceived as a structural, covariational relationship, expressed as $\Delta y = m\Delta x$, that applies to any Δx, including symbolic or infinitesimal values, supporting flexible and general reasoning. Example: "For any change in x, the change in y is 3/4 times that," applied with h or 0.1.

Figure 1 illustrates how generalized, unit rate, and interiorized ratio conceptions influence students' approaches to linear approximation, highlighting how Δx and Δy are conceived as

displacements relative to the reference point $(a, f(a))$ rather than the origin. (Note: I use Δx and Δy instead of dx and dy to reflect students' reasoning in precalculus contexts.)

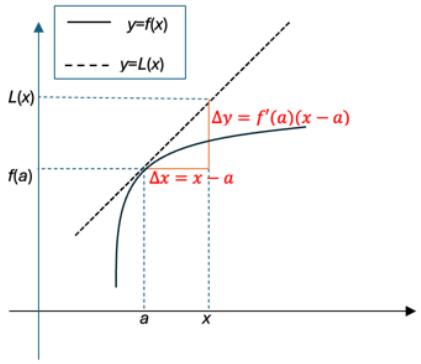


Figure 1. Understanding Linear Approximation from a Covariation Perspective

Results

The results synthesize students' reasoning across tasks on linear equations and linear approximation, organized by a developmental trajectory rather than by participant or chronology. Findings are presented in three parts: (1) pre-abstracted interpretations of rate, (2) emerging abstracted ratios—generalized ratio, unit rate, and interiorized ratio—and (3) conceptions of point as a multiplicative object. Subsections are titled by the transitional strategies observed (e.g., additive, multiplicative) when students' thinking did not fully align with a single level. Tasks were adapted to students' readiness; Tanya contributed the most complete data set, while Zada's more limited mathematical preparation and Sean's reduced participation due to family circumstances yielded partial but informative episodes. Together, these data trace how precalculus students coordinate rate and point and where conceptual fragilities persist.

Slope as a Ratio between Two Points and Construction of Linear Equations

At the outset, all three participants understood slope mainly as the formula $(y_2–y_1)/(x_2–x_1)$ and showed only partial familiarity with "rate of change." When constructing linear equations, Sean applied both slope and point-slope forms, while Tanya and Zada relied solely on slope-intercept form, succeeding only when the y-intercept was obvious. Tanya recognized parallel slopes but struggled to connect tabular data with equations, whereas Zada omitted equation form, misapplied slope in parallel tasks, and added x and y values directly from the table.

In sum, Sean and Tanya displayed stronger reasoning than Zada, yet all three treated slope as an algebraic ratio rather than as an abstracted multiplicative relationship, with little evidence of Cartesian or tabular connections.

Transitioning to Abstracted Ratio

Additive strategies and the emergence of multiplicative reasoning with integer Δx. When asked to find the rate of change between two points $(2, a)$ and $(5, b)$ that lie on the graph of $y = 3x – 7$ (Figure 2A), Zada proposed first calculating a and b and then applying the slope formula. This showed she did not interpret the coefficient 3 as the rate of change for all points on the line but instead viewed slope as something to be computed from particular pairs—an underdeveloped conception of slope as a generalized ratio. To find a and b, she suggested graphing and "counting up by 3" each time x increased by 1.

A. Points $(2, a)$ and $(5, b)$ lie on the graph of $y = 3x - 7$. What is the rate of change between these two points?

B. $(3, b)$ is on the line graph below. What is b?

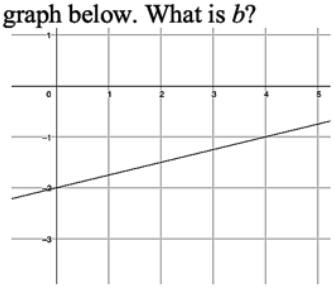


Figure 2. Tasks Requiring a Conception of Slope at an Advanced Level

To move her from counting to multiplication, I asked for the y-value at $x = 10$ on the graph of $y = 3x + 1$. She again counted by 3 until reaching (10, 31) (see Figure 3). For (60, b), however, she hesitated, noting that "it would take forever." With prompts about slope as "a rise of 3 for each run of 1" and the point (60, b) plotted, she initially claimed that b was 6 with a rise of 5, misled by the scale. Only after I clarified the diagram, labeled (60, 1), and reiterated that slope meant a rise of 3 per unit run did she compute the rise as 60 × 3. This episode suggests her multiplicative reasoning with slope was contingent on explicit visual and verbal scaffolds.

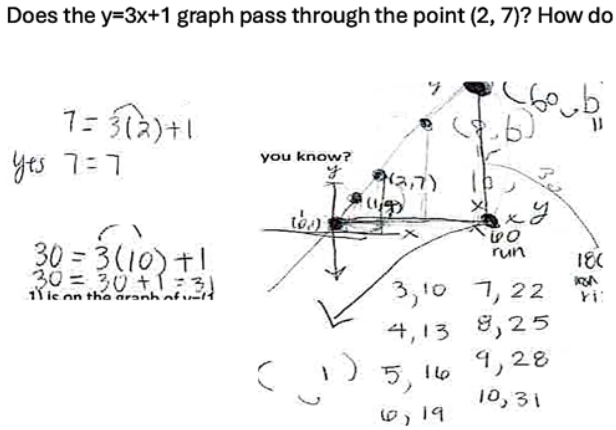


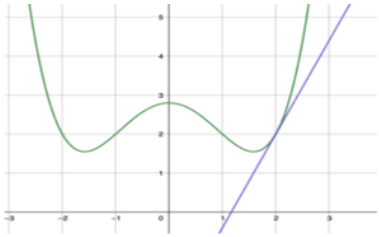
Figure 3. Zada's Transitioning from Counting to Multiplication

From generalized algebraic ratio to generalized geometric ratio. When asked to find b in the point (3, b) (Figure 2B), Sean applied both slope formula and point-slope form, producing $y = ¼ x – 2$. He then set up $\frac{1}{4} = \frac{b-(-1)}{3-4}$ and solved for b—indicating that he viewed ¼ as an invariant relationship, a generalized algebraic ratio. Yet he acknowledged knowing no method beyond algebraic manipulation. When I drew segments between (0, –2) and (4, –1) and asked about $y_2 – y_1$ and $x_2 – x_1$, he recalled from high school that they were "differences in the coordinates."

Sean carried this reliance on point-slope form into the linear approximation tasks. To estimate $f(2.1)$ (Figure 4a), he constructed the tangent line $y = \frac{12}{5}x - \frac{14}{5}$ and substituted $x = 2.1$

to obtain 2.24. When pressed to reason directly with slope, he hesitated until I drew displacements and labeled $\Delta x = 0.1$. Then he wrote $\frac{12}{5} = \frac{\Delta y}{\Delta x}$, substituted $\Delta x = 0.1$, and found $\Delta y = 0.24$—explaining, "x is increased by 0.1 and y is increased by 0.24." He also extended this to $\Delta x = 0.2$, showing movement toward conceiving slope as a generalized geometric ratio.

In a later task estimating $\sqrt{1.05}$, Sean again used the tangent line equation but explained that with slope ½ and $\Delta x = 0.05$, "Δy is ½(0.05)." Still, his reasoning reflected manipulation of the equation ½ = Δy/0.05, signaling a generalized ratio rather than an interiorized ratio.



The y=g(x) graph (in blue) is the tangent line of y=f(x) (in green) at the point (2, 2) and the slope of the line g is 12/5.

a) Estimate the value of f(2.1).
b) Find the equation of the line: y=g(x).
c) Estimate f(2+h).

Figure 4. Constructing Tangent Line and Approximating Function Values from Graphs

From a generalized ratio to unit rate: Additive/multiplicative reasoning with integer Δx. Tanya initially treated slope only as a generalized algebraic ratio, writing $\frac{1}{4} = \frac{b-(-1)}{3-4}$ to solve for b in Figure 2B. When asked what a slope of ¼ meant, she replied, "You go up once and go over four," reflecting an action-based geometric ratio. She also noted, "when the run is 1, the rise is ¼," but could not extend this per-unit idea to find b on the graph, repeating instead, "The rise is 1 and the run is 4."

With prompts connecting slope to displacements—"If x increases by 1, how much does y increase?"—she began using ¼ as a per-unit ratio, including decreases: "x is decreased by 1 and y is decreased by ¼." Using (4, –1) as a reference, she reasoned that (3, b) must be "¼ smaller than –1."

Tanya later used unit-rate reasoning more fluently in other tasks. When $\Delta x = -3$, she stated, "Go down 18," interpreting Δy as (–3)(6) without written calculation. In constructing equations, she often used unit-rate language, "Each time I move left by 1, I need to go down by 1/5." In Figure 4b, she subtracted 12/5 twice from the y-coordinate of (2, 2) to find $b = -14/5$. These episodes show her emerging ability to express accumulated change through unit rate reasoning, flexibly switching between addition and multiplication (cf. Thompson, 1994a).

From unit rate to interiorized ratio: Difficulties with non-integer Δx. Although Tanya demonstrated unit-rate reasoning in earlier tasks, she never directly applied the relationship $\Delta y = m\Delta x$ for non-integer values. Instead, she reconstructed the relationship through explicit equations, suggesting slope was not yet interiorized. For instance, in Figure 4b, she used an additive strategy for $\Delta x = -2$ when locating the y-intercept, but for $\Delta x = 0.1$ she set up 12/5 = Δy/0.1, then solved algebraically: $\Delta y = (12/5)(0.1) = 0.24$.

As approximation tasks grew more complex—shifting from numeric to symbolic slopes and from graphs to purely algebraic contexts—she consistently applied slope to estimate values. Yet

for non-integer Δx, she reverted to writing $m = \frac{\Delta y}{\Delta x}$ and solving for Δy, reflecting a generalized geometric ratio rather than an interiorized ratio. Her reliance on reconstructing proportions suggests she had not yet conceived slope as an interiorized ratio that operates flexibly on any Δx.

Shortened Treatment of Point as a Multiplicative Object

Although students also struggled with conceiving a point as a multiplicative object, this paper focuses primarily on their reasoning about slope as ratio and rate. All three students had difficulties coordinating quantities—specifically, horizontal and vertical displacements—encoded in both rate and point coordinates. While Sean and Tanya showed somewhat greater facility than Zada, all three struggled at times to use a known point and rate to determine a new point in linear equation and approximation tasks. *Due to space limitations, detailed analyses of students' reasoning about point as a multiplicative object are not included here and will be discussed in a separate paper.*

Conclusion and Discussion

This study examined how students coordinated slope in the context of linear equations and linear approximation tasks, with particular attention to how their reasoning reflected different levels of abstraction of ratio and rate. Building on Thompson's (1994a) distinctions between internalized and interiorized ratios, I proposed three analytic categories—generalized ratio, unit rate, and interiorized ratio—as a framework for analyzing students' reasoning. These categories captured important qualitative differences in how students conceived slope, ranging from applying it as a fixed algebraic relationship, to interpreting it as a per-unit change, to treating it as a structural multiplicative relationship applicable to any Δx.

The framework proved useful in distinguishing students' reasoning even when they produced the same correct answer. For instance, Tanya's use of unit rate reasoning allowed her to flexibly handle integer changes in x, yet her reliance on explicit equations for non-integer Δx revealed that she had not fully interiorized slope as an object of thought. Similarly, Sean's progression from algebraic manipulation of slope to its geometric interpretation showed how a generalized ratio could be extended toward more sophisticated reasoning. These distinctions highlight that students' difficulties with tangent lines and linear approximations are not simply procedural gaps in differentiation but reflect deeper issues in how they coordinate quantities through ratio and rate.

By situating students' reasoning within these categories, this study advances prior work on covariational reasoning (e.g., Thompson, 1994a; Johnson, 2015) by offering a finer-grained framework for distinguishing how students engage with slope in algebraic and geometric contexts. The categories of generalized ratio, unit rate, and interiorized ratio clarify not only *where* students encounter difficulties but also *how* those difficulties manifest in relation to symbolic versus graphical reasoning. This contributes to instructional design by highlighting the importance of tasks that explicitly bridge counting strategies, per-unit reasoning, and flexible application of $\Delta y = m\Delta x$. For example, tasks that press students to extend reasoning from integer to non-integer or symbolic Δx may be especially productive in supporting transitions toward interiorized ratio. Future research might investigate how such task sequences influence students' ability to coordinate slope with reference points in linear approximation, thereby deepening their preparation for calculus.

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Student Perceptions of Norms Supported by Definitions in Proof-Based Courses

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Mathematicians' community values and norms play a vital role in shaping approaches to practice, including with respect to definitions. Mathematicians have suggested that in their instruction they portray norms related to the value of clear communication more than those related to freedom in their use of definitions, but limited work has examined what students take from instruction. Here, we examine survey responses from proof-based linear and abstract algebra courses taught by the same instructor. Results include widespread recognition of definitions' role in supporting arguments and facilitating unambiguous communication as well as some support for mathematical doers' agency in creating and using definitions, suggesting it is possible to portray this freedom in instruction.

Keywords: Values, Norms, Definitions, Linear Algebra, Abstract Algebra

Definitions are central to mathematical practice, as they stipulate the objects coordinated in theorems and algorithms. Extensive work has characterized what makes a good definition, including required aspects like being unambiguous and non-contradictory as well as preferences like elegance and minimality (e.g., Harel et al., 2006; Shir & Zaslavsky, 2001; 2002; van Dormolen & Zaslavsky, 2003; Torkildsen et al., 2023), though recent work suggests that the emphasis on minimality may have received more emphasis in the mathematics education literature than mathematicians place upon it (Torkildsen et al., 2025). Work has also considered the pedagogical value of different definitions for different groups, with teachers often valuing clarity and accessibility for their student audience (Leikin & Winicki-Landman, 2000; Torkildsen et al., 2025; Zazkis & Leikin, 2008).

However, comparatively limited work has examined how mathematicians' work with definitions upholds their broader values and norms of practice as a community. Based on interviews with algebraists, Rupnow and Randazzo (2023) found two such values: 'clarity in and for communication' and 'freedom for the use and creation of definitions'. 'Clarity in and for communication' was upheld by norms like *support arguments/get at truth* and *concept names give insight* while the 'freedom for the use and creation of definitions' value was upheld by norms like *mathematicians write definitions purposefully for utility* and *conventions may vary between mathematicians*. Of note, the interviewed algebraists suggested that their research work supported both of these values, but that their teaching gave limited if any support to 'freedom for the use and creation of definitions' and notions of agency in particular, largely because they were unsure how to do so.

Vroom et al. (2024) examined Introduction to Proof instructors' teaching practices for supporting students in revising their own definitions using the lens of Rupnow and Randazzo's (2023) norms. They noted most of the observed teaching practices focused on unambiguous, precise, and minimal definitions, as well as the fact that definitions are stipulated, may be given insightful names, and generally follow certain conventions of the community. Notably, these norms aligned with the 'clarity in and for communication' value. In contrast, only a few

instances aligned with the ideas that people author definitions or contextual conventions supporting the 'freedom for the use and creation of definitions' value. However, because the study focused specifically on the revision of definitions rather than their initial formulation, it is possible that opportunities to support the 'freedom' value arise more naturally in earlier stages of defining. Together, these works suggest that definition values may appear in mathematicians' teaching, and potentially the communication value more consistently. As such, more work is needed to understand how students experience such teaching. Here we address the research question: How do students in a linear or abstract algebra course characterize norms and values of definitions communicated in their course?

Theoretical Perspective

We use the values and supporting norms for mathematical definitions characterized by Rupnow and Randazzo (2023) as our lens. This lens is grounded in Laudan (1984)'s Triadic Network of Justification, which characterizes Axiology (Values), Methodology (Norms), and Factual Claims as mutually dependent in a field of study. Rupnow and Randazzo (2023) adapted Laudan's (1984) framing to the mathematical community's use and creation of definitions. In so doing, they highlighted two values, 'clarity in and for communication' and 'freedom for the use and creation of definitions', along with various supporting norms. Here we especially attend to the norms of practice, which served as codes for the open-response questions and were embedded in the closed-response questions, as described below.

Methods

A survey was administered to the Fall 2024 sections of one author's Linear Algebra (LA) and Abstract Algebra I (AA) courses. 10 students completed and 3 students partially completed the survey in LA (out of 20), and 12 students completed the survey in AA (out of 27). LA and AA were taken by math majors and minors, though in LA most were math minors, whereas in AA, most were math majors. AA required LA as a prerequisite, and five AA survey completers had previously taken the author's section of LA.

Both courses incorporated inquiry-oriented instruction, with linear algebra using the Inquiry-Oriented Linear Algebra tasks focused on span and linear independence (Wawro et al., 2012), linear transformations (Andrews-Larson et al., 2017), and change of basis and eigentheory (Zandieh et al., 2017). These sets of tasks were used to introduce the respective concepts, but more formal treatments and formal definitions of the respective concepts were given on subsequent days in fill-in-the-blank notes during interactive chalk-talk lectures (Artemeva & Fox, 2011). Abstract Algebra tasks focused on motivating and defining a group and group isomorphism (Larsen, 2013). Students were given the opportunity to work with examples of a group to abstract the concept of a group and define it themselves before fill-in-the-blank notes with an interactive chalk-talk lecture formalized what had been discussed in the previous weeks of tasks. Similarly, students worked with Cayley tables to consider examples and non-examples of isomorphisms before defining the concept themselves and subsequently having fill-in-the-blank notes during chalk-talk lectures. As Vroom et al. (2024) note, this inquiry setting opened up the opportunity for meaningful engagement with norms (from the instructor's perspective), particularly *definitions arise from/relate to examples*, *mathematicians make definitions* and *mathematicians write definitions purposefully for utility*.

The survey was available the last two weeks of the semester, and students received extra credit for submitting it. Questions on the survey focused on students' views of definitions and how definitions had been used in the course. Open response questions analyzed here focused on

how mathematical definitions are similar and different from dictionary definitions, the purpose of definitions, ways definitions were used in the course, how their understanding of definitions had changed during the course, and whether the instructor did or could do something more illuminating about definitions. Closed response questions provided simplified versions of the norms from Rupnow and Randazzo (2023) (e.g., 'Definitions allow the use of a single phrase to communicate complex ideas' was provided to ask about the *Definitions as shorthand* norm). Students were given a textbox in which to type a response to each open response question and were asked to drag each stated norm into a "I took these away from the class" or "I did not take these away from the class" box for the closed response question.

We used the existing definition norms from Rupnow and Randazzo (2023) as a starting point for analysis of the open responses but made space to add additional codes if needed, in alignment with codebook thematic analysis (Braun et al., 2019). One non-instructor researcher coded the LA open-ended items independently and another non-instructor researcher coded the AA open-ended items independently. Based on this, three additional norm codes were created: *mathematical definitions are subject to verification/justification*, *mathematical definitions are expressed through symbolic language and notation*, and *mathematical definitions can evolve over time.* We perceived the first two norms to support the 'clarity in and for communication' value, and the third norm to support the 'freedom of choice for the use and creation of definitions' value. Then these two researchers plus the instructor researcher discussed and came to consensus on codes present in each student's response to each question in both courses.

Results

Here we highlight students' responses with respect to the two values separately. Table 1 notes the number of students whose responses received a norm code (Open) or who signified that "I took these away from the class" by dragging the prompted norm into a box (Closed). Throughout we signify norms with italics.

Table 1. Student norm counts by course and question

Value	Norm	LA Open (n=13)	LA Closed (n=13)	AA Open (n=12)	AA Closed (n=12)
Clarity in and for communication	Rigor/precision	4	0*	4	0*
	Clarity/no ambiguity in communication	10	8	10	6
	Axiomatic system/work with definitions	6	10	4	12
	Support arguments/get at truth	11	12	12	12
	Norm of precision perpetuated through teaching	0	0*	4	0*
	Norm of precision irrelevant to laypeople	0	0*	2	0*
	Definitions as shorthand	6	11	2	10
	Concept names give insight	0	8	0	5
	Examples illustrate definitions	2	11	2	10

	Definitions are based on historical motivations	0	2	0	4
	Need more than definitions to understand concepts	4	0*	1	0*
	Definitions scaffolded for undergrads	2	0*	1	0*
	Norms of mathematical communication through formatting choices	1	6	0	7
	Mathematical communication has consistency	2	11	2	9
	Mathematical definitions are subject to verification/justification	0	0*	1	0*
	Mathematical definitions are expressed through symbolic language and notation	4	0*	2	0*
Freedom of choice for the use and creation of definitions	Definitions arise from/relate to examples	0	8	2	9
	Definitions permit inspection of relevant features	0	10	1	8
	Concepts are made through abstraction	0	5**	0	9
	Mathematicians make definitions	1	7	4	8
	Mathematicians write definitions purposefully for utility	7	7	4	7
	Conventions may vary between mathematicians	0	7	0	9
	Norm of agency perpetuated through teaching	1	0*	4	0*
	Mathematical definitions can evolve over time.	0	0*	1	0*

*No question asked on this topic

**Count reduced by one because a student noted they no longer meant to choose this in a later open-ended question

Clarity in and for Communication

A predominant norm for LA and AA students was the role of definitions to *Support arguments/get at truth*. This norm was articulated and selected across both open and closed responses, indicating a strong perception of definitions as foundational for logical reasoning and mathematical communication. Students noted how their instructor clarified this function of definitions, particularly their utility in proofs. One LA student stated, “She showed us how to use definitions to prove theorems.” Another LA student elaborated, remarking, “I feel like I better understand their purpose and am able to utilize them in the context of proofs and problem solving.” Many students emphasized how definitions provide a necessary foundation for constructing valid arguments, arriving at mathematical truths, and developing deeper results. As one AA student described the purpose of mathematical definitions: “To serve as the building blocks for the development of theorems, proofs, and mathematical structures.” This highlights

students' ability to see definitions not as isolated facts, but as interconnected components within a logical framework.

Students also showed awareness of the norm *mathematical communication has consistency*. This norm highlights the mathematical community's expectation for a standard and consistent way of conveying mathematical ideas to promote clarity and understanding. Responses reflecting this norm were noted in open responses but were affirmed more often in the prompted closed responses. One LA student reflected, "I've seen definitions in practice and developed a stronger sense of what they do and don't contain," demonstrating how exposure to definitions helped them notice patterns in how definitions are framed. An AA student echoed this norm, stating, "To establish a consistent rule or mathematical relationship, to describe a proven mathematical principle." Another AA student emphasized its broader significance: "They ensure clarity and consistency which allows us universal understanding and the ability to construct complex mathematical frameworks." These comments underscore that students viewed definitions not merely as isolated facts, but as critical tools for constructing precise, shared meaning.

The norm *Axiomatic system/work with definitions* was highlighted by students in both LA and AA. In open responses, this norm was mentioned by 6 LA students and 4 AA students, while in closed responses, it was selected by 10 LA students and 12 AA students. This pattern suggests that when explicitly prompted, students across both courses largely agreed with this norm as an expectation of practice, though fewer highlighted it unprompted. In the following LA student's response, we interpret species to mean examples of concepts: "This entire course relies on definitions building on each other. Knowing these different definitions allow[s] me to make assumptions as to how certain species will behave and more importantly why." Another LA student recognized the structure of mathematical definitions in describing a word or process: "During this class the way mathematical definitions were presented to me has helped me understand how topics relate to one another better. The use of theorems, lemmas, and corollaries, specifically was very helpful." Similarly, an AA student emphasized the foundational role of definitions: "It is clearer now that definitions play a huge part in the understanding and development of mathematics." This recognition reveals that students understood definitions as foundational, logical building blocks in mathematics.

Beyond their role in supporting arguments and axiomatic systems, students also recognized definitions for their *clarity/ no ambiguity in communication.* This norm was mentioned in both open and closed responses, suggesting that students valued definitions as tools for precise expression. One LA student characterized mathematical definitions as **"**more precise and straightforward**"** than dictionary definitions. They are designed to be **"**unambiguous**"** and **"**to distinguish similar but distinct ideas." Another LA student observed that the wording in mathematical definitions is specific, taking **"**into account every nuance." Nevertheless, this precision can sometimes make definitions long or hard to understand, as one LA student noted, "I personally think mathematical definitions are overly complex when they could be simpler. I understand that they are trying to not be ambiguous, but they are just complex." An AA student also focused on avoiding ambiguity, stating: "The purpose of mathematical definitions is to provide precise and unambiguous descriptions of ideas, forming the foundation for rigorous reasoning and logical deductions. They ensure clarity and consistency which allows us universal understanding and the ability to construct complex mathematical frameworks." These observations reveal that students recognized the communicative function of definitions, understanding their role in ensuring clarity, avoiding ambiguity, and supporting shared understanding across mathematical discourse.

Student responses, particularly in closed questions, revealed an awareness of the role of definitions as shorthand in mathematics. The norm *Definitions as shorthand* was characterized in open responses by 6 LA and 2 AA students and in closed responses by 11 LA and 10 AA students. This indicates that students, particularly when prompted, perceived definitions as concise ways to encapsulate complex ideas. This LA student noted, "Definitions package up entire concepts into concise words; This not only helps with communicating and understanding, it also helps with drawing out theorems." An AA student also reflected that seeing definitions as shorthand was a way "I used to keep the definition in the back of my head and then every now and then used it, but this course has made me study the definition and completely understand it. Made me realize how important the[y] are." Students' responses demonstrate the practical and communicative power of definitions as shorthand in mathematical practice.

Student responses revealed an awareness of certain aspects of the role of examples in mathematical understanding. The norm *Examples illustrate definitions* was not as frequently mentioned in open responses (2 LA students and 2 AA students), but it was more frequently chosen in closed responses (11 LA students and 10 AA students). One LA student noted: "The use of theorems and definitions paired with relevant examples or problems was helpful." Additionally, one Abstract Algebra student illustrated in their open response: "She gave examples of the definitions and how they truly were related to certain problems." Students in both courses indicated that examples help solidify understanding, associate concepts with processes, and demonstrate how definitions operate in practice. This shows that students viewed definitions not only as abstract entities but as tied to concrete examples and intuitive reasoning, though they especially attended to this when prompted.

Freedom of Choice for the Use and Creation of Definitions

Student responses, particularly in closed questions, showed an awareness of the norm *Definitions arise from/relate to examples*, which highlights that mathematical definitions are tied to concrete examples and intuitive reasoning. As one AA student illustrated: "They [definitions] provide a framework for constructing examples that satisfy certain properties or counterexamples that fail to meet specific criteria."

The norm *Mathematicians write definitions purposefully for utility* was consistently mentioned by students, with 7 LA students and 4 AA students in open responses and 7 LA students and 7 AA students in closed responses. Students both spontaneously articulated and recognized with prompting that definitions are not arbitrary but are designed for practical application. For instance, one LA student noted, "I feel like I better understand their [definitions'] purpose in the context of proofs and problem solving." Demonstrating an appreciation for the pragmatic use of definitions in mathematics, one AA student elaborated:

> A mathematical definition defines a mathematical structure or process and how to apply that process / utilize the structure. A definition in the dictionary is more to give meaning to a word. Both give meaning to some abstract object, but a mathematical definition has more nuance than just giving meaning.

Another AA student stated:

> Writing proofs has been extremely beneficial in my understanding of proofs. Before I became more familiar with proofs, I did not think definitions in math had much purpose other than just making rules in math that we have to follow. It is clearer now that definitions play a huge part in the understanding and development of mathematics.

These reflections show students had developed a sophisticated view of definitions not as static rules, but as essential tools in both reasoning and application.

The idea that *Mathematicians make definitions* appeared in open responses from 1 LA student and 4 AA students, and more frequently in the closed responses (7 LA students and 8 AA students), highlighting students' awareness that mathematical definitions do not merely exist but are created by individuals. This LA student expressed, "Mathematical definitions allow us to not only understand the world of math but also create it in a way that is universal," emphasizing mathematicians' active role in constructing tools for the discipline. An AA student further noted:

> My understanding of mathematical definitions is, in general, more developed than it was before because I was exposed to a lot of definitions in this class. I have more appreciation for the process of writing a definition and I know how to read mathematical definitions more accurately.

These reflections point to students' recognition of definitions as thoughtfully constructed tools for shaping and communicating mathematics.

The norm *Conventions may vary between mathematicians* was not spontaneously mentioned in any open responses, but it was selected in closed responses by 7 LA students and 9 AA students. While students did not highlight that mathematical definitions and conventions reflect choices made by mathematicians depending on context or purpose in response to the purpose of definitions, they did agree with this sentiment when prompted more generally.

Discussion

Students in both courses largely highlighted the same norms with fairly similar frequencies. Considering that the same instructor taught both courses and that both courses are proof-based, this perhaps should not be surprising. Across the open and closed responses in both courses, students commonly highlighted *support arguments/get at truth* as a purpose of definitions as well as that definitions support unambiguous rigorous communication (*rigor/precision* and *clarity/no ambiguity in communication*). Students also highlighted *axiomatic system/work with definitions*, *definitions as shorthand,* and *mathematical communication has consistency*, though these arose more in the closed than open responses. This suggests that students agreed with these norms when prompted to consider them but did not immediately highlight them from prompts focused on differences between mathematical and dictionary definitions, the purpose of definitions, or ways definitions were used in the course.

Of note, the norms *mathematicians make definitions* and *mathematicians write definitions purposefully for utility* also arose in the data, suggesting a recognition of mathematicians' (and potentially their own) agency in writing and using definitions. Relatedly, though less commonly, some students did suggest that agency in using and creating definitions was supported through instruction. This contrasts with the algebraists' claims about their teaching practice (Rupnow & Randazzo, 2023) and most of the Introduction to Proof instructors' teaching moves supporting students' definition revisions, though revising definitions comprises a limited portion of instructors' overall work in a course (Vroom et al., 2024). Thus, while supports for the 'freedom for the use and creation of definitions' value may not yet be commonly documented in the literature, it is possible to support this value.

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Interpreting Student Code: Prospective Teachers' Scripted Responses to a Scratch© Coding Task

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We explore prospective intermediate and secondary school teachers' strategies for supporting students in coding and troubleshooting errors while working in Scratch ©. Participants were asked to create an imagined dialogue that addressed student-generated "starter" code, which included bugs, in order to explore the probability of different sums of two dice rolls and present these sums as a graph. Our analysis shows that participants attended to different possible student difficulties with the coding, yet few of them attended to the given starter code as an instance of student thinking when suggesting viable resolutions or ways forward. Instead, participants tended to introduce their own approaches and trajectories for how to advance the code. We suggest that while participants demonstrated useful teaching strategies to support student-characters, their attention was focused on advancing any code, rather than building with or on ideas presented by the student-characters.

Keywords: Probability, Prospective teachers, Computer programming, Scripting, Scratch©.

Recognizing and interpreting student errors as a way to advance student learning and understanding is advocated for in research and policy documents (e.g., NCTM, 2000; Son, 2013). A wide variety of learning theories and studies have emphasized the importance of learning trajectories that build with student thinking (e.g., Hershkowitz, et al., 2001; Simon, 2006). However, researchers have found that prospective teachers commonly prefer providing teacher guidance that mirrors their own thinking as opposed to building on students' thinking. Mamolo and Pali (2014) argued that robust mathematical knowledge is linked with prospective teachers' flexibility in adopting a variety of guidance strategies to support and advance students' thinking, even when the approaches are different from those teachers' own preferences or prior experiences as students. In this research, we are interested in how prospective teachers provide guidance that builds on a hypothetical student's thinking in relation to content for which participants had no analogous prior experiences as students - developing mathematical ideas through coding. Specifically, we address the following research question:

What can a scripting task reveal about how prospective teachers work with student ideas while debugging a coding simulation for rolling two dice?

Background Literature

Block-based coding in programming languages such as Scratch © have become increasingly common in teaching mathematics (e.g., Benton et al., 2017; Burke, et al., 2012; Passey, 2016). Francis and Davis (2018; see also Pei et al., 2018) describe coding as a competency that "can complement and co-amplify mathematics learning," emphasizing its "deep roots in and emergent powers for mathematics" (p.14). When used as a tool to address mathematics problems, coding has been described as transformative to learning (Papert, 1980), with benefits including fostering resilience, creativity, and learner-driven activity. Research has also pointed to benefits such as: self-motivation; experimentation; development of mathematical intuitions and approaches; critical reflection; and working with abstraction and different representations (e.g., Bailey & Borwein, 2011; Henderson, Cortina, & Wing, 2007; Howson & Kahane, 1986; King et al., 2001; Weintrop et al., 2016).

Notwithstanding the positive potential of teaching mathematics through coding, Resnick and Rusk (2020) noted a concern about teachers treating coding with an instrumental instructional approach – such as memorizing definitions, or copying lines of code – instead of fostering student reasoning, experimentation, and active engagement. They warn that the positive potential will be lost if coding is reduced to a memorization exercise. In the jurisdiction where this research took place, coding is newly embedded within the K-12 mathematics curriculum, and teacher education programs have sought ways to support prospective teachers in learning both related content and pedagogy (e.g., Gadandis, et al., 2017). Yet, there is limited research on how to best support prospective teachers in adopting this new-for-them approach (e.g., Gleasman & Kim, 2018, 2020; Mamolo, et al., 2022). Mamolo et al. (2022) suggest that "before coding can be thought of or used as a tool through which to develop and promote mathematics learning, there is a need to establish a foundation of coding practices, content, and confidence" (p.975). They highlight practices related to experimenting and iterating, remixing and reusing, testing and debugging, and stepwise refinement, amongst others.

The need for research in coding education for teachers, along with the concern raised by Resnick and Rusk (2020), compelled us to take a closer look at how prospective teachers attend to student thinking when developing teaching competencies and content knowledge related to coding. We chose the mathematical context of probability since this is a topic for which coding can serve as a useful tool (e.g., Franklin et al., 2015) and is a topic in which students teachers struggle (e.g., Chernoff & Russell, 2012; Shaughnessy, 1977; Stohl, 2005).

Theoretical Underpinning

A theoretical position that supports our research can be found in the work of Carpenter, Fennema, and colleagues, who advocate for cognitively guided instruction that builds on the knowledge of students (e.g., Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1996; Carpenter et al., 1989). In this perspective, teaching is conceptualized as a student-centered activity in which students' ideas, intuitions, and experiences are built upon in the pursuit of new learning. The premise is that by engaging with student thinking, teachers can uncover and expand students' mathematical understanding beyond procedural or rote learning. Researchers working from this perspective primarily research teacher training that focuses on teachers building on students' thinking. We instead use cognitively guided instruction as an impetus for asking, "Is this prospective teacher working to build on student thinking, and if so, how?" This parallels our research question.

Methods and Task

A total of 24 prospective intermediate and secondary teachers participated in this research by developing scripts of imagined dialogues as part of their regular classroom assignments for a methods course in coding education. Scripting or role-playing tasks in which participants imagine an exchange between a teacher-character and student-characters have been used in prior research to explore and develop mathematical knowledge (e.g., Bergman et al., 2023; Kercher et al., 2023; Zazkis & Zazkis, 2014) and pedagogical awareness (e.g., Campbell & Baldinger, 2022; Lajoie & Maheux, 2013; Mamolo, 2021, 2017). In this study, we extend the applicability of these tasks to explore how prospective teachers engage with student thinking in a coding context. Our data includes the scripts of fourteen prospective teachers who elected to work in groups of 2-3 and handed in joint work (referred to as Groups 1-7), and 10 participants who composed their scripts individually (referred to by pseudonyms). The scripting task had two core components: 1) an initial student-generated code, which was to be the starting point for the script (Figure 1), and

2) a goal of building on the provided background to debug the existing code and develop a future plan (Figure 2).

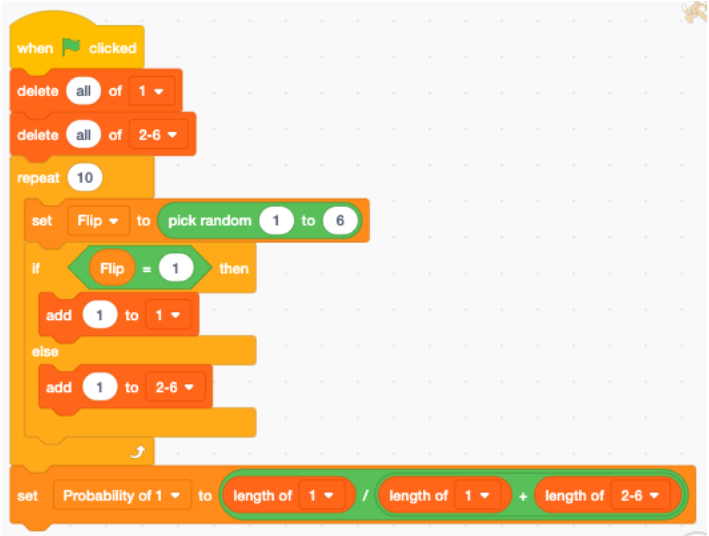


Figure 1. Initial student scratch code for scripting task

Background: A few students are creating a dice simulation for a probability assignment. They want to repeatedly roll two dice, record the sum and then graph the results. They have made a great start with their code: https://scratch.mit.edu/projects/324585345/editor
Bugs: The group notices the program is not recording the sum (it is only recording the most recent sum). They also realize that the program is repeating 10 times- they want the user to choose how many times to roll the die.
Future Plans for Project: The group wants to graph their results but does not know how to do this.

Figure 2. Two Dice Scripting Assignment

In analyzing the data, the authors, first individually and then jointly, carefully documented what occurred in each script. This included the trajectory the code followed from the initial code in Figure 1 to the finalized code that the student-characters in the script created. We also documented which benchmarks were accomplished by the time the script concluded (e.g., was the user prompted for a number of rolls? Did the final code have two dice? Did the code graph the results, and, if so, how?). Finally, we documented which difficulties the student-characters encountered and how these were overcome.

To better acquaint the reader with the scripting task and familiarize the reader with coding in Scratch, it is worthwhile to parse the initial code in Figure 1. The code begins by deleting the content of previously saved lists named "1" and "2-6." Then there is a loop that repeats 10 times. With each repetition, a variable named "flip" is assigned a random integer value between 1 and 6. Then if "flip" is 1, a 1 is added to the list named "1", otherwise, a 1 is added to the list named "2-6." The result is two lists of 1's with a total length of 10. The length of the first list represents the number of times a 1 occurred, while the length of the second list represents the number of times a value other than 1 was rolled.

There are several things to note about the code in Figure 1, given that the scripting task asks that it be interpreted as initial student work. First, only two outcomes are documented within the code, whereas the problem scenario encompasses 11 possibilities, as rolling two dice can yield any integer sum between 2 and 12, inclusive. Next, there is only one variable assigned to a random value. We would expect two variables since there are two dice in the problem scenario. Finally, one generally flips a coin and rolls a die. So, it is unusual to name the variable "flip", as opposed to "roll." A reasonable conclusion to draw is that the code in Figure 1 started as code for

flipping one coin. This explains all three of the above observations (number of variables, number of possibilities documented, and the variable name). So, a natural interpretation of this initial student code is that students started with code for documenting coin flips, began remixing it to account for the differences between coins and dice, and got stuck.

Results and Analysis

Only 4 of the 18 completed scripts included an assessment of the motivation behind the initial student code. All four of those paralleled the "started out as code for a coin flip" explanation discussed above. The other 14 scripts did not acknowledge the origins of the code. Those scripts proceeded by either pointing out the initial code's flaws relative to the task to make progress (8) or replacing the initial code with entirely different code to use as a starting point (6). Below, we exemplify each of these three observed approaches.

Motivating the initial scratch code

As mentioned, only 4 of the 18 scripts included motivation for why students might have generated the initial Scratch© code in the scripting task, as illustrated in Group 5's script:

> *Teacher*: It looks like you have some elements in your program that you don't use. Did you make this code or is it remixed from someone else?
> *Student1*: We used a code that showed the probability for flipping a coin. I made some modifications so that it would work for dice, but we can't get it to repeat 10 times.
> *Teacher*: Can you explain to me what the code is doing?
> *Student1*: Well, it starts with deleting all the previous results, then it flips the die by picking a random number from 1 to 6. Then, if the roll is 1, it adds to column 1, and if it's something else, it adds to column 2.
> *Teacher*: Before you go further with the code, you need to think about the problem you are trying to solve. If you want to remix this code, you need to think about the differences between calculating the probability of a coin flip and the sum of a pair of dice.

The student-character's motivations for the initial code in Figure 1 are evident when he states that the code started as "code that showed the probability for flipping a coin," and that he "made some modifications" so it would work for dice. This simultaneously provides a plausible explanation for why students working on the sum dice task might have generated the code in Figure 1, and provides the teacher-character with a starting point for guiding their progress. In Group 5's script, the teacher-character directs the student-characters to think about the differences between the single coin flip scenario the initial code addressed and the rolling two dice scenario they are working on. Consistent with the tenets of cognitively guided instruction, the teacher-character is building on the imagined students' work, and helping to direct them to the kinds of questions one might ask when trying to self-direct their coding progress.

Notably, a prospective teacher recognizing the code in Figure 1 started as code for flipping a coin does not necessarily mean that their approach is consistent with cognitively guided instruction. For instance, consider Felicity's script below:

> *Teacher:* Ok that sounds like a great idea. Let's see how we can work through the problems you are having. What is the code you currently have doing?
> *Student1:* Well we kind of used a code we found for the flip of a coin and tried to change it around and that didn't seem to work.

Student3: Right now it seems like the dice is rolled 10 times and if the roll is a 1 it gets added to the list of ones and if the roll is between 2 and 6 a 1 gets added to the 2-6 list.
Student1: But we don't want to know if the roll was a 1 or not we want to know exactly what the roll was and do that for 2 dice.
Student2: Does that mean we need to start over and try something else?
Student3: I think we should. [S2 and S1 nod in agreement.]

In the above script, the scriptwriter, Felicity, notes that the initial given code started as code for flipping a coin, when Student 1 indicates that "we kind of used code we found for flipping a coin." So, she has a viable explanation for why the code in Figure 1 might have reasonably emerged from students' work on the two dice task. However, instead of building on this code, Felicity uses a different code that she, as the scriptwriter, generated as a starting point for guiding the student-characters' progress. Although this idea is presented as coming from the student-characters when Student 2 says, "Does that mean we have to start over and try something else?", it is important to keep in mind that this is the work of a scriptwriter, not a student. By presenting the abandonment of the initial code as a student-character's decision, Felicity is, in a sense, avoiding the teacher-character being responsible for not building on the initial code. Felicity is instead using her own code as a starting point. As we discuss later, this is a common move in scripts that do not build on the given code.

Leveraging the flaws in the initial scratch code

A majority of scripts did not provide an explanation for the origins of the initial code in Figure 1. However, 8 of these scripts found other ways to leverage the initial code without such an explanation. This was primarily achieved by analyzing the initial code's functionality in relation to the rolling of two dice task. For instance, Group 6's script included this excerpt:

Teacher: What is happening in this small part of your code?
Student1: That's where we roll the dice.
Teacher: How many dice are you rolling?
Student1: Well, we want to roll 2 dice…
Teacher: Okay, so when you roll 1 die, what are the possible outcomes?
Student2: We can get 1, 2, 3, 4, 5 or 6.
Teacher: Exactly, so in the code here how many dice are you rolling?
Student1: Oh, we're only rolling once! Okay, so how do we roll twice?
Teacher: You've already rolled once, so do that again and combine them however you want. You could also pick a single random number that represents a roll of two dice, but I'd suggest just getting the sum.

Notice that in this excerpt, the teacher-character uses the initial student code as a starting point for guiding student-characters toward code that successfully addresses the rolling two dice task. This is achieved by highlighting the differences between the two dice task and what the initial code does. The above excerpt was used to motivate adding a second variable to record dice rolls, as well as replacing the two lists "1" and "2-6" with eleven lists named 2-12, each for recording one of the eleven possible outcomes of rolling two dice. This, in our view, is similar to Group 5's script, which focused on the differences between flipping a coin and rolling two dice to motivate the same changes to the initial code. Notable in Group 6's work is that the variables

used to record dice rolls are named “flip1” and “flip2”. This suggests that the scriptwriters overlooked the potential inappropriateness of those variable names in relation to the dice task.

Group 6 and Felicity’s scripts present an interesting contrast with respect to how the scriptwriters versus how the teacher-characters appeared to work with student ideas. While Felicity’s teacher-character lets the student-characters take the lead on deciding to shift away from the initial code, as a scriptwriter, Felicity does not attend to building on the student ideas that are represented in the initial code (e.g., the thinking which may have led to the code in the first place). In contrast, Group 6’s teacher-character offers direct instruction in response to a student’s question of “how do we”, suggestive of a model of teacher-centered instruction. Yet, we found Group 6 was attentive to the student ideas represented in the initial code. We suggest that Group 6’s consideration of the thinking that could have resulted in the initial bugs informed their approach in directing the students toward advancing their code, which is consistent with cognitively guided instruction. This contrast helps highlight the tension between how a scriptwriter represents a teacher-student interaction and that scriptwriter's actual attention to the presented student’s thinking.

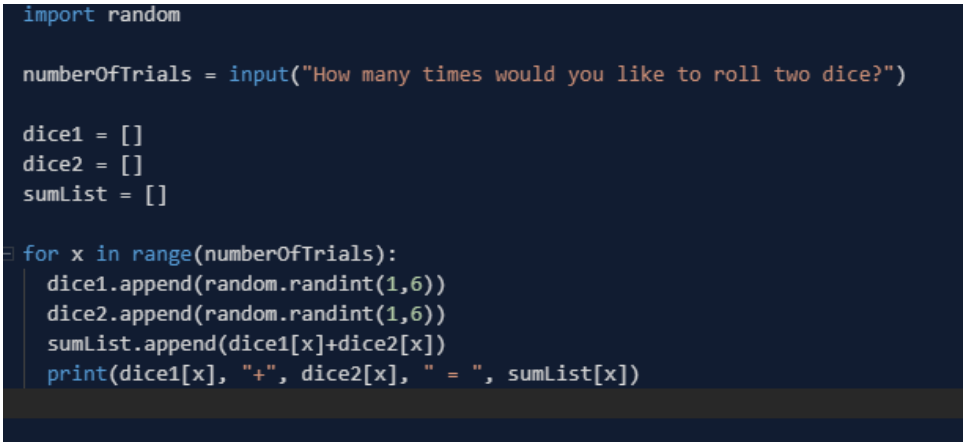
Replacing instead of building on the initial code

Some prospective teachers in our study avoided building on the initial code in Figure 1 entirely. This was commonly done by replacing the code in Figure 1 with scriptwriters own code and using that new code as a starting point for addressing the rolling two dice task. As alluded to in relation to Felicity’s script above, we view this as the scriptwriters avoiding engaging in the thinking and motivation behind the initial code in favor of interjecting their own thinking. Below is an excerpt from Sadiya, which does this in an exaggerated way:

> *Teacher:* Yeah, this code has elements that are necessary for this type of project but it feels like you may have not thought all the way through the problem before starting to code. Good job though on starting early! I like the initiative.
> *Student1:* May we use Python instead of Scratch for this assignment? I feel like for this assignment it would be easier, I remember a lot of how to use it from grade 11 functions and I really like making graphs on it.
> *Teacher:* Sure you can use it. However, Python is not as friendly as Scratch and you might have to google questions if I am not available as well as be prepared to debug things.
> *Student1:* Okay yeah I am prepared for that.
> *Teacher:* Are you okay with Student1 wanting to use Python instead?
> *Student2:* Yes, we already talked about it. I can use either language but I think Student1 might be right that Python could be more straightforward for this project.

In the above, the initial code in Figure 1 is abandoned in favor of a different starting code written in an entirely different programming language, Python. We interpret the teacher-character’s utterance, “it feels like you may have not thought all the way through the problem before starting to code,” as a way to avoid addressing student ideas. It may be that Sadiya was either unable to, or uninterested in, making sense of why a student might have generated the particular code in Figure 1. The Python code Sadiya introduced to side-step the initial code can be found below in Figure 3. As can be seen below, this code already receives user input for how many pairs of dice are to be rolled and records the sum of those pairs of dice rolls in a list called “sumList.” So, all of the “bugs” outlined in Figure 2 have already been addressed by the time the script begins in earnest. The remainder of Sadiya's script is focused on adding features and

graphical representations to the Python code (the "future directions" in Figure 2). We interpret this as Sadiya avoiding building on student thinking entirely. Instead, Sadiya and other similar scriptwriters completed the task themselves without leveraging the initial code or the thought process behind it at all. Then they represented their own personal coding journey in the form of a dialogue. This, in our view, is scriptwriter behavior inconsistent with the cognitively guided instruction perspective which underpins this paper.



```
import random

numberOfTrials = input("How many times would you like to roll two dice?")

dice1 = []
dice2 = []
sumList = []

for x in range(numberOfTrials):
    dice1.append(random.randint(1,6))
    dice2.append(random.randint(1,6))
    sumList.append(dice1[x]+dice2[x])
    print(dice1[x], "+", dice2[x], " = ", sumList[x])
```

Figure 3. Sadiya's Python code

Discussion and Future Directions

This paper provides an illustration of how scriptwriting can be used as a tool for researching prospective teachers' engagement with student thinking. We demonstrated that the scriptwriters engaged in leveraging student thinking did so by directing student-characters to the difference between where they are and their goal. This was done by either discussing the differences between a coin flip and rolling dice, as was seen in Group 5's script, or the differences between the initial code and the code the sum of dice tasks warranted, as was seen in Group 6's script. The prospective teachers who avoided engaging in this student thinking commonly did so by replacing the initial code in the assignment with scriptwriters' own code. This was seen in Sadiya and Felicity's scripts. Commonly, this decision was attributed to student-characters, to maintain the facade that student thinking was being leveraged.

We believe an additional contribution this paper makes is in helping illustrate the utility of parsing code when teaching. Increasingly, teachers are expected to integrate digital technologies and coding into their mathematics teaching. Yet most of these teachers have little if any experience coding and have not experienced coding in their own K-12 education. Tasks that engage teachers in parsing the thinking behind student mathematics have a long history in mathematics education. We suggest that coding tasks, with a similar flavor, which task teachers with parsing, not just what code does, but also the student thinking that might have generated it and how that thinking can be built upon, need to be further researched and studied in relation to teacher education and professional development. The scripting task we discussed here is one such task, and there are myriad others that can include or not include scripting, depending on the specific goals of teacher educators.

Acknowledgments

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Teaching Practices Supporting Students' Awareness of Definition-Related Mathematical Norms

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The mathematics community has a variety of standard norms of practice by which it operates, which are guided by collective values, including in the context of definitions. However, these norms and values are not always shared with students in instruction. Here we examine teaching practices that linear and abstract algebra students perceived as influencing their understanding of norms around mathematical definitions. Teaching practices in linear algebra tended to align more with instructor-centered actions whereas teaching practices in abstract algebra tended to align more with student-centered activities. Implications include the impact of homework and classroom activities on students' perceptions, as well as the long-term use of activities seeming to be more salient in students' perceptions of influential activities.

Keywords: Teaching Practices, Norms, Definitions, Linear Algebra, Abstract Algebra

Mathematical definitions play a central role in both the discipline of mathematics and the teaching of it. They serve as essential tools for naming collections of objects or phenomena, communicating mathematical ideas, and supporting logical derivation (Edwards & Ward, 2008; Ouvrier-Buffet, 2006; Torkildsen et al., 2023). Within the mathematics community of practice (Lave & Wenger, 1991), there are well-established norms for definitions. Broadly, norms are shared expectations about how definitions should function and how they should be crafted. For instance, definitions are expected to clearly identify a property that characterizes a class of mathematical objects, with space for individual agency in how that property is framed, and to be minimal, non-contradictory, unambiguous, and elegant (Edwards & Ward, 2008; Ouvrier-Buffet, 2006; Rupnow & Randazzo, 2023; Van Dormolen & Zaslavsky, 2003; Vinner, 1991).

Understanding definition norms can offer insight into their foundational role in the discipline. However, teaching definitions can be especially challenging since norms in mathematics (generally and specifically to definitions) are often taken-as-if-shared within the community (Yackel & Cobb, 1996; Yackel et al., 2000). That is, they function as tacit expectations that typically go unspoken (Herbst et al., 2011) and thus can remain invisible to students. This could explain findings that identify differences in mathematicians' and students' understandings of the nature of definitions (Edwards & Ward, 2004; Fernández-León et al., 2021; Martín-Molina et al., 2020).

Few studies have directly investigated instructors' teaching of definition norms (Rupnow & Randazzo, 2023; Vroom & Ellis, 2024; Vroom et al., 2024). Rupnow and Randazzo (2023) found that instructors attempt to convey norms aligned with clear communication such as expecting students to use rigorous and precise definitions. Vroom and colleagues (Vroom & Ellis, 2024; Vroom et al., 2024) analyzed classroom video to identify instructional practices aimed at making these norms visible to students as they engaged in revising definitions, for instance, suggesting a particular revision with implicit or explicit reasons related to clear communication. Still, little is known about how students experience and make sense of these efforts. Understanding the student perspective is critical for improving students' opportunities to

engage meaningfully with mathematical definitions' norms. Here we address the research questions: (1) What teaching practices did students highlight as impactful to their understanding of definitions in a linear algebra course?, and (2) What teaching practices did students highlight as impactful to their understanding of definitions in an abstract algebra course?

Theoretical Perspective

We adopt Nachlieli and Elbaum-Cohen's (2021) definition of a *teaching practice* as our lens. They describe a teaching practice as comprised of a task and a procedure; the task is an instructor's educational goal while the procedure is an action or actions taken to accomplish that goal. Here, since we are examining students' perceptions of how understandings of norms related to definitions were supported, we infer the instructor's task from each question, which had a specific norm supporting a definition value of the mathematical community embedded in it. We view procedures as supportable by specified instructor-focused actions (e.g., instructor presenting definitions in a certain way) or specified student-focused actions (e.g., students engaging in a certain type of activity in the course at the behest of the instructor). We additionally highlight students' broad claims about the course as a whole or instruction around particular concepts that they viewed as supporting an instructor's task, but that do not fit the grain size of a procedure.

We drew on Rupnow and Randazzo's (2023) work to identify and articulate norms for definitions. In particular, the following norms increase the likelihood of clear communication with definitions: *mathematical definitions are unambiguous*, *definitions are an important part of an axiomatic system*, *definitions are used to support mathematical arguments*, *definitions allow the use of a single phrase to communicate complex ideas*, *names of concepts give insight into what the concept means*, *examples illustrate definitions*, *definitions are based on historical motivations*, *standard formatting choices in mathematics communicate information about definitions*, and *mathematical communication has consistency*. Additionally, the following norms increase the likelihood of freedom of choice in defining: *examples motivate the creation of definitions*, *the relevant features of a concept motivate the creation of definitions*, *abstraction motivates the creation of definitions*, *people make definitions*, *people write definitions purposefully for ease of use*, and *definition conventions vary between mathematicians*.

Methods

A survey was administered to the Fall 2024 sections of one author's Linear Algebra (LA) and Abstract Algebra I (AA) courses. 10 students completed and 3 students partially completed the survey in LA (out of 20), and 12 students completed the survey in AA (out of 27). LA served as the Introduction to Proof, was taught with a mixture of the Inquiry-Oriented (IO) Linear Algebra activities (Andrews-Larson et al., 2017; Wawro et al., 2012; Zandieh et al., 2017) and interactive lectures, and was taken by math majors and minors though most were math minors. AA had LA as a prerequisite, was taught with a mixture of the IO Abstract Algebra activities (Larsen, 2013) and interactive lectures, and was taken by math majors and minors though most were math majors. Five AA survey completers had previously taken the author's section of LA. Lectures in both courses were organized similarly, with LaTeX fill-in-the-blank notes posted before class and filled during class through interactive lectures where the instructor would lead students through the material, with times for students to ask questions and address examples themselves, in line with a chalk-talk lecture (Artemeva & Fox, 2011). IO lessons in LA focused on introducing concepts like span through concrete examples in which students would work in groups. However, students were not expected to write definitions themselves in those activities.

In contrast, lessons in AA introduced various groups in collaborative activities to scaffold students writing their own definition for group and later for group isomorphism.

The survey was available the last two weeks of the semester, and students received extra credit for submitting it. Questions on the survey focused on students' views of definitions and how definitions had been used in the course. The questions had definition norms supporting values of the mathematical community embedded in them, and the norms were ideas the instructor hoped to inculcate in students. In one question, students were asked to drag each stated norm into a "I took these away from the class" or "I did not take these away from the class" box. Norms that were dragged into the "I took these away from the class box" received a follow-up question asking, "What about the course gave you insight into [that norm]?" Students were given a textbox to type answers to these follow-up questions, and we focus on students' responses to the follow-up questions here.

We used Nachlieli and Elbaum-Cohen's definition of a teaching practice to guide analysis. Specifically, we viewed the definition norm from Rupnow & Randazzo (2023) that was embedded in the question as the instructor's task, since the instructor, as an author, did intend to share the norm in the question and students claimed that was an idea communicated in the course. We created procedure codes by having two non-instructor authors independently open code with the definition of a "procedure" in mind. Then these two authors and the instructor author met to consolidate a list of procedure codes. Next, one non-instructor author coded the LA and one non-instructor author coded the AA responses independently. Finally, the same three authors met to discuss and come to consensus on the procedure codes present in each student's response to the provided open-response questions. This process aligns with codebook thematic analysis (Braun et al., 2019), in that researchers had some idea of the types of procedures that might appear in the data but used the data to guide the creation of procedure codes.

Results

We present results by course, first Linear Algebra (LA) student responses, then Abstract Algebra (AA) responses. The bold heading indicates the procedure whereas italics indicate a task (norm goal, referred to as norms below) supported by the procedure.

Table 1. Procedural Codes Highlighted by Participants in Each Course.

Category	Procedural Code	LA Frequency	AA Frequency
Instructor-Focused Teaching Practice	Instructor presents clear and precise articulation of definitions.	6	3
	Instructor clarifies the function of definitions (e.g., utility for proving).	9	4
	Instructor presents motivating example(s) before definition.	2	4
	Instructor presents examples after the definition to illustrate.	5	4
	Instructor verifies examples fit the definition	3	2
	Instructor abstracts properties from the object to define the object.	2	3
	Instructor formats the notes consistently.	5	5
	Instructor emphasizes the importance of notation.	1	2
Student-Focused	Students verify examples fit the definition.	5	2
	Students write definitions (themselves).	1	4

Teaching Practice	Students practice after learning a new definition.	5	6
	Students explore before defining a concept (themselves).	0	8
	Students work individually on proofs for assigned work (tests/quizzes/homework).	7	10
	Students work collaboratively on activities in class.	1	6
	Students discuss readings or proofs for discussion boards.	1	5
	Students compare definitions.	1	2
	Students learn about mathematicians or mathematics history.	2	10
	Students are exposed to different notations outside class.	2	1
Salient Non-Procedure	Particular concepts support greater understanding.	5	9
	The course as a whole supports greater understanding.	9	9

Linear Algebra

Students identified a wide range of instructor and student-centered practices that contributed to their understanding of mathematical definitions, especially those involving the instructor's presentation of definitions, characterization of the utility of definitions, and formatting of notes.

Instructor presents clear and precise articulation of definitions. This procedure supported students' grasp of the *mathematical definitions are unambiguous* norm. This procedure highlights the instructor's direct presentation and elaboration of definitions, which included stating definitions and using them to prove theorems. Students emphasized that the definitions provided were clear. For example, one student noted, "Definitions that I was given were very clear and precise and left little to no room for misunderstanding/miscommunication." Another student observed, "The way definitions and theorems were written in our notes covered all bases, thus left no room for interpretation which is in an important part of any definition, they are supposed to provide clarity." This emphasis on clear and unambiguous presentation helped students understand definitions as foundational elements for logical reasoning, illuminating how "precise definitions act as the building blocks when proving a theorem." Students valued this precision because it "prevents misinterpretation" and allows for "clear results" and the ability to "distinguish similar but distinct ideas".

Instructor clarifies the function of definitions (e.g., utility for proving). This procedure especially supported the norms *definitions are used to support mathematical arguments* and *definitions are an important part of an axiomatic system*. Students cited instances where the instructor explicitly demonstrated how to use definitions in proofs. One student stated, "She showed us how to use definitions to prove theorems". Another student elaborated more, remarking, "I feel like I better understand their [definitions'] purpose and am able to utilize them in the context of proofs and problem solving." Students noted that definitions were purposeful tools that are used "to reach the end result of the theorem" and "in solidifying the validity of our proof and making it more accessible to people who might be unfamiliar with what we are trying to prove." The consistent explanation of how concepts build upon each other using definitions further deepened this appreciation for utility. This pedagogical choice enabled students to see definitions as tools for logical problem-solving, reinforcing the norm of clarity in mathematical communication and their practical utility.

Instructor formats the notes consistently. This procedure especially reinforced the norms *standard formatting choices in mathematics communicate information about definitions* and *mathematical communication has consistency*. One student noted, "Formatting choices, such as

uppercase letters for matrices communicate information about definitions, making it easier to identify and interpret mathematical objects." The consistent use of symbols and notation, such as "the different notations used, like the vector addition and scalar multiplication symbols, uppercase letters for matrices" aided understanding. Furthermore, students felt that the notes were structured; according to one of the students, "The types of theorems, many with corollaries and lemmas stayed consistent throughout the course and those ideas stay consistent throughout other math textbooks, too". This consistent presentation style, mirroring textbook conventions, illuminated that standardized formatting contributes to mathematical communication.

Students work individually on proofs for assigned work. This procedure directly fostered the norm that *definitions are used to support mathematical arguments.* Students consistently utilized definitions as the foundation for their proofs. As one student put it, "All the proofs I've had to do that have been based on definitions." Another student highlighted this foundational role, stating, "When I wrote proofs I used definitions all the time to support my arguments in each and every case." The consistent application of definitions in these assignments helped students solidify a relationship between definitions and proofs.

Particular concepts support greater understanding. While we did not code this as a procedure, we noted multiple student responses that focused on learning specific concepts related to the norms *names of concepts give insight into what the concept means* and *definitions allow the use of a single phrase to communicate complex ideas*. With respect to the former, one student wrote, "The names of specific concepts that we've learned in this course give insight into what the concept means by hinting at its algebraic or geometric interpretation." Numerous students noted different concepts for which shorthand was salient. One student noted, "Definitions like span, basis, and linear transformation allow the use of single phrases communicate complex ideas and relationships within vector spaces." Others noted examples such as the "invertible matrix theorem list" and the concepts of "one-to-one or onto" functions where a single term encapsulates extensive mathematical meaning. Another student said, "Relevant features of concepts like vector spaces or linear transformations, such as the need for operations to satisfy certain properties (e.g., closure, associativity), motivate the creation of definitions that capture these important characteristics for future reference."

The course as a whole supports greater understanding. Again, while not a procedure, students frequently expressed that the overall structure and approach of the course supported their understanding of definitions as part of a coherent mathematical system. This broad insight was reflected in responses to questions focused on norms of *people write definitions purposefully for ease of use* and *definitions are an important part of an axiomatic system*. This student recognized the utility for their work: "definition[s] are often used to categorize or distinguish concepts, and have been useful in conveying information in a more condensed manner." Another student noted**:** "definitions are critical to understand proofs." This interconnectedness was a key takeaway, as one student observed, "I have yet to see any definitions that contradict each other, only those that build on each other." The consistent use of definitions throughout the course facilitated communication, as students learned to "define terms so that we can talk about more complicated concepts easily." The course fostered the notion that mathematical definitions are "critical to understand proofs" and essential for consistent communication among mathematicians. This holistic view reflects the instructor's goal in conveying definitions not just as isolated facts, but as integral components of mathematical practice.

Abstract Algebra

Students identified a wide range of student and instructor-centered practices, especially those involving active engagement, conceptual exploration, and historical or contextual enrichment.

Students work individually on proofs for assigned work (tests/quizzes/homework). Students referenced proof-based assignments especially with regards to the *definitions are used to support mathematical arguments* norm. Students mentioned working individually on tasks such as tests, quizzes, and homework, where they were required to use definitions as the foundation for formal reasoning. These assignments provided repeated opportunities to see how definitions function as essential tools for constructing logical arguments and developing mathematical structure. As one student noted: "I started every proof by recalling the relevant definitions. This demonstrated that definitions are the foundation upon which all mathematical arguments are built."

Students explore before defining a concept (themselves). Closely tied to this was the theme of student exploration before creating a formal definition, where students engaged with mathematical ideas and patterns before seeing the official class definition. This practice was mentioned in response to several questions, including those focused on the norms *mathematical definitions are unambiguous*, *definitions are an important part of an axiomatic system*, *people write definitions purposefully for ease of use,* and *definition conventions vary between mathematicians*. This practice was valued for helping students build an understanding of a concept before encountering and crafting formal language to describe it. One student remarked: "A lot of the definitions we made in class were motivated by features of the defined object. Usually, these features were discovered first. But ultimately the definition is shaped around these features."

Students work collaboratively on activities in class. Collaborative classroom work was mentioned by students in relation to multiple norms, including *examples motivate the creation of definitions* and *relevant features of a concept motivate the creation of definitions*, highlighting how group activities helped deepen understanding of definitions through peer explanation and comparison. In these settings, students described learning to refine and revise their definitions as they worked through problems with others. According to one student, "Examples can help people understand what is going on and they make a broader explanation like definitions, like we did in activities."

Students learn about mathematicians or mathematics history. Historical and contextual information also played a role in shaping students' understanding of definitions. In connection to several norms, especially *definitions are based on historical motivations*, students referred to learning about the development of mathematical ideas or the mathematicians associated with them. As one student explained: "Historical context was used to explain how definitions evolved to address specific mathematical problems." Another noted, "Looking at the history of some concepts showed to me that there was motivation at one point to develop these definitions." This historical framing showed mathematicians engaged in creating definitions or attending to particular definitions in response to their needs at the time.

Instructor formats the notes consistently. In addition to student-driven practices, several instructor-centered strategies were mentioned by students. The most common of these was the instructor's consistent note formatting, which was highlighted in connection to most of the norms, especially the norms *standard formatting choices in mathematics communicate information about definitions* and *mathematical communication has consistency*. Students reported consistent notes helped them follow the structure of new concepts and clearly identify where definitions were introduced and how they were used. As one student described:

"Definitions were often presented as lists or bullet points, breaking down the properties into manageable components. This modular presentation made complex definitions easier to understand and apply." This suggests that formatting conventions to aid reading were perceived as useful for students.

Instructor presents motivating example(s) before definition. Students also emphasized the value of motivating examples provided before a formal definition, which helped generate interest and guide intuition prior to engaging with more formal language. Students related this procedure to norms of *examples illustrate definitions*, *examples motivate the creation of definitions*, and *abstraction motivates the creation of definitions*. One student noted: "After noticing a pattern in examples, we would try to generalize what we were seeing."

Discussion

Students highlighted a variety of course activities that supported their developing understanding of mathematical norms and values supported by work with definitions. While both instructor- and student-centered activities were highlighted by students in each course, instructor-centered activities were referenced more often by linear algebra students and student-centered activities were referenced more often by abstract algebra students. This result aligns with expectations from the course setup. While students in both courses engaged in inquiry-oriented (IO) activities and attended interactive lectures with organized notes, the IO activities in LA focused on creating intellectual need for learning new concepts and a concrete context to anchor students' developing understandings of span, linear independence, and change of basis (Andrews-Larson et al., 2017; Wawro et al., 2012; Zandieh et al., 2017). However, students were not asked to write new definitions themselves; these were introduced by the instructor in response to the activities. In contrast, students in AA were introduced to examples of groups through concrete activities but then scaffolded through writing a definition for group and group isomorphism themselves (Larsen, 2013). Thus, students in both LA and AA were exposed to ways their instructor formatted definitions and presented definitions in relation to examples, but only AA students were given opportunities to think about definition affordances and conventions as they wrote definitions themselves. This highlights that implementing IO materials is not a monolith with respect to expected student engagement in mathematical practices. Given that many LA courses are not proof-based and serve more non-math than math majors, we recognize that it is potentially beneficial for these materials to center different disciplinary practices.

Moreover, LA also serves as the Introduction to Proof course, meaning the instructor deconstructed how to write proofs and how they relate to definitions often, especially near the beginning of the semester, whereas this was less of a priority in AA, where students were expected to be familiar with this relationship between definitions and proofs already. This aligns with students in LA especially highlighting the instructor's role in presenting definitions and clarifying their function relative to AA. Nevertheless, students in both courses highlighted long-term activities. This included the utility of the homework, especially proof-based problems, in developing their understanding of definitions and viewing the course as a whole as supporting their understanding of definitions.

While some research at the university level has noted the impact of courses on students' beliefs about the nature of mathematics as a whole (e.g., Rupnow, 2023; Szydlik, 2013; Ward et al., 2010), here we instead consider the impact of course activities on students' perception of definitions and associated practices for creating and using them. In so doing, we gain initial insight into how teaching practices can impact students' perceptions of the values and norms held by the mathematics community.

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Analysis Paralysis: On the Potential Overemphasis of Calculus and Analysis in the Undergraduate and Graduate Mathematics Curricula

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In the United States, students in the K-12 system are often adversely affected by a so-called "race to calculus" that we believe to be caused, in part, by the emphasis on analysis found in the mathematics curriculum at both the undergraduate and graduate levels. This originates partially from a historical need for calculus in industry, a need which is no longer as prevalent as it once was. Here, we examine the curricula at 100 of the top graduate and undergraduate programs in the United States and the prevalence of a variety of topics in these programs, including calculus, analysis, linear algebra, abstract algebra, geometry, and statistics. As expected, analysis was much more prevalent than any other advanced topic. We discuss reasons for this prevalence and how such prevalence should be mitigated in the future.

Keywords: Curriculum Development, Calculus, Analysis

Introduction

It is well known that the structure of the mathematics curriculum at the K-12 level is currently based on a "race to calculus," that is, most of the curriculum is heavily influenced by a need to prepare students to take calculus in college (Bressoud, 2021). There are strong historical reasons for this focus, not the least of which include a need for graduates in physics, engineering, and mathematics during the Cold War (Tucker, 2013). However, today's industrial landscape has shifted and now the main mathematical needs in industry are no longer calculus-based, but rather based on the statistics and linear algebra needed for data analysis (Gravemeijer et al., 2017). As a result, there seems to be a significant societal desire for change in the mathematics curriculum and there are many suggestions about what that curriculum might look like (see, for example, Gravemeijer et al., 2017; Stanford Graduate School of Education, 2021). Yet, change in this regard has been relatively slow.

We have sought to better understand the influences that prevent meaningful changes in the mathematics curriculum from happening. One observation we made was the similarity in the undergraduate mathematics curricula we experienced at the various universities we have attended or taught at. Indeed, there were remarkable parallels in the initial courses that undergraduate mathematics majors were required to take at these institutions. These courses also significantly overlapped with the initial mathematics courses that STEM majors were required to take. These courses seem to typically include a single-variable and multi-variable calculus sequence, a course in linear algebra, and a course in ordinary differential equations. This is not too surprising as the calculus sequence for STEM majors is well understood to be the impetus for the race to calculus at the K-12 level (Bressoud, 2021). While it is clear why physics and engineering majors start with calculus, we sought to understand why mathematics programs also begin there when mathematics includes topics like modern algebra, geometry, topology, and discrete mathematics, each of which provide equally valid starting points.

Further similarities in the curricula with which we were familiar provided a possible explanation. We noted that these curricula seemed to continue to emphasize calculus and analysis throughout, typically requiring a considerably larger number of calculus- or analysis-based courses than courses in other topics in mathematics. This seemed to continue even into graduate school where the only course that we had personally observed almost always being required was a graduate level course in real analysis that typically covered measure theory and the Lebesgue integral. This suggested that the race to calculus might continue past K-12 education into the undergraduate and even graduate levels.

To verify whether our personal observations are typical in United States undergraduate and graduate mathematics curricula, we acquired and analyzed data on various mathematics programs throughout the nation. We also sought to better understand the motivations for such similarities if they were found.

As shown in this paper, we found that this emphasis on calculus and analysis in the undergraduate and graduate mathematics curricula is common throughout the schools that we considered. In the remainder of this paper, we explore the historical motivations for why this emphasis on calculus and analysis exists, provide an analysis of the data we collected which shows this emphasis is currently present throughout the United States, and discuss what this indicates about how we can address the race to calculus and modernize the mathematics curriculum at all levels.

Literature Review

Calculus has not always enjoyed its current privileged position within the college curriculum (Tucker, 2013). The teaching of mathematics at the post-secondary level in the United States has evolved greatly throughout the history of the United States. In the early days of the country, American colleges focused mostly on Euclidean geometry and algebra. Mathematics was "studied as a classical training of the mind rather than as the language of science and engineering" (Tucker, 2013, p. 1). Mathematics almost disappeared from college curricula in the later 1800s and early 1900s as colleges moved away from focusing on a classical education to a focus on practical education. Mathematics avoided falling into the same obscurity as other classical subjects, such as Greek and Latin, through the needs of engineers and similar technical professions that required mathematical training in their fields (Tucker, 2013).

In response to this declining emphasis on mathematics, in 1915 the Mathematical Association of America (MAA) began making suggestions on what should be included in undergraduate mathematics curriculum, with calculus as the introductory focus (Tucker, 2013). Additionally, colleges began introducing academic majors and general education requirements, further bolstering the place of mathematics within the college curriculum. During this period, mathematics was a career-oriented major focused on supporting engineers, actuaries, and teachers. However, mathematics gained its modern prominence in the wake of World War II as the national demand for engineers and mathematicians exploded (Duren, 1967; Tucker, 2013).

In 1957, the Soviet Union launched Sputnik, the first manmade satellite into space. This event is widely regarded as the ignition of the space race between the Soviet Union and the United States during the middle and latter parts of the twentieth century (Tucker, 2013). The space race led to an increased demand and respect for collegiately trained mathematicians and other science professionals from math-heavy fields. At the heart of all these suddenly respected fields was calculus, a mathematical subject seen "as a foundation for the modern world and one of science's greatest intellectual achievement[s]" (Tucker, 2013, p. 10).

The newfound demand for mathematically trained college students led to the formation of the Committee on the Undergraduate Program in Mathematics (CUPM), a group tasked by the MAA to set forth standards for what mathematics should be studied by college students seeking degrees in mathematically-based fields of study (Duren, 1967). In 1963, the CUPM published an influential report on the suggested curriculum for undergraduate students who one day wished to become research mathematicians (CUPM, 1963; Duren, 1967; Tucker, 2013). This report pushed for a heavier emphasis of calculus and analysis in the undergraduate curriculum as preparation for learning analysis at the graduate level, (CUPM, 1963). A 1965 report by the CUPM further emphasized the importance of analysis at the graduate level:

> The Panel has consulted many outstanding graduate mathematics departments; a solid grounding in algebra and analysis is what they most want from incoming students. If time and resources permit, it is, of course, desirable to introduce the pregraduate student to a broader range of material, but not at the expense of a depth in algebra and analysis. (CUPM, 1965, p. 450)

Even more influential was the 1972 CUPM report, which did reduce the number of recommended mathematics courses, but led directly to the traditional requirements of calculus 1 and 2, linear algebra, and multivariable calculus seen in many colleges today (Tucker, 2013). This report also acknowledged the preeminent place of analysis in their recommended program and questioned whether such preeminence was appropriate:

> A closely related question is whether the "core" of pure mathematics that all departments should offer is now the same as it was presumed to be a few years ago. As new fields develop, some older fields seem less relevant, and today some mathematicians even question the assumption that calculus is the basic component of all college mathematics. (CUPM, 1972, p. 36).

Despite this questioning of calculus as the basic component of all college mathematics, calculus is still often used as a "gatekeeper" or "filter" for STEM majors, including mathematics (Bressoud, 2021; Bressoud et al. 2015; Tucker, 2013). Many studies have explored how to improve calculus instruction, most notably the 2015 MAA national study of college calculus (Bressoud et al., 2015). These studies have often focused on instructional methods and improving student understanding, with little questioning of calculus as the basis for all college mathematics (see, for example, Berry & Nyman, 2003; Ellis, 2013; Epstein, 2013; Gundlach & Jones, 2015; Jones, 2013, 2016; Jones & Watson, 2018; Mesa et al., 2015; Musgrave & Carlson, 2017; Tallman et al., 2021). However, based on the previously quoted statement from the 1972 CUPM report, mathematics majors should be moving away from calculus or at least be emphasizing analysis and advanced calculus less than in previous years.

In our review of the literature, we also found no examination of the classes mathematics majors take beyond calculus across multiple institutions. In the 1972 CUPM report, undergraduate mathematics majors are advised to include at least three semesters of analysis, while upper-level modern algebra and geometry are only recommended for one semester. Wu (1999) did discuss creating multiple tracks for mathematics majors, with one track specifically for those aimed at graduate school. Still, there was no discussion of what topics would be most appropriate for mathematics majors bound for graduate school, though he did note that many mathematics major curricula are designed specifically with these students in mind.

Research Question

If the place of calculus as the gateway to college mathematics is accepted, yet many fields of modern mathematics do not have calculus as a core component (CUPM, 1972), it stands to reason that analysis should now be less emphasized, or at least equally emphasized with other mathematical topics. While there are many mathematical topics covered even in undergraduate mathematics courses, three main categories tend to be given priority: analysis, algebra, and geometry/topology (Harvard University Department of Mathematics, 2025; Wu, 1999). This led us to the following research question: Of the three major topics of analysis, algebra, and geometry, is there a difference in the number of courses the top 100 graduate and undergraduate math programs require from these three topics?

Significance of This Study

We hope that this study serves as the catalyst for future discussions and research into the most appropriate collegiate mathematics classes, both for mathematics majors and non-mathematics majors. This study seeks to examine two of the major influences on the mathematics curricula offered at colleges and universities: the mathematics major for students seeking graduate degrees and mathematics graduate degrees.

Methods

We compiled a list of the top 100 graduate programs in math, as ranked by US News and World Reports (US News and World Report, 2022, 2025). An initial analysis of required courses for graduate programs was completed using the 2022 list. After this data was revisited in 2025, we then completed a further analysis of undergraduate programs using the updated list. We did this with the intent of utilizing the corresponding undergraduate programs at these institutions, with the intent of seeing what a student preparing for graduate studies in mathematics is required to take. As such, we removed graduate programs at institutions with no corresponding undergraduate program. In place of such institutions, we included the subsequent institutions to complete our list of 100 universities. In this case, the only such institution was the CUNY Graduate Center, which we replaced with the institution ranked 101st at the time of our survey.

For the purposes of our survey, we grouped courses as being in one of the following categories, which we will define hereafter: Calculus, Analysis, Geometry, Linear Algebra, Abstract Algebra, and Statistics. Graduate school courses were only categorized by Abstract Algebra, Analysis, and Geometry. As the goal was to classify courses which are strictly required for graduation, we only counted a course towards a category if a course covering that category was required to satisfy a graduation requirement. We also did not count course prerequisites for courses taken before a Calculus 1 course, as such courses are commonly tested out of by students in a mathematics major.

For instance, if a student had to choose one course from a set including two analysis courses and an abstract algebra course, this would count towards no category, as the student does not need to take one course which fits strictly one category. On the other hand, if a student must take at least one course chosen from among three analysis courses, this would contribute one course to the required analysis courses for that program. In the case that a course covered multiple subjects, we split the course among the subjects based on the course description and course text. This is done while retaining the total value of one between the split, meaning a course covering geometry and abstract algebra would count as 0.5 abstract algebra, and 0.5 geometry, not one abstract algebra and one geometry.

As different institutions offer courses on different schedules, with the most common options being semesters and quarters, we recorded the number of blocks each institution offers in a standard academic year, excluding the summer block(s). This allows us to normalize all data collected into semester equivalents, so that schools on quarter systems are not represented with disproportionately high course requirements. For each undergraduate institution, if multiple emphases were offered with the major, the one most aligned to graduate study was selected.

The graduate data was reviewed by one researcher as such courses were more clearly aligned to a particular subject area. The undergraduate data was reviewed by five researchers, with each researcher reviewing 30 classes, with 20 being shared by other researchers. Situations in which there were inconsistent course counts were flagged and reconciled jointly. Reviewers counted courses as per the guidelines below, which we utilized as descriptions for each type of course.

We classified a course as a calculus course if it covered traditional calculus 1-3 material or differential equations, but with students not being graded or tested on proofs or proof-based material. This includes courses on differential calculus, integral calculus, multivariate calculus, ordinary differential equations, and partial differential equations.

We classified a course as an analysis course if it covered real analysis, complex analysis, or measure theory. Additionally, we classified courses which cover traditional calculus 1-3 material or differential equations with students being graded or tested on proofs and proof-based material as analysis courses. Note this is mutually exclusive with a being considered a calculus course.

We classified a course as a geometry course if it broadly covered the subjects of plane geometry, differential geometry, algebraic geometry, algebraic topology, or general topology. We specifically did not classify analysis courses which covered the topology of the real line or the complex plane as topology courses. This exclusion was so that nearly every introductory undergraduate real analysis course did not end up counted partially as a geometry course.

We classified a course as a linear algebra course if it covered the theory of linear operators, either directly by definition, or by the theory of matrices. We made no distinction between computational linear algebra courses, proof-based courses, or courses which are a combination thereof. We did not count algebra courses which covered matrices as an example space as linear algebra courses, so as not to conflate linear algebra courses with algebra courses, similar to the exemption made with topology courses.

We classified a course as an abstract algebra course if it covered the theory of groups, rings, fields, Galois theory, category theory, or any similar algebraic structures through a proof-based approach. As such, we do not count college algebra, intermediate algebra, or other non-proof-based courses towards this subject.

Finally, we classified a course as covering statistics if it covered statistical thinking, statistical methods, probability theory, or other statistics related subjects. This includes mathematical statistics, and measure-theory based statistics. However, measure-theory based courses were cross-listed with Analysis depending on the particular course description and text.

Results

The initial survey of graduate programs found that exactly half of the institutions examined had strict course requirements for their pure mathematics programs, with the other half only requiring advisor approval or satisfactory scores on qualifying exams. These institutions were excluded from analysis, leaving 50 graduate institutions. In our survey studying the undergraduate programs, an overwhelming majority (84) of the institutions examined worked on a semester system. All schools that worked on a quarter system had their number of classes for

each subject area multiplied by 2/3 to find the equivalent number of semester classes needed for each subject area.

Descriptive statistics for both data sets after norming the undergraduate data are found in Table 1.

Table 1: Descriptive Statistics for Normed Number of Classes by Subject Area

	Undergraduate Schools				Graduate Schools			
Subject Area	Min	Max	Mean	SD	Min	Max	Mean	SD
Calculus	2	4	3.46	0.561	-	-	-	-
Linear Algebra	0	3	1.18	0.496	-	-	-	-
Statistics	0	3	0.33	0.561	-	-	-	-
Algebra	0	2	1.07	0.575	0	3	1.20	1.030
Geometry	0	2	0.22	0.450	0	3	1.02	1.097
Analysis	0	4	1.73	0.900	1	4	2.22	0.840

After data was collected, SPSS was used to perform a one-way repeated measures analysis of variance (ANOVA), as all variables were repeatedly measured on the same subjects (Field, 2018). As the upper-level classes were deemed the most important ones to compare, the ANOVA was only performed on those variables. Data on the other variables is included to give additional context to these more important data points.

As part of this ANOVA, we tested the following hypotheses:

H_0: There is no difference in the number of algebra, geometry, and analysis classes required by the top 100 graduate and undergraduate math programs in the United States.

H_a: There is a statistically significant difference in the number of algebra, geometry, and analysis classes required by the top 100 undergraduate math programs in the United States.

After the ANOVA was run on the graduate data, the Huynh-Feldt estimate of departure from sphericity was $\epsilon = 0.844$. Using this adjustment, the difference between the number of classes offered in different subject areas was found to be statistically significant, $F(1.687, 24.819) = 53.833, p < 0.001, \omega^2 = 1.413$. This is convincing evidence that there is a difference between the number of required classes in algebra, geometry, and analysis for mathematics majors in the 50 of the top 100 graduate math schools that had stated class requirements.

All graduate class categories were then compared using a post-hoc pairwise comparison with a Bonferroni correction. When the two categories that generally had the highest number of required classes, analysis and algebra, were compared, more analysis classes were required ($M = 2.220, SE = 0.119$) than algebra classes ($M = 1.200, SE = 0.146$). This difference, 1.020, 95% CI $[0.699, 1.341]$, was significant $p < 0.001$ with an effect size of $d = 1.118$.

After the ANOVA was run on the undergraduate data, the Huynh-Feldt estimate of departure from sphericity was $\epsilon = 0.874$. Using this adjustment, the difference between the number of classes offered in different subject areas was found to be statistically significant, $F(1.747, 172.978) = 178.087, p < 0.001, \omega^2 = 0.476$. This is convincing evidence that there is a difference between the number of required classes in algebra, geometry, and analysis for mathematics majors in the top 100 undergraduate math schools.

All categories were then compared using a post-hoc pairwise comparison with a Bonferroni correction. When the two categories that generally had the highest number of required classes, analysis and algebra, were compared, more analysis classes were required ($M = 1.7283, SE =$

0.09004) than algebra classes ($M = 1.0700, SE = 0.05750$). This difference, 0.658, 95% CI $[0.461, 0.856]$, was significant $p < 0.001$ with an effect size of $d = 0.812$.

Discussion

From the above analysis, there is a heavy emphasis on calculus and analysis in both the undergraduate and graduate mathematics curricula at most institutions across the United States. Just as the introductory calculus sequence causes a race to calculus at the K-12 level, we believe that the typical requirement of graduate level real analysis is the cause of the emphasis on calculus and analysis in the undergraduate mathematics major programs. Both the introductory calculus sequence and the graduate real analysis course indicate pressure from above against any change in the curriculum. Indeed, one major reason why mathematics programs may find it difficult to move away from requiring the initial calculus sequence is the need to prepare their mathematics majors for that graduate level real analysis course required by the majority of graduate institutions. This suggests that for any meaningful change to happen at the K-12 level, the undergraduate mathematics curriculum first needs to change. But for the undergraduate mathematics curriculum to change, first the requirement of graduate real analysis needs to no longer be the norm.

We hypothesize that the graduate real analysis course plays more of a role in verifying that students are prepared for graduate work than in serving as a prerequisite for graduate mathematics courses or research, as most such courses and research topics simply do not use the content of that real analysis course. This preparation verification can be done with a graduate level course in practically any mathematical topic. Thus, the emphasis on analysis at the graduate level seems more due to convenience than necessity.

If the assumption of basing the curriculum on calculus was questioned in the 1970s because of changing times (CUPM, 1972), how much more should we question that assumption in the 2020s? This raises the question of what should be done with this data. One suggestion is that we replace calculus with data science, including statistics and linear algebra, because data science is more ubiquitous in today's industry (Stanford Graduate School of Education, 2021). However, if we shift to a curriculum that supports data science instead of one that supports calculus, we run into two potential problems. The first is also noted by the CUPM 1972 report, that the landscape of what mathematical topics are needed by society is constantly changing. Data science is the flavor of the day now, just like calculus was 50 years ago, but in another 50 years, it will likely be a new topic in mathematics. However we change the curriculum should be flexible enough to adapt to future changing needs. If the race to calculus should teach us anything, it is that embedding preparation for a specific topic into the curriculum creates a curriculum highly resistant to change. The second issue is that one of the main reasons why a race to calculus is so problematic is not just because calculus is becoming less important, but also because racing to a particular topic encourages a curriculum that focuses on skill preparation and proficiency in algorithms that support that topic rather than a more robust culture of mathematical thinking and exploration (Tallman et al., 2021; Tucker, 2013). Racing to any topic, regardless of how relevant, will continue to have that effect.

In future research, we look to explore ideas of how to develop a graduate, undergraduate, and K-12 curriculum that would be independent of the favored topic of the day, while still providing robust preparation in mathematical thinking that could quickly be adapted to support any mathematical topic at the college level.

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Tugging on Threads: A Loving Self-Critique of our (White) Comforts

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The ACT UP Math project was a multi-institutional research practice partnership (RPP) formed to engage students, faculty, and administrators in collective equity-oriented work for critical change in undergraduate mathematics programs. Through careful critique of our own decisions and results, we identify opportunities with which we could have embraced greater criticality, especially in relation to disrupting our own white comfort. In doing so, we aim to provide guidance on better practices to enact when engaging in critical work. We orient ourselves to this critique with (critical) love, engaging with love for ourselves with the goal of growth as researchers, love for the research we do with the goal of growth for our field, and love for the research participants with whom we partnered with the goal of learning from one another.

Keywords: critical research methods, reflexivity, self-critique, narrative inquiry, research practice partnerships

A growing number of research studies investigate the inadvertent impact of introductory mathematics programs that were *initially* designed to support a predominantly white, masculine, and higher socioeconomic population (i.e. the population of people who initially had access to higher education, especially technically-oriented higher education). As more women, more people of color, more low-income students, and more first-generation students seek higher education, STEM degrees in particular, the mathematics education community increasingly grapples with how introductory mathematics programs are not structurally designed to support this increasingly diverse population of students (Ong et al., 2018). Improvements to existing programs have included tutoring programs, bridge programs, incorporation of active learning, and the creation of co-requisite courses, among other efforts (see Voigt et al. (2023) for many examples).

Some change efforts work to improve the experience and achievement of a more diverse group of students within the current structures of introductory mathematics programs. Other change efforts work to rethink the structure of programs altogether; these efforts often question aspects of programs historically taken as given and work to develop creative alternate ways to provide and conceptualize mathematics education for diverse populations of students. The latter efforts take a *critical* approach to improving equity - they work to change the system - whereas improvement efforts that work within the system take a *dominant* approach (Gutiérrez, 2009). Equity scholars have long argued the value of both approaches, to improve the existing system and simultaneously work to change the system as a whole (Martin et al., 2024). This report discusses the ACT UP Math grant, which engaged in research practice partnerships (RPPs) with three mathematics departments. Each department created a Networked Improvement Community (NIC) of faculty members, students, and/or administrators with the intention of using local data to make transformative changes to their introductory mathematics programs. As a research team,

we worked to support them in making *critical* changes, with the recognition that critical changes are much harder to conceptualize when one works within and has succeeded within the existing systems.

With each department partner, we supported a team of students, faculty, and administrators to develop critical perspectives and apply those perspectives to their introductory mathematics programs. We encouraged the use of data to inform critical perspectives, which included both quantitative and qualitative data on student experiences in the programs. As we supported the three mathematics departments, we also strove to engage in critical scholarship- by which we mean the interrogation of norms and hierarchies in dominant research. For instance, dominant research traditions attempt to adopt "objective" perspectives of observers and value the expertise of faculty over students; a critical approach embraces that all human experiences are subjective, and recognizes the value in hearing from diverse voices and students' lived experiences.

Our research team consists of mathematics education researchers with varied experiences in critical scholarship, and many previous experiences with dominantly-oriented research teams. To support our research team in enacting criticality, we asked (and paid) a *critical friend* with expertise in critical scholarship to attend meetings, view recorded meetings, read meetings notes, and respond to specific reflective prompts related to how we were and were not engaging with our work critically. One transformative part of the feedback that our critical friend gave was that we had centered *white comfort* within our researcher meetings and whole team meetings (which included the research team and the leaders of the department change teams). This specific feedback led to deep reflection from the team related to the ways in which whiteness and white supremacy culture impacted our orientation to our work, and inspires this report of a self-critique of how we engaged in critical scholarship. We orient this self-critique through, taking inspiration from Basile and Azevedo (2022), who used this approach to recognize where they could have done more in working to make transformative progress in STEM education. We also draw upon Foste (2020) who identifies "three reflexive considerations for white scholars beginning to take up the critical study of whiteness: creating contexts for racial comfort, validating racist beliefs, and missing educational opportunities to disrupt whiteness" (p. 132). Therefore, we guide our loving self-critique with the following research question: *How did our research team enact (white) comfort, and what opportunities did we (not) take to disrupt the tensions and contradictions therein?*

Literature Review

Our work follows a growing body of research that is grounded in love throughout theory and methodology. Paris and Alim (2014) critiqued the then-current operationalizations of asset pedagogies with an eye toward moving away from a dominant white gaze. They deliberately centered their critique in love: "we use loving critique to denote the position of deep respect from which we problematize and extend… scholarship and practice" (Paris & Alim, 2014, pp. 1-2). Basile and Azevedo later took up the idea of loving critique and oriented it to the self, in critique of their own "failures" as STEM educators: "Our loving (self) critique entails a willingness and courage to deeply interrogate our own complicity in the perpetuation of some of those same inequities we purport to stand firmly against" (Basile & Azevedo, 2022, p. 1087). We follow this tradition of loving (self) criticality.

Indeed, Clingan (2015) invites us to "consider what might be possible if love were intentionally and specifically identified as a methodology, pedagogical practice, and value in… education" (p. 1). However, as Johnson et al. (2018) remind us, we must use "love" with mindfulness and care. We do not refer to the romantic and often hierarchical love common in

Western culture, but rather we refer to "the type of deep-seated love that is cloaked in pain and that is bounded in action which disrupts… white supremacist patriarchy through the practice of humanizing love" (Johnson et al., 2018, p. 48). The latter kind of love demands self-transformation as we collectively labor for improved education systems. We offer in the next section a perspective for how queer theory supports us in this transformation through self-critical love; also see a theoretical development of *critical love* by Author 1 (under review).

Theoretical and Analytic Framework

After joining for two research team meetings, our "critical friend" – in an act of love for us and the project – shared critiques, the most provoking of which was that our meetings "gave off white comfort." To facilitate our deconstruction of our white comforts, we chose a theoretical lens of critical whiteness studies to "recognize, analyze, and critique the power and privileges associated with Whiteness" (Haviland, 2008, p. 41). Within this lens, we center power to challenge the power-evasive tendencies of whiteness (Haviland, 2008) and to attempt to avoid the confessional, rather than transformational, critique stemming from an overemphasis of privilege (Lensmire et al., 2013).

Within our centering of power, we maintain awareness of the interwoven nature of power and privilege (and the stronger form in supremacy). Therefore, we introduce a weaving metaphor whose structural backbone consists of Okun's elements of white supremacy culture (Okun, 2021). From Okun's work, we highlight white supremacy's *right to comfort*, *only one right way thinking*, *paternalism*, and *fear of open conflict*, as these components of white supremacy resonated with the team during reflections (for a more comprehensive discussion of white supremacy culture, we encourage the reader to visit Okun's website). We appeal to weaving as a metaphor, and consider as vertical threads the structures of dominant cultural traditions (including white supremacy culture), and as horizontal threads the ways we enact institutional and interpersonal norms, policies, and practices as a research team. We therefore consider how we inadvertently wove our project's "horizontal" interpersonal norms of care around the "vertical threads" of white supremacy culture, creating a space of white comfort. We now engage in the ongoing work of "tugging" on the horizontal threads to unweave them, seeking to reweave them elsewhere in critically transformative ways.

Foste (2020) encourages teams of predominantly white researchers, such as ours, to attentively engage in continual reflection and recognize that this work is ongoing. In this work, we both identify comforts and work to deconstruct them, and identify ways in which we are changing behavior and ways in which we still need to change behavior. We root our work additionally in queer theory which holds together the tension of product (this contributed report) and process (our continual engagement in reflection, even after this manuscript is "complete").

Our choice of queer theory also returns us to our namesake ACT UP, a queer grassroots coalition of activists. This coalition "exemplified the pragmatic ability of queer activists of the 1980s and 1990s to join 'deconstructive reading practices and grassroots activism together, laying the groundwork for… queer theory'" (Freeman, 2010; cf. Brim & Ghaziani, 2016, p. 15). We also work to deconstruct – or "*unweave"* – our "white comforts," which we view as inherently indiscrete. Likewise, "Whiteness is not monolithic" (Haviland, 2008, p. 41) and "white racial identities are multifarious messes of thought and feeling… we must remain open and attentive to the pedagogical possibilities of complexity and conflict" (Lensmire, 2010, p. 170). In recognition of this messiness, and the potential for transformation when embracing complexity and conflict, we see queer theory as a natural foundation for our critique--for its ability to deconstruct binaries, critique hierarchy, and engage with uncertainty (see Chapters 2

and 3 of Hall (2003) for what these entail). As such, queer theory offers a supportive frame for our (self-)critical love through the tugging and unweaving of our enacted white comforts.

Data and Method

Our loving self-critique is rooted in our critical friend's feedback and entries from ACT UP Math RPP members' journals reflecting on feedback and tensions felt in the RPP. These journals prompted members to think about their RPP experiences, areas of tension, and progress toward critical transformations. Amorphously, we also bring personal reflections of our time in the RPP -- in part to contextualize written journal entries and in part to dive into how tensions and comforts were addressed.

Our first stage of analysis consisted of identifying the comforts within the reflective journals, asking ourselves "if this space gave off white comfort, what is it that we found comfort in?" We looked broadly for areas of comfort in the data, whether tied to white comfort or not. We employed a reflexive thematic analysis (Braun et al., 2023). Initially, a PI and author of this paper created a preliminary grouping of entries which touch on similar ideas (e.g., white supremacy culture, identity, metaphors for tension, etc.). Then, we used these initial groupings to begin giving names to comforts identified. Next, we individually consolidated comforts under broader categories to allow for each member's unique interpretation of these comforts and how they relate. After comparison and discussion, we collectively reconciled the higher level categories that form the comforts we identified which formed the basis of our critique. The analysis did result in all of the comforts being rooted in white supremacy culture; we used (white) – with parentheses – in our research question to indicate that we were looking for all types of comforts that may have limited our criticality, including those not necessarily attributable to white supremacy culture.

Our second stage of analysis focused on how these comforts were (and are still) both influencing our work and being actively interrogated through our work as the ACT UP Math project team. We found it appropriate to use narrative inquiry (Clandinin & Connelly, 2000) for its ability to *queer* the context of these comforts – that is, queer theory and hence this critique are "perpetually becoming" (Browne & Nash, 2010, p. 1). Moreover, telling stories, especially about mathematics, allow people to unpack held beliefs (Gutiérrez et al., 2023) and heal traumas (Scott & Zuniga-Ruiz, 2024). We utilize the three-dimensional inquiry space (Clandinin & Connelly, 2000) to structure our restory process of comforts as follows: continuity (e.g., time before/after critical friend's feedback), situation/place (e.g., research meetings), and interactions (e.g., members of the RPP). Through these dimensions, we built a narrative that contextualized the things we found comfort in, and thereby giving form to our comforts which could be unraveled. By unraveling each thread, we begin to unravel *our* tapestry of (white) comfort. We share a piece of this narrative below.

Results and Discussion

Structuring our loving self-critique through narrative inquiry required investigation into the overall narrative of the ACT UP Math project. In the prologue to the ACT UP Math narrative, the PIs were involved in multiple past large grant-funded projects, all of which we experienced as rooted in patriarchal models Thus, as the ACT UP Math narrative began, the PI team came together to create an equity-focused project both in process and in focus. We endeavoured to create a non-patriarchal space with a distributed leadership structure that honored the knowledge and experiences of everyone rather than assigning status based on age, gender, or role. We also intentionally scheduled community building activities to begin every meeting, not just the first

few meetings. We set up project norms (later recast as community agreements) that were dynamic, and evolved to align with our growth as individuals and as a community, and as our community membership changed over time. These structures and norms, bundled together with the broader institutional ways of being we have been enculturated in, were woven horizontally, across the vertical threads of white supremacy culture as enacted more broadly in academia. For us, maintaining relational threads in every meeting helped to strengthen our overall weaving, and illuminate the community we were weaving together.

Our narrative then comes to our "critical friend" who, after joining two research team meetings, shared the critique that our meetings "gave off white comfort." Following this critique, we began and still continue the work of unravelling comforts rooted in white supremacy and those rooted in other systems of oppression. We first identified four threads through which comfort manifested in, and sometimes limited, the critical nature of our team's work: *Comfort in (Dominant) Tradition, Comfort in Easing Discomfort, Comfort in Hierarchy,* and *Comfort in Whiteness.* These threads were by no means distinct, but allow us to consider some of the ways in which white comfort worked to limit more critical progress.

In this paper, we share only a piece of the narrative so that the reader has the opportunity to consider one of our unravellings more deeply, rather than more surface-level unravellings of all four threads. We focus on unravelling the *Comfort in (Dominant) Tradition* thread. This thread consisted of ways in which we chose to (or defaulted to) not question research practices and research methodologies that are rooted in white, Western notions of academia. The vast majority of our research team was formally trained in traditional education research methodologies, and carry with us knowledge and comfort in those environments and methods. Both in the structure of our interactions with participants and in the methods we chose, we frequently fell back on what we knew from previous research experiences, rarely challenging taken-for-granted norms in how research is conducted.

Through this loving self-critique, we now recognize missed opportunities for greater power-sharing within our research-practice partnership that we did not take up. In conjunction with the *Comfort in Hierarchy* thread, we did little to dismantle the pedestal upon which research (and therefore, researchers) are often placed in professional development contexts. While we did explicitly encourage NIC members at our participating institutions to see themselves as experts in their own experiences, and to leverage their institutional expertise to inform critical changes, we missed opportunities to invite them to collaborate in research spaces throughout the project. Often, they were left in the dark with regards to *why* we were collecting the data we collected, and they were not asked to influence the types of data we collected.

In a particularly poignant example that stands out to our research team, during a final project interview, one participant explicitly shared some frustration with the fact that we repeatedly asked participants to illuminate "what data felt impactful" during the course of the project. When the researchers conducting the interview shared that we were curious about how participants' perspectives around data changed throughout the course of the project, the interviewee expressed curiosity around why we didn't share that with the local NIC leaders and participants, as it would have mitigated some of the confusion and frustration participants felt around being repeatedly asked the same question. Notably, the participant only felt comfortable sharing this frustration with us after three years of intentional relationship development; as is the case in traditional research and participant interaction, our research team did not make space for critique of our project from the participants, which undoubtedly would have served to enhance the project itself. This is but one reflection of the missed opportunities we had to share the purpose of our research

with the participating NIC members (or, more critically, allow our practitioner partners to shape the directions of our research).

As a result, the research side of our work, especially in the early years of the project, existed in a "black box" which was not visible to our participants, which diminished the *partnership* value of the RPP. As we unravelled this particular thread of comfort during the later months of the project (after our critical friend's critique), some intentional ways in which we rewove this thread included greater intentionality in extending invitations to NIC leaders and members to be a part of research propagation efforts. These invitations were made asynchronously via email and during meetings with NIC leaders at the three institutions, and contained summaries of *all* projects presently being worked on alongside opportunities to indicate interest in other research directions created by them. As more NIC leaders and participants have opted into engaging in the research side of our work, new challenges came to light; in particular, we needed to learn how to share data with participants while protecting the anonymity of other participants in the project, which we have handled on a case-by-case basis. We have found the challenge minimal compared to the immense value of having NIC leaders and participants in conversations that shape (and constitute) our research propagation efforts. While we are still working on many of these efforts, we have seen for ourselves the value that having NIC members and leaders actively involved in shaping propagation efforts has. Indeed, the ACT UP Math grant originally chose the term propagation—rather than dissemination—for its suggestion of growth rather than spreading. The involvement of NIC members has substantially contributed, and continues to contribute, to the growth of our research.

Even so, our propagation efforts continue to reflect traditional education research methods. The PIs of this project intentionally leveraged a grassroot activist organization's acronym with the intention of responding powerfully to the injustices that can be perpetuated by contemporary U.S. mathematics education policies and practices. Our desire was to partake in activism through a research lens—to be activist-scholars as we conducted and propagated this work on how to make change. However, we fell short of this desire, and rarely in the moment considered how the comfort we found in traditional academic and mathematics education research methods continued to feed into some of the systems we had so intentionally set out to disrupt.

For example, the vast majority of our propagation efforts have been filtered through traditional propagation channels: established journals and conference proceedings, which themselves are inaccessible to a broad audience and subject to forms of gatekeeping that maintain traditional academic practices. By choosing to submit our work (and in many cases, be accepted) to such avenues, we have already "played the game" – or at least, a component of the game – of academic prestige associated with publishing in traditional, dominant spaces: a game which a more radical stance might reject entirely. For example, a stance of refusal (Tuck & Yang, 2014) more radically questions and disrupts research as the highest form of knowing, noting that even more inclusive definitions of what research *is* continues to uphold *research* as the eclipsing way to make meaning of human experiences. Even when we use more critical or arts-based methodologies (such as poetic transcription or visual analysis), we continue to call them *research* and seek to publish them in normative mathematics education research spaces. This is done with genuine care and awareness for our colleagues (particularly those in their early career, where citations matter a great deal in terms of career trajectory) and for the research community, who we hope will learn from and build on our work. We continue to submit and publish work in traditional academic avenues of propagation (such as RUME). This care and continuation of practice exists alongside our more recent unravelling of how *Comfort in*

Tradition kept us feeling safe in these propagation practices.

Recently, we have explored other non-academic avenues through which we might share our findings, curiosities, and wonderings, as potential re-weavings of this thread. Drawing from creative spaces of subversion, we distributed a Zine at MAA MathFest 2025 focused on how we came to understand dimensions of creating authentic student partnerships in change-making groups (Bolick, 2025). The Zine featured artwork and poetry from a collaborative team of researchers and NIC participants, many of whom were students. When our project was funded, we also maintained a public blog *Get Lit: Mathematics Education* that discussed literature of interest to our team. Both of these avenues of propagation were not subject to any form of review and highlighted more critical ideas and presentation forms than may have been published in more mainstream journal or conference spaces, and thus allowed us to reflect more of the activist stances we had set out to take with this project, and distribute those thoughts more widely. Ultimately, these remain but two pieces of work, with the vast majority of our work published (even now) in mainstream academic spaces. This does not mean that such traditionally published work does not or cannot have value, but necessitates an acknowledgement that, in filtering through traditional channels, we are not necessarily disrupting research norms in a way aligned with our initial activist intentions. Our unravelling of the *Comfort in (Dominant) Tradition* thread leads us to critique the ways in which our work maintained traditional hierarchies between researchers and practitioners of mathematics education, alongside this recognition that we are continuing to perpetuate traditionally reviewed and published *research* as the highest form of knowing. Within a stance of care for colleagues and awareness of the need to "play the game", we continue to seek ways to reweave this thread.

Implications

We share our loving self-critique with the RUME community for multiple reasons, which bring with them implications for future practice. For many of us, the RUME community is our research home, so the practices we fell back on are exactly those practices shared, cultivated, and valued by this community. Thus, as we lovingly critique our own use of these practices (or comforts), we extend this loving critique to the RUME communities' continued use and comfort in these practices. We encourage our colleagues to consider ways that they default to research practices that uphold dominant tradition (especially in research methodologies and propagation efforts) and hierarchies (based on social and professional identities) – two tenets of white supremacy that have infiltrated our academic culture so much so that we may not notice them.

We also share this loving self critique as an example of the type of vulnerable and reflective work that can lead to growth, which we (and therefore our field) can all always benefit from. We envision a loving self critique of research projects available to all RUME colleagues, asking (and importantly sharing) not only what we believe we got "right", but also how our work could be improved. We do not want this room for improvement to be misconstrued as a robust "limitations" section of a traditional paper. Instead, we remind ourselves and others that courage, humility, curiosity, and a critical interest in *growing* one's own propensity to do meaningful work are all woven into the work of a loving self critique (Black, this volume). We see these values in our RUME colleagues, and recognize many may be doing their own internal loving self critiques. Here, we call for that work to become publicly valuable so that we may all learn from purposeful introspection, and to place those introspections into dialogue with broader societal discourses, such as white supremacy culture. Our hope is that the "deep-seated" and "humanizing love" (Johnson et al., 2018, p. 48) with which we engaged in this work is felt and can propagate within this community.

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Dual Enrollment Students' Perspectives on College-Level Mathematics Instruction

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Dual enrollment (DE) students have simultaneous roles as both high school and college students. DE programs have been utilized to enhance students' access to higher education and post-secondary success. While many studies have focused on the effects of DE programs through quantitative, outcome-oriented frameworks, few studies utilized sociocultural, qualitative approaches to examine DE students' perspectives on their experiences by centering their own voices. Our qualitative case study addresses this need by exploring how three high school students taking a DE Calculus course describe their experiences with college calculus instruction. Student views about the affordances and limitations of college instruction emerged as themes in our study, which have critical implications for undergraduate mathematics education.

Keywords: dual enrollment, college-level instruction, student perspectives, calculus

In the US, dual enrollment (DE) programs, which allow high school students to take college courses, have drawn attention as an educational and policy issue with the goals of enhancing students' college readiness, access to higher education, and interest in pursuing STEM degrees (Cevallos et al., 2016; Hoffman, 2003; Jamieson et al., 2022; Kremers, 2010). Many studies involving DE students focus on the potential benefits of DE programs in relation to these goals through outcome-based approaches (e.g., undergraduate enrollment, matriculation, degree completion, and grade point average measures), often using quantitative perspectives. (e.g., Allen & Dadgar, 2012; An & Taylor, 2019; Blankenberger et al., 2017; Henneberger et al., 2022). While several researchers explored general experiences of US students taking DE college courses such as their adaptation to college environment or self-perceptions as college students (e.g., Cevallos et al., 2016; Kanny, 2015; Karp 2012; Lile et al., 2018), there are relatively few studies examining DE students' experiences with instruction while taking college-level mathematics courses (e.g., Cevallos et al., 2016), particularly through sociocultural and qualitative approaches that bring students' own voices and perspectives to the fore (Ramsey-White, 2012; Williams, 2015). Our research addresses these areas needing more attention by exploring the following research question: How do three high school DE students describe their experiences while taking a college-level calculus course with respect to mathematics instruction?

Theoretical Framework

We use a sociocultural perspective to purposefully frame our work through students' *descriptions* of their experiences rather than framing it via beliefs. While *belief* has been dominantly used as a construct to examine students' views about their experiences and themselves, researchers often treat beliefs as internal, individual, as well as stable cognitive "entities" and predictors of behavior that are detached from social context (Cobb, 2007; Lerman, 2002; Sfard, 2008). Treating beliefs as internal constructs fails to capture the social, discursive, and situated nature of students' practices and experiences (Cobb, 2007; Lerman, 2002). We argue that students' meanings, views, and positions about mathematics, learning, instruction, and themselves always occur within social contexts and are dynamic, contextual, discursive, and participatory rather than merely cognitive (Lave & Wenger, 1991; Lerman, 2002; Sfard et al.,

2001). By privileging student *descriptions* over *beliefs*, we treat talk as a unit of analysis in itself (Edwards, 1993) and align with the sociocultural perspectives that consider what students say not as a window to their inner states but as their evolving discourses and narratives through which they construct meaning (Sfard, 2008). We view students' descriptions and narrations as situated accounts, which are also influenced by students' interactions with us during the interviews, rather than as providing direct access to their internal beliefs about instruction. Consistently, we focus on how three DE students describe and narrate their experiences with their college-level calculus instruction.

Methodology

To examine the students' descriptions of their experiences, we used an exploratory qualitative approach through a case study design (Merriam, 1998). While there are different DE models through which high school students can take college coursework (An & Taylor, 2019; Andrews, 2001), our study focuses on the case in which students take a DE Calculus course (Calculus I) at a college simultaneously with an Advanced Placement (AP) Calculus course at their high school. Focusing our case study on this group of students allowed us to examine their perspectives on college-level instruction while having some prior experiences with calculus.

The three participants, who had dual roles as both high school and college students, were taking a DE Calculus course at a 4-year university called Tech College (pseudonym) located in northeastern US. The DE Calculus course was taught at the Tech College campus and was composed of seven DE students from two local high schools and three traditional college-aged students. All three participants were high school seniors, who had taken DE coursework since 10th grade. Our participants were Billy, Richard, and Catherine (pseudonyms), who all self-identified as Black/African American and attended a high school where 68% of the students were designated as low-income and consisted primarily of students of color. The high school the students were enrolled in offered several advanced coursework options including honors, AP, and DE coursework. The high school utilized DE to provide career-oriented coursework in computer science, engineering, and health sciences. All participants were taking part in the engineering pathway and had previously taken DE mathematics and engineering coursework.

The data we report here is based on two semi-structured interviews conducted with the students individually. The questions in the first interview elicited general information about students' DE program, courses, and overall experiences. The second interview followed up on the student responses to the first interview and provided more information about the experiences they previously mentioned, which resulted in questions related more specifically to mathematics instruction depending on each student's response. For example, if a student mentioned they liked or disliked the instruction in their DE Calculus course during the first interview, we asked the student to elaborate more on this issue during the second interview. Data analysis occurred simultaneously with data collection so that future interview questions could be informed by the prior collected data (Merriam & Tisdell, 2015). Data analysis involved coding the data both deductively and inductively to examine the themes in students' descriptions and narratives (Guest et al., 2012). Due to space constraints, this paper only focuses on the emerging themes about the affordances of and limitations of/barriers associated with DE Calculus instruction because how DE students describe their college-level instructional experiences has critical implications for undergraduate mathematics education.

Findings

Affordances of DE Mathematics Instruction

Learning about applications of mathematical concepts. Participants viewed seeing and learning more about the applications of mathematical concepts as a positive aspect of their DE Calculus instruction.

> *Catherine*: (In) Calculus I, we actually do projects with the stuff that we learn. So, I think the college part of it definitely has more of the aspects of like applying the math, rather than high school, where it's just learning and then seeing if you remember the information to take the tests.

Catherine provided further details about how the inclusion of real-life applications in her DE Calculus instruction helped her appreciate the usefulness and utility of mathematics.

> *Catherine:* The examples and the problems that we're made to solve in Calculus…they were…dealing with like planes and bridges and we were calculating different things that people, especially engineers, use a lot. So, it was good to see math in a practical way and see that it actually is used pretty often.

Similarly, Billy said that taking the DE Calculus course allowed him to "better understand" part of the "fundamentals" of mathematics, an aspect of which was "not knowing just how to solve an equation but knowing how to apply it to the real world." Billy further stated that calculus involved the study of mathematical applications as "calculus ties into basically everything related to engineering." It seemed that, through his exposure to the applications of mathematical concepts during the instruction of his DE Calculus course, Billy started making connections between different disciplines, such as mathematics and engineering. While Richard did not explicitly mention applications of mathematical concepts as a specific affordance of his DE Calculus instruction, he stated that "math is everywhere. I think a lot of people say that but it's just the truth. You need math for everything…math is pretty big in society." We interpret Richard's statement to suggest that he would likely respond positively to instructional approaches that showcased the real-life applications of mathematical concepts in the context of his DE mathematics courses. All participants directly or indirectly indicated that they viewed the focus on applications of mathematical concepts as an affordance of their experiences in the context of DE mathematics instruction, which appeared to have helped them develop a greater appreciation for the usefulness of mathematics, make connections between different disciplines, and learn more about how mathematics can be utilized in real life.

Deeper conceptual understanding of mathematics. Two students' instructional experiences taking DE Calculus seemed to assist them in developing deeper conceptional understanding of mathematics.

> *Billy*: I started to understand how to do math and …(how) a certain concept is the way it is, and not just how to do it…I just understood the fundamentals way more and how to get from A to B and why I'm getting from A to B in this certain way.

Billy elaborated further by stating that taking the DE Calculus course allowed him to go "beyond the world of (routine) problem solving and really seeing why math is that way it is. Like why you do things specifically this way and why and how, if you do it another way, then you know it's not gonna work."

Catherine acknowledged that the instruction in her DE Calculus course made her realize and utilize two types of mathematical skills: the procedural skills "when you see just like an integration sign and a function it's automatic about what you have to do", and the more conceptual mathematical skills, where "they don't just give you these numbers, they give you…a

situation and you have different variables to it…(in) optimization and related rates…you have to paint a literal picture and consider this and this…and you actually have to think about what you have to do." Catherine said she struggled more with the problems involving conceptual skills, jokingly noting that she was "a little traumatized from optimization and related rates", but also noted that she appreciated being exposed to such problems in her DE calculus course, which enhanced her overall understanding of calculus.

Both Billy and Catherine mentioned that the instructional approaches in their DE Calculus course helped them learn how to get from A to B (Billy), why mathematics is the way it is (Billy), paint a larger picture (Catherine), and actually think about what one has to do (Catherine), which are all components of deep and conceptual mathematical understanding (Rittle-Johnson & Siegler, 1998; Skemp, 1987). In addition, these experiences also seemed to help these students distinguish situations involving routine problems (Billy) and automatic applications (Catherine) from those that involve deeper effort and thought, indicating that they were also starting to differentiate between procedural and conceptual understanding.

Although Richard did not directly indicate deeper conceptual understanding through DE Calculus instruction, he noted that his second exposure to calculus concepts in his DE Calculus course (in addition to the AP Calculus course) helped him realize which calculus ideas and topics were important and where he needed to close the gaps in his understanding.

> *Richard*: There would be…this topic we went over in AP Calc and I kind of did not do so good on it. So, the fact that we went over (it) in the dual enrollment class, it kind of made me realize, wow I really need to go over this a ton more.

For Richard, although his DE Calculus instruction might not have led to deeper conceptual understanding, it helped him understand which calculus topics, concepts, and ideas occur again, are important, and where he needs to put more effort to enhance his conceptual understanding.

Limitations of and Barriers Associated with DE Mathematics Instruction

Negative assumptions about and experiences with college instructors. While some of the limitations and barriers mentioned by the students in relation to their DE Calculus instruction were based on their presuppositions, others were related to their actual experiences. All participants entered the DE Calculus course having some negative assumptions about college instructors. Catherine stated she was initially hesitant to enroll in the DE Calculus course until other classmates assured her that the instructor was "nice".

> *Catherine*: (Before taking DE Calculus) I thought if I take this calculus, they're probably not going to be as nice as my AP Calculus teacher but then my two friends that decided to take calculus…I was talking to them, and they were like…the teacher is so nice.

Billy related his viewpoint that his DE calculus instructor was not as lenient and approachable as his high school teachers by saying "it's not like the professor offers a lot of extra credit and a lot of second chances that teachers in high school do. You definitely have to be more careful of how you go about approaching the college calculus." Billy then noted that his perceived need to be more cautious with college instructors prevented him from seeking help from his instructors and said "I'd say, taking college courses, I haven't always asked for help too much."

Richard also stated that he was initially hesitant to ask his DE Calculus instructor for help as college instructors were not nice: "It would just be like me not knowing whether I can ask my college professor for help, because I would feel like college professors aren't really that nice." Richard said his assumption primarily came from the stories he heard from other students when they had negative experiences with their college instructors. In addition to what he heard from

his peers about college professors, Richard further revealed that he enrolled in the DE Calculus course because he was denied from enrolling in a different course he originally wanted to take: "the person who was running the class said they don't want dual enrollment students…. how can you just do that to a whole (group) of kids?" This personal experience, together with the experiences of his peers, likely reinforced Richard's negative views of college instructors. Despite viewing professors as uncaring or unempathetic towards their students, Richard stated that he ultimately became comfortable interacting with his DE Calculus instructor.

> *Richard:* I asked her this one question, she was actually pretty good, nice about it. I just kept asking her more questions and my learning was never stunted…because I felt like I was afraid to ask for help from my professor.

While all participants indicated that they ultimately interacted with their DE Calculus instructor, especially when they needed to ask questions, their assumptions about college instructors based on their own as well as their peers' experiences were initial barriers they had to work through in order to fully participate in the DE Calculus course.

Inability to attend office hours. All students brought up the issue of not being able to attend their DE Calculus instructor's office hours due to the scheduling of their high school courses. Billy and Catherine explicitly stated how scheduling prevented them from attending office hours.

> *Billy*: …office hours, we can't even attend those because it's during high school.
>
> *Catherine*: A difficult aspect of it, though is office hours. Because they're always at 11 am on a Thursday and I'm in (high) school then.

Richard also acknowledged his inability to attend the instructor's office hours and stated that he would work around this limitation by completing his DE Calculus assignments early so that he could ask any questions he had in a class meeting before the assignment was due.

> *Richard*: I would just wait until I had class again and then asked my teacher about it in class to get the additional help. Because I would always prioritize early college work over everything…So I'd always start on (the assignment) early so I had time to ask (the instructor) in class about the work.

Billy said he would utilize outside resources, like online math videos, to get help with calculus since he did not have the opportunity to get assistance from his instructor during office hours.

> *Billy*: I didn't ask for help too much. I think self-studying did help me a lot. Like taking time to use resources outside of class to understand (the course material) and maybe watch some math videos.

Catherine revealed that on the school days in which students did not have the DE Calculus course, the DE students would meet in their high school's library and assist one another as peers.

> *Catherine*: When we don't have class that day, we have like a free hour, which is supposed to be our last block of school, where we're in the library altogether and we take that time to do the homework and also projects (together)…we definitely use each other a lot for support.

While the DE students devised strategies to get around the challenge of not being able to attend their professor's office hours and find ways to get assistance with the DE Calculus material, they were unable to interact with their instructor regularly, which was a barrier to clarifying their questions, getting assistance from, and establishing a rapport with their instructor.

Time- and expectation-related challenges of DE students. During the interviews, the students did not directly attribute this challenge to DE Calculus instruction but mentioned it as a time commitment challenge. However, college instructors might be unaware of time- and

expectation-related challenges often faced by DE students. College instructors' potential unawareness of and lack of accommodation for DE students who operate in two spaces (high school and college) and work with at least two different sets of requirements at once (high school and college mathematics) can hinder their learning opportunities and success. So, we interpret this issue also as a potential pedagogical barrier.

During the interviews, all students expressed their difficulties in meeting the time and effort requirements for their DE studies in addition to their existing high school coursework. Richard said DE coursework was a time commitment as students would need to be transported to the college campus, which resulted in their DE coursework ending in the evening hours.

> *Richard*: You have to take the bus to go to Tech College…(and it's a) time commitment because at the start we literally got at Tech College at like two (p.m.), and then (class) would end at five (p.m.)…It was kind of crazy, honestly… We don't have that much time as high school students, because we have a whole day's worth of work we have to do.

Since the DE students finished their school days late, they had limited time in the evenings to then complete the homework required for all their courses. Billy stated: "Be prepared to set a lot of time aside after school and when you get home to do your college work and do your high school work." While Billy and Richard spoke more generally about the issue of not having enough time after school to do their studies for their coursework, Catherine was more explicit about her struggles:

> *Catherine*: I would get home at six (p.m), I had three AP (classes), and two regular (high school) classes, and (the DE) class, and to get all the homework done from six (p.m) to the time I have to sleep and eat and take a shower and do everything else in the middle of it…I was getting like five hours of sleep.

As expressed by Billy, these time limitations and different requirements of the courses impacted how much time the DE students could dedicate towards their DE Calculus studies and improving their understanding of the mathematics content.

> *Billy*: I was putting in so much work and putting a lot of time outside of class…to understand what's actually going on…It was something that was new to me so I didn't know how to exactly go about approaching the class and the material.

In the context of this theme, Richard mentioned an additional challenge regarding the faster pace of instruction in their DE Calculus course compared to the AP Calculus course, which further complicated catching up with the DE Calculus instruction, content, and assignments: "AP Calculus…they teach at a slower pace than college. So, I think there's more time to get the material." Richard's sentiment about the fast-paced nature of DE Calculus instruction was echoed also by Billy and Catherine, who said the DE Calculus course "is definitely faster, and it covers a lot more". The pacing of the DE Calculus instruction added another layer of complexity regarding students' time commitments as they needed to understand more material faster to meet the requirements of their DE Calculus course.

Discussion

Our findings help illuminate some affordances of college DE Calculus instruction when students have concurrent calculus instruction in their high schools. In particular, by taking the DE Calculus course, our participants stated that they enhanced their conceptual understanding of mathematics, studied calculus concepts more deeply, and targeted their studies to the topics with which they still struggled. Moreover, since the DE Calculus course included the study of mathematical applications, the students viewed calculus as useful in everyday life and made connections to other fields such as engineering. On one hand, this type of DE model (where

students take a calculus course at a high school and college concurrently), might be seen as wasteful in terms of institutional resources and students' time since students are essentially taking the same coursework twice. On the other hand, our findings indicate that there might also be some benefits to this type of DE model in terms of improving mathematical understanding of calculus, making connections, and appreciating the utility of mathematics.

Our participants also highlighted certain barriers impacting how they carried out their college mathematics coursework and limited their opportunities to seek assistance from their DE Calculus instructor. Typical systems utilized to support college students, such as office hours, were completely inaccessible to the DE students due to the time periods in which they were scheduled to be on the college campus. Our findings have implications with respect to the instructional practices utilized in DE mathematics courses offered at universities. Colleges and instructors offering DE coursework need to devise solutions and find ways to accommodate the needs of DE students to provide equitable access to course assistance and content support services. For instance, post-secondary institutions could develop agreements with their high school partners that DE students must have regularly scheduled time on the college campus to utilize college support services like tutoring centers. College instructors teaching DE students could structure their office hours so that they are more accessible to DE students such as offering evening or weekend office hours via video conferencing technologies.

Our findings also reveal the importance of college instructors' roles in building positive rapport with DE students to help students overcome any negative presuppositions they may hold towards college instructors and assist students in feeling more comfortable interacting with their college professors. One way college instructors can help build positive student rapport would be to acknowledge the time limitations that DE students face and work with them to help minimize any challenges in relation to DE coursework. For instance, college instructors might restructure assignment due dates so that they occur on weekends or some other day that is more accommodating of their DE students' schedules.

It is also important for college instructors to acknowledge that DE students have to balance various demands of high school and college courses. If college instructors do not recognize this challenge, are not willing to accommodate the students, and are inflexible, they might unintentionally create barriers for DE students' mathematical learning and success. This, together with potential emotional and physical burnout DE students can experience in the process, would negate the intended goals of DE programs and mathematics courses to provide students with enhanced access to higher education.

While some findings of this study align with the results from prior research such as DE students reporting difficulties in balancing their DE coursework and their high school courses (Lile et al., 2018) and taking DE coursework supporting students' time management and other study skills (Felder, 2020; Karp, 2007; Ramsey-White, 2012), some findings require more examination. Further questions to think about include: what specific aspects of DE Calculus instruction assist students in developing conceptual understanding of mathematics? How can college mathematics instructors develop instructional practices that would make DE students feel more welcome and feel a sense of belonging in their college courses? What are the additional needs of DE students taking college mathematics courses to feel emotionally and academically supported to achieve the aims of their DE programs in terms of access to higher education and interest in pursuing STEM degrees? A key issue our study underlines is the need to examine such questions through more studies that prioritize students' own voices and perspectives.

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Making Upper Division Math Courses More Relevant for Prospective Teachers

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In this report we address the longstanding problem of the perceived relevance of upper division math courses for prospective secondary school teachers. Through a multi-year design-based research approach we are reimagining a Dynamical Systems and Modeling course and investigating ways to make the course more relevant to prospective teachers. In this work we report on the impact of these efforts on students' relational understanding of why a particular function is a horizontal shift to the left of another function, a high school topic that is often only instrumentally taught and understood. Analysis of beginning and mid-semester assessments indicate significant growth in our students' ability to provide meaningful explanations. We also report on students' perspectives on classroom activities that influenced their reasoning.

Keywords: function translation, university-secondary connections, prospective teachers

A longstanding international problem in the preparation of secondary school mathematics teachers is the perceived relevance and usefulness of upper division mathematics courses for the work of teaching (e.g., Álvarez et al., 2024; Klein, 1932). In recent years, progress has been made on this pervasive and concerning issue. For example, to explore and make strong connections to secondary mathematics, Cook (2018) conducted four Abstract Algebra teaching experiments with two students and Wasserman et al. (2022) redesigned a Real Analysis course.

Consistent with these and other efforts, we are redesigning an upper division mathematics course for prospective secondary school teachers and investigating ways to make the course more relevant to their future work as teachers. More specifically, we are engaged in a multi-year design-based research (DBR) project focusing on an Inquiry-Oriented Dynamical Systems and Modeling (IODSM) course. We have previously reported on the positive impacts of this work on future teachers' beliefs about meaningful learning and teaching (Goodchild et al., 2021; Qiu et al., 2025) and their participation in the Standards for Mathematical Practice (Cruz et al., 2024).

In this report we investigate the impact of leveraging IODSM content to revisit related secondary school mathematics. In particular, we address prospective teachers' ability to explain *why* the graph of $g(x) = (x + k)^2$, for example, is a shift k units left of the graph of $f(x) = x^2$ (when $k > 0$). This is a classic example of the need for future teachers to have an understanding that goes beyond a rule-based explanation for why the graph of g is a horizontal shift to the left of the graph of f so that they are in a position to teach their future students this particular transformation and related function transformations with meaning.

Our two research questions are: (1) What classroom discussions about the horizontal function translation do IODSM students report as being influential on their understanding? and (2) To what extent do the students in the redesigned IODSM course evidence growth in their ability to explain *why* a particular function is a horizontal shift to the left of another function?

Background Literature

In this section we examine the ways in which prospective secondary school teachers understand why the previously mentioned quadratics are horizontal shifts of each other in a particular direction. We therefore focus our literature review on studies that have examined students' and teachers' understanding of this seemingly contradictory relationship.

Researchers consistently acknowledge that horizontal translations of functions are conceptually challenging and counterintuitive (Yim & Lee, 2021; Zazkis et al., 2003). These challenges stem from multiple sources. Studies show that secondary school and university students often demonstrate a limited understanding of function (Yiğit Koyunkaya & Boz-Yaman, 2023; Yim & Lee, 2021). Zazkis et al. (2003) found that students reduce the abstraction or difficulty in accounting for why there is a horizontal translation by drawing on their prior experiences with arithmetic and number lines. Their intuition tells them $x + c$ is located c units to the right of x (i.e., adding means moving to the right). Yiğit Koyunkaya and Boz-Yaman (2023) found that future teachers relied on procedural knowledge like plugging in values and using memorized rules. Students' symbolic struggles with f, x, and y can also be a source of difficulty when working with composition functions like $y = f(x)$ and $y = f(x + 2)$. Misconceptions often arise when students treat f as just a "shape," x as a general label for points on the x-axis, and y as solely an output rather than a function x (Yim & Lee, 2021).

Both in-service teachers and students tend to rely on tables and graphing applets to notice patterns and the location of graphs. These approaches show what happens to the graphs but do not justify why the changes occur (Yao, 2024; Zazkis et al., 2003). Hence, students make generalized rules from observing patterns but cannot provide a conceptual explanation, so the transformations remain counterintuitive. Zazkis and colleagues (2003) found that many in-service teachers often cite rules like the "law of opposites" where one does the opposite of what you think and that many in-service teachers admit that they feel their explanations are insufficient, but they have never seen or developed for themselves a better explanation.

Theoretical Background

Consistent with the focus of our research questions on future secondary school teachers' content understanding, we detail here our theoretical framing for understanding mathematics and the importance of content knowledge for teachers. We draw on Skemp's (1976, 1978) construct of instrumental and relational understanding. Instrumental understanding is when one knows what needs to be done but does not have any reason or backing for why it is done. Whereas relational understanding considers both knowing what to do and why it is done. Knowing what to do and why to do it is critically important for prospective secondary mathematics teachers.

As our students are future secondary teachers, mathematical knowledge for teaching also plays important roles in our theoretical foundations. Shulman (1986, 1987) provided one of the initial delineations between content knowledge and pedagogical content knowledge. Similar to instrumental and relational understanding, Shulman argued that a "teacher need not only understand that something is so, the teacher must further understand why it is so" (1986, p. 9). Ball and colleagues (2008) elaborated on what they called domains of mathematical knowledge for teaching (subject matter knowledge and pedagogical content knowledge). Koehler and colleagues (2013) describe the technological pedagogical content knowledge framework. Most relevant here is that they view pedagogical content knowledge as the intersection of pedagogical knowledge and content knowledge.

The differences between the previous constructs are often described theoretically. Researchers in Europe have also investigated this empirically. For example, Krauss et al. (2008)

found that the degree of connectedness between pedagogical content knowledge and content knowledge in secondary mathematics teachers is a function of the degree of mathematical expertise. They, along with colleagues Baumert et al. (2010), argue that content knowledge supports the development of pedagogical content knowledge. Put another way, preparation programs for prospective teachers should focus on the importance of building relational (in the sense of Skemp) content knowledge. We align with this conception of content knowledge; that is, it is an important precursor to pedagogical content knowledge. In the next section, we briefly describe our course and the content learning opportunities that prospective secondary mathematics teachers had to expand their content knowledge related to function translation.

Mathematical and Course Background

In the IODSM course students frequently analyze the relationships between the context being modeled, the symbolic form of a differential equation (DE), a slope field, graphs of solution functions, and the general solution. Autonomous DEs play a central role in the course. Autonomous DEs are ones where the rate of change of the dependent variable depends only on the value of the dependent variable itself. Many real-world contexts are modeled well with autonomous DEs. The symbolic form of an autonomous DE is $\mathrm{d}y/\mathrm{d}t = f(y)$. Slope fields for autonomous DEs are horizontally invariant. Graphs of non-equilibrium solution functions to autonomous DEs are horizontal shifts of each other and the general solution manifests this structure. Importantly for this analysis, the general solution to an autonomous DE contains the term $(t + \mathrm{c})$, which parallels the quadratic equation $g(x) = (x + c)^2$. DEs where the rate of change in the dependent quantity only depends explicitly on time have the form $\mathrm{d}y/\mathrm{d}t = f(t)$ and the general solution contains the term "$+ c$", which parallels the quadratic function $h(x) = x^2 + c$.

Many of the relationships noted above occured a third of the way through the course in Unit 3 where students derived the general solution for separable DEs and then explored the relationship between the slope fields for an autonomous DE, a time-dependent DE, graphs of their respective solution functions for a given initial condition, and their respective general solutions. In an embodied cognition (Edwards & Robutti, 2024) inspired activity, students were given slope fields for the DEs $dL/dt = 0.5(1 - L)$ and $dh/dt = -t + 1$ and asked to sketch a solution for a particular initial condition on each slope field, make a copy of each solution graph on patty paper, and demonstrate how they would move the copy of the respective solution graph so that it was the solution for a different initial condition. The task also prompted them to: (1) discuss what feature of the differential equation justifies your approach, (2) find the two respective general solutions; $L(t) = 1 - \mathrm{e}^{-0.5(t+c)}$ and $h(t) = -0.5t^2 + t + c$, and (3) explain how their idea for moving the patty paper solution copy can be understood from the general solution.

This IODSM content laid the groundwork for the in-class end-of-unit section with heading "Translations: Connections to Secondary Mathematics." In this section, which took approximately 30 out of the 75 minutes of class time, students were first asked to explain why the graph of $f(x) = x^2 + 5$ is a vertical shift up five units of the graph of $g(x) = x^2$. Similarly, they were asked to explain why the graph of $f(x) = (x+3)^2$ is a horizontal shift three units to the left of the graph of $g(x) = x^2$. After developing their own why explanations, they were prompted to complete the following sentence to further justify the vertical shift: "For every input x, the output $f(x)$ is…" and for the horizontal shift they were prompted to finish the sentence "In order for every output of $f(x)$ to be the same as the output of $g(x)$, the input…".

As was the norm, student ideas and explanations on these tasks were the substance of robust classroom discussion and sharing of ideas. It was also the norm that after each class, students submitted, via their course learning management system, a response to the following end-of-class

reflection: "Who, either in small group or whole class discussion, said something that influenced your thinking and what did they say? Please provide a little bit of detail."

Methods

The IODSM course was implemented at two different universities; one large Hispanic-serving public institution in the Western United States (US) and one mid-sized predominantly white public university in the Southern US. The class sizes were 31 and 7 students, respectively.

The data for the first research question were students' end-of-class reflections from the day they completed the "Translations: Connections to Secondary Mathematics" section of the materials, as previously discussed. There were two sources of data to answer our second research question. The first was an ungraded beginning of semester assessment where students were given functions $f(x) = x^2$, $g(x) = x^2 + 3$, and $h(x) = (x+4)^2$, and asked to state how the graphs of f and g are related and why, and how the graphs of f and h are related and why. Note here that the questions themselves did not explicitly mention anything about shifting functions. The second data source was a question on the midterm exam that asked students to explain why the graph $f(x) = (x+4)^2$ is a shift of 4 units to the left of the graph $g(x) = x^2$.

All data were organized in a spreadsheet. Each row contained separate columns to identify each student, their responses to all data sources, and our coding. For each data source, subgroups of authors did initial coding which was then brought to the full research team for vetting and discussion until 100% agreement was reached. Following our initial coding, we engaged in thematic analysis (Braun & Clarke, 2006) to analyze all data sources.

Results

To answer our first research question, we analyzed the 33 submitted end-of-class reflections. 10 students reflected on a part of class that focused on something other than horizontal shifts and hence here we focus only on the 23 reflections that addressed horizontal shifts. The fact that 23 of the 33 students (70%) chose to write about horizontal shifts is an indicator of how impactful this class was for them regarding this particular concept. While the assignment asked students to discuss who influenced their thinking and what they said, 14 of these 23 students went further and explicitly mentioned a significant enrichment in or change in their understanding of why there is a horizontal shift in what for many is an unexpected direction. For example, Onyx (all names are pseudonyms) said,

> In all honesty, I had no idea how to answer that question when it was asked on the survey we took at the beginning of the semester. As mentioned in class, it was just a rule that we had learned and then implemented from then on out. Even in our individual groups, I was having a hard time coming up with an explanation for why this happened. So, when that group said to think about it as the transformed function being "ahead" of the other and needing to have its input "reduced" by 3 in order for the output to be the same, I felt like I actually understood for the first time what it was that that transformation did.

Similarly, Alina said, "Overall, I found this discussion of horizontal shifts to be very engaging because it's a rule that most people know by heart, but we're rarely taught *why* a horizontal shift $g(x + h)$ is a shift to the left" [emphasis in the original]. Heidi, like Onyx and Alina, reflected on the change in her understanding but went even further and commented on the relevance of this new understanding for her future teaching.

> I have known this shortcut since I was taught it many years ago but never fully understood its explanation. His discussion … made me reflect on how the shifting works and a better way to explain it to my future students.

Heidi's reflection is consistent with the assertion that content knowledge supports the development of pedagogical content knowledge (Baumert et al., 2010).

We next report on insights from the 23 students into *why* the graph f, where $f(x) = (x + 3)^2$, is a horizontal shift three units to the left of the graph of g, where $g(x) = x^2$. By far the most frequent insight taken from the class activity was that the same output for the graphs of f and g occurs or corresponds to when the input is "compensated." In other words, maintaining the same output necessitates that the input be altered in some way. 15 of the 23 student reflections included some form of the insight that the same output corresponds to a compensated input. We see this as a shift in perspective on the function machine metaphor. Typically, the function machine metaphor starts with an input and then examines the corresponding output. In accounting for the horizontal shift, it can be more insightful to focus on the output first and then backtrack to see what inputs led to the same output. Jude explicitly called out this shift in perspective in his end of class reflection, using a different pair of functions.

> When we were discussing about how to explain why the graph of $(x – 3)^2$ is shifted to the right 3 units, someone mentioned how instead of focusing on the input, we focus on the output. This made me realize how this particular graph will behave 3 units earlier than the parent function x^2, and thus will be shifted to the right by 3 units. This perspective of focusing on the output was something that I had not even considered, and I found very interesting.

In addition to calling out this shift in output first perspective, Jude uses the term "earlier." The term "earlier" relates to a race analogy discussed in class. Of the 15 that discussed the insight that the same output corresponds to a compensated input, 11 of these students also mentioned the race analogy. Skye's reflection captures nicely the various ways that the race analogy was discussed by her peers.

> While [my] group discussed how $f(x) = (x + 3)^2$ was a shift to the left, we had a few different ideas that emerged, and it was starting to click with me, but not fully. It wasn't until Glen explained to the class how their group used $g(x) = x^2$ to describe that $f(x) = g(x + 3)$ and seeing it in this notation really helped me fully comprehend it. In class we used words such as "ahead", "earlier", "behind" and "later" to describe these transformations. This vocabulary doesn't help everyone understand it, but it really helped me understand the transformations that are happening. I like the phrase that $f(x)$ as being 3 units "ahead" of $g(x)$, and it would need 3 units in order to catch up to $g(x)$. This can also be seen in the notation that $f(x – 3) = g(x)$ which helps me really see that it would be a shift 3 units to the left.

In addition to the race analogy, Skye's end of class reflection brings in composition notation, $f(x) = g(x + 3)$ and $f(x – 3) = g(x)$. The composition notation provides a generalized symbolic way to say that the outputs of f are the same as the outputs of g when the inputs are compensated or altered. Three students included composition as part of their reflection.

Lastly, 3 of the 23 students focused on the different location of the vertex to account for why the graph of f is a shift of 3 units to the left of the graph of g. For example, Lina's reflection, which also highlights the race analogy, foregrounds Skye's focus on the location of the vertex for the two parabolas.

> Glen had a good explanation of the shifted function being "ahead" three units in comparison to the original function, and I thought that was helpful. But the idea that really gave me a better understanding was Jamie's idea of setting $x + 3$ (vertex) = 0.

> Being that $x = -3$, that is sufficient enough to display why it's not shifting RIGHT three units.

A focus on the vertex *implicitly* centers an output first approach because for both f and g, their respective vertexes have an output value of 0. In Lina's reflection, it is implicit that the output values for both f and g need to be zero as this is the likely logical reason Skye set $x + 3 = 0$ because when $(x + 3)$ is zero, so is the output of f.

To answer our second research question we compared student responses on the beginning of semester assessment to a similar question on the midterm, which asked students to explain why the graph $f(x) = (x + 4)^2$ is a shift of 4 units to the left compared to the graph $g(x) = x^2$. In short, there was a significant improvement in students' ability to explain *why* the horizontal shifting relationship exists. On the beginning of semester assessment, 20 of the 39 students identified the correct horizontal shift, but of these 20, only 4 provided an explanation that went beyond a focus on a single point with no other supporting explanation. That is, most students started the semester knowing *that* the horizontal shifting relationship was true, but very few could explain *why*.

In contrast, 30 of the 39 students provided a meaningful explanation for why the horizontal shift exists. The remaining nine students either had a rule-based, incomplete, or incoherent explanation. We developed two themes to characterize students' meaningful explanations. One we refer to as an Explicit Output First approach and the other we refer to as an Implicit Output First approach. We alluded to both of these themes in answering our first research question. Next we provide operational definitions for each theme and give illustrative examples.

The Explicit Output First approach, which characterizes 21 of the 30 explanations, begins the explanation with a focus on the outputs (which Jude noted was a new perspective in his end of class reflection) and then shifts to explain how the inputs need to be compensated so that the outputs are the same. For example, Darrin's explanation began by comparing the outputs via a composition formulation. More specifically he said, "We can think of $f(x) = (x + 4)^2$ as $g(x + 4)$, where $g(x + 4) = (x + 4)^2$." This statement uses composition to make explicit the condition when the output of f is the same as the output of g. Darrin then continued to explain how the inputs need to be compensated so that the outputs are the same. Specifically, he said,

> When we look like this, in order to get $g(0)$ or any original $g(x)$ values, we will need to subtract 4 from x to compensate for the added 4. In this sense, we 'shift' $f(x)$ left 4 to compensate for the added 4 to the x (when compared to $g(x)$).

A total of nine students had similar composition followed by input compensation arguments. Two of these nine also included the race analogy.

For the other students whose explanations we characterized as an Explicit Output First approach, four began their explanation with a table of values followed by input compensation and five students began their explanation with a verbal description of the same output followed by input compensation. For example, Ann said, "In order for $f(x)$ to obtain the same output value of $g(x)$, your input value for $f(x)$ has to be …". The three remaining students whose explanations fell under the Explicit Output First approach had a more proof-like explanation, with one of these including the race analogy. Space constraints prohibit us from providing examples that made use of a table or a proof-like explanation. We do note, however, that all of these explanations spoke to the generality of the relationship as opposed to focusing on a single point, such as the vertex, as the sole justification for why there is a horizontal shift in a particular direction.

We characterized the remaining nine students that provided meaningful explanations as employing an Implicit Output First approach. The main difference between the two

characterizations is that in the latter explanations, students began their explanation with a focus on how the inputs need to be compensated to achieve the same outputs but the reason for doing so was so that the outputs would be the same. For example, Damien explained that the graph of f is a 4 unit shift to the left of the graph of g

> because 4 units more are needed from $f(x)$ to reach $g(x)$, meaning that it is constantly behind. Rewriting this with x in mind, shows this => $x = \sqrt{(g(x))}$, $x = \sqrt{(f(x))} - 4$. Here we see that focusing on the input rather than the output (x rather than y), the shift is negative, or to the left.

Damien alludes to the fact that f and g have the same values ("$f(x)$ to reach $g(x)$") but maintains a focus on the relationship between the inputs.

Another common meaningful explanation that we characterized as an Implicit Output First approach started with a general statement about how the inputs are related and then shifted to a specific example, typically the vertex. For example, Rhea gave the following explanation:

> The graph of $f(x) = (x + 4)^2$ is a shift 4 units to the left of $g(x) = x^2$ because the function of $f(x)$ is changing an input, rather than an output. So, say we want both of the graphs to equal 0. We see that our graph of $f(x)$ would get there 4 units earlier that our parent function when put side by side [sketches graphs of f and g with the respective vertexes circled]. Differently, our vertical translations such as $h(x) = x^2$ and $b(x) = x^2 + 3$, vertical translations affect our output, rather than our input.

Rhea's explanation begins with a focus on the inputs and then shifts to the conditions necessary for an example output to the same, using the vertex as an example. The phrasing "say we want" suggests that Rhea is using the vertex as a more generic example of the same output. This move to a more generic example stands in contrast to the explanations on the beginning of semester assessment that much more strongly used the empirical evidence of a single point, the vertex, as sufficient explanation for why the two graphs have the claimed horizontal relationship.

Conclusion

In this report we provided insights into what aspects of class discussion students found influential on their understanding of horizontal translations and the corresponding growth in their relational understanding, as evidenced by their meaningful explanations for why graphs are related in the way that they are. Our findings move beyond the many descriptions in the literature that focus on misconceptions and difficulties and instead highlight two asset-oriented themes characterizing their reasoning, both of which focus on the relationships between outputs and inputs, with a focus on the outputs first, which was largely a new perspective for students.

The next steps in our analysis are to analyze the classroom video recordings that we collected as part of our DBR as well as additional student responses to related homework problems. This will include analysis of participation patterns that might be correlated with growth of individual student reasoning. We are particularly interested in exploring how students' growth in relational understanding of secondary school content is related to their meta learning about what constitutes a meaningful explanation. This line of inquiry has both theoretical and practical implications. Lastly, we are currently making revisions to the curriculum based on this analysis and will implement these revisions in our next DBR iterations.

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Approximation, Riemann Sums, and Coordination Class Theory

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Researchers have recognized Adding Up Pieces (e.g., Jones, 2013) to be a critical interpretation when applying definite integrals to problems in physics and other domains. At the same time, researchers have not investigated as closely students' moment-to-moment reasoning when constructing pieces of an approximation and how that reasoning might vary across situations. We interviewed 11 students enrolled in calculus-based physics courses using tasks that asked for approximations, not definite integrals. Individual students approached approximations in more than one way—depending on the problem situation and how it was presented—and they did not always make connections among their different approaches. Based on this observation, we conjecture that approximation in service of Riemann sums is a good candidate for a coordination class (e.g., diSessa, 2002; diSessa & Wagner, 2005). We summarize the theory, present the case of one student, and draw out implications for research and instruction.

Keywords: Approximation, Riemann Sums, Coordination Class Theory

A growing body of research has reported on students' interpretation and construction of integrals, especially when solving problems in physics and other contexts outside of pure mathematics (e.g., Cui et al., 2006; Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Thompson, 1994a, Von Korff & Rebello, 2012). One key area of interest has been whether and, if so, how students connect definite integrals with adding up small amounts of a target quantity (Ely, 2017; Jones, 2013, 2015). At the same time, research has not investigated in detail students' moment-to-moment reasoning when constructing pieces that contribute to an approximated target quantity. We report on results emerging from an on-going study that examines how undergraduate students enrolled in calculus-based physics courses respond to tasks that ask for approximations, not definite integrals. Individual students approached approximation in different ways from one task to the next, leading us to conjecture that Riemann sum approximation is a good candidate for a coordination class (e.g., diSessa, 2002; diSessa & Wagner, 2005).

In the sections that follow, we review key strands of research on students' reasoning when applying definite integrals to solve problems, primarily from physics. We follow with a brief description of Coordination Class theory. Then we present our research questions and summarize our methods. Finally, we present the case of one student, Daner, as an exemplar of empirical phenomena we observed across participants.

Literature Review

Applying definite integrals to physical situations requires reasoning in terms of quantities such as lengths, time, work, electric charge, etc. Past research has provided a variety of perspectives on such reasoning, much of it grounded in Thompson's account of quantitative reasoning (e.g., Thompson, 1994b, 2011). In one line of such work, Jones (2013, 2015) has identified several ways in which college students can interpret the $\int_a^b f(x)\,dx$ notation. One of these, the Adding up Pieces interpretation, consists of (a) partitioning a physical situation into small pieces, (b) approximating the target quantity within each partition piece, and (c) summing the small amounts of target quantity across all partition pieces to approximate the total amount.

The Adding Up Pieces interpretation is particularly useful when applying definite integrals to physics and other domains (e.g., Stevens & Jones, 2023; Oehrtman & Simmons, 2023).

In further work, researchers have identified reasoning about local relationships as a key challenge for many students (e.g., Ely, 2017; Jones et al., 2017; Oehrtman & Simmons, 2023; Sealey, 2014; Thompson & Harel, 2021). As one example, Oehrtman and Simmons (2023) proposed *Emergent Quantitative Models* as a tool for tracing how students' attention shifts across three scales when constructing definite integrals. They named these scales the Basic, Local, and Global models. A Basic model relates quantities with constant values, a local model restricts a basic model to small regions where varying quantities are approximately constant, and a Global model is an accumulation based on local models. Oehrtman and Simmons highlighted the interaction and co-evolution of the three scales.

Past research has established *that* students can have trouble reasoning with local approximations but has stopped short of examining *how* students reason in contexts that afford constructing approximations in ways compatible with Riemann sums. We take this next step.

Theoretical Framework

Coordination Class theory (e.g., diSessa, 2002; diSessa & Sherin, 1998; diSessa & Wagner, 2005) provides an account of certain concepts within the knowledge-in-pieces epistemological perspective (e.g., diSessa, 1993, 2006). diSessa (2002) motivated the notion of coordination classes with the following description of expert accomplishment: "The fundamental assumption behind the idea of coordination classes is that information is not transparently available in the world. Instead, we have to learn how to access different kinds. Indeed, in different circumstances, we may need to use very different means to determine the same kind of information" (p. 43). This description implies that (a) a collection of knowledge resources supports "seeing" the same concept-specific information across situations, a phenomenon referred to as *span*, and (b) different subsets of concept-related resources might be needed in different situations, a phenomenon referred to as *concept projections* (diSessa & Wagner, 2005; Wagner, 2010). Researchers have begun applying coordination class theory to gain insight into students' reasoning about several topics in mathematics and science (e.g., Izsák et al., 2021; Levin, 2018; Parnafes, 2007; Wagner, 2006), but not to approximations or Riemann sums.

We sketch how Riemann sum approximations could be interpreted in terms of Coordination Class theory. First, we acknowledge that researchers have taken different positions on whether one can always interpret the integrand, $f(x)$, as a rate in a definite integral, $\int_a^b f(x)\,dx$. If one does interpret $f(x)$ as a rate, then proficient reasoning about Riemann sums involves approximating $f(x)$ using locally constant rates over small changes in x. Furthermore, if one interprets areas of rectangles as products of constant rates and small changes in x, then such areas can serve as generic representations of small pieces that can be accumulated to approximate target quantities. Students might use such reasoning to approximate a target quantity in one context but then use quite different reasoning when approximating target quantities in other contexts. Employing different reasoning to construct approximations across contexts is consistent with using different concept projections. Furthermore, seeing that quantities across situations can be interpreted as rates need *not* be transparently available. As an example, students might have experience thinking of locally constant speeds when estimating distance travelled in a straight line over an interval of time, but not see density as a locally constant property that can be used in a similar way to estimate mass. Constructing parallel reasoning across situations is an example of increasing span.

Research Questions

1. What knowledge resources do students engage when constructing approximations of target quantities across problem situations?
2. To what extent are students' approaches to constructing approximations compatible with Riemann sum?

Methods

We recruited 11 students from second semester, calculus-based physics (Five in Fall 2024, six in Spring, 2025) at a selective university in the northeast United States. Each student participated in a series of three, semi-structured interviews (e.g., Bernard, 1994; Ginsburg, 1997) and was paid $50 per interview. The interviews lasted 60 to 70 minutes each and were spaced a few weeks apart. Students worked on tasks that afforded opportunities to construct approximations through addition of pieces, consistent with constructing Riemann sums. We began with approximation in contexts familiar to students (e.g., length • width = area, speed • time = distance) and then moved to approximation in contexts we conjectured would be less familiar (e.g., density • volume = mass, pressure • area = force). These choices were deliberate: We wanted to see whether and, if so, how students would draw on prior experiences constructing approximations in what, for them, would be novel contexts. We omitted formulae that related quantities, because they could short cut students' reasoning about multiplication when forming pieces of target quantities. In addition to text, several of the tasks included diagrams or graphs. Students worked on two or three tasks per interview, pacing that allowed for multiple follow-up questions to explore their reasoning in detail.

We recorded the interviews using two cameras, one to capture the student and interviewer and one to capture close ups of students' written work. After each interview, we collected all written work, combined the two video recordings into a single file, and transcribed verbatim. We analyzed each interview individually and met weekly to discuss our observations. We used students' utterances, inscriptions, and hand gestures to infer the knowledge resources they relied on moment-to-moment for each task. After group discussion we often rewatched videos to refine our analyses. We constructed accounts of reasoning for each task and looked for similarities and differences in the knowledge resources and approaches to approximation students used across tasks.

Results

Our top-level result is that all students approached approximation in different ways as they moved from one problem situation to another, and we conjecture that students' reasoning about approximation may be well captured using constructs from Coordination Class theory, as discussed above. Given space constraints, we have room to present key features of one case. Daner (a pseudonym) self-reported receiving a 4 on the AP Calculus BC exam, was enrolled in second-semester calculus-based physics (Spring 2025), and was majoring in mechanical engineering. The second author led all three interviews with Daner.

Interview 1: Multiple Concept Projections When Constructing Local Approximations

During the first interview, Daner demonstrated different approaches to approximating area from lengths and widths, distance from speeds and times, and mass from densities and volumes. The Lawn task asked students to approximate the area of a lawn whose boundaries were presented through a diagram that showed two perpendicular line segments and a third, curved line. Daner immediately recognized the relevance of an integral but was stuck initially without

"the equation of the line." He then partitioned the lawn into two regions, a parabola and a rectangle (Figure 1a). Here, his sense of approximation relied on guessing an algebraic expression for the parabola, choosing five as an upper bound for the definite integral, and treating the second region as a rectangle. He then wrote $\int_0^5(-x^2-3)dx$ for the area under the parabola (Figure 1b). When asked to explain the meaning of dx, he described a second method for approximating area more compatible with a Riemann sum:

> Like this is the process that does it, uses the equation of the parabola that is given, takes the height, multiplies by *dx* and it's like a small rectangle, multiplies by *dx*, and does it infinitely many times until five….The integral, what it does, it adds it up together.

When asked to illustrate his explanation, Daner drew a generic "small rectangle" (Figure 1b). Thus, he demonstrated two approaches to approximating the area of the lawn, and only the second was well-aligned with Adding Up Pieces and constructing a Riemann sum.

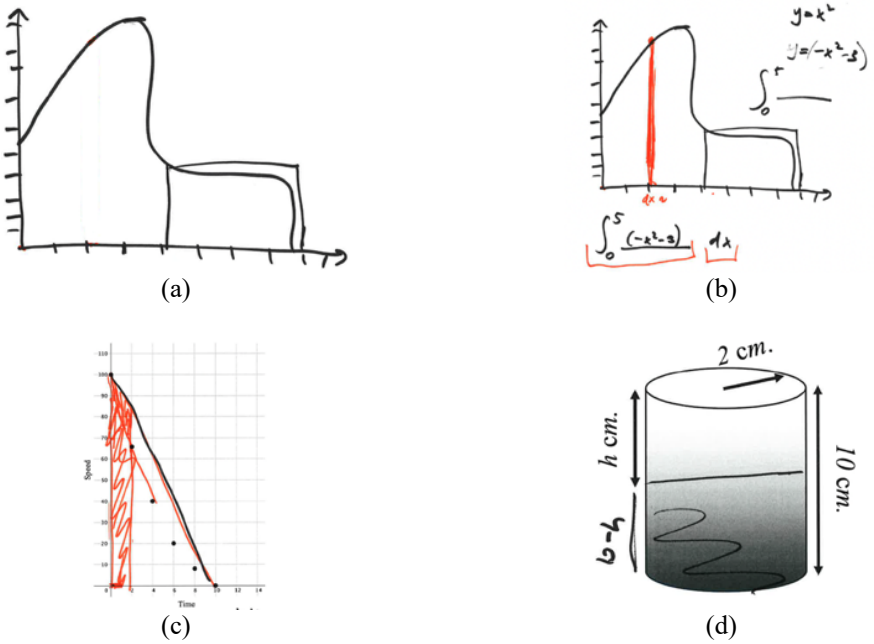

Figure 1. Daner approximates (a) Area using a parabola and rectangle, (b) Area using small rectangles only, (c) Distance using trapezoids, (d) Mass using basic models.

The Car task presented the speed of a car every 2 seconds as it slowed to a stop. The data were presented in a table with columns labeled "Time" in seconds and "Speed" in feet per second. Similar to the Lawn task, Daner commented that he did not have an equation with which to work. When he tried to recall an equation, he multiplied speed not by time, but by distance. At this point, the interviewer introduced the same data presented as discrete points on a graph whose axes were labeled "Speed" and "Time." Daner stated that "the distance is the area under the curve" and explained how to estimate using averages for each 2-second interval:

> I would find like, the velocity, the mean, like the mean velocity between the first one and the second one, and do the same like here, here, here, here, and multiply it by 2 seconds, and add them together.

Daner then used the given data to compute two examples of average speed correctly, (100 + 66) ÷ 2 = 83 and (66 + 40) ÷ 2 = 53. He multiplied both results by the duration of 2 seconds but, at first, said he could not "visualize" the corresponding 166 and 106 values on the graph. Then he changed his mind, drew a curve that connected the graphed points, and said that 166 would be

the area "under this curve from 0 to 2." Daner then clarified that 166 would not be exact, drew a straight line connecting the points (0, 100) and (10, 0), shaded the resulting trapezoidal region for the interval from 0 to 2 seconds (Figure 1c), and stated that 166 would be exact if the deceleration was linear.

From the data, we could not tell why Daner switched his focus from lengths and widths of rectangles on the Lawn task to average speeds and trapezoids on the Car task. Several interviewed students expressed a preference for midpoint Riemann sums over lefthand or righthand Riemann sums because they are often more accurate. The associated calculations can be associated either with areas of trapezoids or with areas of rectangles. The important point is that Daner maintained a focus on Adding Up Pieces but used different knowledge resources in each task, ones related to areas of rectangles versus ones related to averages and trapezoids.

The Sand task presented a cylinder containing sand suspended in water. The cylinder had a height of 10 cm and a radius of 2 cm. The density of sand in water was given by $\rho(h) = \sqrt{h}$, where h measured distance from the top surface. A diagram showed the cylinder with graduated shading to reinforce that the density of the sand varied continuously from the top to the bottom of the cylinder (Figure 1d). Daner recalled the formula "density is mass over volume" and the formula for the volume of a cylinder. At the same time, he interpreted the $\rho(h) = \sqrt{h}$ function to give a single value of density that applied to the top h cm of the cylinder. He then considered whether the sand would be in the top region with a height of h or in the bottom region with a height of $10 - h$. He wrote two expressions for mass based on basic models, one for each case: $m = \rho \bullet 4\pi h$ and $m = \sqrt{10 - h} \bullet 4\pi(10 - h)$. Daner explained the meaning of density using qualitative comparisons, "if there's a lot of density, that means that there's a lot of mass in a more compact volume." He then relied on this meaning to explain:

> In this case, if we're looking at sand, the lower part is more dense than the upper part, because there's less sand in it for the same amount of volume compared to the other one where there's more sand for a specific volume.

Finally, when asked if he interpreted $\rho(h)$ as the density for the entire top h cm or just for a thin slice of the cylinder, Daner reported, "I would think of it, yes, from the top to the bottom of h, not in terms of layers."

Daner's approach to the Sand task both resembled and differed from his approach to the Lawn and Car tasks. Parallels included both a correctly stated relationship among mass, density, and volume that paralleled a correct relationship among distance, speed, and time and a description of density in terms relevant for conceiving a unit rate—an amount of sand "for a specific volume." At the same time, he gave no evidence of constructing small pieces of a target quantity. It is plausible that his interpretation of $\rho(h)$ as a fixed value for a cylinder of height h obstructed his attention to small pieces of mass. We conclude that Daner drew on different sets of resources across the Lawn, Car, and Sand tasks tasks and that features of task presentation (e.g., diagrams, graphs, explicit variables) shaped which resources Daner employed. Such data are consistent with concept projections of resources relevant to approximating small pieces of a target quantity in some situations but not others.

Interview 2: Coordinating Knowledge Resources Toward Increasing Span

The second interview began with the Rod task, a task similar to the Sand task that involved approximating mass given a varying density. The rod was a horizontal cylinder of length L, radius R, and density $\lambda(X) = X^2 + 4$, where X was the distance from the left-hand end. The first version of the task used uniform gray shading on the cylinder, the second version used gradated

shading to help communicate that density increased continuously from left to right (Figure 2a). Daner reused knowledge resources evident in his reasoning on the Car and Sand tasks: He interpreted density, $\lambda\,(X)$, first to be a single value for a cylinder of length X and then reasoned about midpoints and averages.

(a) (b)

Figure 2. Daner approximates mass (a) Partitioning the rod into fourths, (b) Using a graph.

Daner began by recalling density equals "mass over volume" and the formula for volume of a cylinder. Once again, he treated density as a single, fixed value and used a basic model to multiply density and volume for the entire rod: $M = (L^2 + 4) \bullet \pi R^2 \bullet L$. At this point, the interviewer sought to help Daner perceive density as varying. He began by presenting a second image of the cylinder in which gradated shading reinforced density increasing from left to right. Daner stated that the new diagram did not change his interpretation of density as a single value: "It just doesn't make any sense why the density would increase if it goes to the right."

When the interviewer asked to view density as varying anyway, Daner partitioned the rod into fourths (Figure 2a) and explained that he would "subtract the distances from each other and plug it into L again." He subtracted $\frac{3L}{4} - \frac{2L}{4}$ and generated: $M = (\frac{L}{4} \bullet \pi R^2) \bullet ((\frac{L}{4})^2 + 4)$. Thus, he interpreted $\frac{L}{4}$ as a new length and reasoned in ways that paralleled his work on the Sand task and his initial work on the current task. When the interviewer asked about the mass of the segment between $\frac{L}{4}$ and $\frac{2L}{4}$, Daner saw that his work implied that the mass of two regions would be the same while the actual masses should differ. At this point, he was stuck: "The answers are going to be the same for mass. So I don't really, I don't really know how to find the different masses."

Realizing that he needed a new approach, Daner turned his attention to the graph of $\lambda\,(X) = X^2 + 4$. He partitioned the horizontal axes from zero to L into six pieces, considered the interval between $\frac{2L}{6}$ and $\frac{4L}{6}$, and asked how "to find a like density between two lines" (Figure 2b). After considering and rejecting using differences in densities at two points, he returned to notions of midpoints and averages. First, he used the example $\lambda\,(X) = 2X$ to explain that using a midpoint would work for a density that increased linearly: "Like if it was a linear, because it goes linearly the average density would also, like it would be just the half of, the half value." This reasoning resembled his reasoning about average speed in the Car task. Then he computed midpoints *for distances* and generated expressions for mass in the third region from the left, $M = (\pi R^2 \bullet \frac{L}{4}) \bullet ((\frac{5L}{8})^2 + 4)$, and in the second region from the left, $M = (\pi R^2 \bullet \frac{L}{4}) \bullet ((\frac{3L}{8})^2 + 4)$. Notice that these expressions could contribute to a normatively correct midpoint Riemann sum approximation. Daner did *not* draw anything that looked like rectangles or trapezoids that approximated area under a curve.

At this point, the interviewer asked Daner to identify similarities or differences across the Lawn, Car, Sand, and Rod tasks. Not surprisingly, Daner saw similarities between the Sand and

Rods tasks and found the Lawn task to be less similar to the other three tasks. Daner said that the Car and Rod tasks were similar but, an exchange later, said they were not similar. When the interviewer asked about these two points of view, Daner described in more detail parallels that he saw. First, he compared interpretations of area under the graph in the Car task (Figure 1c) and in the Rod task (Figure 2b). He knew the former represented distance and after a 75-second pause discussed whether the latter could represent mass:

> It has to be density multiplied by length. Because [in the Car task] this region is speed multiplied by time, which gives us distance. And so this region is distance. Whereas [in the Rod task] it also has to be density multiplied by length. But I was thinking like why would you multiply length with density, like what would that ever give? But then I just understood that that's basically, that's basically the volume. Like multiplying by length can be the same as multiplying by volume for a shape. So that will give you mass.

On the one hand, Daner drew sensible parallels where units on axes (speed and time, density and length) determined units represented by areas under curves. On the other hand, he knew that in the Rod task "it has to give you basically mass." He then included πR^2 on the horizontal axes:

> So you can just give pi r squared to L, multiply L by pi r squared, which will give you volume. And when you multiply volume by density, it gives you mass.

Although Daner articulated normatively correct coordination of density, volume, and mass represented as area, he had trouble maintaining that coordination when he wrote: $\lambda\,(X) = (\pi R^2 L)^2 + 4$. In response to further questions, Daner explained that the Lawn, Car, Sand, and Rod tasks were all similar based on the solution process: "what you do basically at the end of each task is you find the area underneath of the, of any like equation that you get." At the same time, when he stated that "density is not similar to speed here at all" we could see that the parallels he perceived between constructing approximate distances in the Car task and approximate mass in the Rod task were incomplete. Thus, he made partial progress towards increased span.

Discussion

In the Background section, we organized past research on definite integrals into two strands, one that has identified different interpretations of integrals and one that has delineated layers of attention (e.g., Basic, Local, and Global models). Our tasks combined with semi-structured interview methods allowed us to investigate in more detail how students construct approximations moment-to-moment within and across problem situations. We presented Daner's case as an exemplar of our overall finding that the students we interviewed often approached approximation in different ways as they moved from one problem situation to next. This suggests that constructing parallel reasoning across situations that include speed, density, and other quantities is not straightforward and involves developing disciplinary perspectives on what counts as a rate—for instance, the independent variable does not always have to be time.

Compared to past research, our results suggest that more precise theory is necessary if we are to account for students' reasoning about definite integrals in ways that can ultimately inform instruction. Tools from Coordination Class theory have helped us describe with some precision the complexity of constructing approximations across situations in ways compatible with Riemann sums. In this regard, we build on Wagner (2016), who took a first step in this direction when arguing that students' various interpretations of definite integrals could be understood as different concept projections. As noted by diSessa (2002), relevant information is not always transparently available, and students may need different means to construct approximations when assembling target quantities and solving problems in areas outside of pure mathematics.

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Intersections of Active Learning and Instructor Care: Undergraduate Women's Perspectives on Their Instructor's Role in Supporting Their Sense of Belonging in Calculus

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One's sense of belonging, or their sense of feeling like an accepted member of an academic community, has been identified as a major contributor to women's decisions to stick with or leave their STEM major. Prior work has examined ways in which pedagogical practices such as providing opportunities for students to engage in active learning in undergraduate courses might influence women's sense of belonging in STEM. However, the focus had been on students' perceptions of the practices themselves rather than the person enacting those practices – the instructor. With this study, I explore ways in which women describe the role their instructor plays in supporting their sense of belonging in an active learning Calculus course. Women's survey responses suggest several ways in which their instructor supported their sense of belonging by portraying care both through their enactment of active learning as well as other pedagogical and personal attributes.

Keywords: calculus, gender, instructor care, sense of belonging

Fewer women than men choose to major in STEM early in their college education (Bressoud, 2023; Eagan et al., 2016), and women leave STEM majors at higher rates than men (Chen, 2013; Seymour & Hunter, 2019). Calculus is a common exit point for all STEM students, but especially for women (Ellis et al., 2016; Rasmussen et al., 2019; Seymour & Hunter, 2019). Their reasons for leaving are not for lack of interest or ability, but rather women, especially those who leave, report feeling a low sense of belonging in STEM (Seymour & Hunter, 2019; Shapiro & Sax, 2011). Prior work has examined ways in which engaging in active learning can support women's sense of belonging in Calculus (Griffin, 2023, 2025). The current study aims to investigate women's perspectives on the role their instructor, rather than the pedagogical practices, plays in supporting their sense of belonging in an active learning Calculus course.

Literature Review & Theoretical Framing

Feeling a sense of belonging has been equated with feeling accepted and valued within a particular environment as opposed to isolated or excluded. Good et al. (2012) studied students' sense of belonging to an academic domain, which they define as "one's personal belief that one is an accepted member of an academic community whose presence and contributions are valued" (p. 701). This definition of sense of belonging (SB) is most relevant to the current study. Feeling a SB has been linked with persistence in STEM (Seymour & Hunter, 2019; Shapiro & Sax, 2011), however women are less likely than men to develop a strong SB (Good et al., 2012; Rainey et al., 2018; Seymour & Hunter, 2019; Shapiro & Sax, 2011).

One major factor that can influence one's SB in an academic domain is the classroom learning environment. The learning environment consists of several features, including (but not limited to) the physical space, the ambience or climate, and learning opportunities (Evans et al., 2009). Researchers have identified links between the types of learning opportunities instructors provide, the ways in which instructors portray care, and students' SB (Fraser, 1998; Griffin, 2021; Lahdenperä & Nieminen, 2020; Rainey et al., 2019).

Active Learning

The pedagogical contexts of the learning environment, such as the learning opportunities the instructor provides during class, have been shown to influence students' engagement and persistence in STEM. Over the last few decades, there have been efforts to reconsider traditional modes of undergraduate STEM instruction to work toward improving student engagement, performance, and retention (Chickering & Gamson, 1987; Freeman et al., 2014; Mesa et al., 2015; PCAST, 2012), specifically by providing opportunities for students to engage in active learning (i.e., during which students have the opportunity to do or talk about mathematics during class rather than passively listening and taking notes while the instructor lectures).

Some research has indicated that engaging in active learning can support students', especially women's, SB, engagement, performance, and retention (Griffin, 2025; Aji & Khan, 2019; Freeman et al., 2014; Fuchs & Sahmbi, 2024; Lahdenperä et al., 2019; PCAST, 2012; Rasmussen et al., 2019; Theobold et al., 2020). However, the benefits of active learning may depend on its enactment (Deslauriers et al., 2019; Johnson et al., 2020). Johnson et al. (2020) found that the instructors in an active learning class posed conceptual questions to women at lower rates than men and highlighted women's contributions less frequently than men's contributions, which they suggest may have contributed to men outperforming women in the course. Meanwhile, participants in Rainey et al.'s (2019) study reported feeling more cared for by their instructor in an active learning environment. And as previously reported in Griffin (2025), women found that interactive learning opportunities like group work supported their feelings of belonging. Therefore, given its potential to influence women's experiences, active learning classrooms provide a rich context in which to explore women's SB.

Instructor Care

The notion of caring has been equated with concepts such as compassion, empathy, and helping (Eisenberg, 1992; Ferreira & Bosworth, 2000; McCroskey, 1992). Care is relational, meaning there is always someone who is doing the caring and someone who is being cared for (Noddings, 1984, 1992). Thus, when considering instructor care, it is necessary to identify ways that students perceive care from their instructors; otherwise, the caring relation (Noddings, 2005) is not complete. Straits (2007) noted that there are elements of instructor care that are learning-centered (i.e., emphasizing content mastery and higher-order thinking) and learner-centered (i.e., emphasizing mutually trusting and respectful relationships). Researchers have identified several ways that students perceive that their instructors care about them and their learning, including learning-centered portrayals of care like involving students in lessons, checking for comprehension, providing help to students, being available to students; as well as learner-centered portrayals of care like establishing a safe and encouraging learning environment, showing a personal interest in students, and reflecting a kind disposition through actions, recognizing students' learning differences, and preparing lessons and enacting them with enthusiasm (Bandura & Lyons, 2012; Garza, 2009; Straits, 2007).

Scholars have identified links between students' perceptions of instructor care and their SB (Cheng, 2004; Rainey et al., 2019). Cheng (2004) studied factors that influence undergraduate students' sense of campus belonging and found that students' feelings of being cared for contributed directly to their SB. Further, Rainey et al. (2019) found that students who felt they belonged in STEM were more likely to report their instructors cared about them than students who did not feel like they belonged in STEM. Thus, demonstrating care to students might be a way instructors could support their students' SB.

Purpose & Methods

Prior work (Griffin, 2023, 2025) has shown that women in this study found engaging in active learning to be supportive of their SB. However, as discussed in Johnson et al. (2020), active learning can have a negative impact on students' experiences depending on its enactment. The purpose of the current study is to explore women's perspectives about the role their instructor played in supporting their SB in their active learning Calculus class, thus placing more emphasis on the instructors than on the active learning practices they employed. For this study, I focus on women's experiences without comparing them to men's experiences because women's experiences are worthy of study in their own right (Du Bois, 1983; Kitzinger & Wilkinson, 1997). The current study is qualitative in nature with open-ended survey responses being the primary source of data.

Research Context

This study was conducted at a midsized, mid-Atlantic R1 public university during the 2022-2023 academic year. The university offers a two-semester Integrated Calculus course as an alternative to the university's standard one-semester Calculus course for STEM majors. The course and sample have been described in detail in prior work (Griffin, 2023; Griffin et al., 2023). Students enrolled in the course are typically first-year students who have declared or are considering a STEM major.

The course was designed to incorporate frequent opportunities for students to engage in active learning. The course is highly coordinated in that instructors teach from collaboratively designed lesson plans, which include recommended active learning opportunities. Specifically, instructors follow a cycle of brief lecture, followed by students working on problems in groups or individually, followed by whole-class discussion. While the course was designed to incorporate opportunities for active learning, students do not know that upon enrolling.

Data Collection and Analysis

At the time of study, there were two sections of the course offered, taught by two different instructors, both committed to active learning. There were 63 and 64 students enrolled in each section. Students enrolled in the course were invited to complete a survey via email during the first week of the course. Those who completed the initial survey were asked to complete a survey during Weeks 21 and 28 of the year-long course (i.e., middle and end of spring semester). The survey included an item asking participants to respond to the following prompt in an open-ended response box: "Give 1-2 examples of what your instructor has done that has impacted your SB (either positively or negatively). If your instructor has had no impact on your sense of belonging, give 1-2 examples of what they could do to support your sense of belonging."

To analyze the open-ended descriptions, I read all the responses multiple times and recorded emerging themes. It became apparent that instructor care seemed to be an overarching idea conveyed by participants. This inspired the use of student perceptions of instructor care as potential codes, while still allowing other themes to emerge. With the instructor care literature in mind, I clustered responses by theme and read them multiple times to determine subcategories of responses within each larger theme. The larger themes and subcategories together comprised the codebook, which was used to code all responses. Data that could not be coded with the codebook were identified and analyzed to determine whether they represented a new category or a subcategory of one of the pre-existing codes. I engaged in an iterative process of coding the data and revising the codebook four times. Each code was binary, and each response could be assigned multiple codes. To achieve reliability, 10% of the responses were coded with three

colleagues not involved in the study. Together, 78% interrater reliability was achieved. Once reliability was achieved, I coded the remaining responses a final time.

Results

At Weeks 21 and 28 of the course, participants responded to a prompt asking them to provide examples of times they felt their instructor supported their SB. Women's responses tended to align with themes from the literature on instructor care, describing ways their instructor demonstrated care such as showing a personal interest in students and being available to provide help (Bandura & Lyons, 2012; Garza, 2009). Table 1 shows the six different portrayals of instructor care that emerged from women's responses along with the frequencies of women who discussed each theme. Reported *n* counts refer to the number of distinct women who provided a response within each category at either Week 21 or 28, as the goal was to capture a landscape of ways in which women articulated their SB being impacted by their instructor rather than to explore changes over time. Corresponding percentages are based on the total number of distinct women who responded to the prompt at either Week 21 or 28 (N=40). Responses could be double-coded, so percentages do not add to 100%. In the sections that follow, I provide several examples of each theme, highlighting the various ways in which the instructors supported women's SB. The sections that follow are presented in order from most prevalent to least prevalent theme, as in Table 1.

Table 1. Type of Perceived Instructor Care that Emerged from Women's Descriptions of How Their Instructor Supported Their Sense of Belonging

Portrayal of Care	Example response	Frequency, *n* (%)
Involved students and checked for comprehension (Bandura & Lyons, 2012)	"My instructor has approached me during group work to make sure I am on track with the material."	19 (48%)
Provided help during a teaching episode (Garza, 2009)	"[The instructor] has come to our table several times to make sure we understand the problems and explains it when we do not."	19 (48%)
Established a safe and encouraging environment (Bandura & Lyons, 2012)	"[The instructor] doesn't make us feel bad about getting something wrong and treats everyone the same."	14 (35%)
Showed a personal interest in the students' well-being (Garza, 2009)	"[My instructor] wants to get to know all of her students. She has spoken to me not only on math, but also empathized with me when I said that I miss home."	14 (35%)
Was available to students (Bandura & Lyons, 2012; Garza, 2009)	"[The instructor] is always available to answer questions."	8 (20%)

Reflected a kind disposition through actions (Garza, 2009)	"Our instructor is always kind and welcoming. She is also very approachable and never yells at students."	8 (20%)

Involved Students and Checked for Comprehension

When describing the impact of their instructor on their SB, many women described the ways that the instructor involved students in the lesson. For example, one woman stated, "[My instructor] encourages everybody to participate and share their ideas." Another woman noted, "[The instructor] usually engages with the class as he lectures, which keeps us more attentive and feeling valued." More specifically, other women reported their instructors encouraging them to interact with their classmates during group work. One woman said, "Our math instructor encourages us to work with each other regularly. She encourages us to talk." Another reported, "I think the group work she has us do does a lot for my social interactions with my classmates and I also think she tries her best to make sure everyone is engaged in class."

Women also identified ways in which their instructor would check to see if they were understanding the material. For example, one woman reported, "[My instructor] asks us questions constantly," referring to interactive lecture, and another stated that her instructor "checked our work and made sure we understood what was going on." Some women described this in the context of group work – as students worked on problems together, the instructor would circulate and check in at the different tables to see how students were doing. As one woman noted, "[My instructor] approaches me during group work to make sure I am on track with the material." Another woman reported, "[My instructor] is great at getting to every table and speaking with students while we're doing group work." These checks for comprehension made participants feel like the instructor cared whether they understood the material or not.

Provided Help During a Teaching Episode

In many cases, those checks for understanding led to another common theme. Many women described their instructor's efforts during class to help them learn as supportive of their SB. Women reported that when they had questions about the material, their instructor provided clear and thorough explanations to help them understand. For example, one woman said, "One time, I was struggling to solve a problem in class and I asked my instructor to help me. He made me understand the concept the second time around and told me he would help me understand the concept as many times as needed." Another woman echoed this idea: "[My instructor] makes sure we feel the content in our hearts and does not just throw content at us... She explains it very thoroughly and makes sure that you understand it."

Some women (n=5, 13%) reported on receiving help during group work specifically. One woman described the instructor providing help after checking for comprehension: "During group work, the instructor goes around and makes sure everyone is on the right track, and if not, they go and help them." Further, another woman said:

> [My instructor] always walks around to talk with us during group work and doesn't give us a hard time if he sees something wrong. He sticks there and works with the student even if it means giving a little extra group time.

This one-on-one attention during group work is something that made participants feel supported.

Establishing a Safe and Encouraging Environment

Some women explained that the safe and encouraging environment the instructor established contributed to their SB. In their descriptions, some reported feeling comfortable and not feeling

stupid or judged. For example, one woman shared, "She explains and teaches things in a way that makes me feel more comfortable being in the class." Other times, they mentioned that their instructor reassured them when they were incorrect. One woman said, "She doesn't make us feel bad about getting something wrong." Another noted, "He explained why I did poorly without making me feel like an idiot." The instructors made them feel safe to make mistakes and kept them from feeling stupid or inadequate.

One of the ways instructors can establish a safe and encouraging environment is by continually providing words of encouragement (e.g., Banduras & Lyons, 2009). In describing what their instructor did to support their SB, several women commented on the encouragement they received from their instructor. Participants reported their instructors communicating that they believed in students' potential and ability to succeed in the course. For example, one woman explained, "[My instructor] encourages us and sees our potential even if we don't see it ourselves." Another reported, "[My instructor] assured us on the first day of class that he believes anyone can learn this material and we can all pass the class."

Showed a Personal Interest in Students' Well-being

Some women reported that their instructor supported their SB by showing a personal interest in them and getting to know their students as people. For example, one woman stated, "[My instructor] wants to get to know all of her students. She has spoken to me not only about math, but also empathized with me when I said that I miss home." Another woman reported, "[The instructor] remembers who I am and details about myself so I feel like I am seen despite being in a larger sized class." Others reported times that the instructor would relate to them by sharing their experiences as students. One woman said that her instructor "related his experiences back to ours and explained that even he had struggles with the material that we do," which helped her feel included. Another woman reported that her instructor was "understanding of what we are going through in and out of class." Relating to students and forming personal connections with them, despite the larger class size, made these women feel like they belonged.

Was Available to Students Outside of Class

Some women explained that their instructor supported their SB by being accessible and available to students. These responses were similar to those coded as *provided help during a teaching episode* but specifically mentioned the instructor as "always there" for them or described the instructors' availability outside of class. For example, one woman reported, "[My instructor] is always available to answer questions and will spend lots of time making sure you understand." Another woman noted, "My instructor is very supportive whenever I contact him. He provides me with any answers to any specific questions that I have." Other women stated that their instructor was "open to helping outside of class," and "there if we do have questions."

Reflected a Kind Disposition

Finally, some women reported that their instructors supported their SB by being kind, approachable, or positive. For example, one woman reported, "[My instructor] is super approachable and cares about students." Some women described the instructor as kind in conjunction with some other way they showed care. One woman said, "[My instructor] is a positive person and really pushes for everyone to get an A or B in the class. This shows she really cares about her students' well-being and learning." Here she both describes the instructor as positive and reports that the instructor encourages students that they can succeed in the course, which is part of *establishing a safe and encouraging environment*. Another woman reported,

"[My instructor] is a sweet human who just genuinely wants you to understand the content and makes sure you understand it." This woman describes the instructor's kindness in conjunction with the way they *provide help* and make sure students understand the material.

In summary, when asked to provide examples of what their instructor has done to support their SB, women tended to describe different ways that their instructor showed care. Considering Straits's (2007) learning-centered versus learner-centered care, participants seemed more likely to discuss ways their instructor portrayed learning-centered care such as getting them involved in the lesson and providing help during a teaching episode or group work. Being encouraged to speak with their classmates and share their ideas, while also knowing they could go to their instructor for help made them feel like their instructor was on their side. Some women provided context in which the instructor showed care, which sometimes occurred during more interactive learning opportunities like group work and interactive lecture. Further, on occasion, participants seemed to view active learning opportunities and instructor care as overlapping rather than the context in which instructor care can be conveyed. For example, women reported that their instructor supported their SB by involving them in the lesson, which is the very essence of active learning. Women's responses not only highlight benefits of employing active learning opportunities, but they provide details into how these opportunities can be enacted in a way that makes students feel cared for and like they belong in the class.

Discussion

Prior work has claimed that students can have varying experiences with active learning. On one hand, some scholars have found that women who engage in active learning report a higher SB (Griffin, 2021) and feel more cared for by their instructor (Rainey et al., 2019) than students who do not engage in active learning. On the other hand, other scholars have found that active learning classrooms can have unintended negative impacts on student experiences depending on its enactment (Deslauriers et al., 2019; Johnson et al., 2020). With this report, I aimed to explore ways in which instructors can support women's SB outside of the learning opportunities they provide. Women's descriptions of the ways in which their instructor supported their SB aligned with actions other scholars have identified as ways that students perceive instructor care.

Interestingly, some of these ways of showing care suggest specific actions instructors can take in facilitating active learning opportunities. For example, giving students time to work on problems in groups or individually might support their SB because this is one way to involve students in the lesson. As students work, the instructor can circulate the room to monitor students' comprehension, make themselves available to students with questions, and provide help when needed. Interactive lecture or whole-class discussion is another context in which instructors can involve students, check for comprehension, and provide help. In the context of any pedagogical practice, even traditional lecture, instructors can establish a safe and encouraging environment by providing words of encouragement and reassuring students that making mistakes is part of learning. While prior research has linked SB with instructor care and instructional approaches (Rainey et al., 2019), the current study contributes concrete actions Calculus instructors can take in the enactment of their instruction such as responding to wrong answers with encouragement rather than judgment and sharing belief in students' potential.

Future work should explore potential gender differences in portraying care via instruction. The current study included two instructors, one man and one woman, but all descriptions of ways they supported women's SB were pooled. While both instructors were mentioned within each code category, it would be interesting to see if any portrayal of care was more or less prevalent across instructors, especially in terms of learner-centered or learning-centered types of care.

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Exploring the Relationship Between Group Roles and Participation Patterns in an Inquiry-Based Mathematics Course

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This report examines how group roles and instructional practices shaped student participation in an inquiry-oriented undergraduate mathematics course. Using classroom video with thematic analysis and equity analytics, findings indicate alignment with the four pillars of Inquiry-Based Mathematics Education (IBME). The role responsible for reporting to the class (called reporter) most clearly influenced participation, while other roles were taken up more fluidly as students responded to the needs of the group's mathematical exploration. The results suggest that IBME design fosters collaborative learning while also highlighting opportunities for future research on how group roles and instructional practices can be refined to support equitable participation.

Keywords: group roles, equity quantified in participation, inquiry-based mathematics education

Collaborative group work is central to many active learning, inquiry-based mathematics classrooms, where the goal is not only to solve mathematics problems as an active participant, but also to collectively engage students in making sense of mathematics. To support equitable participation, instructional strategies from Complex Instruction (Cohen & Lotan, 2014) suggest assigning students group roles to distribute group work labor and responsibility (Lotan, 2022; Reinholz, 2023). A substantial body of research in elementary and primary education contexts has shown that assigning group roles can positively influence participation and collaboration (Cohen & Lotan, 1997; Herrenkohl & Guerra, 1998), but relatively little is known about how such roles are taken up in undergraduate mathematics classrooms. Some scholars have argued that roles can feel artificial, as students may ignore assigned roles, self-assign roles that align with ability or status, or rely on teachers' reminders rather than developing autonomous collaboration (Good et al., 1989). While the idea of assigning group roles for group work is not new, questions remain about how students take up these roles and how the roles may (or may not) influence participation patterns in undergraduate classrooms.

This report examines how students in an upper-division undergraduate mathematics course engaged with their assigned group roles during small group work, and how these roles shaped patterns of participation, collaboration, and mathematical thinking across both small-group and whole-class discussions. To study participation through an equity lens, I used the Equity Quantified in Participation (EQUIP) tool to analyze participation disaggregated by student demographics such as program and pronouns (Reinholz & Shah, 2018; Reinholz et al., 2024). Instructional practices and peer interactions were also considered, drawing on the four pillars of Inquiry-Based Mathematics Education (IBME) as a guiding framework. Specifically, I address the following research questions: (1) How did assigned group roles shape students' participation in small-group and whole-class discussions? and (2) How did the interplay of group roles and instructional practices shape students' engagement with mathematics?

Theoretical Background

This study draws on three areas of literature: IBME as a framework that informs course design, group roles as a method to foster collaboration and equitable participation, and equity analytics, particularly the EQUIP tool, to examine who participates and how students participate.

IBME, as a theoretical framework for instructional practices, consists of four pillars: (1) Students engage deeply with coherent and meaningful mathematical tasks, (2) Students collaboratively process mathematical ideas, (3) Instructors inquire into student thinking, and (4) Instructors foster equity in their design and facilitation choices (Laursen & Rasmussen, 2019). These principles position students as active participants and instructors as facilitators who shape classroom norms and structures. Norms such as explaining one's reasoning and making sense of others' ideas support inquiry-based learning (Yackel & Cobb, 1996). Instructors further sustain these norms by attending to group composition, crediting student contributions, and using students' reflections to adapt instruction (Kuster et al., 2018; Rasmussen & Marrongelle, 2006), thereby centering student voice and fostering equitable participation.

One strategy for structuring equitable group work is group role assignment. Featured in instructional strategies from Complex Instruction (Cohen & Lotan, 1997), Reinholz (2023) describes four roles—starter, reporter, moderator, and spy—to support collaborative inquiry and distribute responsibility. The starter initiates the task by reviewing the prompt, asking clarifying questions, and helping the group establish a shared starting point. The reporter communicates the group's progress or ideas to the whole class. The spy visits other groups to gather insights and strategies, whereas the moderator ensures that all group members have opportunities to contribute. These group roles aim to scaffold meaningful collaboration and foster equitable participation by splitting the responsibilities among the individuals in a group. Prior studies show that roles can boost engagement (Herrenkohl & Guerra, 1998), increase talk and joint activity, and clarify group responsibilities (Cohen & Lotan, 2014).

However, the influence of group roles on student participation is unclear, especially in university classrooms. Students may reinterpret, disregard, or remain peripheral to assigned roles (Good et al., 1989). As Esmonde (2009) argued, equitable learning depends not only on what teachers design but also on how students interact, and a single cooperative structure may be taken up in ways that both support and constrain equity. Since assigning group roles has the intention to influence who speaks, who justifies, and whose ideas are taken up, it is essential to study how group roles actually shape student participation. To examine how strategies like group roles work in practice and whether they promote equitable participation, equity analytics tools like EQUIP can be used to analyze the discourse patterns in the classroom (Reinholz & Shah, 2018). EQUIP allows researchers to analyze classroom discourse data disaggregated by social markers such as race, gender, programs, or language background. This disaggregation helps identify participation patterns that might otherwise be invisible, such as the dominance of certain voices or the marginalization of others. Each imported contribution to EQUIP is coded based on multiple dimensions, including solicitation method (called on or volunteered), length of talk, and contribution type, to categorize the nature of a student response (Reinholz, 2023).

EQUIP has been applied in both research and professional development to support instructional change. For instance, Reinholz (2023) showed how an instructor used EQUIP data to adopt strategies like wait time, structured turn-taking, and inviting quieter students to share. Reinholz et al. (2024) also demonstrated how participation analytics can support assigning competence by highlighting overlooked contributions from marginalized students, thereby reshaping classroom status dynamics. Extending this work, Yeh et al. (2025) examined how students with multiple marginalized identities engage through embodied modes (e.g., gestures, movement, and manipulatives), showing that mathematical participation extends beyond speech. These studies position EQUIP as a powerful tool for analyzing equity in classroom participation.

In this study, I use EQUIP to examine how group roles and instructional practices shaped who participated, how they participated, and what forms their mathematical contributions took.

Methods

This study was a design-based research project aimed at redesigning an inquiry-oriented undergraduate mathematics course on dynamical systems and modeling at a large Hispanic-serving public university in the Western US. The course enrolled 31 students: 4 graduate students in the Master of Arts in Teaching Service (MATS) program, 14 prospective secondary teachers in the single-subject teaching credential (SSTC) program, 2 in mathematics education, and the remainder in programs such as mathematics, applied mathematics, biology, computer science, or computer engineering. Collaborative group work was central to the course, with four group roles introduced at the start of the semester and rotated in every class session.

Data for this report comes from a mid-semester class session consisting of two 75-minute video recordings. In this inquiry-oriented class, students spent most of the time working in groups, with the rest participating in whole-class discussion. During the small-group discussion, each camera focused on a small group to capture their discussion and shared whiteboard work. During the whole-class discussion, one camera recorded the full class, while the other centered on the speaker (instructor or presenting students). This setup produced two synchronized recordings of the session (see Figure 1 for class layout). In addition, demographic information (academic level, program, and self-reported pronouns) was collected. Recordings were transcribed using Rev AI and then manually reviewed for accuracy.

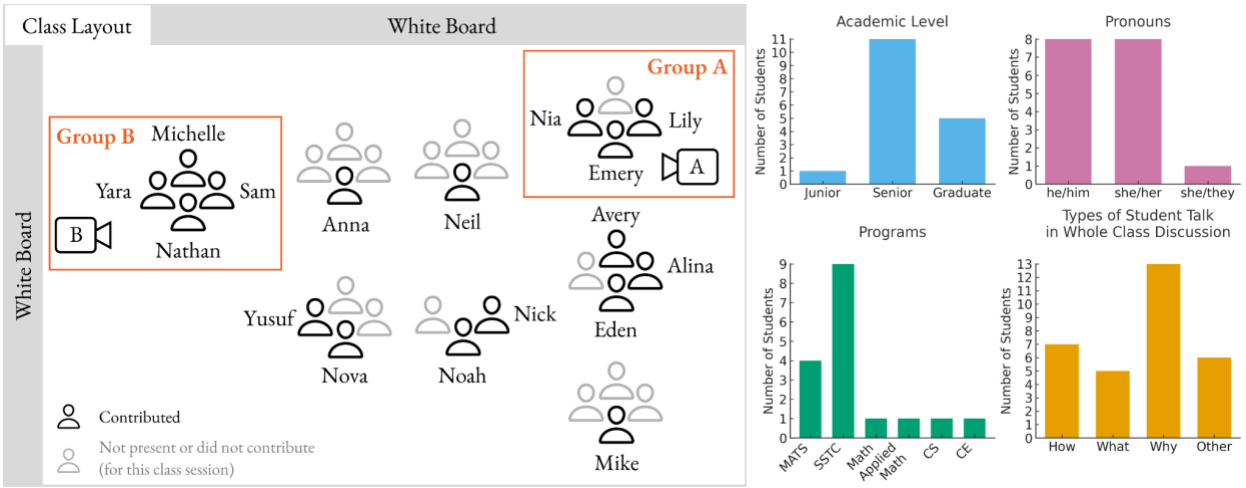

Figure 1. The class session layout, students' demographics, and types of student talk in whole-class discussion.

The class session focused on modeling coffee cooling with autonomous derivative graphs. Given temperature data over time, students approximated and fitted a differential equation for the cooling rate graphically in order to model the phenomenon of coffee cooling. They then analyzed the fitted differential equation by interpreting equilibrium behavior with slope fields and phase lines, connecting the symbolic equation, graphical representation, and situational context of coffee cooling to deepen understanding of autonomous differential equations.

I conducted a thematic analysis (Braun & Clarke, 2006) to examine students' participation patterns under the influence of group roles and mathematical engagement. Themes were iteratively developed through both inductive coding and deductive coding informed by the IBME framework, with attention to how students contributed mathematical ideas and how group roles

shaped (or failed to shape) these patterns. To systematically investigate participation patterns, I used the EQUIP tool (Reinholz & Shah, 2018), which allows researchers to import demographic and classroom data and code participation at the level of individual contributions quantitatively. Following the EQUIP protocol, contributions were classified as talk types of why (explaining or justifying disciplinary ideas), how (describing steps or processes), what (providing factual answers or recall), or other (e.g., asking questions). Contributions were then analyzed quantitatively and disaggregated by demographic variables such as academic level, programs, and pronouns. By integrating thematic analysis with EQUIP's metrics, I examined how group dynamics and participation varied across group roles and individuals, including whether certain roles were more likely to involve justification or engagement with peers' ideas. For this report, only students who made at least one contribution during small-group or whole-class discussion were included in EQUIP (17 of 31 students; see Figure 1 for demographics). All student names are pseudonyms.

Results

RQ1. How did Assigned Group Roles Shape Students' Participation?

Students in both featured groups rotated their group roles from the previous session and wrote them on the corners of their whiteboards. Group A did this even before the instructor's reminder, while Group B did so afterward. It was explicit that rotating group roles was an established class norm.

The reporter role most clearly shaped participation in this session. Lily (reporter) regularly asked clarifying questions in anticipation of reporting, and Sam (reporter) summarized his group's mathematical ideas. Other roles were taken up more fluidly. For example, Nia (starter) often initiated discussions and clarified goals, while Emery (spy) prompted her group to revisit instructions in starter-like ways. In Group B, Yara (starter) frequently restated problems and suggested approaches, while Michelle (moderator) posed questions and alternatives without explicitly signaling her role. Nathan (spy) did not participate in the spy role, likely because the instructor did not prompt spy time. Instead, he was among other students who focused on sense-making and problem-solving the modeling task within their group.

EQUIP metrics indicated how participation was distributed within groups. In Group A, Nia contributed 23 of 45 talks (51.1%), Emery had 13 (28.9%), and Lily had 9 (20%). Based on talk types, Nia (8 how, 7 what, 4 why, and 4 other) and Emery (8 what, 2 why, and 3 other) contributed more during the solving and reasoning phases, and Lily (1 how, 8 other) contributed primarily through questioning. In Group B, participation was more evenly distributed: Yara had 28 of 96 talks (29.2%), Michelle had 27 (28.1%), Sam had 24 (25%), and Nathan had 17 (17.7%), with a different mix of talk types. These data complement the qualitative analysis, showing that the amount of participation and the specific ways students contributed to group work by talk types were not dictated by roles alone; students adapted and blended roles based on group needs organically to contribute to discussions.

Overall, group roles provided a structure that occasionally influenced participation patterns, most notably through the reporter role. More often, roles functioned as background structures, as students focused on sense-making and problem-solving, with responsibilities shifting naturally to the needs of their mathematical exploration.

RQ2. How did the Interplay of Group Roles and Instructional Practices Shape Students' Engagement with Mathematics?

By comparing the thematic analysis on the class recordings with the EQUIP analysis, I observed evidence that students' engagements indicated context-rich mathematical discourse and collaborative sense-making, whereas the instructor's moves supported students' mathematical thinking and participation. The results demonstrated strong alignment with the IBME pillars.

Pillar 1: Students engage deeply with coherent and meaningful mathematical tasks. Students in both groups consistently engaged with the real-world problem that involved interpreting empirical data of cooling coffee, constructing differential equations, and reasoning about equilibrium solutions. The use of technology (e.g., GeoGebra) enabled dynamic visualization and deeper interpretation of their models. Yara used the software to empirically determine the rate of change from the data, noting that "This form [of the symbolic representation of the differential equation] is more important because you can clearly see the equilibrium solution", whereas Emery suggested that the equilibrium solution occurs at room temperature contextually. Students further contextualized the mathematical model in relation to their daily experiences. For example, Michelle discussed whether the differential equation should depend on temperature, time, or both with other group members: "Just because you leave your coffee hotter, it doesn't mean it's going to cool down faster... it's more just going to take longer to cool down to whatever your room temperature is." These episodes exemplified the kind of mathematically meaningful activity outlined in Pillar 1.

Pillar 2: Students collaboratively process mathematical ideas. Pillar 2 was evident that the two featured groups collaborated effectively to interpret coffee cooling data, calculate average changes, and create different representations of the cooling coffee phenomenon. Both Group A and B collectively reasoned about whether time or temperature was involved in the rate of change. The group roles also gave students clear responsibilities and positioned them to understand the group's mathematical progress. For example, Sam asked for clarification being the reporter when the instructor visited their group and asked the reporter to report back for the whole-class discussion, and his group supported him by helping interpret the slope of the dy/dt vs. y graph. In addition, Michelle and Yara elaborated on Sam's comments during the whole-class discussion, showing that group members saw themselves as co-contributors even when not assigned to report. Group A's Lily also interacted with the other two peers similarly.

Students even collaborated with students outside of their small group without any intervention from the instructor. In particular, Emery initiated a "spy" visit to Group B without prompting, reflecting a culture where students felt empowered to gather ideas from not only group members but also other peers. Similarly, Avery from a group near Group A visited Nia and Lily for help. These self-initiated interactions across groups further reflected a shared classroom culture that supports collaborative learning organically.

Pillar 3: Instructors inquire into student thinking. The instructor played a facilitative and responsive role throughout the class session, aligning well with Pillar 3. The instructor inquired into students' thinking, often prompting elaboration or clarification. When one group's explanation lacked clarity, the instructor asked, "Can you say it again without using the word it?", encouraging students to articulate their ideas more precisely. The instructor also made efforts to build on student-generated ideas during whole-class discussions. As the instructor walked around to answer questions or observe the ongoing exploration during small group work, they noticed students' confusion or struggle. The instructor always asked a student, usually the assigned reporter, to present their group work strategically to address struggles. The instructor

also built up ideas from students' thinking by crediting their mathematical ideas collected from small-group discussions or whole-class discussions. For example, the instructor revoiced students' reasoning: "Your table has a way of making sense for you, and I think Neil and Avery have articulated why at least it could make sense to put it in the first row", and "I love [Anna's hand gesture]… You may not have seen Anna's hand gesture because the rate of change is going this [way], almost shifting the graphs over."

Pillar 4: Instructors foster equity in their design and facilitation choices. Evidence of Pillar 4 emerged through both the classroom design and teaching practices. The use of rotating group roles distributed responsibilities and aimed to support participation across students, as discussed earlier. In addition, the instructor intentionally composed small groups to ensure there are at least two women in a group of 3 or 4, suggesting particular attention to the underrepresented voice of women students in small groups (e.g., to not be outnumbered). A counterargument about the structure of calling on reporters is that it might limit participation to one student per group. However, while reporters like Noah (he/him) and Neil (he/him) were the primary speakers during this class session's whole-class discussion (see Figure 2(b)), other students who were not reporters also supported or supplemented their ideas without solicitation. For example, Nia (she/her) helped Noah (he/him) answer a question from Avery (she/her). Another example of this was when Sam (he/him) presented their group's thinking, Michelle (she/her) jumped in to elaborate on the direction of the slope and the sign of it on the graph. These contributions suggest a culture of shared contribution, even within a possibly constrained called-on reporters structure. EQUIP metrics also aligned with these qualitative observations, as for not-called-on students, more students with she/her pronouns contributed to questions or answers during the whole-class discussion (see Figure 2(a)).

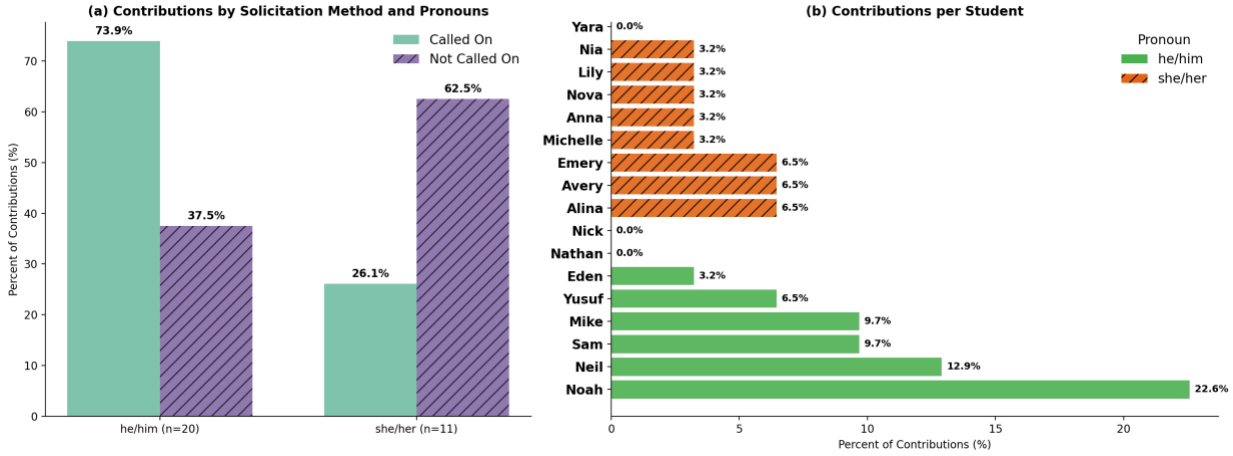

Figure 2. The percent of contributions in the whole-class discussion disaggregated by (a) solicitation method and pronouns and (b) per student.

Discussion and Conclusion

There was limited evidence that roles directly shaped student participation, aside from the reporter role. Both reporters in group A and B consistently summarized group ideas in preparation for whole-class discussions, aligning with the role descriptions and the class norm of the instructor calling on reporters. Broader participation appeared driven more by individual dispositions, class norms, and task demands than by assigned roles. For example, Nia's frequent contributions in Group A might reflect her enthusiasm and confidence rather than her role as a

starter (a pattern I observed consistently across multiple class sessions). Group B students participated more evenly, with a mix of talk types regardless of their assigned roles. In both groups, supportive group dynamics enabled students to contribute, with roles functioning more as background structures than primary drivers of participation.

Rather than viewing the fact that the majority of the group roles did not directly shape participation patterns as a disadvantage, I argue that group roles' purpose is to support collaboration and classroom management, not to dictate or prescribe students' intellectual contributions. This interpretation aligns with prior studies (Bianchini, 1997, 1999). In other contexts, such as efforts to elevate students' status (which is students' perceived academic ability and popularity), group roles may need to be assigned more strategically to promote equitable participation (Cohen & Lotan, 1997). These findings suggest that the reporter role, in particular, can be assigned strategically since it most clearly shaped participation.

Data analysis showed alignment with all four IBME pillars that foster effective mathematical discourse. Students engaged with a meaningful, real-world modeling task on coffee cooling, using empirical data, differential equations, and technology to connect math models to everyday contexts (Pillar 1). Collaboration emerged both within and across groups, supported by group roles and peer-to-peer interactions (Pillar 2). The instructor facilitated by probing and revoicing student reasoning in whole-class discussions and by visiting groups to elicit and inquire into students' thinking (Pillar 3). Equitable participation was encouraged through intentional group composition and role rotation, which created a supportive space where even non-reporters, especially women students, contributed voluntarily (Pillar 4). The classroom norms supported students to actively construct, share, and refine mathematical ideas collaboratively.

The strong engagement observed in this study was likely supported by both the design of the course and the particular context of the class. Many students were seniors or graduate students in teaching credential or math-related programs, and the instructor brought extensive experience with inquiry-based teaching. In other settings, structures like group roles may play a more central role in shaping participation, especially for less experienced students or instructors newer to inquiry-based teaching. Future research should explore how different student populations and instructor experience levels mediate the impact of group roles on students' participation, engagement, and mathematical thinking.

While some norms like calling on reporters directly shaped students' participation, the findings indicate that contributions from students with she/her pronouns were more common among those not called on. This suggests that calling on reporters alone did not ensure comparable visibility in whole-class discussion for this group of students. Future instructional designs could invite co-reporting or rotating roles multiple times per class to provide more called on chances. Another limitation of this study is the focus on a single class session, which may not capture the full variability of participation over time. Future work will compare multiple class sessions to provide more insights on how participation patterns evolve.

In conclusion, this study found that the reporter role most clearly shaped participation patterns, while other group roles blended into collaborative dynamics, and overall participation and mathematical engagement were further influenced by the interplay of group roles and instructional practices. Together, these findings highlight the promise of IBME as an instructional framework that empowers instructors to design equitable and meaningful learning experiences. For instructors, this means treating group roles and other IBME pillars as evolving instructional guidelines, refining them in response to student feedback and class dynamics to sustain equitable participation and deepen students' collaborative engagement with mathematics.

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A Comparative Case Study of Student Reasoning about Planes and Vertical Change in Multivariable Differential Calculus

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This is part of a larger APOS-grounded study where we investigate students' reasoning about and between multiple components of multivariable differential calculus. We report on the different reasoning of the strongest interview participants from two sections taught by the same instructor. One section used the locally linear approach; the other was taught traditionally. The locally linear approach leverages the geometric understanding of the point-slopes equation of a plane to help students develop key ideas for the differential calculus of two-variable functions. We used semi-structured interviews to investigate students' understanding of multiple components of differential calculus and the relations established between those components. In this report, our analysis focuses on the relations that two high-performing students established between the components of plane and vertical change. Findings suggest that the locally linear approach may help students obtain a deeper understanding of vertical change on a plane than traditional instruction.

Keywords: multivariable calculus, plane, locally linearly approach, APOS, differentials

Introduction

Multivariable calculus is an important tool to model different phenomena in science and engineering. Typically, its study begins with the simplest case, two-variable functions. The study of the didactics of two-variable functions has gained traction in the past twenty years (for a survey, see Martínez-Planell and Trigueros, 2021). In an early paper, Tall (1992) proposed how students may visualize differentials in two and three dimensions. He proposed a model to enable students to think of a two-variable function as locally approximated at a point $(a, b, f(a, b))$ by its tangent plane, with the change in function values modeled by the total differential

$$[df(a,b)](dx, dy) = \frac{dz_x}{dx}dx + \frac{dz_y}{dy}dy$$

where *dx* and *dy* are independent variables representing horizontal change at (a, b) and dz_x, dz_y are the corresponding vertical changes along the tangent plane (Figure 1) for $\frac{dz_x}{dx} = f_x(a, b)$ and $\frac{dz_y}{dy} = f_y(a, b)$. Martínez-Planell et al. (2017) built upon Tall's idea to describe the notion of vertical change on a plane (Figure 2, left) which may be symbolized as

$$\Delta z = \Delta z_x + \Delta z_y = m_x \Delta x + m_y \Delta y.$$

From this idea, you can obtain

$$z - c = m_x(x - a) + m_y(y - b)$$

the point-slopes equation of a plane,

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

the equation of a tangent plane at a point, and directional derivatives (Figure 2, right). While the idea of vertical change appears when interpreting directional derivatives, students find it

challenging to construct this notion on a plane. Students do not readily generalize slopes to three dimensions and have difficulty combining rates of change in the *x* and *y* directions into a single rate of change (Martínez-Planell et al., 2015, 2017; Moore-Russo et al., 2011; Weber, 2015; Yerushalmy, 1997). McGee & Moore-Russo (2015) observed that students tend not to think of partial and directional derivatives as slopes of lines on the tangent plane. This led to the design of activities to help students construct a locally linear approach to the differential calculus of two-variable functions.

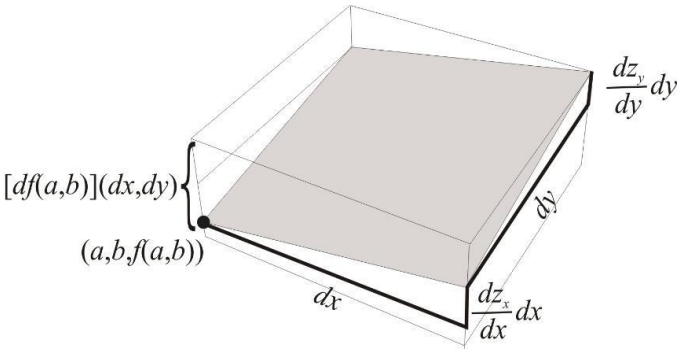

Figure 1. Total differential at a point.

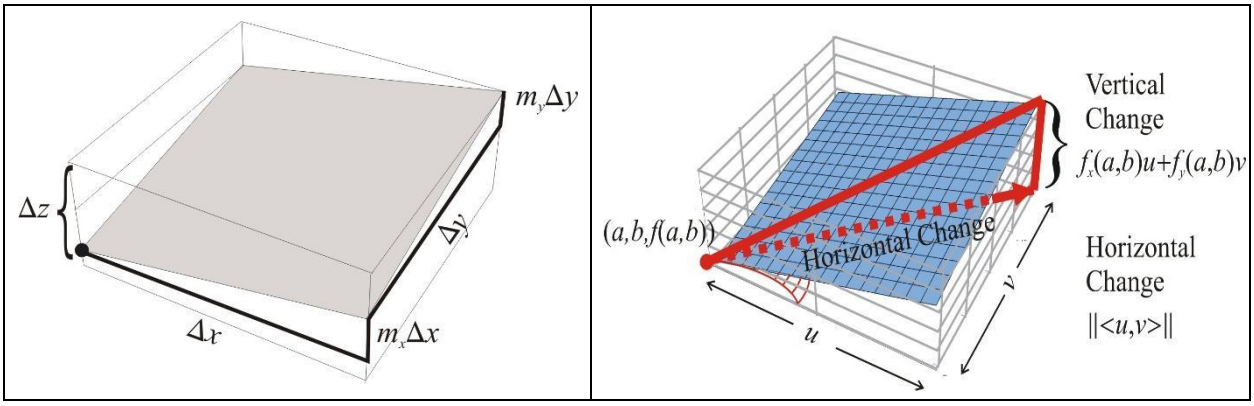

Figure 2. Vertical change on a plane, $\Delta z = \Delta z_x + \Delta z_x = m_x\Delta x + m_y\Delta y$ *and directional derivative,* $D_{\langle \Delta x,\Delta y\rangle} f(a,b) = \frac{f_x(a,b)\Delta x + f_y(a,b)\Delta y}{|\langle \Delta x,\Delta y\rangle|}$.

Borji et al. (2023, 2024) used previously documented activities to observe improved student understanding of partial and directional derivatives. The present study adapted some of these activities for an interview protocol. In this paper, we focus specifically on the relations students established between plane and vertical change. The research question driving this study is: How do two high-performing students, one taught with the locally linear approach and one not, differ in the types of relations they demonstrate between two key components of multivariable differential calculus, plane and vertical change, when solving a series of tasks involving geometric interpretation?

Framework

APOS Theory (Arnon et al., 2014) considers how students carry out Actions (step-by-step manipulations) mechanically until they are able to internalize these as Processes where they can imagine actions without actually performing them. These processes are then encapsulated into a mental Object that in and of itself can be acted on and manipulated. Finally, a Schema of a

mathematical concept is a structured collection of Actions, Processes, Objects, and previously constructed Schemas. These components are interconnected so that an individual can recognize when a problem fits within the Schema's scope (coherence) and can activate it to address a task that is presented. Schemas constantly evolve as the students face new problems and learn new mathematics.

In this paper, we use the notion of types of relations between Schema components, introduced by Trigueros (2019), as helpful in describing this evolution. A *correspondence* (C) relation exists between components when, out of habit or repetition and with limited external prompting, the individual correctly relates one component to another without being able to justify the relation. A *transformation* (T) relation between two components exists when a student is able to explain or justify how one component relates to another, possibly through the use of other mediating components. An *equivalence* (E) relation exists between components if properties of one component are conserved or maintained in another such that changes in one component can fluently be interpreted as changes in the other. The result is that a person is able to move easily and efficiently from one component to another, in a consistent manner, across tasks despite the particulars of any given task.

Methodology

One of the researchers taught two multivariable calculus sections for engineering students in Spring 2024. Both sections used the Stewart (2012) textbook and had the same homework exercises with only a few pre-class assignments that differed. Both sections dedicated the same amount of class time to differential multivariable calculus and had common assessments. One section with 19 students was taught traditionally. In particular, the point-normal equation of a plane was underscored and used to develop the differential calculus. This is convenient algebraically but can mask the geometry. With the traditional section, the point-slope equation of a plane was introduced later but was not emphasized. The experimental section with 40 students used activities based on those discussed by Martínez-Planell et al. (2017) and adapted them for use with 3D printed surfaces and the digital application CalcPlot3D (Seeburger, 2016). Within these activities, students developed an approach to compute vertical change on a plane using a 3D-printed surface and later used this to construct the point-slope equation of a plane, prior to exposure to the point-normal equation.

All students were invited to participate in two 45-minute semi-structured interviews at the end of the semester; ten agreed to participate. Most of the 27 questions posed were taken from Borji et al. (2023, 2024). Two of the authors independently coded each participant's response for each question; then, all three discussed and analyzed codes until a consensus was reached. The coding process involved first identifying the components of differential calculus the student was relating in their response, including Function (F), Slope (S), Partial Derivative (PD), Directional Derivative (DD), Gradient (G), Vertical Change (VC), and Plane/Tangent Plane (P). We then decided on the type of relation (i.e., no relation (NR), correspondence (C), transformation (T)) between components, if any, the student was demonstrating in their answer to that particular question. Finally, we looked at each pair of components and identified for which problems that pair had been coded. Looking across the problems and the type of relation coded for each problem, we were able to assign a type of relation for a pair of components. An equivalence (E) relation was assigned to a pair of components only if it was established consistently and fluently across the different problems involving the components.

For brevity, we report only on our coding for the relations established by two students between the components plane or tangent plane (P) and vertical change (VC). Note that the VC code refers to a process conception of vertical change on a plane or tangent plane when there is horizontal change in both the x and y directions.

Results

In this report, we compare the cases of Jackson and Daphne. They were the strongest performers on the interview tasks in the experimental and traditional sections, and they received course grades of A and B+, respectively. We focus specifically on the relations Jackson and Daphne established between plane/tangent plane and vertical change and the components they used to mediate these relations. Below, we present and discuss transcripts from the three interview questions which provided opportunities for students to relate the plane or tangent plane to vertical change.

For question 1, students were given Figure 3. After finding slope of the line for question 1a, in 1b students were asked: *Starting at any point on the plane, find how much the z coordinate changes if the x coordinate increases by 4 units and the y coordinate increases by 5 units*.

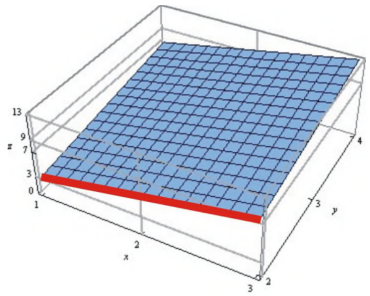

Figure 3. Plane for question 1b.

Jackson: So, I have m_x, to complete the problem I need to find m_y [he correctly finds m_y] so, the change in x is equal to 4, change in y equals to 5. So, with that in mind … change in z is equal to 3 times 4 plus 2 times 5, 12 plus 10, 22

Without prompting or assistance, Jackson used slope and a given amount of horizontal change to determine a resulting change in z in one direction. Additionally, he was able to coordinate the change in z resulting from movement in two directions to find vertical change. This was coded as a transformation relation because Jackson was able to explain how he moved from plane to vertical change, using slope as a mediator (P-S-VC). We now consider Daphne's response to question 1b.

Daphne: So, $\Delta x = 4$, $\Delta y = 5$ … I'm setting up the equation of a plane… that won't work…Okay, I don't really know how to, like, think about this one.

Interviewer: Let me give you a hint. What would be the change in the z coordinate if the x coordinate increases by 4 units? Only the x coordinate.

Daphne: If the x coordinate increases by 4 units, the z would increase by 12

Interviewer: Good, and what would happen if the y coordinate increases by 5 units?

Daphne: If the y coordinate increases by 5 units, then the z coordinate will also increase, but I don't know by how much… I guess I could find the … entire equation of the plane.

Interviewer: Just like you found how much z increases when x changes, could you use the same idea to find out how much z changes when y changes?

Daphne: … I guess; I don't know.

Interviewer: Let me give you another hint, can you find a slope in the y direction?

Daphne: [She was able to find the slope in the y direction.] … delta x would equal 3 times 4, and delta y would equal 5 times 2, that's 12 and this is 10, so then, in that case, could you square those two values, add them together? … I just don't know what to do with this information now that I have it.

Interviewer: Okay, so you are standing at a point, you move in the x direction, and you go up 12 units. Then you move in the y direction, and you go up 10 units. So, how much did you go up overall?

Daphne: Oh! It's just 22.

We see that Daphne eventually related the given plane component (P) with the requested vertical change (VC), using the slope component (S) to mediate. However, this happened only with significant scaffolding from the interviewer. We therefore coded "no relation" (NR) for P-VC for Daphne on problem 1b, despite her eventually producing the correct answer of 22. Daphne demonstrated a tendency to rely on habit or repetition, more than once suggesting finding the plane's equation without any indication that she understood its meaning. She first needed to be prompted to compute the slope in the y direction and the changes in z due to moving solely in the x or y direction. Moreover, Daphne showed difficulty coordinating changes in the x and y directions, as Yerushalmy (1997) and Weber (2015) have observed in their work.

Problem 1c asked students to find the equation of the plane in Figure 3. They each did so correctly. Then, they were asked to relate this equation to the previous calculations in part b.

Daphne: I mean, yeah, it includes both of the slopes, and, like, it shows us how x and y, like, change independently of each other …

Daphne produced the point-slope equation of the plane with no assistance. This appears to be a memorized process for Daphne. When asked to relate the equation to prior calculations, she focused on the symbolic calculations in the formulas without any geometric interpretation. Therefore, this was coded as a correspondence relation (P-S-VC). For Jackson:

Jackson: I guess it's, yeah, related in the sense that if I plugged in an (x, y) point, an x point of, x change by four, x equal to 5 [from the initial x equal 1] and a y equal to 7 [from the initial y equal 2], I'd get a change in z of 22. So, depending on where I start, if I were to change x by 4 and change y by 5, my final z value would be z initial plus 22.

For problem 1c, Jackson was coded as establishing a transformation relation between plane and vertical change (P-S-VC). In his explanation, we can see that Jackson assigned geometric meaning to specific algebraic expressions within the plane equation. Specifically, he identified $(x\text{-}1)$ and $(y\text{-}2)$ as horizontal changes. When Jackson said "depending on where I start," this suggests that he was aware that vertical change is independent of initial point, but depends only on the slopes and horizontal changes. He also went further to show how he used his equation to produce a vertical change of 22 units to correspond with what he calculated in Problem 1b.

Problem 4a (in Figure 4) presented a different plane as the plane tangent to a surface at a particular point and asked, "What can you say about the change in the value of the function if x increases 0.02 units and y increases 0.03 units?"

We note that problem 4a is computationally the same as 1b. The difference is that in 4a, the plane is presented as a plane tangent to a surface and the problem asks about the change in z for

the surface, rather than the plane itself. Both Jackson and Daphne first attempted the problem very similarly to one another, using a visual approach.

Jackson: We can expect if *x* slightly increases in the positive *x* direction and *y* slightly increases in the *y* direction, the *z* value should increase.

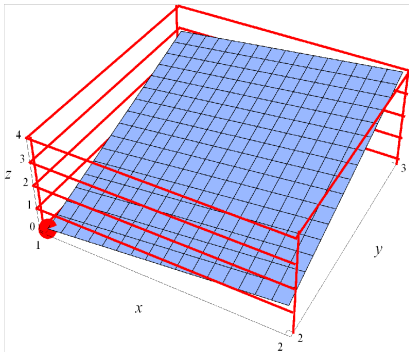

Figure 4. Plane for question 4a.

Daphne, in answering question 4a, first made a connection to problem 1b.

Daphne: This is asking a similar question to, I think, one of the ones on the first or second page. And, then, so if *x* increases 0.02 and *y* increases 0.03 units, the change in the value of the function would be positive, cuz, like, this way it goes like that [starting at the base point, pointing along the edge of the plane shown in the figure, corresponding to increasing *x*], and that way it goes like that [pointing along the edge of the plane, corresponding to increasing *y*], so the net vector would be positive.

Each student made a claim that the vertical change would be positive. The interviewer then prompted each student to calculate, or explain how to calculate, a value.

Jackson: I'm going to find the equation of the plane, m_x times the change in *x* plus m_y times the change in *y* equals *z*. Next, I was gonna try and find m_x . To find m_x I just fix it at the plane *y* = 2 and see how *z* changes when I move from *x* = 1 to maybe *x* = 2. And same thing for m_y. I'm in the plane *x* = 1, and see how *z* changes when I change *y*. And my change in *x* and change in *y* is gonna look like *x* - 1 and *y* - 2 and when I plug in the points, if *x* increases, okay, my change in *x* is given to me. They told me 0.02 and my change in *y* is 0.03, so the main work is just finding m_x and m_y and I should get my final *z* value.

Here we see consistency between Jackson's approach and a geometric interpretation of algebraic expressions in problem 4 with those in problem 1. For example, he identified "*x* - 1" as "change in *x*" and "*y* - 2" as "change in *y*." Jackson recognized that .02 could be used instead of *x* - 1 and .03 instead of *y* - 2, showing that he could give geometric meaning to the algebraic expressions in the formula. When the interviewer asked Daphne to calculate a value for the vertical change, she did not make any further connections to problem 1b.

Daphne: Yes, I could find, um…I know there is a way to do this. I'm just gonna say that it's around 1 for the *z* value.

Daphne ultimately abandoned this question with no further explanation or attempt at computation. In contrast to Jackson, Daphne did not identify that she needed to find slopes in the *x* and *y* directions, despite having been prompted to do so in Problem 1b. It appears that the process she had used for computing vertical change in problem 1b had not been interiorized; her actions had been guided externally by the interviewer. It was not apparent to Daphne that she

could use the equation of a plane to compute vertical change. Therefore, for problem 4a, we coded Daphne as not demonstrating a relation between tangent plane and vertical change.

We see evidence that using the locally linear approach with CalcPlot3D and 3D-printed surfaces enables Jackson to construct a geometric understanding of ideas in differential calculus of two-variable functions in a way that makes sense for him and that he can explain. Jackson, possibly as a result of the classroom activities, was able to give meaning to movements along a plane, slopes, and vertical change. Throughout the interview, Jackson was consistent and fluent in relating the plane, slope, and vertical change components. He considered that properties of the plane component are conserved in properties of the vertical change components, so we considered Jackson to have established an equivalence relation for P-S-VC.

Daphne was quick to produce equations of planes when a problem involved a plane, whether helpful or not. She was not able to relate the algebraic expressions in her plane equations with geometric representations. Overall, we considered that she established the relations P-S-VC as a correspondence relation because she relied on a memorized procedure to produce equations of planes and needed significant scaffolding to relate components in unfamiliar question formats. The relations established by Jackson and Daphne are summarized in Table 1.

Table 1. Comparison of coding of PTP-S-VC relations for Jackson and Daphne

Problem with Description		Coding for Jackson		Coding for Daphne	
		Code	Justification	Code	Justification
1b	Compute VC on plane	T	Computed and explained computation	NR	Computed with significant scaffolding
1c	Create equation of plane and relate to computation of VC	T	Created equation and explained connection between algebraic expressions and geometric features	C	Created equation but could not interpret computation geometrically
4a	Estimate VC on function using tangent plane	T	Made visual estimate and was able to explain how to compute VC	NR	Made visual estimate but was not able to compute VC

Implications and Limitations

This study provides further evidence that constructing vertical change on a plane and geometrically interpreting components of the point-slope equation of a plane are non-trivial for multivariable calculus students. Jackson's understanding of the components of plane and vertical change is more robust than Daphne's, in the sense that he can interrelate the components through slope flexibly across different problems and justify his solutions based on his geometric insight. We hypothesize that the activities using digital visualization and 3D-printed surfaces in Jackson's section supported him in developing these relations. However, we note that greater similarity of classroom activities to the interview tasks in Jackson's case, and the focus on only two participants of all interviewed, limits the conclusions we can draw about the impact of these activities relative to traditional instruction. The study also provides a theoretical contribution by considering the types of relations between Schema components, which is a relatively new idea in APOS theory that can be better understood through further applications to empirical research. Future studies can report on the analysis of all participating students and the relations between all the differential calculus schema components.

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"I Can Make a Difference Through Mathematics": Mathematics Majors' Conceptualizations of Justice in the Context of Teaching Mathematics for Social Justice

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There has been a call to emphasize how mathematics can be used to tackle social issues in ways that create a more just world. Such efforts can take the form of an explicit topic, entire course, or as a call-to-action impacting communities in various manners. This study analyzed course reflections of 15 mathematics majors enrolled in a Teaching Math for Social Justice course. We used open coding and constant comparison analysis to analyze pre-class and post-class reflections to understand the ways that students conceptualized the notion of "justice". Eight themes emerged from the data, with notable differences between the ways students described justice before and after the course, with students' conceptualizations demonstrating change or becoming solidified after completion of the course. These findings contribute to the larger body of research that continues to investigate how mathematics students engage with incorporating social justice contexts into mathematics teaching and learning.

Keywords: Teaching Math for Social Justice, Justice

In the past 40 years, there has been a call to emphasize how mathematics can be used to tackle social issues in ways that create a more just world. Social justice pedagogy "can be used in mathematics classrooms to help students interpret and apply mathematical knowledge to answer questions that will potentially empower their lives and their communities" (Leonard et al., 2010, p. 264). Scholars like Bob Moses (2002), Rico Gutstein (2006), and Rochelle Gutiérrez (2013) are a few in a long list of scholars who have committed themselves to using mathematics to engage in and address real-world issues. Grounded in critical pedagogy (Freire, 1970/1996), which asserts that education can be used to transform and improve society, using mathematics to understand and address issues of social justice is a way to understand and "engage in political and economic realities of everyday life" (Kirylo et al., 2010, p. 332). With a recent sociopolitical turn in the mathematics education field (Gutiérrez, 2013), there has been a more focused attention towards issues of social justice. Gutstein (2006) defines reading the world as using mathematics to "understand issues of power, resource inequities, and disparate opportunities between different social groups and to understand explicit discrimination based on race, class, gender, language, and other differences" (p. 25-26), which he argues is necessary for students.

Teaching mathematics for social justice (TMfSJ) has become an explicit topic in teacher preparation programs, courses in mathematics departments, and as calls to action for college mathematics faculty. We believe that the education system is one of many institutions in the United States that has been historically oppressive, violent, racist, sexist, ableist (and more). Asking students to engage and learn about these facets in a safe space in the context of mathematics is one way that we can begin to acknowledge these oppressions in ways that we can make important shifts towards change. Gonzalez (2012) shares four components of TMfSJ: 1) access to high-level, quality mathematics instruction for all, 2) recentering the curriculum around the experiences of students from marginalized communities, 3) use of mathematics as a critical tool for understanding social life and injustices within it, and 4) using mathematics to actively advocate for social change toward a more equitable society (p. 128-129). There are multiple authors who have published materials to support the conceptualization of using real-world social

issues to engage in mathematics learning. Gutstein and Peterson's (2013) book provides context and "examples of how to weave social justice issues throughout the mathematics curriculum and how to integrate mathematics into other curricular areas" (p. 1). Berry III et al. (2020) developed a textbook that helps mathematics teacher educators support future high school teachers to understand relevant social justice standards in the classroom context while also providing ready-to-use lesson plans. Karaali & Khadjavi (2019/2021) provide examples of lessons for the college classroom, in which they point out that college mathematics instructors have "the opportunity and the responsibility to help…students develop strong analytical skills and empower them to tackle complex issues" (p. 3).

In recent years, more emphasis has been placed on preparing future mathematics teachers to incorporate social justice contexts within their mathematics lessons. This has been challenging, however, given that many students have not experienced instruction that utilizes societal issues as context for the mathematics content. Leonard and Evans (2012) found that it is difficult to change preservice teachers' beliefs and dispositions around social justice via short-term interventions (e.g., five-week module or after participating in a social justice activity), indicating that it may take more time to assist in developing a commitment to social justice. Bartell (2012) asked preservice mathematics teachers to articulate how they conceptualize the notion of TMfSJ, in which students felt they better understood their community, were more motivated to be more active citizens, and that they saw mathematics as a tool to change the world. However, it has also been challenging to find appropriate examples that connect to the content in the mathematics classroom (Leonard et al., 2010). Bartell (2013) found that preservice teachers struggled to find cohesion between the mathematics content goals of the lesson to the social justice goals, with students often focusing on mathematics content in the start of the lesson and finalizing the lesson by returning back to social justice goals, not finding ways to negotiate both at the same time.

Theoretical Framing

This study is framed by the notion of social justice pedagogy (Leonard et al., 2010), which allows students from all backgrounds to engage in meaningful mathematics grounded in relevant contexts which develops their mathematics identity. Social justice pedagogy provides the opportunity for students to develop an awareness of their world, while also developing critical mathematics literacy to accomplish goals relevant to them (Martin, 2000). Larnell and colleagues (2016) argue that it is important for a student to have agency within the mathematics classroom while also developing their mathematics identities – but that their identities are impacted by social structures. That is, a student's mathematics identity develops within the context of understanding the world around them. We believe that part of the development of mathematics identities through social justice pedagogy is that students have the need to develop a personal understanding of what justice means to them within the contexts of their own lived experiences. Parsons (2005) argues that oftentimes, the literature presents justice and caring as incompatible ethics. On the one hand, justice is seen as addressing moral problems through previously established rules (Parsons, 2005). However, from an ethic of caring, individuals approach moral dilemmas differently, considering "responsiveness to others and the nurturance of interpersonal relationships as paramount in determining action" (Parsons, 2005, p. 26). Parsons argues that "justice and caring must come together in the act of teaching; if not, systemic inequities and inequalities are preserved and perpetuated by the teacher's actions" (p. 26). Su (2020) defines primary justice (treating people with dignity and care) and rectifying justice (spotting something wrong and trying to make it right). Su argued that if primary justice was the only type of justice present, then rectifying justice would not be needed. Defining justice is a nuanced task–Gutiérrez

(2018) believes that concepts such as equity are difficult to engage in when people hold different conceptualizations of it—we argue the same is true related to the concept to justice. We believe that a student's engagement in social justice pedagogy can shape their understanding of the ethic of justice such that their definition of justice can take on a more caring perspective, and that this can impact their mathematics identity. Our study investigates how mathematics majors enrolled in a TMfSJ course: 1) conceptualize the phrase "justice" prior to and upon completion of the course and 2) change/develop their perspectives on what "justice" means to them.

Methods

This study took place at a large, public Hispanic- and Asian American Native American Pacific Islander-serving institution (HSI & AANAPISI). A total of 15 mathematics students enrolled in an upper-division mathematics course intending to engage students in using mathematics as a tool to investigate issues of social justice. Within this course, students engaged in social justice lessons and were expected to develop and implement their own lesson. Students were labeled with identifiers S1-S15. Eleven students majored in pure mathematics, three majored in applied mathematics and one majored in electrical and computer engineering with a minor in math. There were a total of nine men and six women in the course. Three students identified as Asian (one woman; two men), six identified as Latine (four women and two men), two identified as Latine and White (one woman; one man), three identified as White men, and one student identified as a Latine, Asian, & Black man. Six students spoke another language at home, including Spanish, Mandarin, Vietnamese, and Bengali. Two-thirds of students were first-generation status while 80% of students received financial aid. Two-thirds of students worked while enrolled in the course, with two of those students working a full-time job. Not all students who enrolled in the course intended to go into the field of teaching–only half of students had this as a goal, whereas others planned to work in industry.

All course assignments were collected consisting of homework, reading reflections, students' lesson plan, and a final portfolio. The data for this paper is pulled from two course assignments. Prior to the first day of class, students were asked to read Chapter 1 and 10 from Su's (2020) book *Mathematics for Human Flourishing* and to reflect on what the phrase justice means to them. At the end of the semester, students were asked to submit a final portfolio where one component asked students to reflect on their first write-up and to re-evaluate their definition of justice. Students were not obligated to change their conceptualization and were encouraged to utilize any experiences in the course in their reflection. For sake of analysis, we will call these pre-class (PreCR) and post-class reflections (PostCR). One student did not submit her PostCR and therefore will not be included in the change analysis.

The PreCRs and PostCRs were analyzed in two rounds using open coding (Merriam, 2009) and constant comparison analysis (Corbin & Strauss, 2008). First, two authors coded each assignment individually, developing and grouping themes that came from the data. All authors met to discuss these themes, developing a set of eight themes (discussed in the findings). Next, two authors analyzed the differences in the students' conceptualization of justice. We categorized these differences as 1) moderate change, 2) profound change, or 3) solidified. We defined a change in their conceptualization based on 1) whether the students explicitly stated that their definition had changed, 2) students indicated that they wanted to expand their definition, or 3) when a student did not make explicit mention, all authors discussed the students' PreCR and PostCR accounting for nuances in their writing (in which we utilized the presence of the eight themes developed from the initial coding to determine whether the change was moderate or

profound). A student's conceptualization of justice was labeled solidified if the student explicitly stated that their definition had been "solidified" from their initial writing, indicating no change.

Given the nature of TMfSJ, we recognize the importance to share the authors' positionalities. Author 1 is a cisgender, white, disabled, queer woman who is a tenured professor at a HSI in a large urban city. She is also the instructor of the TMfSJ course analyzed in this study. Author 2 identifies as a white, able bodied, neurodivergent, cisgendered woman. She is a doctoral student at a large public HSI. Author 3 is a cisgendered Hispanic Chicano male who is a masters student at the same institution as Author 1. He was enrolled as a student in the TMfSJ course analyzed in this study. Due to proximity of both Author 1 and Author 3 to the course, both authors consistently reflected on and discussed possible subjectivity during the analysis process. All data were de-identified, yet due to the small sample, full anonymity may not always be possible.

Findings

We will first discuss the students' conceptions of *justice* in their PreCR and PostCR through the themes that developed from their responses. Then we will discuss the changes we saw in students' conceptualizations from start to end of the semester. Table 2 shows the eight themes demonstrating the total counts from the PreCRs (270) and PostCRs (172) across all responses; each theme contains a set of subcodes that made up that theme. An excerpt could receive more than one subcode when it contained multiple ideas (e.g., "justice means fairness and equality" was coded as "fairness" and "equality" which both fell under the theme JFEO). Repeated mentions of the same term within a single statement/sentiment were not double-coded. It is important to note that different themes could apply to the same statement. For example, the statement, "[Justice] is actively working to give voice to individuals who do not have one or need help being heard." which would be coded under themes JP and JRV.

Table 1: Themes Related to Conceptualizations of the Word "Justice"

Justice as…	PreCR Freq (N=15)		PostCR Freq (N=14)	
Fairness, Equity, & Opportunity (JFEO)	48	(17.8%)	27	(15.7%)
Systemic/Institutional (JSI)	40	(14.8%)	26	(15.1%)
Contextualized (JC)	98	(36%)	50	(29%)
Care, Dignity, & Ethical Guidance (JCDE)	19	(7%)	8	(4.7%)
Rectification (JR)	18	(6.6%)	8	(4.7%)
Recognition & Valuation (JRV)	24	(8.8%)	15	(8.7%)
Identity, Positionality & Empowerment (JIPE)	9	(3.3%)	12	(7%)
Praxis (JP)	14	(5%)	26	(15%)

Pre- and Post-class Conceptualizations of *Justice* by Theme

Of the eight themes, we will discuss four that demonstrated most variance from PreCR to PostCR. The most frequently coded themes across both reflections were Justice as: Fairness, Equity, and Opportunity (JFEO), Systemic/Institutional (JSI), and Contextualization (JC). The first, JFEO, refers to students' associations between justice and familiar terms such as fairness, opportunity, equality, equity, access, and support. JFEO captured moments when students emphasized equal opportunities are a core component of justice. JSI reflected the ways students connected justice and injustice to broader systems, institutions, and structures. JC encompassed the ways that students drew on specific examples, metaphors, and lived experiences to define justice. Reflections in JC stressed that justice is dependent on context and is up for interpretation. We noticed significant decrease in these three themes from pre to post with one theme

demonstrating a notable increase, Justice as Praxis (JP), which represented views as justice as action through advocacy, responsibility, and efforts to redress injustice.

Justice as Fairness, Equity, and Opportunity. JFEO was coded 48 times (17.8%) in PreCR and 27 times (15.7%) in PostCR. Within the JFEO theme students most often invoked concepts of fair/fairness and opportunity. Early responses framed justice in individualistic and concrete terms, equating justice with fair and equal chances. There were cases in the PreCRs where students used the terms equity and equality interchangeably, S6, a first-gen, white man wrote "justice means to treat everyone with equitability and fairness...upholding the idea of equality". This illustrates how students collapsed equity, equality, and fairness into a single cluster of ideas. By the end of the course, however, references to fairness and opportunity had declined, and some students began to articulate clearer distinctions between equity and equality. S10, a first-gen, Latina woman explained that "equitability isn't the same thing as equal". Such statements signal a progression from broad, interchangeable framings towards more precise definition of justice.

Justice as Systemic/Institutional. JSI was coded 40 times (14.8%) in the PreCR and 26 times (15.1%) in the PostCR. The PreCRs highlighted the absence of justice within systems through references to discrimination, prejudice, bias, and stereotypes. By contrast, PostCRs contained fewer references to these individual or interpersonal manifestations and instead demonstrated an increased emphasis on a larger view of social justice and systemic injustice, and the notion of "more than individual" appeared. S1, a first-generation, Asian, Bengali-speaking man reflected in their PostCR, "Justice, I now realize, is a concept that extends far beyond individual fairness." Another student, S7, a white man, admitted that in their PreCR, "I believe I focused too heavily on what justice means for me". Such statements capture students' movement away from describing isolated manifestations of injustice toward recognizing the structural mechanisms that sustain injustice. In PostCRs, students began to reflect on larger, systemic oppressions that may inhibit justice. S13, a Black/Asian/Latine man shared the value of TMfSJ, indicating that "If we want to give justice to those that have been systematically oppressed by the U.S government, then teaching mathematics for social justice will be a necessary step."

Justice as Contextualized. JC was coded 98 times (36%) in the PreCR and 50 times (29%) in PostCR. The subcode contextual most often applied in PreCRs, marking when students made up an example to ground their definition or indicated that the definition was dependent on the context of a scenario or situation. Students also frequently relied on lived experiences (moments when a student would recall a time in their life or retell a story from someone they knew) and connections to education (when a student mentioned anything related to the presence of justice/injustice within education) to make sense of justice. While JC decreased in frequency from the PreCRs to PostCRs, various subcodes shifted in density; all three subcodes above decreased by more than half. A code that increased significantly in the PostCR was *connection to math*. This was coded whenever a student mentioned mathematics or the math classroom, oftentimes referring directly to the power that mathematics has when addressing issues of social justice. S15, a white man, shared, "I can make a difference through mathematics" and S5, an Asian Vietnamese-speaking man indicated that he recognized that "the intersection of mathematics and social justice reveal[s] how math education can be a powerful tool to address and understand social justice issues." This shift indicated that students were formulating a more attuned perspective of justice within their world as mathematicians.

Justice as Praxis. JP was coded 14 times (5%) in the PreCR and 26 times (15%) in the PostCR, illustrating a significant increase in responses related to collective action. There was an increase in the subcode *practices, advocacy and action*, which was applied any time a student

mentioned the importance of justice-oriented practices, recognized injustices, called for action, or provided examples of how they or others could engage in justice work and the subcode *responsibility of justice*, captured when moments when students described a personal or collective responsibility to act, reflecting on the roles individuals play in correcting injustices, or contributing to justice. In their PostCRs, S7 and S15 (both white men) shared this shift to action. S7 shared "You instead need to look at the greater efforts and ask yourself how you can help in such efforts". S15 viewed his role as one who influence his community, "I will continue to challenge those around me to speak out on injustice and take the necessary actions to bring positive change to the community I serve." Taken together, this theme showed notable growth and suggests that students' conceptualizations of justice shifted from abstract ideas toward enactment and commitment to transformative action. The rise in responsibility language highlights a deepened sense of both personal and collective obligation.

Development of Conceptualizations of *Justice* at the End of the Course

Of the 14 students who submitted both PreCRs and PostCRs, five students showed moderate change, seven students exhibited profound change, and two students' views were solidified.

Moderate Change. Students that demonstrated moderate change mostly indicated adjustments they would make to their original conceptualization. For example, S2, a Latina Spanish-speaking woman, initially described justice as "a staircase with many exit points" indicating that the world has little respect for justice of fairness. She chose the staircase analogy because "you can step off when you're done or tired of fighting…where some people keep fighting till the top, some people fall off the stairs and some sit on the stairs instead of getting off". In her PostCR, she slightly modified her analogy to indicate that "everyone is actually on their own ladder regardless of what they want to be." S9, a Latino/White Spanish-speaking man, initially stated that, "justice is a response to make matters right as much as possible for victims of wrongdoings and for consequences to ensue for the perpetrators of the wrongdoings." Later, he indicated that overall he still agreed with his first definition but with a slight change in what "the scope for which the word justice could be used for" indicating that initially he only saw it as used purely in legal issues, however, now he sees "that justice is not only for legal cases but for issues of social inequity beyond the lens of legality."

Profound Change. Seven students indicated profound change in their conceptualizations of justice, indicating complete re-definition of the concept. For example, S6, initially described justice as "treating individuals equitably and fairly, regardless of their background, identity, or circumstances." Yet, in his PostCR he indicated that he sees it as much bigger than being equitable and fair, indicating that using the word "fair" was too ambiguous. He expanded his thoughts by sharing, "Justice to me is ensuring that everything is taken at face value from person to person, circumstance to circumstance, regardless of any assumptions or pre-existing notions. However, this course hasn't just changed my understanding of justice, it's changed the way I approach and think of issues and broadened my mind about how people think." S10, initially conceptualized justice as "when everyone gets a chance to experience life equally." Later, in her PostCR, she shared that justice "means making sure everyone is given the tools they need so that everyone is at the same level or in other term everything is equitable for everyone. The reason for this is because equitability isn't the same thing as equal because equal just means to give everyone the same number of things while equitability means to make sure everyone is at the same starting point."

Solidified. Two students were labeled solidified. Both students utilized this word in their write-up, indicating that this course reaffirmed their initial view of justice, making it a more

stable conceptualization for them. S8, an Asian Mandarin-speaking woman, initially indicated that "justice isn't just righting a wrong but showing the true qualities of something." Later, she indicated that her understanding had been solidified after the 15-week course. S13, a Latino/ Black/Asian man, initially described justice as "righting any wrongs that have been done upon another and stopping those wrong doings from happening again. I think I understand justice more as rectifying justice due to how much discrimination I see everyday from the news to social media." Upon completion of the course, he shared that the course solidified his definition.

Discussion

This study sought to understand how mathematics majors enrolled in a TMfSJ course conceptualized the notion of *justice* before and after the course experience. Our analysis of students' reflections highlights several important shifts in how students understood, described and located justice. The findings suggest that engaging undergraduates in explicit study of social justice pedagogy both broadened conceptualizations and elevated their identities as mathematicians and future educators. Our data points out that conceptualizing *justice* is a complex and challenging task, yet being able to begin defining such a nebulous word is important to rehumanize mathematics learning and its impacts in the classroom and on society (Gutiérrez, 2018). However, we did see that most students developed and expanded their conceptualizations of the phrase, which shows promise that a semester course can have impact.

We saw four notions arising from the data: 1) movement from individualized justice to collective views, 2) a shift from the view of fair/equal to more equitable stances, 3) an overall increase in viewing justice as action or a way of being, and 4) more direct connections to how students see the power of mathematics contributing to justice. In initial descriptions of justice, students often gave individual experiences as examples to demonstrate the concept. By the end of the term, students spoke in much larger perspectives, indicating that achieving justice was part of a sociopolitical movement (Gutiérrez, 2013), of which many students saw themselves as one part of a bigger process. Next, students initially discussed justice with many words that were used interchangeably, including *fair*, *equal*, and *equitable*. Students end of term reflections suggested a deeper understanding of the differences between these words, including direct acknowledgement of how they felt prior to the course they did not necessarily understand the differences between them. We noticed a large decrease in use of the word *fair* at the end of the course, with students demonstrating an awareness that there is a lot to consider when determining fairness. The theme of Justice as Praxis demonstrated a significant shift in students' perspectives of action. Students often indicated ways to advocate and enact justice, reflecting views of rectifying justice (Su, 2020). We noticed that students began to write more about *who* is involved in the concept of justice, but also *where* and *when* justice (or injustice) takes place. Students also indicated more about *how* power and resources tie into justice. These tie into the notion of *reading the world* (Gutstein, 2006), in which students think about these types of issues related to power and identity. Finally, students reflected more on how mathematics is a powerful tool with which to use to consider justice, which is an important tenet to transitioning to TMfSJ (Gonzalez, 2012).

We note that, generally, students did not explicitly discuss issues of race or gender, often referencing injustices of oppressed or marginalized groups without naming the oppression. Milner (2003) argues that preservice teachers specifically need to be comfortable discussing issues of race within larger contexts. Due to the small sample size, we were unable to find any trends in the data related to students' demographics (e.g., race, gender). We plan to investigate this further with future data collection.

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Characterizing Instructor Questions Intended to Support Quantitative Reasoning

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Meanings for the definite integral are most productive when rooted in quantitative reasoning. However, little is known about how instruction may effectively support these meanings. This study examines the discourse actions of one instructor who intended to promote students' quantitative reasoning in a lesson exploring ideas foundational to the definite integral. We draw upon video data of the instructor's conversations with three undergraduate students who were working together during class. We analyzed whether his conversational turns could promote quantitative reasoning and characterized them according to a discourse action typology. Results suggest most of his discourse actions did opportune quantitative reasoning and that different types of discourse actions had the potential to elicit quantitative reasoning in distinct ways. We also identify two additional discourse actions not in the original typology. We conclude with implications for practice and suggestions for future research.

Keywords: quantitative reasoning, discourse, definite integral, instructional practice

A burgeoning body of research suggests quantitative reasoning (QR) is crucial to developing productive meanings for the definite integral (see Jones & Ely, 2023). This work has led to rich descriptions for meanings about the definite integral that are useful across various situations as well as documentation of the strengths and limitations students bring to bear as they develop these meanings (e.g., Oehrtman & Simmons, 2023; Sealey, 2014; Stevens & Jones, 2022). Given most studies investigate the development of students' meanings via teaching experiments or clinical interviews which center only one or two students at a time, additional work is needed to understand how these findings can inform effective instructional practice in classroom settings.

Research suggests larger groups of students have developed these meanings in classroom settings (Ely, 2019; Lehmann, 2024), but little is known about the instructional practices that may have supported students to develop these meanings. It is important to highlight that each class in these studies was taught by that study's author, and we presume that their instructional practices are influenced by a deep background knowledge about this body of research. To help prepare instructors who are less familiar with this literature to effectively facilitate students' developing QR related to the definite integral in whole class contexts, it would be useful to study the practice of instructors who intend to foster QR in this area.

With this goal in mind, we explored an instructor's practice during a lesson on accumulation from rate of change (one approach to developing QR with the definite integral; e.g., Thompson, 1994). This lesson took place within an undergraduate math course for preservice middle school math teachers that has developing quantitative and covariational reasoning as a central aim. Specifically, we address the research questions (RQs): *To what extent does an instructor's conversation with students have potential to elicit QR? In what ways does this occur?*

Conceptual Framework

Instructor discourse is widely accepted as an important facet of math instruction and has been studied extensively from various perspectives (e.g., Morgan, 2014; Ryve, 2011). We adopt Gee's (1996) definition of discourse as "connected stretches of language that makes sense" (p. 127), and we define discourse actions as the comments and questions an instructor verbalizes.

Researchers have proposed a variety of frameworks to characterize the kinds of discourse actions instructors pose at both the K-12 (DeJarnette, 2020) and post-secondary (e.g., Paoletti et al., 2018; Viirman, 2015) levels. Common across these frameworks and other research (e.g., Reiten et al., 2025) is that the types of discourse actions instructors enact are related to their goals for students. Hence, we hypothesize an instructor with a goal of promoting QR may pose certain types of discourse actions that they expect could elicit this type of reasoning from students. In line with Thompson (2011), we consider QR to include the mental construction of quantities, (i.e., measurable attributes in a situation), the relationships between quantities, and the process of solving problems based on quantitative operations.

In this study, we adopt Drageset's (2014) teacher discourse actions framework to identify trends in instructors' discourse actions that have the potential to elicit QR. This typology consists of 13 distinct discourse actions organized into four categories. Redirecting Actions (RAs) include questions or statements intended to redirect current ways of thinking. Progress Actions (PAs) include actions that seek to move problem-solving forward. Focusing Actions (FAs) include actions that draw students' attention to details in the problem. FAs considered Requests for Student Input (RSI) aim to elicit these details from students. In FAs considered Pointing Out Information (POI), the teacher provides these details. For brevity, we present discourse actions that constitute these categories in Table 1. See Drageset (2014) for more detail and examples.

Table 1. Drageset's (2014) Teacher Discourse Actions Framework

Discourse Action		Description
RAs	Put Aside	Dismissing a student's response
	Advising a New Strategy	Explicitly suggesting an alternative approach
	Correcting Questions	Asking a question that affirms something *and* suggests a deficiency
PAs	Demonstrating	Explicitly indicating the next step(s) in a process
	Simplifying	Providing hints or scaffolds
	Closed Progress Details	Asking how to move a process forward with a next step in mind
	Open Progress Initiatives	Asking for students' ideas of how to move a process forward
FA-RSI	Enlighten Details	Prompting students to elaborate elements of their explanation
	Justification	Prompting students to justify elements of their explanation
	Apply to Similar	Prompting students to replicate thought process in related situations
	Request Assessment	Prompting students to comment on a classmate's response
FA-POI	Recapping	Restating student's response and emphasizing important detail
	Noticing	Restating student's response and adding information

Methods, Analysis, and Tasks

Data collection occurred during an undergraduate math course for third-year, pre-service middle school math teachers at a large, public university in the Mid-Atlantic United States. Each course session was three hours long, and there were ten students enrolled. The second author, Dr. Paoletti, served as the course instructor. His research interests include QR, and this expertise deeply influences his teaching practice. Indeed, the primary goal of Dr. Paoletti's course was to support QR about core concepts spanning middle school to secondary and even post-secondary mathematics.

This report focuses on the fifth session of the course, which was a lesson on accumulation from rate. During this session, students worked together in pairs or trios to address the task sequence. The data for this study consists of video and audio recordings of the conversations and written work of one trio, Meghan, Amber, and Lydia (pseudonyms), all of whom consented to

participate in the study. Relevant to this report, the recordings also captured the conversations the trio had with Dr. Paoletti, who circulated the room to assist and provide feedback to groups.

Analysis

We first isolated segments of the recordings during which Dr. Paoletti was interacting directly with the trio. We transcribed these segments and identified the conversational turns (i.e., a period of speech bounded by another's; Ford et al., 1996) in which Dr. Paoletti responded to student thinking. We coded this set of turns for discourse action type and potential to elicit QR.

To analyze the *extent* to which these turns had the potential to elicit QR, we used a combination of provisional (Saldaña, 2021) and open coding (Braun and Clark, 2006). That is, we began with a codebook that outlined criteria for whether a conversational turn had the potential to elicit QR. When existing criteria was not sufficient, we inductively developed additional criteria. To analyze *how* Dr. Paoletti promoted QR, we deductively coded (Saldaña, 2021) each turn using Drageset's (2014) framework. Throughout this process, Dr. Paoletti shared his "perspectives of practice" (Lampert, 2010) as the course instructor, which provided additional insight to the data. As we coded, we discussed discrepancies to come to a consensus and refine our codebook. This process also led us to generate two, new discourse action codes: Endorse and Counter-Justification. These codes capture elements of Dr. Paoletti' discourse that were not described by the existing framework. We describe these further in the Results. Finally, within each code, we identified patterns in the criteria with which conversational turns had been identified as having the potential to promote QR (or not).

Tasks

To provide additional context for the interactions described in the results, we briefly outline the tasks from this class session. The main purpose of these tasks was to extend the meanings for rate students had begun developing earlier in the course and to provide a foundation for QR with the definite integral. Each of the three tasks in this session leveraged relationships between speed, distance, and time by asking students to make more accurate estimates of distance using various representations of speed and time data. Students began with the *Which Racetrack?* task in which they determined the track a car may have completed based on speed and time data and the total length of each potential track. In *Where's the Car?,* students used instantaneous speed information from a video of a racecar's dashboard to track its location over time. Finally, in *Drag Race,* students used a model of a drag racecar's speed, i.e., $v(t) = 71 \times (1 - (1/3)^t)$, to write an accumulation function for the car's approximate distance from the starting line. See Olshefke-Clark (2025) for more detailed task information.

Results

We address our first RQ by summarizing the frequencies of the different discourse actions Dr. Paoletti employed as he responded to the trio's mathematical thinking and the percentages of turns with potential to elicit QR. We address our second RQ by describing trends in how Dr. Paoletti's different discourse actions had the potential to support QR.

Frequency of QR

Dr. Paoletti contributed 128 turns across 17 small group conversations with the trio. These conversations ranged in length from five seconds to three minutes and totaled approximately 24 minutes of the three-hour lesson. Table 2 summarizes the frequency of Dr. Paoletti' discourse actions organized by category as well as the percentage of turns in each category that we

identified as having potential to elicit QR. Overall, the majority (62.1%) of Dr. Paoletti' turns had the potential to elicit QR. Moreover, the potential to elicit QR was substantial across all four of the discourse action categories; three of the four categories had over 55% of its turns tagged as having potential to promote QR, and the fourth had 45%.

Table 2. Breakdown of Dr. Paoletti's discourse actions

Category	Frequency	Frequency of Quantitative
Redirecting Actions	20	9 (45%)
Progress Actions	38	21 (55.3%)
Focusing Actions- Request for Student Input	48	34 (70.8%)
Focusing Actions- Pointing Out Information	26	18 (69.2%)
TOTAL	**132**	**82 (62.1%)**

To further characterize Dr. Paoletti' responses to the trio's mathematical thinking, Figure 1 organizes turns by code and potential to elicit QR. Notably, all discourse actions except Request Assessment from Other Students had at least one quantitative instance. Moreover, 10 of the 14 discourse actions had over 50% of instances coded as quantitative. In what follows, we describe ways Dr. Paoletti promoted QR by outlining trends in specific discourse actions that had especially high and low proportions of turns with potential to elicit QR.

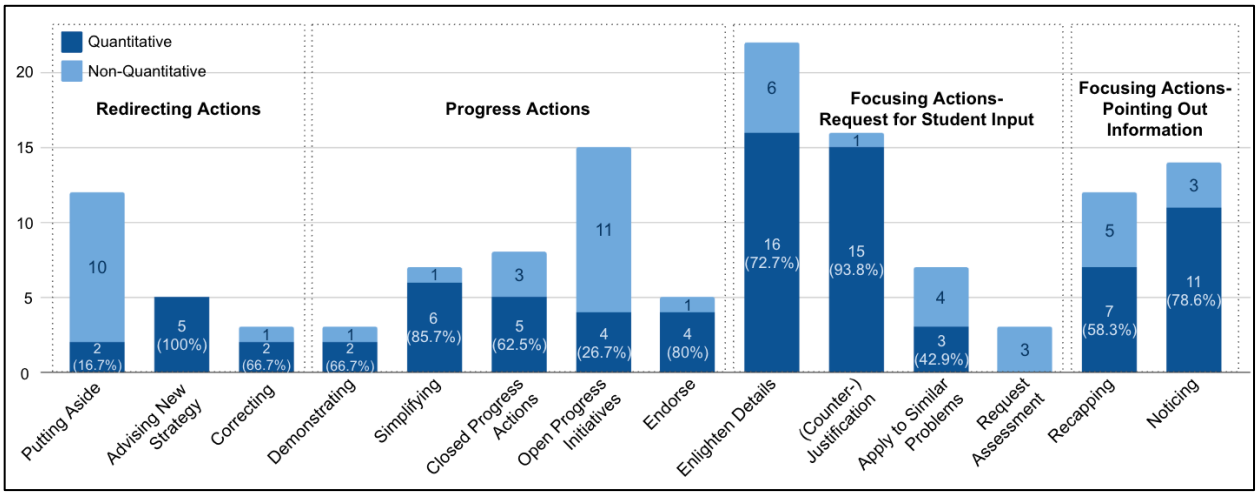

Figure 1: Frequency of (non-)quantitative conversational turns by specific discourse action

Redirecting Actions

RA was the only category with less than a majority (45%) of turns coded as potentially eliciting QR. This was also the smallest category overall, representing only 15.2% of total turns. RA also contained Advising a New Strategy, which was the only type of discourse action type in which 100% of turns had the potential to elicit QR. In these five turns, Dr. Paoletti urged the trio to consider the meaning of their processes in the situation, rather than calculate with no rationale. For example, in certain instances where the trio did not have a justification for their process, he advised them to change their strategy, exhorting, "don't just randomly multiply and divide things!" and "we can't just average things for the sake of averaging things." With these statements, Dr. Paoletti told the trio that their current strategy was insufficient and implied that

they should pursue a process that was not "random" or done "just…for the sake of" it. These kinds of statements have the potential to elicit QR because, from Dr. Paoletti's perspective, a more intentional strategy should include considering how process related to the quantities in the situation and the relationships between them.

Progress Actions

PA turns made up over one quarter (28.8%) of total turns, and over half (55.3%) had the potential to elicit QR. Specific actions ranged from having 80% or more (Simplifying and Endorsing) to only about 25% (Open Progress Initiatives) of turns with potential to elicit QR.

Endorsing. During the process of coding Dr. Paoletti' turns, we identified a type of discourse action that was not captured by Drageset's (2014) existing framework. These turns explicitly encouraged students to continue progressing through a particular process by providing confirmatory feedback (i.e., endorsing) their current approach. We considered this discourse action to be a type of PA because they "serve to move the process forward" (Drageset, 2014, p. 298). This type of discourse action included statements like "I think now you're on track. Follow that line of reasoning further and see." Most (80%) of these codes had the potential to promote QR because Dr. Paoletti often endorsed the trio's explanations when they made suggestions for a next step that was motivated by quantities in the situation. In one instance, when the trio were proposing an idea to improve their estimate of the car's distance in *Drag Race,* Meghan shared, "We were talking about going, like, in between the two, like one and a half seconds." Dr. Paoletti replied, "So I think that's a good idea," thus endorsing a line of thinking that referenced situational quantities and encouraging them to progress along this path.

Open Progress Initiatives. Although Dr. Paoletti encouraged the trio's QR with endorsing turns, we found he demonstrated less potential to elicit QR when he asked the trio general questions about how they might progress. Similar to Put Asides, Open Progress Initiatives were often too vague or non-specific to be considered having potential to elicit QR. For example, on multiple occasions during *Where's the Car?* and *Drag Race*, Dr. Paoletti simply asked, "how can we get more accurate?" Though a potential (and perhaps ideal) response to this question could involve considering the relationships between quantities, we do not consider these questions of this nature to be clear in communicating to the trio that they should consider quantities instead of just calculation. Because Open Progress Initiatives are, by definition, open to students' authentic suggestions, it makes sense that these turns may not be dense in potential to promote QR. Dr. Paoletti was soliciting the trio's ideas for moving forward, not explicitly guiding them.

Focusing Actions – Request for Student Input

FA-RSI turns were both the most frequent (36.4% of total turns) and had the highest percentage of quantitative turns (70.8%). Interestingly, FA-RSI contains discourse actions that had both the highest (93.8%; (Counter-)Justification) and lowest (0%; Request Assessment from Other Students) densities of potential to elicit QR.

Request for Student Assessment. No turns that requested the trio to provide feedback on another member's response were identified as having the potential to elicit quantitative reasoning. However, very few (3) turns were coded as requests for student assessment, and it is possible that with more turns of this type, the potential to elicit QR could be observed. As a hypothetical example, if Dr. Paoletti had asked the trio to assess a group members' QR-rich explanation, this could be considered as having the potential to elicit QR.

(Counter-)Justification. This was the most common type of turn that Dr. Paoletti contributed, and all but one had the potential to elicit the trio's QR. Almost all of Dr. Paoletti

prompts to justify were phrased as a "why?" question and asked the trio to explain how a process they had completed worked based on the relationships between quantities in the situation. Notably, when coding for Justification, there were multiple instances in which Dr. Paoletti asked the trio to explain why a process didn't work, which is slightly different from Drageset's (2014) definition of Justification as "questions about why the answer found or method used is correct" (p. 295). We consider Counter-Justification to be a special case of Justification because it elicits QR in a similar way but is specifically focused on why an answer or method is not correct.

As an example of how Counter-Justification was intended to promote QR, consider when the trio was explaining to Dr. Paoletti how they combined speed and time data in *Which Racetrack?* to come up with an estimate for distance which they believed was too large:

Meghan: I multiplied these two together for all of them and then added them.

Dr. Paoletti: Why shouldn't that work?

Meghan: Why shouldn't-? Because they're not equal speeds. Like, you're not doing 13 per- Well, 'cause you're multiplying meters by seconds by seconds. That's meters per second per second. And that's not what we're looking for.

Dr. Paoletti responded to Meghan's first explanation (which only referenced arithmetic calculations) with a request for Counter-Justification, asking why her process *would not* make sense. Although her response was not normative, it did explicitly reference quantities in the situation. As such, Dr. Paoletti' Counter-Justification both promoted and successfully elicited QR because it required Megan to interpret her arithmetic calculations in terms of the situation, which helped her consider why her process did not produce an expected result.

Enlighten Details. Almost three-quarters of Enlighten Details turns (72.7%) provided opportunities to engage in QR. The details that Dr. Paoletti asked the trio to enlighten were often prompts to connect their calculation to relevant quantities in the situation. In some cases, he paused the trio in their explanation of their process and prompted them to add units to a number they referenced (e.g., "3.7 what?"). In others, he asked more generally what a number represented in the context of the task (e.g., "what do these numbers represent again?"). In either case, these prompts encouraged them to keep quantities at the forefront of their explanations.

Focusing Actions – Pointing Out Information

Almost 70% of FA-POI turns had the potential to elicit QR, and both types of discourse actions that compose this category had a majority of turns with the potential to elicit QR. Recall that in FA-POI turns, the instructor himself is pointing out important information. It makes sense, given his overarching goals, that Dr. Paoletti would frequently point out information related to quantities and their relationships, thus giving students opportunities to engage in QR.

With Noticing actions, that is, instances where Dr. Paoletti paused the trio's progress to emphasize an important detail in their explanation, he often reiterated and/or expanded upon elements of QR present in their explanation. For example, consider the exchange:

Lydia: Like, the difference between it is 24.

Dr. Paoletti: The difference of what? [Enlighten Details]

Lydia: The difference in speed.

Dr. Paoletti: It's the difference in speed, right? [Noticing]

Dr. Paoletti used a Noticing action to emphasize the quantity that Lydia had added to her response to his Enlighten Details question. As such, this provided the potential to elicit QR by further pointing out that this detail of Lydia's explanation was important. We note that although in some cases, this kind of discourse action may provide implicit endorsement of their approach, we consider this distinct from an Endorsing actions because they are not explicit affirmations.

Dr. Paoletti's way of opportuning QR was similar across instances of Recapping as well, except Recapping actions focused on the conclusion of a process rather than intermediate steps.

Discussion

This report provides an account of the discourse actions posed by one instructor in a whole class setting who had a goal of promoting QR about ideas foundational to the definite integral. Specifically, we explored the extent to and ways in which Dr. Paoletti's conversational turns had the potential to elicit QR for a small group of students in his class. We found that the majority (62.1%) of Dr. Paoletti's discourse actions had the potential to elicit QR, suggesting that his discourse did align with his overall goal. Moreover, we illustrated different ways that Dr. Paoletti presented opportunities for the trio to engage in QR across various types of discourse actions.

Our results suggest there is not a single type of question or statement that instructors need to rely on to elicit students' QR; Dr. Paoletti posed 14 types of discourse actions, and he provided opportunities for the trio to engage in QR with all but one. However, FAs may be a particularly important skill for instructors to hone. We highlight FA-RSI and FA-POI had the highest densities of potential to promote QR, and together these made up the majority of Dr. Paoletti's turns. Dr. Paoletti used FAs to direct the trio's attention to relevant quantities in the situation and pushed them to explicitly include the quantities and the relationships between them in their explanations of calculations and results. Prior research (e.g., DeJarnette et al., 2020) indicates that teachers have particular difficulty posing questions and comments that "mak[e] specific connections or relationships explicit" (p. 9). Practicing incorporating FAs, which can be as simple as asking students a Counter-Justification, "why not?," is one way that instructors can promote QR in class by pushing students to explicitly consider relationships and connections.

Additionally, our identification Counter-justification and Endorsing actions contribute to research on instructor discourse more generally by extending Drageset's (2014) framework. We do not believe these two discourse actions are particular to classes emphasizing QR and conjecture both discourse actions could be leveraged by teachers with other foci.

Implications and Future Directions

It is important to emphasize this study does not draw a link between the types of questions Dr. Paoletti poses and their impact on the trio's QR. Rather, we intended for this study to provide an illustration of how instruction focused on promoting QR may be enacted and to begin a conversation on bridging existing research on students' QR about the definite integral with instructional practice in the undergraduate math education space. Additional work is needed to understand what types of questions and other learning opportunities actually result in development in students' QR related to the definite integral.

Though understanding the questions that an instructor poses to support QR may support other instructors in doing the same, DeJarnette et al. (2020) remarked that "even if teachers learn to pose questions that should elicit meaningful explanation, justification, and mathematical relationships, these questions will not make productive contributions if students do not recognize or respond to the demands" (p. 10). Thus, future research should attend to how these QR-focused discourses are co-constructed with students and consider questions such as: How do students respond to questions intended to elicit QR? How do instructors respond to normative and non-normative student responses? How might students' take up instructors' QR-promoting discourse actions in their discourse with one another? Investigating questions such as these in whole-class settings can provide important insights about how instructors can support students in developing meanings about the definite integral that are productive in their future coursework and careers.

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Visualizing the Work of Critical Transformations: Processes, Positions, and Emotions

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What does it mean to engage in equity work and critical transformation efforts? We asked mathematicians who were part of research-practice partnerships addressing inequities in their departments to create or find an image to represent what it meant to them to engage in critical transformation work. We interviewed them about their images, and categorized images in overlapping domains of the processes of critical transformation efforts, emotions, and relative positioning of members. Many participants illustrated and voiced the process of building community, which often included an inclusive power dynamic and positive and productive feelings. Images without both community and inclusivity present instead centered loneliness and discouragement. A key implication for engaging in critical transformation efforts includes the need to center building inclusive communities.

Keywords: equity, arts-based research, critical transformation, research practice partnership

Despite the values of diversity, equity and inclusion being demonized politically in the United States in 2025 (e.g., Executive Order, 2025), mathematical teaching and learning in United States K-12 and institutions of higher education do not yield equitable student outcomes, largely attributable to structural inequities and racism (e.g., Tatapudy et al., 2024; Womack et al., 2024). As part of the larger *Achieving Critical Transformation in Undergraduate Programs of Mathematics* (ACT UP) project (NSF #2201486, 2022-2025, we established a research-practice partnership with three university mathematics departments that each acknowledged their inequitable student outcomes and sought to make critical transformations to improve (Bolick & Voigt, 2023; Hagman et al., 2024; Quaisley et al., 2025).

What does it mean to engage in equity work and critical transformation efforts? We asked mathematicians who were part of research-practice partnerships addressing inequities in their departments to create or find an image to represent what it meant to them to engage in critical transformation work. We asked them about their images, and also learned about their processes and emotions. Multiple people mentioned experiencing discomfort with the request, or being uncomfortable trying to represent their work visually. However, the overall responses were positive, and people found the exercise valuable and enlightening.

One way we as researchers can challenge dominant traditions and enact critical transformations is by questioning what counts as research, rejecting reductionist methodologies that suppress individual experiences, and engaging in multiple representations of research data. As a note, we use Gutièrrez's (2012) conceptions of non-orthogonal axes of equity comprising the dominant (access, achievement) and the critical (identity, power). We considered discomfort with arts-based research (Boyd & Barry, 2024)--for those providing data and those analyzing data--to be analogous to the way we might encourage our students to embrace discomfort and struggle in learning mathematics.

Within the larger ACT UP project, mathematicians in three different departments engaged in a research practice partnership with a RUME research team, in which each department formed a

networked improvement community (NIC; Bryk, 2020; Bryk et al., 2015) comprised of faculty, students, and sometimes administrators (the community), working toward a common aim of critical transformation to address inequities (the improvement). By connecting the three NICs with the research team in a broader research practice partnership, the improvement communities could learn from each other (the network). Each NIC selected their own focus, which spanned improving coordinated (lower-division) courses, revamping placement processes, better understanding student experiences, and confronting varying assumptions about mathematical rigor. The ACT UP project launched in 2022; federal grant funding was terminated in spring 2025 by the U.S. Department of Government Efficiency. For this analysis, we focus on the research questions:

- What are the images, emotions, metaphors, positioning of self to NIC, and conceptualizations that people describe when engaging with equity-focused critical transformations?
- What is the utility of using visual imagery to make sense of critical transformations?

Methods

As part of the larger ACT UP project, we interviewed 22 NIC members and NIC leaders in early spring 2025; interviews occurred via Zoom and lasted up to one hour each. NICs were comprised mainly of mathematics instructors (faculty, graduate students), and undergraduate students; NICs had 8-13 members each. Prior to the interviews, we shared the full interview protocol and asked the interviewees to create or take a picture related to their experience of engaging as a NIC member working toward critical changes. During the interview, we then asked them, "Please describe this image - what feelings or emotions does it invoke for you?" The larger project utilized a mixture of arts-based (Boyd & Barry, 2024) and more traditional forms of data collection and analysis.

Our analysis team includes some members of the original ACT UP research team and some members of the ACT UP department-based NICs. Salient dimensions of our identities and researcher positionalities we bring to this work include being women, queer, and/or non-white in ways that are "othered" in mathematics settings; being part of families; holding university positions (e.g., postdoc, non-tenure-track, pre-tenure, administrative responsibilities); being neurospicy; and being mathematics instructors, mathematicians, and mathematics education researchers.

Our analysis team took the combination of images and explanations and used a thematic arts-based analysis process (Boyd & Barry, 2024) to tag the images with hashtags. We iteratively considered how to group and name the hashtags, and ultimately decided to group the hashtags as having to do with critical transformation processes; emotions; positioning relative to the NIC; metaphors; and other (which included mathematical connections). We used the full list of hashtags to create a word cloud (Fig. 1) to represent both the range and relative frequency of the ways we categorized images and the ways participants discussed the meaning behind the images they created or selected. We then created separate word clouds to capture the critical transformation processes (Fig. 2), positioning relative to the NIC, and emotions (latter two images omitted for space constraints). We did not create a word cloud for metaphors, as each metaphor was unique; similarly, we did not create a word cloud for the mathematical connections ($n = 2$) nor pop-art references ($n = 2$).

Findings

Our findings center the themes of engaging in critical transformations as a process, emotions, and how NIC members felt they were positioned in their NICs. Mentions of the NIC as community were the most frequent comments overall, as well as participants using metaphors, discussing connections to teaching (omitted here for space constraints), and sharing how the NIC helped them feel connected to each other (see Fig. 1). Using imagery, NIC members were able to convey the complexities and nuances of engaging in critical transformation work in mathematics departments.

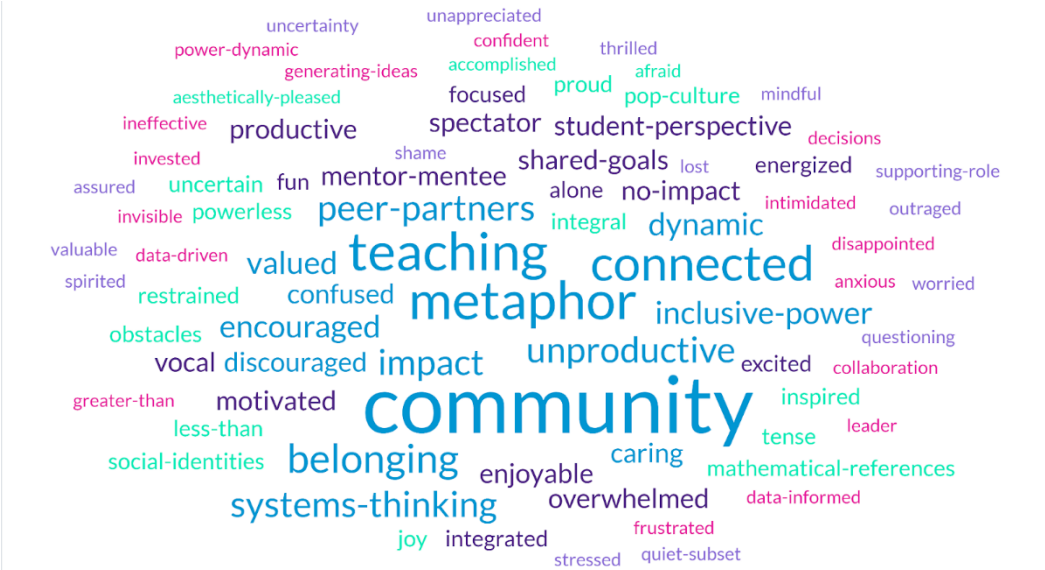

Figure 1. Word cloud illustrating the relative frequency of the hashtags we applied to the images

Figure 2. Word cloud illustrating the relative frequency of hashtags related to processes of transformation

Engaging in Critical Transformations as a Process

Engaging in critical transformations is a process. For our NICs it involved establishing and nurturing collaborative communities and shared goals, making data-informed decisions, questioning systems and generating ideas, trying to make a difference or impact within their mathematics community, and reflecting on ways to change systems and navigate barriers to change across ecological levels (e.g., institutional, departmental, personal, etc.) (see Fig. 2). Many of these processes were illustrated using visual metaphors (Fig. 3). For instance, people collaborating to put together a jigsaw puzzle represents how individual ideas can fit together to form shared goals, and a guide dog represents the NIC's shared goal to guide the direction of mathematics departments, and at the same time, empower students to be puppeteers with more autonomy over their own journey within a mathematics program. As another example, one NIC member used a chaotic cloud of question marks and colorful arrows pointing in different

directions to illustrate questioning systems and generating ideas. Concurrently, this cloud hovers above NIC members engaging in systems-thinking as a process similar to deconstructing and rebuilding a brick wall.

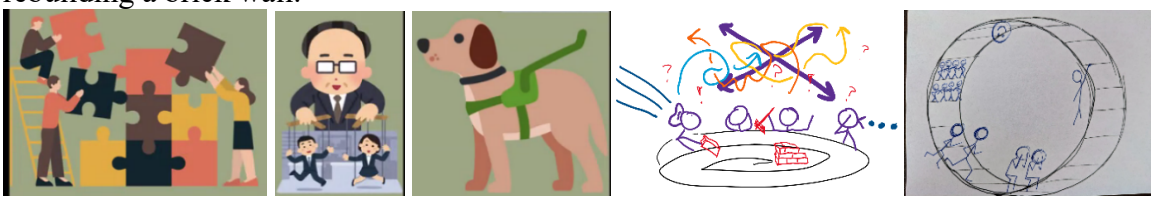

Figure 3. From Left to Right: Shared Goals as a Jigsaw Puzzle; Student as Puppeteer; NIC as a Guide Dog; Systems-Thinking as Communal Questioning, Deconstructing, and Rebuilding; and Making an Impact as a Cycle of Emotions on a Hamster Wheel.

These transformation processes were deeply connected to emotions, whether the NIC member voiced them or depicted them visually. In general, many of these processes conveyed and evoked mixed emotions. For instance, when showing or describing the process of trying to make a difference or impact within the NIC's mathematics community, some NIC members described entirely positive emotions like feeling proud, thrilled, and invested, whereas others described feeling either positive and negative emotions, such as integral, but invisible and unappreciated, or entirely negative emotions, such as discouraged, disappointed, overwhelmed, powerless, frustrated, unproductive, uncertain, lost, or confused. One NIC member specifically portrayed making an impact as running on a hamster wheel and cycling through a multiplicity of emotional states, which were both vocalized and captured in the positions and angles of the stick figures' bodies. The hamster wheel drawing in Fig. 3, depicts energized figures with elbows and knees bent to run, which transitions to worried, afraid, and shameful figures hunched over and covering their faces or dangling off the wheel with one hand. A smiley face returns the NIC member back to a state of enjoyment and fun, followed by groups of figures with arms around each others' shoulders to indicate a sense of belonging. As described in the next section, however, there were processes that consistently represented and evoked positive emotions.

Community and Belonging Fosters Inspiring, Joyful, and Accomplished Emotions

Examining the processes of transformation word cloud (Fig. 2), the relative frequency of the hashtags indicate a shared perspective that building community is one of the most important aspects of what it means to engage in the work of critical transformation. Processes of building community were conveyed through a multiplicity of media, such as real-life photographs of NIC members, including faculty and students, smiling together; AI-generated images of intently focused NIC members huddling together to make sense of and discuss data; found images of excited officemates; or mixed images of inspired NIC members generating ideas (Fig. 4). The AI-generated image and mixed image in Fig. 4 additionally demonstrate the ways in which other processes, in these cases, making data-informed decisions and generating ideas, often intertwined with images depicting a community of NIC members.

Community and Belonging Mitigates Feeling Alone, Discouraged, and Overwhelmed

A few of the images that did not center both community and an inclusive power dynamic instead highlighted feelings of being alone, alongside other negative emotions such as feeling discouraged, lost, or overwhelmed. In Fig. 5, we see an image (Starfish Students) of a young girl attending to one of many starfish spread over a beach. This NIC member described an initial and lingering feeling throughout their time on the project, stating, "I just felt so obtuse, like this was

impossible.” Although this NIC member also expressed that they felt they finally understood what a critical change was – “some sort of massive structural change within the university” – they expressed a view of limited impact, stating, “I can't say that I pulled that off… I didn't really necessarily pull off a big, huge structural change. But I think I made a significant impact to some students in my classes.”

Figure 4. Images Depicting Collaborative Communities.

Fig. 5 also shows an image (“No Tracks in the Ground”) of a person standing alone, proportionally smaller than the rest of the winding labyrinth-esque landscape that seemingly leads nowhere in particular. During the interview, the creator of “No Tracks in the Ground” described their process of deciding on an image. At first, the NIC member considered Sisyphus’s unending punishment of “pushing a rock up a hill in Hades and having it roll back down and then pushing it back up and then having it roll back down” but ultimately decided on “multiple, confusing paths, not knowing which way to go.” They elaborate on why they chose this image:

> there's no tracks in the ground to see where anyone has gone before.. I thought people were going to tell me what to do. And now I realize that no one knows. Now I have to try to navigate. I have to pick a path and see if it brings me right back where I started, or if it brings progress... I feel alone sometimes.

Figure 5. From Left to Right: Starfish Students; “No Tracks in the Ground”; and Community is the Best.

In contrast, NIC members who illustrated and vocalized both community and inclusive power dynamics highlighted positive and productive feelings, such as caring, motivated, and energized. Consider Community is the Best in Fig. 5, and this NIC member’s explanation for the choice of their image:

> The best part of participating in this has been just having the excuse to get together with my faculty that care so much about student engagement, and are just so interested in helping their students. Those monthly meetings are just really recharging, and just a great

reminder that as you're working through issues with your students, you're not alone. There are other people that have ideas you can use, and that are experiencing the same thing. And the community is just really the best part of the whole thing.

Discussion

Asking people to create or find images to represent their engagement in critical transformations led to nuanced representations of critical transformation processes, emotions, metaphors, and relative positioning. Engaging in critical transformation work as a community was the overwhelmingly cited process captured by nearly every participant; indeed, every image that conveyed transformation processes involved people. This stance reinforces approaching critical changes systemically, and the need to do this work together. Being in community was also associated with varied positive emotions and feelings of belonging and connectedness. At the same time, when the work has a transformative aim to change systems built to maintain the status quo, people experienced mixed emotions over the slow pace of change or the barriers they encountered. Trying to make systemic changes can understandably lead to feelings of confusion, being lost, and discouragement. Feelings of community and belonging helped to mitigate feelings of isolation, discouragement, and being overwhelmed.

Schnall et al. (2008) found that people's perceptions of the steepness of a hill were altered whether they were alone (more steep) or with a group (less steep). ACT UP participants who expressed more feelings of being alone also expressed other negative emotions related to the difficulties of critical transformation work. Conversely, participants who expressed feelings of belonging also expressed other positive emotions and feelings of impact and productivity. Generally, NIC members' visual imagery of the processes of engaging in critical transformations included people, implying that these processes are people-centric and subjective, rather than an abstract process of coming to some objectively ideal conclusion. Furthermore, we found that images representing community were intimately linked to positioning and emotions. Specifically, all NIC members who centered community and an inclusive or non-hierarchical power dynamic in their images also only described positive emotions, such as motivation, inspiration, and encouragement; excitement, joy, and fun; and productivity, focus, accomplishment, and pride.

Beyond the images, emotions, metaphors, positioning, and conceptualizations, one might ask what the utility is of asking people to visually represent their critical transformation efforts. Indeed, multiple people mentioned feeling discomfort with the visualizing request, or being uncomfortable trying to represent their work visually. However, in expressing these feelings, they also expressed finding value and being challenged in a productive way through the reflective exercise. We saw ample evidence that the reflection process was useful and helpful, even when those same participants were initially uncomfortable with the task. It can be easy to focus narrowly on the "now" and not stop to see how far one has come; some participants appreciated the invitation to make sense of the process and their progress with critical transformation efforts. As researchers, moving outside the typical reflection question to an arts-based reflection enriched our understanding of how participants viewed critical transformations as a complex, difficult, and rewarding endeavor. Arts-based responses captured not only critical transformation processes but also the power dynamics of participants' positioning in their NICs as well as a wide range of emotions and metaphors.

Implications

We see several implications for researchers and for those seeking to engage (or already engaged) in critical transformation efforts. One of the key implications of this research is that

emotions and perceived power dynamics are intertwined. When a person feels alone versus feels like they belong, this is accompanied by differing perceptions of agency. Knowing that sustaining change efforts takes passionate change agents (e.g., Koh et al., 2023), such passion is more easily maintained when the change agent feels more positive emotions. Further, mathematicians rarely have training on establishing and sustaining foundational trust and professional relationships (e.g., Kaschner & Neihaus, 2025). Yet, when NICs prioritized relationship building, NIC members perceived themselves more centrally in the change efforts, which is connected to higher perceived agency and more positive emotions (all of which can sustain change efforts).

Another key implication is to expect uncertainty, confusion, and tensions. Engaging in critical transformation work is more analogous to mathematical modeling than mathematical proof or problem solving. Critical transformation works from what is known and uses data to inform next steps, but there is not a single "best" path, nor a "right" answer. There are approaches that can improve outcomes (or not), but since change efforts revolve around people (with complex identities, beliefs, attitudes, skills, knowledge, power dynamics, positional authority, etc.) outcomes are also tied to the complexities of the individuals and cultural settings involved (e.g., Hagman et al., 2024; Kezar, 2014; Kezar & Gehrke, 2015). The images selected or created by our participants expressed a wide range of emotions, including conflicting emotions from single individuals (e.g., feeling both "proud" and "unproductive"). Further, by experiencing uncertainty and confusion, some instructors found themselves more empathetic to student confusion in class (similar to findings in Tremaine et al., 2024; Quaisley et al., 2025).

Engaging in visualizing critical transformation work also helped participants consider what it meant to prioritize values like diversity, equity, and inclusion, while those concepts are under increasing political attack (e.g., Harper, 2025) and being used to stoke fears (e.g., McElvane, 2025). The visuals created by reflection on critical transformation can provide concrete evidence of how people approach critical transformation, centering student outcomes, building community, and seeking to improve educational systems and policies.

Our research highlights the importance of community. When working toward difficult goals, various emotions are evoked; people cannot change systems alone. NIC members' modal feeling expressed through their visual depictions was that of community: being in community, feeling supported by community, and knowing they are not alone. Community is a remedy (perhaps the only remedy, e.g., García Peña, 2022) to feelings of isolation and overwhelm; community is the only way to change systems to address systemic inequities. NIC members' experiences of community through this project showcase what it means to combat negative connotations of DEI work, centering student outcomes to transform education systems.

Finally, we do not make generalized claims that all NIC members experience critical transformation work the way those in the ACT UP project did. Rather, we use our arts-based methods (Boyd & Barry, 2024) and findings to invite others to engage in similar efforts. We suspect that many researchers focused on undergraduate mathematics education, like us, were trained in traditional (reductionist/positivist) research methodologies that position researchers as observers of participants, and research as being a way to find out "truth." We challenge our community to seek out and try more critical methods, such as arts-based methodologies (Boyd & Barry, 2024), as ways to engage collaboration with participants in understanding unique experiences. By telling our stories, we can invite others into concomitant reflection, ideally in ways that can inspire people to initiate or expand local critical transformation efforts.

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Power Implicit in Discourses of Mathematical Rigor Within an Equity-Oriented Improvement Community

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"Rigor" is often used as a benchmark to determine whether mathematical work is sufficiently high-quality, and, thus, has significant bearing on students' course success and continuation in STEM majors. However, judgements about what constitute rigor are deeply subjective and may function as a vehicle for social bias. To interrogate possible connections between disciplinary standards and exclusionary gatekeeping, we analyze references to rigor by members of an "elite" mathematics department's equity-oriented change community. Our findings exhibit how participants' perceptions of rigor unveil a culture of power operating in undergraduate mathematics. We conclude with implications for research and practice.

Keywords: rigor, discourse, culture, power, equity

Introductory college mathematics courses, such as calculus, function as core pre-requisites for many university STEM majors and therefore impact STEM attrition rates (Weston et al., 2019). Given national disparities in tertiary STEM retention along axes of race, gender, and first generation status (e.g., Costello et al., 2023; Maloy et al., 2022; Park et al., 2020; Ortiz-Martínez et al, 2023; Suárez et al; 2021), advancing equity in the context of calculus curricula and instruction has become a central concern in undergraduate mathematics education. Mathematics departments and instructors often use "rigor" as a standard for the quality of their courses. Despite its role as a widely-used standard there is no agreed-upon definition of rigor; the term carries different meanings across individuals and mathematical contexts (Authors, under review). As an ambiguous standard that mediates inequitable STEM retention, it is possible that conceptualizations of rigor—and their implementations through instruction—embody unexamined disciplinary biases.

To further our collective understandings of how rigor and equity may interact in undergraduate mathematics instruction, we analyze an equity-oriented community of mathematics faculty and students at an elite institution, who adopted rigor as a focal issue to guide equity-intended change initiatives. Leveraging critical discourse analysis (Fairclough, 1989, 2013), we investigate implicit understandings of the term "rigor" and how those understandings get operationalized to maintain or disrupt hegemonically-focused practices in mathematics classrooms as a gateway to STEM degree attainment. Specifically, we guide this contributed report with the following research question: *How do discourses around rigor reveal a culture of power in undergraduate mathematics education?*

Conceptual Framework: Culture of Power

Delpit (1988) theorized that educational values are shaped by a culture of power (CoP), evidenced through five characteristics:

1. Issues of power are enacted in classrooms.
2. There are codes or rules for participating in power; that is, there is a culture of power.
3. The rules of the culture of power are a reflection of the rules of the culture of those who have power.
4. If you are not already a participant of the culture of power, being told explicitly the rules of that culture makes acquiring power easier.
5. Those with power are frequently least aware of—or least willing to acknowledge—its existence. Those with less power are often most aware of its existence (p. 282).

To elaborate on these characteristics, we leverage the original context of antiblackness in K-12 literacy education as an illustrative example. Delpit describes how issues of power are enacted in classrooms through the differential recognition of some forms of language as 'correct' while others are deemed incorrect or informal. Linguistic conventions deemed 'correct' are codes for participating in power. For example, students who speak and write 'correctly' generally have more access to careers that yield financial power and societal standing. These linguistic conventions align more closely with those commonly used in white households than in Black households, a reflection of cultural norms held by those with more societal power. Students, therefore, must be taught these linguistic codes of power when they do not acquire them through their households. However, many liberal/progressive white teachers oppose teaching these conventions directly in favor of approaches that encourage more freedom for students to use the language style that they prefer. Consequently, education systems may unjustly perceive Black children as requiring linguistic remediation upon entering formal education, which can have long-term ramifications for their access to power. In this way, teachers' desires to cultivate deep thinking and creativity can inadvertently perpetuate racial stratification through language instruction by centering children who have already assimilated to the CoP.

In the context of undergraduate mathematics education, we recognize that rigor plays an analogous role to language standards within K-12 literacy. Correctness is determined through rigor (however it may be interpreted by instructors/authorities), which can favor mathematical grammar over substantive reasoning. Rigor is capital; those who do mathematics with 'rigor' gain more access to success in their courses and STEM degrees. Here, we explore how rigor may function to uphold a CoP within undergraduate mathematics education.

Literature Review

Articles as early as 1920 argue for and also question the privileged role 'mathematical rigor' plays within mathematics education (e.g., Pierpont, 1928; Kitcher, 1981; Kleiner, 1991). These writings emphasize how conceptualizations of rigor, and thus the associated aims of enhancing such rigor, vary considerably. In particular, rigor is highly context- and position-specific; and thus mathematics students and faculty define general academic rigor differently (e.g., difficulty versus higher-order thinking; Cristobal et al., in press; Draeger et al., 2013; Draeger et al., 2015).

This ambiguity in what constitutes rigor raises questions about its potentially biased role in classroom practice and policy. Scholars have viewed rigor as completely separate from, or even in opposition to, care, equity, and inclusion in higher education (Marsellas, 2021; McNeill & Gresalfi, in press). Rigor can be operationalized through instruction to perpetuate existing inequities. Leyva et al. (2021) describe how instructors' standards for rigor can be experienced as part of a "weedout logic," through which students who are deemed "not cut out" for STEM are discouraged or prevented from continuing in mathematics coursework. One Latino student directly interpreted an instructor's remark as "t[a]k[ing] a shot at [his] self-esteem" through imposing "very rigorous" standards that activated racial stereotypes of mathematics ability. In

this way, standards of rigor may lend a false sense of legitimacy to the categorization of certain (groups of) students as mathematically incapable, akin to Delpit's (1988) observation of racial stratification justified by marking young Black students' language as incorrect.

Discussing the elitism of 'rigorous' mathematics within the engineering discipline, Riley (2017) notes that definitions of rigor are socially constituted; conceptualizations of rigor reflect existent social hierarchies and dominant presumptions about who can conduct 'rigorous' science. They assert that invoking rigor often functions to maintain exclusion through its assertion of white masculinized heterosexual power, drawing boundaries around what competent disciplinary practice and perspectives are, and academically punishing students to discount interdisciplinary or innovative scholarly work, including work which is oriented toward benefiting marginalized populations. They come to the conclusion below:

> "While many successful programs in mentoring and support for students of color and white females have been established over decades, these change efforts will be self-limiting as long as our standards of rigor work to reinforce gender, race, and class hierarchies in engineering, and further maintain invisibility of queer and disabled engineering students" (p. 261).

In response to this conclusion, Riley provides tenants upon which alternative conceptions of rigor might be built, one of which is facilitating abundance instead of the competition that common conceptions of rigor rely on. They highlight that this involves knowing that strengths can come from a variety of sources and working toward facilitating greater opportunity for broader engagement, as well as building up colleagues (and students) through "criticality in a caring process" (p. 262). Riley simultaneously acknowledges that coming up with an inclusive definition of rigor is an immensely daunting task, and straightforwardly advocates for "doing away with the concept altogether so we can welcome other ways of knowing" (p. 263).

However, in many institutional realities, rigor—as a word and as a concept—remains central to many conversations in undergraduate (mathematics) education. While departments work to enhance or maintain rigor in our undergraduate mathematics courses, it is necessary to think critically about how we define rigor, as well as what functions the invocation of rigor might perform. Towards this end, Gutiérrez (2013) argues that "making dominant discourses more apparent is a necessary step toward both recognizing how those discourses dis/advantage individuals and in challenging those discourses and their associated practices so as to put new ones into place" (p. 50). We respond to this call through a critical discourse analysis of rigor, and how an implicit CoP surrounding mathematical rigor may contribute to such dis/advantages.

Data and Method

This analysis comes from a larger study on Networked Improvement Communities (NICs) under the Achieving Critical Transformations in Undergraduate Mathematics Project (ACT UP Math). Through this project, education researchers with equity specializations partnered with three mathematics departments to initiate equity-intended changes, supported by equity-oriented professional development, quantitative and qualitative data, and a broader network of support across the three departments. We selected *Kappa University* as a focal institution for the present study in alignment with their NIC's focus on addressing rigor. Perhaps related to the NIC's identification of rigor as a change priority, Kappa is a highly selective institution where students and instructors alike expect difficult mathematics courses; Kappa students characterized the courses as "unnecessarily hard." As such, the Kappa NIC context serves as a fertile ground for reckoning with tensions between the perceived value, inherent ambiguity, and operationalized impacts of rigor.

The Kappa NIC was composed of four faculty affiliated with the introductory mathematics program, two graduate students who had taught in the introductory mathematics program, and two undergraduates who had taken classes within the introductory mathematics program. Research participants opted into the NIC with interests of contributing to equity-oriented change and were all equally paid by ACT UP Math. Faculty (Anna, Senna) were invited by the two NIC faculty co-leaders (Tiersa, Skylar), and the graduate (Parker, Kiara) and undergraduate students (Ricardo, Timothy) were purposefully selected out of an application pool. The NIC predominantly identified as white women, but included one woman of color and two men.

Data consists of journals and structured interviews from the eight NIC members, collected from Spring 2023 to Spring 2025. Journals were collected intermittently and prompted participants to reflect on their experience with the NIC, with some journal prompts asking specifically about participants' understandings of rigor and their progress towards addressing rigor in their undergraduate program. Interviews were conducted once per semester, typically within the latter half of the semester.

Data Analysis

We selected a subset of data for analysis by locating instances of the word "rigor," taken together with relevant context. We leveraged critical discourse analysis—a method that examines how ideologies related to societal power relations shape discourse, both implicitly and explicitly (Fairclough, 1989, 2013). Our multi-stage analysis proceeded as follows, methodologically aligned with reflexive thematic analysis, which generally includes six phases: familiarizing yourself with the dataset; coding; generating initial themes; developing and reviewing the themes; refining defining and naming the themes; and writing up the analysis (Braun et al., 2023). First, we read through all responses to the journal prompt "What does mathematics rigor mean to you?" as a basis for our preliminary codebook. The second stage consisted of developing our codebook using the subset of data which mentioned "rigor" and its surrounding context. As the codebook developed, each excerpt was coded by at least two authors, wherein different interpretations of an excerpt were reconciled within partner-to-partner or full group discussions. Multiple excerpts were coded with multiple codes. As we organized these codes into themes, we recognized the underlying CoP and organized codes according to the five characteristics of CoP to address our research question: *How do discourses around rigor reveal a culture of power in undergraduate mathematics education?*

Findings

To structure our investigation of rigor as imbued with implicit power in mathematics, we organize our findings according to Delpit's (1988) five characteristics, with brief empirical illustrations provided for each.

Issues of Power are Enacted in Classrooms

NIC members learned from student feedback on a survey and through a focus group that many students experienced Kappa mathematics courses as "unnecessarily hard." The NIC members attributed these perceptions of the course as unnecessarily hard to how rigor was enacted in classes, and led to the NIC's decision to explore rigor as their shared goal: "This idea of being unnecessarily hard… really informed that decision to interrogate rigor" (Tiersa). Given that rigor was a widely-held value in the department, including among members of the NIC, participants grappled with emphasizing rigor through their teaching. Tiersa commented that, "Instructors are feeling like 'rigor is the key component of my class', and students are feeling

like 'rigor is the thing in class that is destroying me,'" while recognizing that for students, the emphasis on rigor "can feel like gatekeeping." While participants recognized the power exerted by instructors in some cases, at other times power was implicit in their own values around rigor. For example, Skylar shared, "My dream would be to change the narrative of Kappa math, so that it is no longer perceived as 'unnecessarily hard.'" In desiring to change how students *perceive* rigor rather than the ways rigor is *enacted* in classes, Skylar left intact the assumption that rigor is inherently valuable and students must adjust themselves accordingly.

There are Codes or Rules for Participating in Power; That is, There is a Culture of Power

When discussing rigor, participants described many codes and rules associated with rigor, which were especially pronounced in relation to proof and notation. However, they also recognized these codes and rules were largely unrelated to students' academic development. For example, Parker noted,

> There are times where [rigor] is important when you're, like, writing up a proof because… you're publishing it. And someone on the internet is going to read it… [and] you won't necessarily be given the option to explain yourself later, and you want to be rigorous with how you write it up to do your very best at being understood correctly. But that's not necessarily why our students are learning math.

Parker frames adhering to written conventions related to rigor as important for mathematicians who may be publishing their work, but less important for students. In this way, codes and rules surrounding rigor can be seen as cultural values that arise from professional imperatives among mathematicians rather than conventions that are necessary to support students' learning. Supporting this view, Kiara suggested that adhering to rules and codes surrounding rigor may be antagonistic to goals of preparing students to apply their mathematical understanding to their major discipline when teaching Calculus II:

> Mathematical rigor at a course level sometimes loses sight of the point… to equip students with tools that they will hopefully use in whatever discipline they want to… it's not always most appropriate for us to go through proofs very carefully… On some level it does feel useful for students to see… what a proof looks like… to carefully check details starting from a definition and moving towards a proposition or a statement. And yet at the same time, maybe that's not the key skill that we want to be moving through.

Kiara's description references specific codes and rules associated with establishing rigor (e.g., "starting from a definition"). However, they perceive these codes as in tension with the goals of mathematics learning. This reinforces a view that the codes and rules of rigor are the product of cultural values within the mathematics discipline, functioning as a CoP.

The Rules of the CoP Reflect the Rules of the Culture of Those Who Have Power

Participants framed the adherence to rigor conventions as culturally rather than pedagogically motivated. Their discussions of rigor suggested, both implicitly and explicitly, that these cultural mathematical values aligned with racial and socioeconomic power differentials. For example, when discussing which marginalized group to focus on to guide instructional improvements, Skyler shared the NIC considered "students of color and students who come from… less rigorous high schools," because this population experienced "culture shock" when moving from schools that "didn't even offer calculus" to being in "classrooms with people who took linear algebra already." Given the tendency for 'rigor' in secondary schools to align with socioeconomic status (Palardy, 2013), Skylar's description draws an implicit correspondence between rigor and race and class inequality, suggesting those who value and are familiar with rigor constitute a racially

and financially privileged group.

Racial inequities were named infrequently but explicitly when participants discussed qualities of rigor. For example, pertaining to notation, participants discussed, without naming, the "worship of the written word"—a quality of white supremacy culture (Okun, n.d.) which can dismiss the perspectives of marginalized people due to syntax. NIC members questioned the precise grammar of rigor, claiming that instructors "making sure that they're getting the exact correct notation every time is not necessarily in their [students'] best interest" (Parker).

Being Told Explicitly the Rules of that Culture Makes Acquiring Power Easier

In line with Delpit's (1988) work, participants advocated for explicit teaching about rigor when rigor was deemed necessary. For example, when discussing notation, Tiersa considered changing the way she evaluated student work to validate "different dialect[s]" and to be explicit about expectations for notation when she could not.

> I don't want to be picky about notation, and I wanna make sure that I'm trying to understand what my student is saying, even if they're saying it in a way that's different from how I would say it, when I'm evaluating their work. And if I am gonna be picky about something, I better be really clear about whatever it is I'm being picky about.

Beyond the specific example of notation, participants challenged the rendering of rigor as an unquestioned, implicit expectation in calculus classrooms. As an example, Parker shared,

> I… tell my students… when it's okay to not be as rigorous… There's so much that we don't tell students, that we just say… "You have to learn this because it's important." …[I'm] taking that away a little bit… [at] times I'm like, "This is correct. I would like you to use this [notation], but I also understand if you don't"... showing a little bit more of that flexibility with the ultimate goal of them learning math… and put it together for themselves, instead of just doing what they've been told, because they were told "Do it."

Parker describes being explicit about when and why rigor was required, distinguishing between students' performance of rigor and their mathematical comprehension. These instructional aims recognize that explicit instruction helps students acquire rigor as a form of power in mathematics, while also decentering rigor as the "right way" to do mathematics. It is possible such explicit forms of instruction mitigate the racial and socioeconomic stratifications that arise through students' differential pre-college exposure to rigor.

Those with Power are Frequently Least Aware of Its Existence

Although participants occasionally acknowledged links between rigor and social power tied to race and class, more frequently they revealed a lack of certainty about rigor's relationship to broader inequities. As an example, Anna shares how "interrogating rigor [in the NIC]… was student driven" and that "if it were just [her] and [her] own silo, [she] would say 'I'm not looking at that [student work] because it's not rigorous enough'", which we interpret as maintaining rigor as a benchmark for evaluating hypothetical student work. She does recognize that she does need to "keep figuring out how to maintain [rigor] in the class while still keeping things accessible or making things more accessible and inclusive," implicitly contrasting rigor with accessible or inclusive classes. The NIC's tensions with recognizing potentially exclusionary aspects of rigor was also notable when Skylar brought a discussion topic to the NIC about possible connections between white supremacy and rigor, operationalized through rigid notation expectations.. Skyler noted that when they raised possible connections to white supremacy, the NIC "veer[ed] away from criticality," consistent with an avoidance of recognizing racial power—a common move among those who benefit from it (Bonilla-Silva,

2006).

Participants often expressed a lack of clarity and sometimes concern about how rigor and the team's explicit broader goals of advancing equity were related, particularly because they engaged a critical perspective that explicitly connected equity to identity. For example, commenting on students' common perception of rigor as "unnecessarily hard," Tiersa questioned, "what's the particular population or group that feels like things are unnecessarily hard?" After reflecting on how rigor relates to issues of power mediated through mathematical notation, they also shared,

> I don't know where the diversity, equity, and inclusion are in what I've just said…. I'm coming back to this 'for all' kind of language… I guess I'm not sure how that [notation] interacts with this idea of helping include people who haven't been well included.

Similarly, Skylar reflected "we really haven't talked much about student identity at all in this group, not even the sort of students with less rigorous high school experiences." Hence, although participants were attentive to power dynamics in the instructor-student relationship, they demonstrated a limited awareness of how rigor was tied to power in broader society. Participants' separating rigor from equity reveals their lack of awareness that rigor itself is part of the CoP they inhabit—and from which they benefit.

Discussion and Implications

Riley (2017) highlights the parallelism between the dual meanings of rigor—as both academic credibility and hardship—and of discipline—as both an academic field and corrective control. The CoP revealed through our findings demonstrates the confluence of these multiple meanings of disciplinary rigor, linking widely-held and seemingly-neutral mathematical values with the exclusionary impact of introductory undergraduate mathematics courses. While some work has importantly identified aspects of classroom practice that may contribute to racialized and gendered exclusion from STEM (e.g., Leyva et al., 2021), our analysis contributes to an emerging body of work that interrogates mathematical values as a root from which exclusionary teaching practices emerge (McNeill & Gresalfi, in press).

Delpit (1988) concludes by naming a number of ways to move beyond the limiting and exclusionary confines of the CoP within K-12 literacy education, including to give students opportunities to be authentic and creative in their writing while also explicitly teaching the cultural convention of writing. This resonates strongly with many current debates around modern pedagogical approaches within undergraduate mathematics and thus has implications for both our practice and future research. Mathematics educators seeking to improve student experiences by cultivating deep thinking and creativity within mathematics may see this as at odds with directly teaching mathematical conventions of rigor, similarly centering students who have already assimilated to the CoP within undergraduate mathematics. However, through the lens of CoP we recognize rigor as cultural capital, and thus see a need to make its value and form explicit, especially for students who have not yet assimilated to the CoP. We call for future research to take up and explore the role of rigor across undergraduate mathematics education to further theorize how instruction can move towards equitable mathematics learning within a CoP.

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Student Experiences of Active Assessment in Undergraduate Mathematics

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Active-learning-based instructional practices are well-established in undergraduate mathematics. What it means to align assessment with those active learning methods is an emerging area of research and development. Here we present student experiences of "active assessment," assessment strategies grounded in the values and practices of active learning. As part of a larger study, we gathered undergraduate student responses to survey and semi-structured focus group interview questions about their experiences with various active assessments. Their instructors all were part of a year-long Assessment Community of Practice focused on active assessment. In this paper, we focus on student-identified benefits of active assessments, as well as student perceptions about how the assessments were equitable or potentially inequitable. We conclude with suggestions for instructors and discussion of considerations for moving forward.

Keywords: active assessment, active learning, student experience

What does it mean for assessment to be aligned with the values and practices of active learning? The benefits of active learning for students in college level mathematics courses are well-documented (Freeman, et al., 2014; Theobald, et al., 2020; Uhing, et al, 2021). However, little is known about how to align assessment practices with the methods promoted for active teaching and learning. *Active learning* has been defined broadly as instructional methods involving students in doing things and thinking about what they are doing (Bonwell & Eison, 1991). Our consideration of active learning includes Laursen and Rasmussen's (2019) four pillars of inquiry-based mathematics education (students engage deeply with meaningful tasks, students collaboratively process ideas, instructors inquire into student thinking, and instructors foster equity in their choices). Consonant with these methods, we have defined *active assessment* based on a consolidation of ideas including active learning, formative assessment, and alternatives to timed, high-stakes individual exams. Our working definition of active assessment was developed across a year of work by an instructor Assessment Community of Practice.

Active assessment captures student progress and facilitates growth in the journey through mathematics. Active assessment is a process that sets high expectations and supports meeting those expectations. The process is rooted in

- engaging student understanding more equitably,
- reinforcing communication, reflection, and collaboration, through mathematically rich tasks,
- valuing diversity in expression while being asset-minded, and
- giving students the opportunity to share, learn, and expand on ideas as a community.

Examples of active assessment include group projects and presentations, exams with a group or collaborative component, variations on standards-based grading, and portfolios. This paper presents student experiences of several of these forms of active assessment. The driving question for the larger project is: What are the benefits that students and instructors identify in the

implementation of specific active assessment strategies? Here we report on analysis of a subset of data aimed at addressing a particular question about student experience. *Research Question:* Among students who participated in at least one group-based active assessment, what did they identify as benefits and what did they point to as equitable (or not) in working with others to complete the assessment?

Conceptual Framing

We draw from Black and Wiliam's (2009) definition of formative assessment and Laursen & Rasmussen's (2019) definition of inquiry based mathematics education in framing this study.

Formative Assessment

Black and Wiliam (2009) give the following definition of formative assessment which highlights how teachers, learners and their peers are involved in formative assessment:

> Practice in a classroom is formative to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited.

They elaborate on this definition by introducing five strategies that were developed by crossing the three different agents (teacher, peer, learner) with Ramaprasad's (1983) three key processes for learning (see Table 1).

Table 1. Aspects of formative assessment from Black and Wiliam (2009).

<table>
<tr><th></th><th>Where the learner is going</th><th>Where the learner is right now</th><th>How to get there</th></tr>
<tr><td>Teacher</td><td>1. Clarifying learner intentions and criteria for success</td><td>2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding</td><td>3. Providing feedback that moves learners forward</td></tr>
<tr><td>Peer</td><td>Understanding and sharing learning intentions and criteria for success</td><td colspan="2">4. Activating students as instructional resources for one another</td></tr>
<tr><td>Learner</td><td>Understanding learning intentions and criteria for success</td><td colspan="2">5. Activating students at the owners of their own learning</td></tr>
</table>

While elements of active assessment are formative, assessments also may serve as summative for a course or unit. Of particular interest to this study is strategy 4, *activating students as instructional resources for one another.* We explored the ways in which students are "activated" in this way by examining the elements of the active assessment process that students identified as beneficial, equitable, and potentially inequitable.

Four Pillars of Inquiry Based Mathematics Education

The Assessment Community of Practice was conceived in an institutional context that has a strong commitment to active learning. The active learning values of the departmental community were strongly aligned with the four pillars of inquiry based mathematics education (Laursen & Rasmussen, 2019). These four pillars are:

1. Students engage deeply with coherent and meaningful mathematical tasks.

2. Students collaboratively process mathematical ideas.
3. Instructors inquire into student thinking.
4. Instructors foster equity in their design and facilitation choices.

The four pillars did not originate in a context where assessment was considered explicitly. Nonetheless, they deeply influenced the ACoP's definition of active assessment. In this paper, we focus on the second and fourth pillars: students' reported experiences of how they collaboratively process mathematical ideas and their self-reports about perceived equity and inequity in the active assessment process.

Methods

Data for the study was collected through a two-stage process during the Fall 2023 through Fall 2024 semesters. In the 2023-24 academic year, we received a total of 278 survey responses from students in 13 different lower and upper division undergraduate mathematics courses, taught by 8 participating instructors who implemented active assessment strategies. This represented 59% of all students enrolled in those courses. Stage 1 data consisted of written responses to a survey administered during class time towards the end of the semester. Questions asked the students to share their experience with the active assessment strategy implemented in the class that they took, and to identify potential benefits or inequities of the strategy. Stage 2 data were from semi-structured focus group interviews with a subset of 18 of the 278 students who had completed surveys. At least one student from 9 of the 13 courses surveyed was in one of the four focus group interviews. The group interviews were completed across Spring 2024 (5 students and 2 students) and Fall 2024 (6 students and 5 students). Each interview lasted approximately 60 minutes. The interview questions were designed as follow-ups to the stage 1 survey and discussions focused on students' experiences with active assessment.

All data analyzed for this report were qualitative, from open-ended survey and interview prompts. The unit of analysis was a "student response." For the survey, this unit of analysis was the entirety of a student's answer to a survey question. For the focus group, a "student response" was everything a student said in response to a prompt (from the interviewer or from another student). Analysis of student responses was through the inductive constant-comparative method commonly used for qualitative data interpretation (Patton, 2014; Creswell & Creswell, 2018). The inductive hypothesis was that the categories for coding came from the prompts (e.g., the categories of benefits and equitable/inequitable experience). For the surveys, thematic coding was completed by three authors. Each person coded a subset of responses (about 20), then they compared and calibrated their coding. This cycle continued with larger subsets until all responses had been coded for the surveys. A different subset of three authors completed a similar coding and calibrating cycle for the transcripts of the focus group interviews. Next, key themes were identified across coding of student responses from the written surveys from stage 1 and the interview transcripts from stage 2. We report here on this analysis.

Researchers

All of the researchers have experiences of active assessment as students, in undergraduate mathematics courses, graduate courses and/or post-graduate professional development. In the context of undergraduate courses, some of the researchers have experience implementing active assessments as instructors of record and some plan to in the future. Our research team includes one faculty member and three current/former graduate students, each of whom worked on the project for at least one year and contributed to analysis for at least one of the stages.

Results

We present student experiences of active assessment strategies first from the survey and then from the focus group interviews. Students shared benefits and challenges about active assessments that involve collaborative group work, offering insight into how they saw these strategies as impacting their learning, confidence, and sense of equity. The key themes touch on collaboration, access, group dynamics, and equity, highlighting strengths and limitations of active assessment from the student point of view.

Student Survey Results

The two main survey questions we focused on for this report were:

1. Please give an example of something about the [assessment strategy] that you think is beneficial for students? and,
2. Please give an example of something about the [assessment strategy] that you think might be unfair or inequitable? Why?

Of the 268 responses to the first question, the most frequently identified benefit of an active assessment strategy was collaboration, with 85 student responses (31%) mentioning it. For example, one student wrote, "They allow students to share their knowledge and come together to solve problems. Multiple sources of input helps because maybe one section is particularly challenging." The next most frequently identified benefits were applications and reflection/review, each with 21 student responses (8%) mentioning these aspects as beneficial. Other identified benefits included less stress, building experience, feedback, and the opportunity for demonstrating knowledge. The vast majority of responses identified only one benefit, but several longer responses identified more than one benefit.

On the other hand, among those who responded to the second question, the most commonly identified potential inequity of the active assessment strategies was unequal work, with 76 responses (27%) identifying unequal work distribution as a potential source of unfairness. These comments were similar to the student who wrote, "With any group, there tends to be at least one person who doesn't do much work while there is someone who does too much." The second most frequently identified inequity was format/accessibility, with 23 student responses (8%), and the third most frequent response was group dynamics (not related to unequal work distribution), which appeared in 18 responses (6%). Other identified inequities included time/pacing and workload. Finally, 44 students (16%) said the strategy they experienced was not inequitable.

Student Interview Results

Students' perspectives were also gathered through focus groups. The interviews offered glimpses into how students defined equity, perceived the benefits of group-based assessments, and how they identified the assessments as equitable or potentially inequitable. Students also suggested improvements for future implementation. The following sections present key themes that emerged from the focus group interviews.

Student definitions of equity. During the focus group interviews we asked students to define equity in their own words. This gave additional context and meaning to their replies to later questions regarding the ways the assessment strategies they experienced were equitable or not. The common themes that arose in their responses were access and accommodations. As one student stated, "[Equity means] a level playing field for everyone." This was echoed by a student in a different focus group stating "The definition of equity is being able to allocate more toward those who are in need." In another group, a student said,

> I think equity has a lot to do with I guess access for me. Like, if you have access to these certain things or certain… Not learning strategies, like… Access to these different modes or these different styles and different learning. Different kinds of learning, right?

which was then backed by another student who said,

> For my definition of equity it would probably be, to be able to accommodate as large of an audience as possible. So pretty much not leave anyone out, be beneficial to everyone.

Students' definitions of equity provide a meaningful lens through which to interpret their experiences. Some students saw benefits as being an indicator of equity, and there was significant overlap in responses to questions about benefits of active assessments and what students indicated was equitable.

The beneficial impact of active assessment. A main theme that emerged from the interviews was the academic benefit of learning from peers. Students often described how working collaboratively exposed them to different ways of thinking.

> And also you could learn more from the group, because sometimes like you think you're doing it right, for example a lot of graphing for that one and you know sometimes it's just like those small things where someone could actually walk through it and be like, oh, actually I think this, you know, two brains are better than one, you know?

Another benefit students identified was the shift in students' mindset and confidence in problem solving. Rather than viewing assessments with anxiety or dread, some students reported that group assessments had changed their perspective to be more agentic:

> Students are not thinking of [math course] as like… it's a terrible exam. 'How am I going to do? What am I going to do?', that perspective changed to 'Okay, I'm going to figure it out. I got this exam.' So that change of perspective is what I observed within me, my friends in that class and all other people as well.

Group dynamics also played a positive role for many students. Several students emphasized how collaboration allowed for a safe environment to make mistakes and learn from them.

> When you do it in a group environment, you're allowing for that mistakes that happen. And with those mistakes, they get more confident in their answers, they're like, oh I know exactly what I'm doing now, I learned from my mistakes.

Furthermore, shared responsibility was reported as beneficial:

> I really found that effective uh because like for me as well as for others, because people... felt confident they feel comfortable in that setting and then that eventually helped them.

also serving to motivate students to engage more deeply with the material:

> I also observed that a few people from my group, they actually started putting more efforts in the group when they became a part of this… since everyone is answering questions, I need to put more efforts on myself. Maybe I need to study more so that even I can share answers or even I can help others to get solved and then maybe a few doubts which I have, others can help me with that.

These themes, exposure to other ways of thinking mathematically, an agentic perspective, increased support for risk taking, and the motivational benefit of shared responsibility captured what 14 of the 18 students (78%) said related to benefits. The students whose responses were not represented by these themes focused on reduced stress around grades, opportunities to "think outside the box" when demonstrating knowledge, and instructor support.

The inequities of active assessments. In addition to benefits, students also identified aspects of the assessment strategies that were potentially inequitable. Students identified challenges that raised questions about fairness, accessibility, and the burdens placed on certain individuals

within groups. One theme that arose was the shared responsibility of group work and having to rely on one's peers, particularly in groups where interaction was strained.

> For the project I did, a big barrier was not working well with team members, if they don't show up or you don't get along, it's hard to do the work. Unlike an exam where you rely on yourself, the main challenge here is working with others.

Another inequity that many students identified was the discomfort that group interactions might have for students with social anxiety.

> I've noticed that some people maybe they want to like, say their point of view out loud, but they just don't feel comfortable. Or, or yeah I guess they don't feel comfortable enough to talk about it as a group, maybe they might be afraid of how they might be judged. Maybe if it's wrong, they might overthink it and assume the entire group is gonna think they don't know what they're talking about.

Additionally, some students also acknowledged the potential to "ride the coattails" of their peers in a group setting.

> For mine, I like for sure rode the coattails of one of the dudes in our group, like I mean because math in like upper division takes a lot of time to study and some of these I feel like should be year long courses. So when I was able to study and prepare, I was contributing well.

These themes, interpersonal dynamics, discomfort and social anxiety, and unequal contributions to group work captured what 14 of the students (78%) responded in relation to inequities. While other concerns brought up included fairness of grading and time commitments, not all of the 18 students identified any inequities for their assessment strategy.

Student Suggestions for Next Time

As part of the survey, we asked students "If you could give advice to an instructor who was going to use this strategy in a future class, what would you want them to know?" Students identified areas of opportunity for instructors to combat the challenges that may arise with active assessment strategies. Student suggestions included:

1. Develop and communicate clear rubrics outlining specific expectations for active assessment tasks, as students noted that the unfamiliar nature of these strategies often introduced ambiguity in grading criteria compared to traditional assessments.
2. Integrate structured icebreaker activities to support the development of positive group dynamics and mitigate the social complexities associated with collaborative work.
3. Allocate additional in-class time for group collaboration, which students identified as a means to foster stronger peer connections and reduce the logistical challenges of coordinating meetings outside of class.
4. Incorporate a system of accountability within group assessments, such as peer evaluations and periodic check-ins with the instructor, to ensure equitable participation and sustained engagement.

In the focus group conversations and survey responses, student suggestions conflicted or students identified tensions in their preferences for how the active assessments were structured or implemented. For example, some students preferred randomly assigned groups while others said it "felt more comfortable" working with people they chose to work with. Similarly, some students preferred keeping the same group throughout a term, while others noted benefits from working with different peers. One-third of the focus group participants and about 9% of survey respondents indicated that they would like to be able to choose between assessment strategies (e.g. project or traditional exam for a particular unit).

The survey results indicate that despite the potential inequities they identified, students overwhelmingly preferred active assessments to timed, individual exams, with 79% of students stating that, given a choice, they would choose the active assessment compared to 16% who would choose a timed exam. Other students would choose both forms of assessment, or neither.

Discussion and Recommendations

While the majority of students reported meaningful academic and motivational benefits from active group assessment, their reflections also revealed important concerns. Despite the potential for deeper learning and confidence, experiences in collaborative settings were not universally positive. Student reflections highlight the complexity of group-based active assessments. Though some students reported finding them academically enriching and confidence-building, others encountered challenges that raised concern about equity. These tensions point to a need for thoughtful implementation, reflection, and ongoing improvement.

Some concrete suggestions from students to instructors look to alleviate the social complexity of working in groups and develop stronger relationships among peers. At the same time, students came to active assessments more familiar with individual assessments. Thus, their suggestions emphasized the importance of being graded for their specific contributions and being able to hold groupmates responsible for doing a fair share of work.

Not all challenges in implementing active assessment have straightforward solutions or recommendations. We note that different grouping strategies might work better for particular courses and contexts. Instructors may choose to poll their class to give the class more agency in determining group structures, or give some choice (e.g. choose one person you want to work with) within a more randomized system. Also, student preference for the option to choose among active assessment formats may raise additional concerns about equity in grading and opportunity.

A major ongoing topic of discussion for higher education is how to accommodate and support students who have extensive outside of class obligations or other personal reasons for being unavailable to engage in collaborative assessment activities that call for outside-of-class work. This is particularly relevant at our institution where many students work full time outside of the university, are nontraditional students, and commute to campus. The challenge of collaborating with peers who are not always present came up repeatedly for students, even as many of them found collaboration to be a primary benefit of active assessments.

Topics not raised by students in the interviews, but important for continued research and development of active assessments include: strengths and contexts of multilingual students, considerations for accessibility and accommodations, and adaptability to changing technology.

Each group of students is different. There is no one right way to make an active assessment perfectly equitable. Our conceptualization of active assessment is aspirational, and implicit in its definition is ongoing revision and response focused on making assessment equitable in a specific context. In the words of one student focus group participant: “Maybe it's impossible to have like, equitable classroom[s]. Lovely in theory, but it's just almost impossible to do. Okay. But I appreciate the efforts.” Student participants overwhelmingly preferred active assessments, though imperfect, and identified a number of benefits. This research suggests that active assessment has potential to imperfectly move toward equity.

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PSTs' Symbolic Forms During Mathematizing: The Case of Mia and Ava and the Laundry Task

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This study investigates the symbolic forms pre-service teachers (PSTs) drew upon while mathematizing a real-world laundry task. Drawing from Sherin's (2001) framework of symbolic forms, we analyzed how two PSTs, Mia and Ava, coordinated mathematical symbol templates with conceptual reasoning to model when clothes should be washed based on bacterial load thresholds. Our analysis identifies both singleton forms: threshold and exponentially growing, and networks of forms: rate, contributing factors, and exponential growth with a threshold. We end with implications for research on student cognition during modeling.

Keywords: symbolic forms, mathematical modeling, mathematizing, pre-service teachers, student thinking

Introduction

Mathematical modeling entails using mathematical structures and relationships to represent real-world situations. As such, modelers are required to engage in certain processes to have successfully modelled (Niss, 2010). Within the mathematical modeling (hereafter referred to as modeling) education research literature, the process of using mathematical structures to formulate a mathematical model for real-world situations is referred to as *mathematizing*. In essence, mathematizing is a process by which an individual uses mathematics to make sense of reality (Freudenthal, 1968). While there exist various modes of mathematical representations that can be used during mathematization, such as graphs, diagrams, and tables (Duval, 2006), modeling from first principles often requires modelers to represent real-world situations using symbols, expressions, and equations. When a modeler models from first principles, the goal is to align a mathematical representation with a scientific interpretation of the situation (Greca & Moreira, 2002). As mathematizing has been documented as notoriously challenging for learners of all levels (Stillman et al., 2010; Roan, 2024), many research programs have developed around the central goal of supporting students' mathematization. Examples of these research programs include theoretical frameworks for what it means to have successfully mathematized (Niss, 2010), how to support students through scaffolding to mathematize (e.g., Stender & Kaiser, 2015), identifying the blockages students' experience during mathematization (Klock & Siller, 2020; Schaap et al., 2011), and cognitive constructivist perspectives to construct accounts of students thinking around mathematizing, which is relevant to our study.

The primary goals of research programs that adopt a cognitive constructivist perspective to study students' mathematizing are to (a) build asset-based accounts of how and why a student did a certain mathematizing move, and (b) use those student-centric accounts to guide instructional design and intervention. In this study, we aim to contribute to the knowledge on goal (a). To do this, we adapt Symbolic Forms as a theoretical construct that is rooted in Knowledge-in-pieces (KiP) (diSessa, 1993). Particularly, in this paper, we contribute to the literature surrounding modeling and cognitive constructivist theories by documenting the symbolic forms our participants, pre-service teachers (PSTs), drew upon as they constructed a mathematical model for the Laundry Task scenario.

Theoretical and Empirical Background

Mathematizing

We situate our work within the cognitive perspective of modeling (Kaiser, 2017), where the focus is on investigating modelers' cognitive processes during model construction. Within this perspective, mathematizing is typically considered as transforming a real model (i.e., a modeler's idealized conception of the situation) into a mathematical model (i.e., a modeler's organization of their conception of the real-world in mathematical terms). While the process of modeling includes other competencies during modeling, such as understanding the situation, simplifying the situation by making assumptions, validating the process and results, and interpreting the results, the model construction process can be clustered into two major phases: pre-mathematization and post-mathematization (Niss, 2010). The compartmentalization of the modeling process, as before and after the model is formulated, implies the centrality of mathematization to modeling. For successful mathematization, it is suggested, modelers have to engage in both cognitive, meta-cognitive, and anticipatory meta-cognitive activities (Galbraith, 2017; Niss, 2010; Stillman, 2011). Yet, despite this centrality of mathematization to modeling, the process of mathematization has only been investigated to the extent of the kinds of mathematical approaches modeler's use during mathematization (e.g., Anhalt et al., 2018; Larson, 2013), the cognitive blockages students' experience during mathematization (e.g., Klock & Siller, 2020), and the modes of representations through which modelers externalize their mental representations (e.g., Duval, 2006). While this research illuminates important facets of students' mathematization activities, it does not attend to the cognitive resources (Hammer, 2000) that students draw upon during mathematization. Understanding these resources is critical for identifying instructional interventions that go beyond instructing students what to do to diagnosing students' blockages and prescribing ways where students can overcome the blockages to progress in their modeling. In this study, we focus on the cognitive resources PSTs drew upon to mathematize the real-world situation through equations.

Symbolic Forms

The fundamental premise of the KiP framework is that one's knowledge is a complex network consisting of knowledge ontologies and knowledge systems that work in coordination to make sense of a given situation (Swanson, 2025). One example of knowledge ontology—fine-grained knowledge elements that get activated when a situation is presented—that is particularly useful for mathematizing, is symbolic forms (Sherin, 2001). Symbolic forms build on the idea that scientific construction is an epistemic game in which the players' (or learners' or modelers') moves are guided by pre-existing knowledge templates (Collins & Ferguson, 1993). Thus, symbolic forms constitute both a symbol template and a conceptual schema. These constructs have been neatly illustrated by Jones (2013) with the following example. Consider the equation $V = V_0 + \Delta V$. This equation has the symbol template ■ = ■ + ■, and the conceptual schema associated with this template is total equals based plus change. At the same time, consider the equations $T = P + K$, where T represents the total energy of an object and P and K represent potential and kinetic energy of an object, respectively. While this equation also has the symbol template ■ = ■ + ■, it has the conceptual schema parts of a whole. In his work with students working on physics concepts, Sherin (2001) documented the various symbolic forms his students used while constructing equations to represent physical situations.

Few scholars have used symbolic forms and an analytical tool to construct explanatory models of students' mathematizing activities. For example, Roan and colleagues (Roan, 2024

Roan et al., 2025) used symbolic forms together with quantitative reasoning (Thompson, 2011) to explain the many meanings of multiplication students attribute to ■ × ■, while using it to model real-world scenarios such as interaction between species. The underlying premise is that symbolic forms blended with quantitative reasoning (Brahmia & Thomson, 2025; Roan, 2024) are powerful tools to go beyond describing what the students did to explaining why the students did what they did, providing prescriptive capabilities for educator intervention (Czocher, 2025). With this hope, in particular, we address the following research question: *What symbolic forms did the PSTs draw upon as they mathematized the laundry task?*

Methods

Participants and Setting

The participants were recruited from a mathematics content course required for K-8 pre-service teachers at a large public university in the western United States. All participants had completed prerequisite coursework in college algebra. A total of four students participated, working in two pairs. The research team met with each pair for 60 to 90 minutes for a total of ten sessions. Sessions were held in a quiet room on campus and were audio and video-recorded. The PSTs were given the freedom to conceive the task however they wished and use any mathematics they thought best represented their conception of the task. The researcher's role was to ask clarifying questions and probe the PSTs' thinking. In this paper, we focus on the symbolic forms of one pair of PSTs' (pseudonyms Mia and Ava) working on the Laundry Task. The Laundry Task is presented in Table 1. Mia and Ava's work on the laundry task was reminiscent of what Collins and Ferguson (1993) would call an epistemic game. First, the PSTs conceived that bacterial growth could either occur linearly or exponentially. Next, they modelled each linear and exponential growth, abiding by the "rules" of how each growth occurs. For this reason, we chose to explore Mia and Ava's work, the Laundry Task, for further analysis. In this paper, we report on a portion of those findings.

Table 1. The Laundry Task (abridged version). Task description taken directly from Smith (2023).

On January 10, 2023, TikTok user Allison Delperdang sparked a lively debate when she shared a video about wearing the same pajamas multiple times before washing them. The video garnered millions of views and countless comments, with some agreeing that frequent washing isn't necessary, while others argued it was unhygienic. Experts weighed in, explaining that how often clothes should be washed depends on several factors, including personal hygiene habits, activity levels, and the type of clothing. Dermatologist Dr. Anthony Rossi emphasized that our cultural norms often lead to "over-washing," even though not all clothes need frequent cleaning. At the same time, wearing hygiene clothes can cause skin infections due to bacterial overgrowth in the unwashed clothes. Research indicates that bacterial loads exceeding 1000 colony-forming units per square centimeter (CFU/cm^2) can pose a risk, particularly for pathogenic bacteria like Staphylococcus aureus. Develop a mathematical model for determining how long clothing can be safely worn before washing?

Data Analysis

The data were transcribed and analyzed using MAXQDA, a qualitative data analysis software, to support the identification of PSTs' symbolic forms. Our analysis proceeded in three stages. First, we used Sherin's (2001) framework on symbolic forms as a foundation to develop a shared criterion for identifying symbolic templates and their coordination with conceptual reasoning. We conducted an open reading of the transcripts, supplemented by repeated viewing

of video recordings to become familiar with the data while attending to moments where PSTs expressed their reasoning through both inscriptions and verbal utterances. During this round, we used Sherin's framework (see Sherin, 2001, p. 532–537) to identify any symbolic forms. For this analysis, we sought evidence of symbolic forms through (a) symbolic templates that were explicated either verbally or through notations and (b) PST's justification for their proposed templates. For example, when Mia and Ava wrote the expression $(c + h + a + s)^{\#days}$ as their model for exponentially growing bacterial load, we coded this as the symbol template ■▪. When the interviewer probed for justification for their representation, Ava stated, "I don't know if you need to put it to a power or something…I feel like when I've done exponential kinds of problems in the past…I mean I think it makes sense in the fact that as more days there's more growth." We took this as evidence of a growing exponentially conceptual schema. This is because the PSTs reasoned that the bacterial load grows "with some parameter that appears raised to a positive power in the exponent of the exponential" (Sherin, 2001, p. 536). Both authors independently coded the transcripts for such instances, marking segments where symbolic forms appeared and assigning them to one of Sherin's categories (e.g., *parts-of-a-whole*, *canceling out, exponent*). If there were any forms were not part of Sherin's framework, we created a new code based on the inferred PSTs' conceptual reasoning. We then met regularly to compare interpretations, discuss discrepancies, and refine the coding scheme together. Finally, the authors applied the refined framework to the entire video to ensure consistency.

Findings: PSTs' Symbolic Forms

Our analysis revealed that PSTs used a variety of forms as they worked on the Laundry Task. These forms fell into two categories: *Singleton Forms,* which involved the use of a single symbolic template tied to a conceptual schema, and *Network of Forms,* which reflected the coordination of multiple symbolic templates to organize the situation. Within the singleton forms, we identified the threshold and exponentially growing forms. Within the network of forms, we identified the rate form, the contributing factors form, and the exponential growth with a threshold form. Table 2 summarizes these symbolic forms, their associated templates, and the conceptual schemas that PSTs drew upon. Due to space constraints, we elaborate on the threshold, rate, and contributing factors in symbolic forms.

Table 2. Overview of PSTs' symbolic forms

Form	Template	Conceptual Schema
	Singleton Forms	
Threshold	■ < ■	One side cannot exceed the other
Exponentially growing	■▪	The amount is growing exponentially with time
	Network of Forms	
Rate	■/■ = ■	Instantiating a rate
Contributing Factors	■ = ■ + ■ + ■	Factors that contribute to a whole
Exponential growth with a threshold	■ ≤ ■▪	The amount that is growing exponentially with time cannot exceed the other

The Threshold Form

The threshold symbolic form consists of the symbol template ■ < ■ and the conceptual schema that one situational attribute cannot exceed the measure of another. For example, Mia and Ava reasoned that if the bacterial load reaches 1000 CFU/cm^2, then the clothes must be washed. The following conversation between Mia, Ava, and the interviewer illustrates this form.

Mia: I feel like one side of the equation would need to equal this, though [referring to the 1000 CFU]. Because that's a max that you can go.

Ava: Yeah, because that's a risk.

Mia: But like something equals 1000 colony for me. I don't know. *Interviewer:* What did you mean by that's the max we could go?

Mia: Because it says any bacterial loads exceeding this amount need to be washed basically, because they pose a risk.

Here, Mia and Ava reasoned that once the bacterial load reaches or exceeds 1000 CFU/cm^2, the clothes need to be washed, implying the bacterial loads on clothes must be less than 1000 CFU/cm^2, for the clothes to be safely worn. They initially mathematized the situation as $clothes < 1000\ CFU/cm^2$. Once the interviewer probed on what the PSTs meant by "clothes", Mia and Ava modified their model as shown in Figure 1. Mia and Ava's expressions, "$clothes < 1000\ CFU/cm^2$" and "$bacterial\ load\ on\ clothes/cm^2 < 1000\ CFU/cm^2$", and their reasonings reflect their organization of the Laundry Task scenario by drawing upon the threshold form.

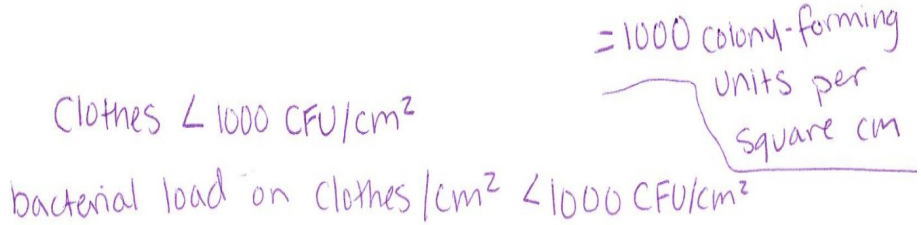

Figure 1. Mia and Ava's threshold symbolic form

The Contributing Terms Forms

Contributing terms symbolic forms consist of the symbol template ■ = ■ + ■ + ■and the conceptual schema that distinct quantities (left side of the equation) contribute to the whole (right side of the equation). This form is considered a network of symbolic forms because it integrates two key components of Sherin's forms: *parts-of-a-whole* form, which represents additive components (■ + ■ + ■) and the *balancing* form (■ = ■) which represents that both sides of the equation should be equal.

Mia and Ava demonstrated this symbolic form as they mathematized the factors that contributed to the rate of bacterial growth on specific clothing pieces. First, they began by identifying specific factors that influence bacterial load, such as type of clothing, personal hygiene, level of activity, and sickness (Figure 2). Mia justified the inclusion of sickness as a contributing factor to bacterial growth as "I just feel like when you're sick, it's like you're trying to get rid of all the germs, so you're trying to wash your clothes more." Next, they conceived that the rate of growth of bacteria per day is composed of the rate of growth of bacteria per day due to the contributing factors. Building on this reasoning about what might cause the bacterial load on specific clothing to reach the threshold of 1000 CFU/cm^2, Mia and Ava mathematized the

situation as shown in Figure 2. In this form, the overall rate of growth of bacteria is a whole composed of additive parts, for example, $rate\ of\ bacterial\ growth\ per\ day =$ $rate\ of\ bacterial\ growth\ due\ to\ Factor\ A\ +$ $rate\ of\ bacterial\ grwoth\ due\ to\ Factor\ B\ +$ $rate\ of\ \ bacterial\ growth\ due\ to\ Factor\ C.$

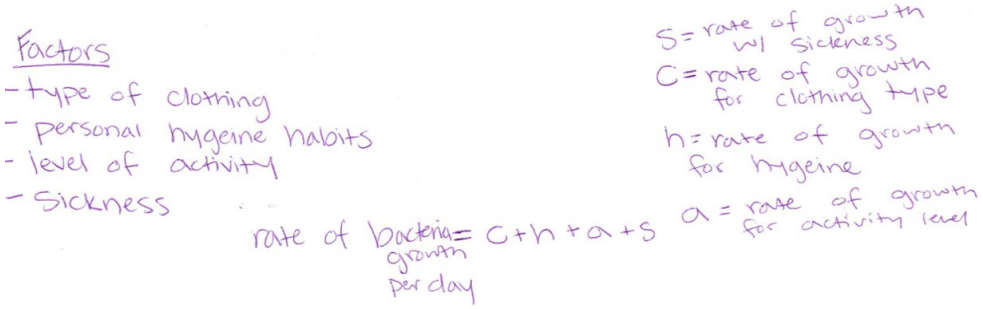

Figure 2. Contributing terms listed by Mia and Ava

The Rate Forms

The PSTs mathematized the number of days specific clothing could be worn as below:
$1000 \div rate\ of\ bacterial\ growth\ (measured\ in\ days) =\ days\ of\ wear$ (1)

We identified this mathematization as PSTs drawing upon their rate form. A rate symbolic form consists of the symbol template ■/■ = ■ and a conceptual schema instantiating a rate. We identified this as a network of forms as PSTs combined multiple symbolic forms simultaneously: (i) canceling form with symbol template ■/■ and conceptual schema that identical symbols appear in the numerator and denominator, and which represents the rate relationship, and (ii) balancing form with symbol template ■ = ■ and conceptual schema both sides of the equation are equal. We interpret that the PSTs drew on both these symbolic forms to mathematize the number of days one could safely wear a clothing piece before it needs to be washed. We elaborate on this interpretation below.

For the PSTs', one side of the equation was number of days. We make this conclusion due to the task prompt and the models PSTs constructed for this task. So, for the PSTs, the game was: *if one side of the equation was number of days, what should the other side be so that both sides are equal*. To attain this, as shown in Figure 3(a), the PSTs simplified the scenario by assuming that the total rate of bacterial growth was 100 CFU/cm^2 per day. Next, drawing on their balancing form that two sides of an equation should equal, the PSTs sought out what operation would produce number of days to balance the other side of the equation. Given that, 1000 CFU/cm^2 was the threshold bacterial load before the clothes required washing, to balance both sides of the equations to number of days, the PSTs drew on their canceling form and divided the threshold bacterial load by the bacterial growth per day to effectively cancel identical symbols. They verified their actions through considering different values for bacterial growth per day, such as 200 CFU/cm^2 (see Figure 3(b)). Finally, as shown in Figure 3Figure 3(c), the PSTs generalized their reasoning into equation ($1000 \div rate\ of\ bacterial\ growth\ (measured\ in\ days) =$ $days\ of\ wear$ (1).

We nominalized this form as a rate form because Mia and Ava purposefully operated on the threshold bacterial load, 1000 CFU/cm^2, and the bacterial growth per day, for instance 100 CFU/cm^2 per day, to yield the number of days. This operation mirrors Thompson's (1994)

notion of the quantitative operation instantiation rate. In a similar manner, we hypothesize, Mia and Ava constructed to a network of symbolic forms to mathematize their conception of the laundry scenario.

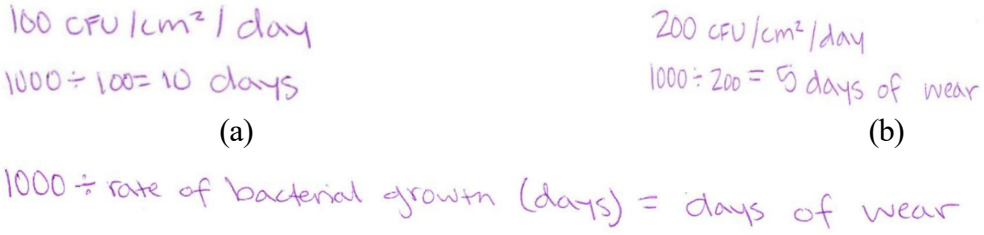

Figure 3. Network of symbolic form examples

Discussion

In this study, we found that our PSTs drew on symbolic forms in multiple ways, alone or in coordination as a network of forms. Our study is a first step towards building a repertoire of modelers' constructive resources (Sherin, 2000)—cognitive resources used by modelers as they mathematize real-world scenarios such as the Laundry Task. As mathematizing is not limited only to the construction of equations, our study's focus does not provide a complete picture of students' constructive resources for modeling. Researchers interested in modeling-centric cognition should draw on other representational forms, such as graphical forms (Rodriguez & Jones, 2024), to understand modelers' representational activities during modeling. Building such fine-grained accounts of modelers' knowledge enables educators and researchers to (a) diagnose the blockages modelers may experience during modeling (Roan, 2024), (b) provide modeling support that attends to and responds to modelers' cognition, and (c) develop modeling task characteristics from the perspective of the students (Roan et al., 2025). This repertoire of students' thinking can be used in modeling specific professional development programs to support teachers in designing and implementing modeling tasks that are guided by modelers' cognition (Franke & Kazemi, 2001).

Investigating students' symbolic forms and other constructive resources will pave the way to conceptualize modeling as a goal oriented, epistemic game, where students have the autonomy to choose their next move based on the resources available to them. In this view, modelers' actions are driven by "*I need to get X, what can I do to get X?*" For example, in the threshold form we illustrated, Mia and Ava's modeling actions were guided by the question, "*how can I mathematically represent that one side of an expression cannot exceed* 1000 CFU/cm^2." Similarly, in the rate form, Mia and Ava's actions were driven by "*what operations can perform to make one side of the equation the same as the other, which is number of days of wear*." Thus, adapting this perspective implies that facilitating students' modeling activities entails creating opportunities for students to activate certain resources, as well as designing learning ecologies that occasion the construction of certain resources which can later be activated for model formulation. Students' cognition during modeling is idiosyncratic and complex; we contend that using one theoretical framing alone is inadequate in capturing the intricacies in modelers' mathematics. Therefore, we call for more work in connecting multiple theoretical framings (Radford, 2008) that work synergistically to provide rich descriptions of modelers' cognition.

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Connecting Multiplicative Structures Across Contexts: Students' "Product Layer" Reasoning in Water Pressure and Electricity Tasks

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Students often struggle to interpret the multiplicative structure within integrals—what we call the "product layer" ($f(x) \cdot dx$)—when reasoning about applied problems. In this study, we analyze the reasoning of Melinda, an undergraduate in a calculus-based physics course, as she works on two tasks: estimating the total force on a tank wall (water pressure) and the total voltage across a capacitor (electricity). Using conceptual blending theory, we examine how Melinda connects symbolic, mathematical, and physical ideas when setting up integrals. Melinda frequently interprets integration as summation of small pieces or as area under a curve but struggles to coordinate multiplicative relationships among varying quantities (e.g., pressure· area, current· time). These findings suggest that challenges with product layer reasoning persist across contexts, highlighting the need for instruction that explicitly supports blended multiplicative reasoning.

Keywords: definite integral, conceptual blending, introductory physics, knowledge resources, product layer

Reasoning about integrals in applied contexts is central to success in mathematics and physics, yet students often struggle to connect symbolic procedures with physical meaning (Brahmia et al., 2016; Wilcox & Corsiglia, 2019). These difficulties are especially pronounced when interpreting the product layer—the multiplicative structure within an integral (Hu & Rebello, 2013a; Sealey, 2014). Applied contexts such as fluid mechanics and electricity require students to coordinate two simultaneously varying quantities and relate them to physical phenomena, rather than merely computing antiderivatives.

Despite its importance, students' reasoning about the product layer remains poorly understood, particularly across different contexts. The present study addresses this gap by examining how an undergraduate student constructs and coordinates symbolic, mathematical, and physical ideas when working with integrals in two applied settings. By comparing reasoning patterns across contexts, we aim to identify both persistent challenges and context-specific features that influence product layer reasoning.

This study is guided by two research questions:

1. *How do students reason about the product layer within integrals solving water pressure and electricity tasks?*
2. *How do symbolic, mathematical, and physical spaces interact across these contexts, and what roles do blended spaces play in shaping students' reasoning?*

Background and Literature Review

Students frequently struggle with applied integration, particularly in understanding the multiplicative structure of infinitesimal elements ($f(x) \cdot dx$), known as the product layer (Hu & Rebello, 2013a; Sealey, 2014). Mastery of this structure is critical for modeling quantities that depend on multiple varying measures, such as *pressure· area* or *current· time*.

Research consistently shows that students often interpret integrals as summing small pieces without attending to the multiplicative relationships among quantities (Jones, 2013; Sealey,

2014). These difficulties appear in a variety of physics contexts. In electricity, learners may fail to conceptualize $\lambda \cdot dx$ in continuous charge distributions, relying instead on memorized formulas (Nguyen & Rebello, 2011), while interpreting dq or dE requires coordination of symbolic, mathematical, and physical ideas (Meredith & Marrongelle, 2008). In fluid mechanics, students often decompose integrals into additive pieces, rarely interpreting *pressure* $\cdot$ *area* as a meaningful product, revealing fragile connections between notation, reasoning, and accumulation (Sus & Izsák, 2024).

Frameworks such as Multiplicatively Based Summation (MBS) (Jones, 2015) and Emergent Quantitative Models (Oehrtman & Simmons, 2023) describe productive ways of reasoning about integrals. Conceptual blending theory offers tools for understanding how students dynamically integrate symbolic, mathematical, and physical ideas (e.g., Hu & Rebello, 2013b). Despite these advances, the coordination of product layer reasoning across multiple applied contexts remains underexplored.

Theoretical Perspectives

In this section, we present theoretical perspectives and a methodological approach which best spans the results obtained in the study. It includes two components—the knowledge-in-pieces cognitive perspective and the modified dynamic conceptual blending diagram (Sus & Izsák, 2025), coming from the dynamic conceptual blending perspective (e.g., Van den Eynde et al., 2020).

The Knowledge-in-Pieces Epistemological Perspective

The perspective was first developed in science education on conceptual change (diSessa, 1993, 2006) and has since been applied to topics in mathematics, such as whole-number multiplication (Izsák, 2005), functions (Izsák, 2004; Moschovich, 1998), and integrals (Jones, 2013). It posits that reasoning is supported by diverse, fine-grained knowledge resources, which evolve from novice to expert through the construction, refinement, and reorganization of these resources. In this study, we focused on the knowledge resources students use when coordinating multiplication with a definite integral in physics contexts.

Conceptual Blending Perspective, Modified Dynamic Conceptual Blending Diagram

Fauconnier and Turner (1998) introduced conceptual blending to model how people create new meaning by combining information from different experiential domains. Studies (e.g., Bing & Redish, 2007; Hu & Rebello, 2013b; Wittmann, 2010) have applied this framework to describe students' use of mathematics in physics. In this study, we take the modified dynamic conceptual blending diagram (Sus & Izsák, 2025) to analyze Melinda's dynamic reasoning about the product layer in applied integration contexts. We construct the modified dynamic conceptual blending diagram (MDCBD) first by identifying and coding knowledge resources, listing selected resources in apparent order of activation, and assigning resources to one of four mental spaces—*mathematics* (mathematical concepts without new physical input, including drawings and equations), *physics* (physical concepts and quantities without introducing new mathematical input, including drawings), *symbolic* (technical roles of mathematics), and *blended* (combinations of symbolic, mathematical, and physical ideas, including equations and drawings). We then connect selected elements with coded links. Resources and connections are coded as more confident (e.g., "I remember") or more tentative (e.g., "I guess"), with more confident elements/connections shown as bold text/green solid lines and more tentative as regular text/green dotted lines. Activated/deactivated resources are marked with squared blue/red boxes,

while connections are marked with blue/red solid or dashed lines, respectively (Figure 1). By analyzing the elements and connections of these mental spaces, and their dynamic changes, we captured Melinda's moment-to-moment reasoning about the product layer as she blended ideas from symbolic, mathematical, and physical input spaces.

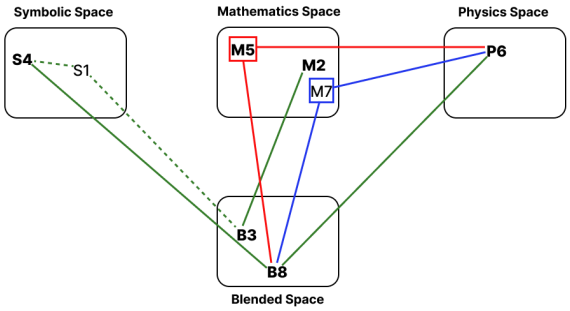


Figure 1. Schematic representation of MDCBD.

Methods

We conducted three 1-hour, one-on-one, semi-structured think-aloud interviews (e.g., Bernard, 1994; Ginsburg, 1997) with three students in a second-semester calculus-based physics course at a selective university in Fall 2023 to examine how they reasoned about integrals and constructed a product layer. The first interview focused on multiplication in applied problems that required summing products, without mentioning integrals. The second and third interviews explored connections between definite integrals and applied problems. Interviews were recorded with two cameras to capture both the student-interviewer interaction and the student's written work. We then synchronized the two videos, transcribed verbatim, and analyzed the knowledge resources students used in building the product layer.

Melinda was selected as an ideal case for studying conceptual blending and product layer reasoning across contexts. She reported an A- in Precalculus, a 5 on AP Calculus AB, and a 4 on AP Calculus BC, indicating strong preparation. In both tasks, she consistently attempted to link symbolic, mathematical, and physical representations, even when her reasoning was incomplete. In the Water Pressure task, she progressed from approximating sums to constructing integral notation, reflecting developing ideas of accumulation and infinitesimals. In the Electricity task, she articulated relationships among voltage, current, and charge, built expressions from definitions, and worked through integral and multiplicative relationships. Her thinking demonstrated productive strategies (e.g., visualizing contributions of small increments) alongside challenges in coordinating two varying quantities within an integral.

Results

This section presents an analysis of how Melinda reasoned about the product layer when calculating the total force exerted from the water pressure on a tank wall and determining the total voltage across a capacitor. The analysis focuses on how she constructed and coordinated symbolic, mathematical, physical, and blended spaces highlighting both productive reasoning strategies and challenges.

Water Pressure Task

Data. Melinda began by reading the problem aloud (Figure 2a) and paraphrasing the key details—the pressure increases with depth according to the law $P = 15x$, where x is the depth of the water—and described an approximation method using sampled points. Melinda then

produced work presented in Figure 3 and explained that she would add pressures at different depths to get a total pressure, not a total force, on the wall.

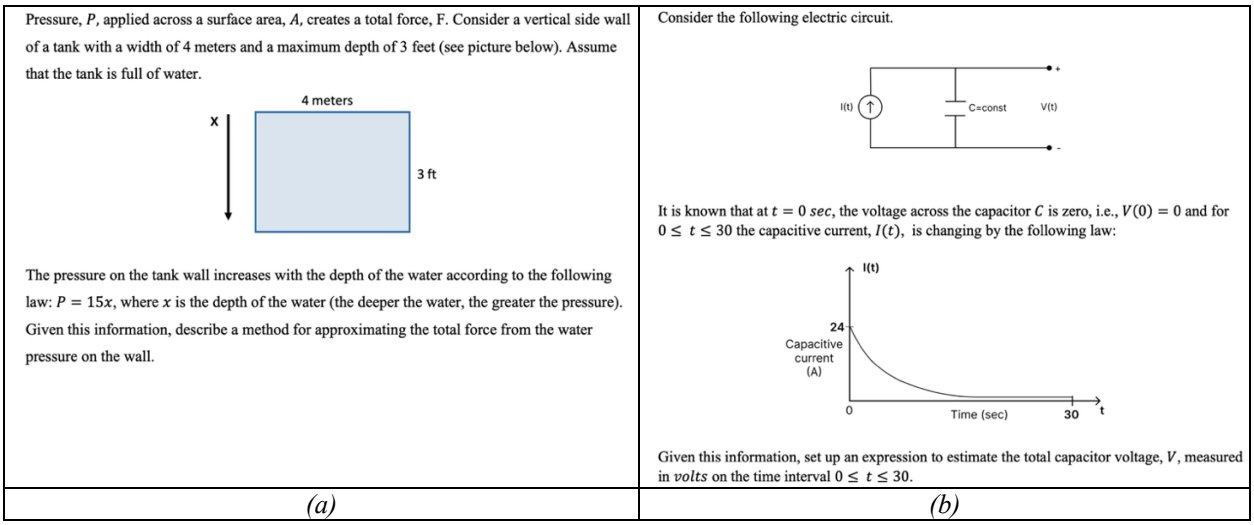


Figure 2. (a) Water Pressure task statement, Interview 2; (b) Electricity task statement, Interview 3.

Melinda then drew the red horizontal lines in Figure 3a that correspond to the red vertical lines in Figure 3b. She continued by stating that the area under the curve in Figure 3b represented the total pressure over the entire tank wall. Shortly after, she admitted uncertainty about finding the force from the pressure: "I don't really know how to get the force from the pressure though." She reiterated that to estimate the total pressure, one would sum all pressure values from 0 to 3 meters, represented by rectangles in Figure 3b.

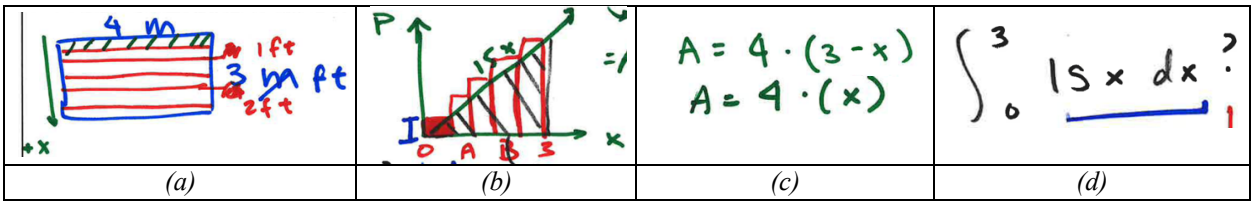


Figure 3. Melinda's reasoning about the definite integral structure in the Water Pressure task.

When asked to clarify her representation of the tank wall in Figure 3a, Melinda explained that she used the labels A and B on her diagram (erased red letters in Figure 3a) as arbitrary points along the depth rather than as areas. These letters A and B in Figure 3a correspond to letters A and B in Figure 3b. Melinda continued by explaining her interpretation of the surface area (Figure 3c):

> Surface area though … is like four, because the width doesn't change as you increase the depth, but the depth changes at three minus x because three is like the total three feet is the total depth minus wherever you are on the x axis or how far down you are.

The shaded green strip in Figure 3a represents the piece of area described by the first formula in Figure 3c. Melinda then changed labels A and B to $1ft$ and $2ft$ and corrected herself: "Your area would be four times x, you don't need the three minus. That's really counterintuitive actually." Ultimately, she represented the area at a given depth with 4 • (x).

Melinda produced two sketches during the task: one representing the tank wall with depths labeled at one-foot intervals (Figure 3a), and another showing pressure as a function of depth

(Figure 3b). She observed that both drawings share the variable x: "as you increase x, your pressure also increases. And here, as you increase x, your surface area also increases." She further suggested that the two quantities could be related by eliminating x: "If you do like $x = P/15$ and then $x = A/4$ and set them equal to each other, you could find the pressure as a function of area instead of x." Melinda clarified that the area of a shaded green strip in Figure 3a represented the surface area and was different from the area of a rectangle in Figure 3b where the height represents pressure.

As the discussion continued, Melinda reasoned about the integral notation $\int_0^3 15x\ dx$ (Figure 5d). She stated, "I kind of interpret it as from 0 to 3 … and you have the function of pressure as it relates to the depth, and then dx, which is like a super small amount of change in x." She explained further that the integral represented summing many small pieces: "Because you are taking the integral of it, the piece becomes like super small instead of like being as large." Finally, Melinda articulated the arithmetic operations involved in the integral expression:

> For the integral, I think it's like broken down into addition and multiplication.…you multiply like the height times the change in x, and then you add all of those like really small pieces together.

She related this multiplication to finding the area of a rectangle and recognized that the base and height corresponded to dx and $P(x)$, respectively."

Analysis. Melinda's reasoning included elements from four mental spaces (Figure 5a). She began by noting that pressure increases with depth (P1) and linking the $P = 15x$ relation (S2) to the physical behavior of the fluid. To quantify this, she approximated the total pressure by summing contributions at sampled points along the tank wall (B3) and interpreted the corresponding rectangles in Figure 3b as small pieces contributing to the total pressure, blending symbolic and physical understanding (B4). She clarified the calculation of surface area, noting the width of the wall is constant while the depth varies, so that area equals $width \cdot depth$ (P5). Using sketches, she visualized the tank wall and the pressure function, observing correspondence of the x-axis between the two (B6). She then expressed the integral symbolically $\int_0^3 15x\ dx$ (S7) and interpreted dx as a small change in depth (B8), describing the arithmetic operations involved: multiplying height by dx and summing contributions (M9).

Electricity Task

Data. Melinda began by reading the Electricity problem (see Figure 2b) and paraphrasing the key details, emphasizing that the voltage across the capacitor is zero at $t = 0$ and that the current varies with time. At first, she expressed uncertainty about how to connect the given quantities: "I don't really know how to relate voltage and current." Drawing on familiar ideas from resistive circuits, she considered Ohm's law but quickly dismissed it as irrelevant: "V equals IR … that really has nothing to do with the capacitor, that has to do with resistors."

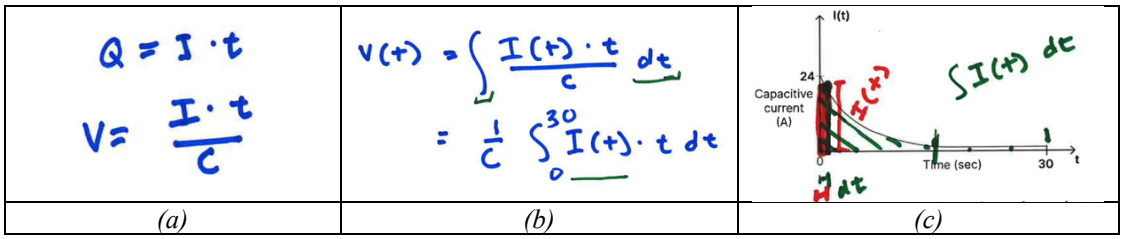
To move forward, Melinda recalled the capacitor equation, $V = Q/C$, and explained: "That means the voltage is equal to the charge that is pushed through … by the battery and then over the C, which is a capacity, like a property of the capacitor." She then attempted to connect current and charge, reasoning, "I don't know how to relate the charge Q, I current. Is it $I = Q/t$?" She elaborated, "I feel like the current is like the amount of charge that's moving through a wire at a certain time t," but admitted, "I don't know how Q, I, and t relate." She then tentatively expressed Q in terms of current and time and substituted into the voltage formula,

arriving at $V = \frac{I \cdot t}{C}$ (Figure 4a). She noted that both current and voltage depend on time, signaling awareness of the dynamic nature of the situation.

At this point, Melinda suggested integration as a tool to account for the varying current: "Then you could, to estimate the total capacitor voltage, you could take the integral from 0 to 30 seconds t …" (Figure 4b). She interpreted the integral of $I(t)$ only as the area under the curve of $I(t)$ (green shaded region in Figure 4c) and stated: "That would be the total current that passes through between the times of 0 and 30."

When asked about the visualization of the expression $I(t) \cdot t\, dt$ on the graph Figure 4c, Melinda responded that it was easier to visualize $I(t)$, the given line. She expressed uncertainty about the t term, which built on her uncertainty about $Q = I \cdot t$. Regarding dt, she said: "dt is like a small change in t" and indicated it on the drawing (Figure 4c). Explaining the full integral notation (Figure 4b), she stated: "The [green] shaded region would be the integral of $I(t)$ only. But I don't know how to multiply that if there is an additional t on the inside."

To interpret the expression $I(t)dt$, Melinda reasoned: "If this is dt, then $I(t)$ is here [width and height of the red rectangle in Figure 4c] … multiplying them together is like this little rectangle … that would represent the total current between zero and a small amount of t … then using that current you could, I guess, do like V equals I times t over C." Finally, she reflected on the meaning of the differential in the integral, noting: "dt would be in changing seconds because t is in seconds … dt kind of falls out of the equation when you take the integral" and added that the total voltage is measured in volts.



(a) (b) (c)

Figure 4. Melinda's reasoning about the definite integral structure in the Electricity task.

Analysis. Melinda's reasoning unfolded across four interrelated mental spaces—symbolic (S), physics (P), mathematics (M), and blended (B)—revealing both tentative explorations and confident connections (Figure 5b). Initially, she expressed uncertainty about how voltage and current were related, briefly recalling Ohm's law (S1) before dismissing it as irrelevant. This tentative symbolic recall illustrates her search for applicable prior knowledge. She then recalled the canonical capacitor relation, $V = Q/C$ (S2), and connected it to the physical property of the capacitor (P3), noting that C characterizes the capacitor's ability to store charge. Tentatively, she attempted to relate charge and current as $Q = I \cdot t$ (S4), combining this with the capacitor formula in blended reasoning (B5). To account for the time-varying current, Melinda proposed using an integral from 0 to 30 seconds (M6) to estimate the total capacitor voltage. She visualized the integral of $I(t)$ only (S7), not $I(t) \cdot t$, as the area under the current-time curve (B8), translating a dynamic physical process into a symbolic mathematical form. She interpreted dt as a small change in time (M9) and conceptualized $I(t) \cdot dt$ as a small rectangle representing current over an infinitesimal interval (B11), while expressing difficulty visualizing $I(t) \cdot t\, dt$ and its integral (B10). She also recalled the units for voltage (P12) and noted that after finding the total current, one could plug that value into $V = \frac{I \cdot t}{C}$ to compute voltage.

Discussion and Conclusion

This study examined how Melinda reasoned about the product layer—the multiplicative structure within an integral—while working on two applied problems. Across both tasks, Melinda demonstrated productive strategies, such as interpreting differentials as small changes and recognizing that integration involves combining contributions over an interval, but she also showed persistent difficulties in constructing the infinitesimal products that define the product layer.

In the Water Pressure task (Figure 5a), Melinda initially treated pressure values as directly summable and interpreted the area under the pressure-depth curve as "total pressure," overlooking the role of surface area. Although she later expressed the integral symbolically and described multiplication of height and a small change in depth, her interpretation remained primarily additive, with limited emphasis on how pressure and area together form an elemental force. In the Electricity task (Figure 5b), she represented $\int I(t)dt$ graphically as the area under a current-time curve and identified dt as a small change in t, but she struggled to connect this representation to the capacitor voltage formula.

These patterns indicate that interpreting the integral as summing contributions or finding an area does not guarantee an understanding of how multiple varying quantities combine within an infinitesimal element. The Water Pressure problem offered geometric anchors and visual cues that supported partial blending of symbolic and physical ideas, while the Electricity problem relied on abstract symbolic relationships and some visual representation, resulting in more tentative and fragmented blending.

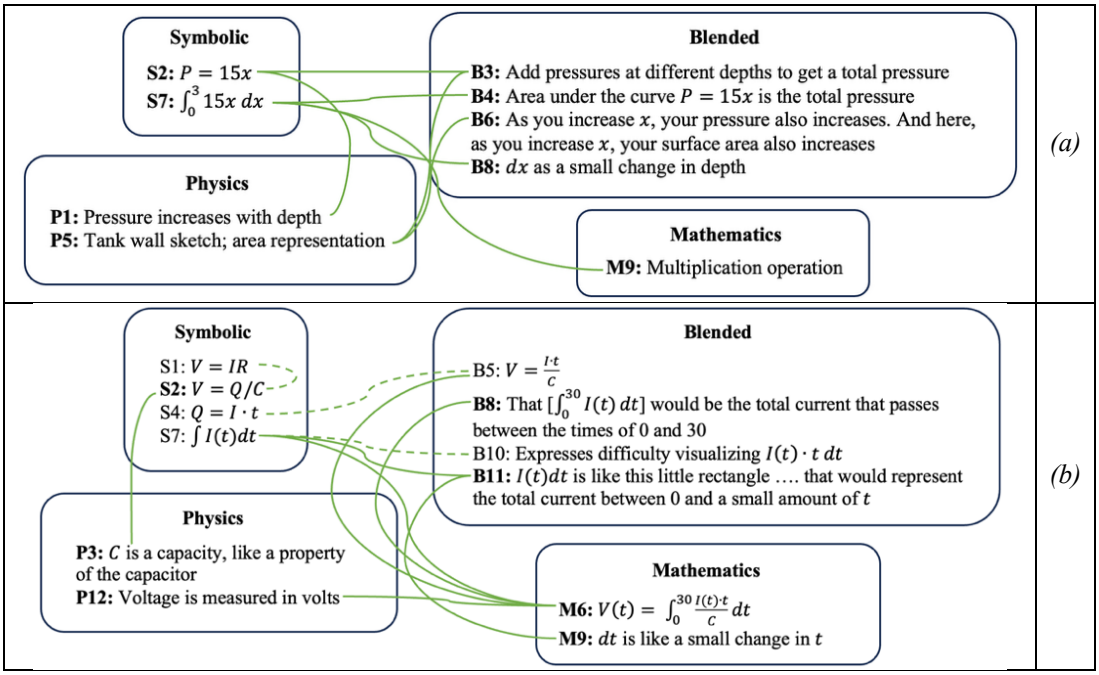


Figure 5. MDCBD representation for (a) Water Pressure and (b) Electricity tasks.

These findings suggest that instruction should explicitly foreground the blended meaning of the multiplicative structure in applied integrals and support students in coordinating symbolic, mathematical, and physical representations. Emphasizing how these quantities combine to form infinitesimal elements may enable students to move beyond additive interpretations toward a robust understanding of the product layer across contexts.

Acknowledgments

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Enacting Role Identities: Insights Into Mathematics Graduate Students' Professional Identity Development

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This study examined the professional identity devel0opment of six mathematics graduate students by exploring how they enacted and constructed meanings around five role identities: teacher, researcher, mathematician, mentor, and peer. Drawing on role identity theory, I highlighted the socially mediated nature of identity formation and the ways it was shaped by program structures, departmental cultures, and disciplinary expectations. Findings indicated that while graduate students identified the roles of teacher and researcher as most central to becoming mathematics faculty, the program offered more formalized opportunities for teaching than for research, suggesting an imbalance in preparation. This work contributes to understanding the liminal space between graduate student and professor, where multiple roles are navigated. The study points to the need for graduate programs to intentionally design structures that expand opportunities for development as researchers, mentoring, and community building.

Keywords: Graduate Students, Professional Identity, Role Identity

Recent scholarship on graduate students' professional identity development underscores the importance and impact of disciplinary and departmental contexts (Solari & Ortega, 2020; Tran et al., 2011). Graduate students' immersion in their fields can shape how they understand academic work (Clarke et al., 2013; Colbeck, 2008), with discipline-based cultures serving as a primary source of identity development and formation (Anderson & Louis, 1994; Clarke et al., 2013). In mathematics, the absence of laboratory work and the emphases on abstraction, correctness, and rigor (Gutierrez, 2017; Herzig, 2004) may uniquely shape students' roles and self-conceptions. While many experiences, such as teaching and research, are common across graduate programs, the meanings attached to these roles are shaped by departmental, sociohistorical, and personal contexts. Given that graduate students are deeply embedded within undergraduate mathematics education, understanding their multiple roles and identifying best practices to support their identity development has notable implications for postsecondary mathematics.

Across many mathematics departments, graduate students are often at the front lines of interacting with and supporting undergraduate students, meaning that their roles can directly influence undergraduate learning experiences, departmental climates, and students' sense of belonging in mathematics (e.g., Olarte et al., 2024; Herzig, 2004). Graduate students' pathways into faculty careers also mean that how they come to see themselves during graduate school has long-term implications for the practices, values, and representations that shape undergraduate mathematics education. Given the fragmentation of higher education and evolving institutional expectations (Nixon, 2006), graduate students' enactments and understanding of their roles may diverge from historical notions of the professoriate, raising questions about how graduate programs equip students for faculty careers (Austin & McDaniels, 2006; Zhang & Wang, 2024). In this study, I adopt a role identity perspective to examine how mathematics graduate students developed their professional identities through enacting and interpreting multiple roles, exploring which became most salient and how disciplinary and departmental contexts shaped these processes. The study is guided by the following research question: How did mathematics graduate students enact and make sense of their role identities as prospective faculty?

To examine graduate students' professional identity development, I draw on identity theory (Burke, 2003; Stryker, 1968; Zhang & Wang, 2024) and role identity theory (Colbeck, 2008; McCall & Simmons, 1978; Jazvac-Martek, 2009). These perspectives, aligned with sociocultural views of identity development, acknowledge the salience of others and context in shaping identity formation. Identity theory frames the "self" as constructed through performing activities and behaviors, while also emphasizing the meanings and expectations associated with the roles that individuals perform. Zhang and Wang (2024) wrote, "In identity theory, the core of an identity is the categorization of the self as an occupant of a role, and the incorporation, into the self, of the meanings and expectations associated with that role and its performance" (p. 353). For graduate students, navigating various roles such as teacher and researcher during graduate education contributes to their evolving professional identities as prospective faculty.

Role identity theory especially emphasizes how graduate students' internalized meanings of roles are shaped by social interactions, institutional contexts, and personal beliefs and values (McCall, 2003; Olarte & Street, 2025). Graduate students enact multiple roles such as researcher (Davis & Lester, 2016), teacher (Beisiegel & Simmt, 2012), and mentor (Brown & Sheerin, 2008), depending on situational demands, creating a salience hierarchy in which some roles become more central to their professional selves (Stryker, 1968; Zhang & Wang, 2024). When roles are affirmed through sustained enactment, peer recognition, or institutional validation, they stabilize and may anchor as enduring dimensions of identity. Alternatively, when affirmation is absent or roles conflict with personal commitments, tensions arise that disrupt identity development (e.g., Austin & McDaniels, 2006; Herzig, 2004). These negotiations underscore that professional identity is not a fixed construct but a continual process of (re)construction across multiple, and sometimes conflicting, role identities (Bale & Anderson, 2022; Olarte et al., 2024; Winstone & Moore, 2017). Understanding how graduate students navigate these dynamics provides insight into the interplay of personal, disciplinary, and institutional factors shaping their preparation for faculty careers. Figure 1 below visualizes this perspective and process of professional identity development through sustained and negotiated role enactments.

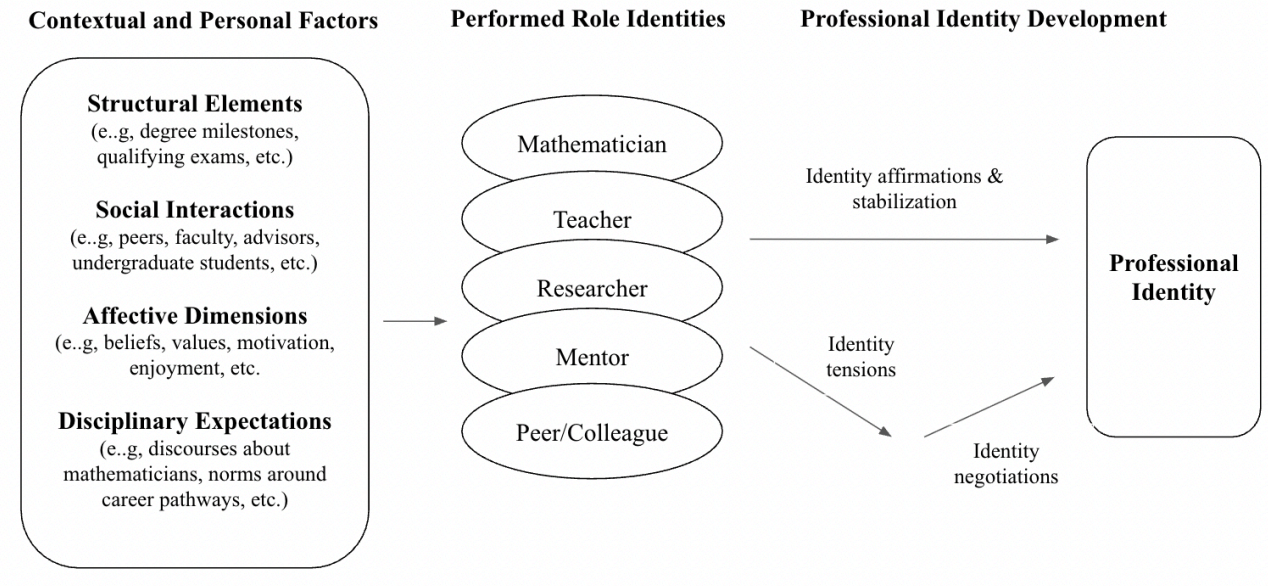

Figure 1. A Process of Professional Identity Development in Graduate Mathematics Education

Method

I employed a thematic analysis approach (Miles et al., 2020) that combined inductive and deductive approaches to examine the role identities that emerged from participants' narratives of graduate education.

Context and Participants

This study was conducted at Spruce University (pseudonym), a large, research-intensive university in the West Coast of the United States. The mathematics department at Spruce University confers graduate degrees (i.e., Master's and PhD) in applied and theoretical mathematics and completion of the doctoral program is expected to take about 5-6 years. The first two years of the program involves completing program requirements and graduate-level sequences for three content areas (e.g., Algebra, Analysis, etc.). Students take comprehensive exams (i.e., qualifying exams) for each area after the required coursework. After fulfilling these three area requirements, doctoral students select a focal research area, secure a dissertation advisor, and take an oral examination given by their respective doctoral committees to discern if the student can reasonably complete a dissertation on their proposed topic. This marks advancement to candidacy. At Spruce University, doctoral students are expected to advance to candidacy within four years of starting their graduate studies. The final milestone is to successfully conduct research and write and defend their dissertation.

I employed a convenience sampling (Miles et al., 2020) to recruit graduate students in the mathematics department at Spruce University in Spring 2023. 16 doctoral students completed a survey that elicited self-described or self-selected academic information (e.g., year in program, prior education, etc.), inclusive demographic information (e.g., gender, race, etc.), and career aspirations information. Then, purposeful sampling (Yin, 2016) was used to recruit students who indicated that they had considered or were actively pursuing careers as mathematics faculty. This study focuses on the six students who participated in an interview about their experiences. Their brief profiles are provided in Table 1 below.

Table 1. Participant Profiles

Participant	Year in Program	Gender	Race	Research area
Connor	3rd	Male/Masculine	White/Caucasian	Data-driven methods of learning dynamical systems
Kayla	1st	Female/Feminine	White/Caucasian	Geometric group theory, combinatorial group theory
Kendall	5th	Female/Feminine	White/Caucasian	Topology
Landon	2nd	Male/Masculine	White/Caucasian	Quantum topology
Martin	1st	Genderqueer/ Genderfluid	Latinx/Chicanx	Applied mathematics and math bio
Morgan	5th	Female/Feminine	White/Caucasian	Low-dimensional topology

Data Collection and Analysis

Data collection occurred within a 60-minute semi-structured interview (Rubin & Rubin, 2011), conducted over Zoom, about their experiences in graduate school, their understanding of the mathematics professoriate, the kind of faculty they hoped to be in the future, and key features of doctoral education. Some questions from this semi-structured interview included: (a) In what ways have you seen yourself as a mathematics professor or faculty while in graduate school?; (b) What aspects of your graduate education have most contributed to your development as a professional in higher education mathematics?; and (c) Tell me about your relationships with your peers in graduate school. Guided by Saldaña's (2013) system of open coding, I conducted multiple rounds of coding that involved close reading of the transcripts to identify emergent themes, actions, and meanings participants associated with their roles. In the first round, I coded for the roles and responsibilities participants reported enacting, such as serving as teaching assistants, collaborating with faculty on research, and engaging in mathematical work. The analysis was informed by existing literature identifying common graduate student roles (e.g., teacher, researcher, mentor), which served as sensitizing concepts. Codes were then organized into broader categories that reflected salient role identities and the contexts in which they were enacted, stabilized, or challenged. Using constant comparison (Miles et al., 2020), I examined similarities and differences in how participants enacted and made meaning of their roles, for instance, seeing teaching as both delivering mathematics content and supporting students' broader development. This process culminated in thematic findings that illuminated how graduate students enacted role identities during graduate school.

Findings

Participants' reflections of their experiences revealed five notable role identities that could qualitatively be organized in a salience hierarchy (Stryker, 1968; Zhang & Wang, 2024) based on the frequency and depth of their reflections on these role identities. I found that their identities as teachers and researchers were more central to their broader professional identities (See Figure 2).

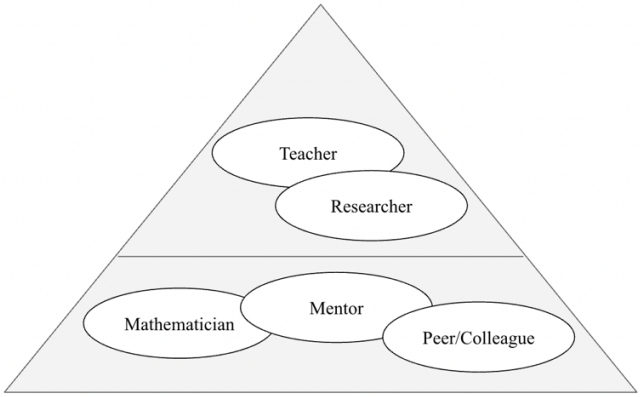

Figure 2. Participants' Salience Hierarchy of Role Identities

Teacher

The role identity participants most often enacted, and often most enjoyed, was that of *teacher*. All served as teaching assistants at Spruce University, with some, like Kayla and Morgan, also teaching as instructors of record at prior institutions, experiences that solidified this role identity. Teaching was not only central to how they envisioned the work of mathematics faculty but also a primary motivator for pursuing faculty careers. As Kayla shared, "I think I've

always known that I've wanted to teach, like that has been an unquestionable thing," while Kendall explained, "I still really love [teaching], and I don't want to stop. That's part of the reason why I want to pursue a career as a professor." Participants emphasized that positive classroom moments and relationships with students helped them see teaching as more than content delivery, but also about humanizing mathematics and supporting students' broader development. Kayla, for instance, described wanting to "make mathematics a more welcoming community" and be the kind of professor who both teaches math and "makes a difference in someone's life." Similarly, Connor said his goal was to be "approachable," while Kendall emphasized guiding students "in more of an emotional sense and a math sense." Many participants distinguished TA positions and instructor-of-record opportunities, with the latter providing greater agency and autonomy that deepened their sense of selves as teachers. As Morgan explained, "Sometimes over the summer, you're able to teach your own class, [and] that probably helps the most of preparing me to be a math faculty member." Overall, teaching was the role through which participants felt most connected to their career aspirations, grounding their professional identities in both pedagogical practice and relational commitments to students.

Researcher

For all participants, the *researcher* role identity, characterized by curiosity, agency, and independence, was understood as equally central to the professoriate as teaching. Many explicitly articulated the faculty role as a dual responsibility: teaching *and* contributing to the field through research. Morgan captured this duality, noting that being a math professor "means that you get to teach students pretty much during the whole school year," while also contributing through "researching and publishing papers." Kendall described research as "really hard, but super fun" while Landon particularly valued the freedom "to study whatever you want… and think about interesting problems all day long." Yet, unlike the role identity of teacher, which was formally structured early through TA positions, the formal path into conducting mathematics research was more fragmented, often depending on student initiative or chance faculty support. Kendall reflected on how a professor she "cold called" during her first year became a supportive mentor who "never made me feel stupid," and Connor shared how presenting in a faculty member's class led to a serendipitous advising relationship. Such mentorship and scholarly community were pivotal in affirming students' researcher identities, with Morgan emphasizing that "my advisor is super great… just a really good mentor." Across their accounts, being a researcher was not simply about producing results but about cultivating values of persistence, creativity, and collaboration, qualities that both grounded their sense of self in graduate school and shaped how they imagined themselves contributing to the field as future faculty.

Mathematician

While connected to teaching and research, the role identity of *mathematician* carried distinct meanings, often tied more to the process of engaging with mathematics than to producing publishable results. Participants characterized being a mathematician as involving deep thinking, intellectual exploration, and a love of learning mathematics for its own sake. Landon expressed this sentiment, saying, "I guess doing math seems like a really enjoyable way to use my brain's processing power, and I guess I kind of fit in, in that sense." Many participants said they saw themselves as "math persons" even before graduate school. Successes during their undergraduate years sparked curiosity and enjoyment that motivated them to pursue advanced study. Landon reflected, "There was never any doubt in my mind that I was going to study math, and taking college level math classes certainly didn't do anything to dissuade me from studying math." They

also associated being a mathematician with qualities like persistence, curiosity, and determination, especially in facing difficult problems without immediate solutions. Kendall noted, "I guess I don't give up very easily. That helps with being a mathematician….A lot of it is determination and resilience." This role identity was most often enacted during coursework, grappling with advanced concepts, and navigating qualifying exams, which became experiences that both prompted and challenged their identities as mathematicians. The qualifying exams, in particular, caused significant challenges and self-doubt. Several participants described this period as the most difficult time of their graduate careers. Morgan recalled, "It was definitely the qualifying exams. That's the only time where I've slightly wavered on, 'Is this [career] right for me?' That was the only time." These moments of uncertainty resulted in tensions with their mathematician identities, which were not grounded in research output but in a sustained connection to the discipline through learning and shared participation in mathematics.

Mentor

The *mentor* role identity captured how participants positioned themselves in relation to undergraduates, often describing mentorship as deeply fulfilling and distinct from teaching or research. Kendall shared, "Being a mentor to undergrads has always been a lot of fun… you basically get to be an advisor to them in some sense," while Landon framed it as "a great way to pay that forward in some sense and make that experience good for other people." Their approaches to mentorship were shaped by their own educational journeys, with many aiming to provide the support they had received, or wished they had, through structured opportunities like Directed Reading Programs (DRPs) and Research Experiences for Undergraduates (REUs). Landon described mentoring undergraduates in an REU as his "favorite thing to ever happen in math," and Morgan similarly shared that mentoring through a DRP and an REU affirmed her belief that "being a professor would be good." Notably, for the three women-identifying participants, mentorship also deeply intersected with gender identities and personal values. Kayla emphasized showing care and compassion to counter stereotypes of mathematicians as distant, while Kendall aspired to "be a woman in a field where I can make a trail for future girls." She shared that this commitment "was in the back of my mind for a lot of the time." Beyond formal programs, participants also viewed office hours and casual conversations as spaces where they could support students academically, emotionally, and personally. These meaningful interactions anchored their mentor identities and underscored the role of mentorship in shaping both their professional trajectories and more inclusive visions of undergraduate mathematics education.

Peer

Participants primarily enacted the role identity of *peer* through relationships with fellow graduate students, centering this identity on mutual support, peer mentorship, and advocacy. Grounded in values of collegiality and community, this role was often the most significant source of belonging in graduate school. Kendall summarized this sentiment: "I feel a sense of belonging in the field, but only because of the other graduate students, I feel a sense of belonging in my graduate students at Spruce." Rather than stemming from formal institutional structures, participants described informal spaces and social activities as essential to cultivating academic and emotional support. Connor shared, "Things like game night that we run have been really helpful in actually meeting friends in the math department. I was invited to go climbing with a couple of the math department people." These relationships then made it easier to ask each other for academic help, discuss challenges, and feel connected. For some, community was forged through shared identities. Morgan, for example, described bonding with a close group of women-

identifying graduate students, while Landon reflected on early social moments like playing video games and getting pizza with peers that helped ease his transition into graduate school. Peer mentorship across cohorts was another important facet of this role identity. Kendall noted how observing senior students navigate the job market helped her understand and anticipate future steps in her career: "Watching them, like older grad students, decide what to do and watch them apply to postdocs… Talking to older graduate students, talking to graduate students in my year and below, because you're all navigating the same career trajectories." This exchange of knowledge, support, and empathy was reciprocal and sustaining.

Discussion

Throughout graduate school, students deeply engage in learning, teaching, research, and academic life, while concurrently developing an understanding of themselves and the profession's values, norms, and practices (Clarke et al., 2013; Tierney & Rhoads, 1993; Trede et al., 2011). Participants' reflections revealed how their role identities were informed by program milestones, social interactions, personal commitments and affective dimensions, and the broader disciplinary expectations of higher education mathematics. Drawing on sociocultural perspectives of professional identity and role identity theories, I found five role identities that mathematics graduate students enacted: teacher, researcher, mathematician, mentor, and peer. While prior work has studied roles such as teacher (e.g., Deshler et al., 2015; Winstone & Moore, 2017) and researcher (Davis & Lester, 2016), the present work adds to this literature by examining how mathematics graduate students develop their professional identities within a specific departmental and disciplinary context. The findings revealed some of the meanings students associated with their roles, the ways departmental and institutional norms shaped professional identity development, and how pivotal doctoral program experiences impacted the salience of these identities. Interestingly, participants identified the roles of teacher and researcher as most central to what it means to be a mathematics professor, suggesting these dimensions were more strongly anchored (Mockler, 2011) and higher in their salience hierarchies (Stryker, 1968; Zhang & Wang, 2024). As Morgan noted, these were the "two aspects of being a math professor." Their prevalence indicates that Spruce University emphasized teaching and research as the most important roles for doctoral students, though the program's structure afforded more formal opportunities for teaching and fewer structured opportunities for research, creating an imbalance in preparation for the two salient roles communicated as essential for mathematics faculty.

Implications and Conclusion

This study offers both theoretical and practical implications for understanding professional identity development and preparing graduate students for faculty careers in mathematics by examining how students enacted and constructed meanings around five role identities. This work contributes to broader conversations about the multifaceted and dynamic nature of professional identity formation in graduate school (Clarke et al., 2013; Solari & Ortega, 2020; Trede et al., 2012). Moreover, the identification of critical experiences that support or hinder role enactments (Jazvac-Martek, 2009) points to the need for graduate programs to intentionally design structures, policies, and practices that expand opportunities for early research development, mentoring, pedagogical development, and community building. Ultimately, attending to graduate students' role identities provides valuable insights into how we might better support their growth within mathematics departments and the broader academic profession.

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Math Anxiety Data Sources: Capturing Individual Differences Through Diverse Lenses

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This multiple case study explores the affordances and limitations of four data sources: a validated math anxiety (MA) scale, written narratives about one's relationship with mathematics, semi-structured interviews, and recorded classroom observations. Analysis of three preservice elementary teachers revealed that each data source provides a unique and valuable perspective. Our findings demonstrate the need for researchers to leverage a diverse range of data sources to develop a more robust understanding of each individual's MA experiences, their MA triggers, responses to these triggers, and how these things may change over time.

Keywords: math anxiety, preservice teacher education, observational data

Identifying ways to reduce the math anxiety (MA) experienced by preservice elementary teachers (PSTs) is a persistent concern in education. Defined as "a feeling of tension, worry, and/or fear in situations involving math-related activities" (Bjälkebring, 2019, p. 1), MA is particularly prevalent among PSTs (Hembree, 1990), and is linked to a variety of negative outcomes for their future students (e.g., Mizala et al., 2015). Although relatively well studied, the vast majority of what is known about PST MA has been established through research conducted primarily using self-reported data sources. We aimed to investigate the affordances and constraints of three such data sources (a validated MA scale, written narratives about each PST's relationship with mathematics, and semi-structured interviews), as well as a fourth data source that does not rely on PST self-report, namely classroom observations.

Literature Review

Research on PST MA is primarily conducted using four data sources: validated survey scales, written narratives, interviews, and classroom observations. Validated survey scales are, by far, the most widely used data source, and therefore underpin most of what is known about PST MA. These surveys ask individuals to rate the amount of anxiety they associate with various mathematical settings, such as "listening to another student explain a math formula" (Alexander & Martray, 1989, p. 147). These surveys are easy to administer, take little time to complete, and can be helpful in identifying broadscale shifts in MA experienced over the course of a semester (e.g., Barrett, 2013). Despite their usefulness, these scales are limited to the situations referenced in the scale items, which tend to emphasize testing scenarios (Pizzie & Kraemer, 2019).

In contrast to surveys, written narratives give respondents the freedom to share about their MA experiences in their own words and to highlight the mathematical situations that are most relevant to them. Such narratives can help identify classroom aspects that either exacerbate or reduce individual PSTs' MA (e.g., Lipscomb & Lorenzen, 2023). Similarly, interviewing PSTs enables respondents to share about aspects of their MA that they believe are important and allows researchers to ask clarifying questions. This increases the chances that the researcher will build a nuanced understanding of PST's MA experiences and the kinds of scenarios that trigger their MA (e.g., Stoehr, 2017). The fourth data source, observational data, is rarely leveraged in published research; however, this data source has the potential to yield unique insights because it does not rely on PST self-report. That is, observational data can provide researchers with access to aspects of PSTs' MA that they are unaware of themselves (e.g., Daniel & Bailey, 2024).

Theoretical Perspective

We drew on Ramirez et al.'s (2018) *interpretation account* of MA because it acknowledges the individualized nature of MA by asserting that "students' development of MA is largely determined by how they interpret … previous math experiences and outcomes (rather than the outcomes themselves)" (p. 151). This perspective explains why certain situations, such as receiving a failing exam grade, might give rise to MA for some students and not for others. If a student interprets the failing grade as evidence that they have a fundamental inability to succeed in mathematics, that interpretation may provoke MA in future mathematical settings. In contrast, if the student interprets the failing grade as the product of their instructor's strict grading policies, this interpretation may give rise to anger instead of MA. We selected interpretation account because it centers the ways in which PSTs make sense of their own MA, the unique situations that elicit it, and the kinds of experiences that might help to alleviate it.

Leveraging the individualized focus of interpretation account, we define *MA triggers* as the unique mathematical situations and/or interactions that elicit tension, fear, or worry for an individual PST. Interpretation account suggests that MA triggers are unique to each PST because such situations and/or interactions will be interpreted differently by individuals. Although PSTs will write and speak about their MA triggers in a variety of ways, researchers have identified some MA aligned language commonly used by math anxious individuals. For example, Stoehr and Olson (2023) found that math anxious PSTs described their mathematical experiences using words or phrases such as *embarrassed*, *math is not my thing*, and *dreaded math class*. PSTs will not necessarily write or talk about all of their MA triggers (or even be aware of them); however, noting the commonly used MA-aligned language can help researchers to identify such triggers, thereby gaining a fuller understanding of each PST's unique MA experiences.

Because interpretation account asserts that the development and experience of MA is dependent on individualized interpretations of events, it also implies that MA experiences depend on a complicated tangle of factors, such as past experiences, personality, and culture. This suggests that multiple data sources are necessary in order to develop a robust understanding of the various ways in which MA is experienced by different PSTs within a single class. The primary aim of this work is therefore to explore what can be learned using four different data sources in characterizing the MA experiences of PSTs over the course of a single semester.

Methodology

The purpose of this multiple case study (Yin, 2018) is to understand the strengths and limitations of four different MA data sources: a validated MA scale (sMARS; Alexander & Martray, 1989), written narratives about each PST's relationship with mathematics, semi-structured interviews, and classroom observations. The three cases each consisted of a single PST enrolled in a terminal mathematics content course. Of 25 study participants, we selected three PSTs with high sMARS scores and different classroom dispositions; Mari was reticent, Eva confidently participated in class, and Velma was loquacious but appeared to lack confidence in class (all pseudonyms). We aimed to answer the research question: What do we learn about PSTs' MA using the sMARS, written narratives, interviews, and classroom observations?

Data was collected in Spring 2024 and included the sMARS (pre/post; Alexander & Martray, 1989), short written narratives about each PST's relationship with mathematics (pre/post), semi-structured interviews (pre/mid/post), and recorded classroom observations (pre/post). To characterize each PSTs' baseline MA, we used a priori theoretical coding for MA-aligned language (e.g., Stoehr & Olson, 2023) on the pre-narratives and interviews. We coded the pre-observation for evidence of their MA triggers and trigger responses and open coded the same

data for any additional MA triggers and trigger responses beyond the MA-aligned language. To track any MA changes, we performed descriptive statistics on the sMARS data and open coded the post-semester narratives, interviews, and observations for any changes in PSTs' MA triggers, responses, and potential rationale for those changes. After writing up case descriptions, we performed a cross-case analysis to answer our research question.

Findings

Our sMARS data showed that Mari and Eva reported notable decreases in their MA levels at the end of the semester (3.7 to 2.0 out of 5 and 3.6 to 2.9 out of 5, respectively). Velma's pre- and post-sMARS scores were nearly identical, with average item scores of 4.1 and 4.0, respectively. Considering these scores alone, it appeared that Mari and Eva had experienced a marked reduction in their MA levels, while Velma did not. However, considering the PSTs' written narratives, their interview responses, and our classroom observations revealed that all three PSTs' MA experiences had changed substantially over the course of the semester. Further, we observed that these changes were unique for each PST—not just in magnitude, but in nature.

Mari: "I can't figure it out on my own"

As identified in her pre-narrative and interview, Mari's baseline MA triggers were shaped by a belief that her peers were more mathematically capable than her: *working without support* and *peer comparison*. With respect to her first trigger, Mari expressed a desire to avoid working alone, writing in her pre-narrative that "I work better with a partner or in groups." In her pre-interview, she indicated that she not only felt more comfortable learning mathematics with her peers but believed that she *could not* do so without them. For example, Mari said, "I don't like doing stuff on my own. … I can't figure it out on my own." Despite her affinity for group work, Mari indicated that comparing her mathematical capabilities to theirs was also a source of MA. In her pre-interview, Mari was asked to clarify why she had scored the sMARS item *listening to another student explain a math formula* as eliciting more MA than *watching a teacher work on an algebraic equation on the board* (5 and 3 out of 5, respectively). Mari explained:

> When I hear another student, like, "oh, I did all of this to get this answer," I would just think, "I didn't even do it that way. I didn't even think the way she thought." … I get anxious thinking about that. But when I see a teacher doing it, it makes me feel better.

This explanation made it clear that, in addition to working alone without support from her peers, comparing herself to them mathematically was also a prominent MA trigger for Mari.

In the pre-interview, Mari described remaining inconspicuous as her primary response to an MA trigger. She stated that "I rather just sit quiet in math since I don't understand at all," and, when asked about sharing her thinking with the class, Mari said "I'd be deadly afraid to go up to that board and write everything down." Mari's pre-classroom observation was consistent with this characterization, as we saw her stay silent the entire class period, often engaging in behaviors that may have indicated anxiety (e.g., arms crossed, staring off, chewing her lip).

By Mari's mid-interview, we found emerging evidence that her levels of MA (connected to both MA triggers) were decreasing. This was most apparent when Mari discussed her experiences during the midterm exam and a subsequent email she received from her professor. Although the professor made it clear that students would have ample opportunities to correct any mistakes they made on the midterm, Mari reported that taking the exam was a stressful experience because she spent most of the exam comparing her experience to that of her peers:

> I heard everyone going, "Oooh!" like they under-, they said, "Oh!" out loud. I was like, what do you mean, "Oh?" I don't know what they're "oh"-ing about. … I was like, …

> maybe I'm not getting this. I was doing it … and I kept moving on. But then they're like, "Oh, I get it." And I was like, I'm not getting this … because I was not "oh"-ing.

Although her lack of having an "oh" moment made her fear that she had performed poorly, Mari reported that the professor sent her an email that had helped change her perspective:

> She sent me an email saying that I did really good in the midterm and that she was proud of me. And I was like, I did good on the midterm? … I just hoped that I passed, … but she was like, "You did really well. I'm so proud of you." … I was like, that means I should not think low of myself. I can actually learn this. So, next time … I could be like, hey, Mari, you can do this. You did well, … you can do it again now.

This suggests that this interaction with the professor helped Mari to make progress with respect to both of her MA triggers, as she seemed to recognize that she was capable of doing mathematics on her own, and that she had done so without "oh-ing" like her peers.

Mari's progress in overcoming her MA triggers appeared to continue throughout the semester. In her post-narrative, she wrote that her relationship with mathematics was "completely different now than it was before." She identified her realization that mathematical problems can be solved in multiple ways as a source of this change, which seemed to free her from her peer-related MA triggers. In her post-interview, she said, "Now that I know there's multiple other ways to do it, … if I can't do it one certain way, I could try another way to do it, because maybe that way works better for me than the other person."

Although Mari reported feeling much more comfortable in class during her post-interview, this change was only marginally observable. Mari was still mostly silent in her post-observation, and she did not share her mathematical thinking with her group or during whole-class discussions. However, we did observe her smiling and sharing a brief high school math experience with the whole class. For Mari, the observational data indicated some change in how she responded to her MA triggers but did not reveal the same profound MA transformation evident in her sMARS scores, written narratives, and interviews.

Eva: "It really was more dependent on the teacher"

Unlike Mari, Eva's MA experiences seemed unrelated to peer comparison or beliefs about her capabilities. Eva said early on that her MA was almost solely triggered by *teachers requiring particular solution methods* and *inadequate teacher support*. Eva wrote about her first trigger in her pre-narrative by describing a high school class: "I didn't like that I couldn't argue my point or think creatively in any sense. … I was frequently told I was wrong, even though I had gotten the correct answer, simply because of the method I chose to solve the problem." Similarly, in the pre-interview she described asking for mathematical explanations and being rebuffed: "No one could explain to me why things were the way that they were. … Like, two plus two equals four, just because two plus two equals four. There was no rhyme or reason." When asked whether the content of her high school math classes influenced how she felt about the class, she replied, "I think it's not so much the content that affects the math course … it really was more dependent on the teacher." Therefore, Eva's baseline data revealed that her MA was almost entirely dependent upon the classroom environment created by the course instructor.

Eva's pre-interview indicated that, in the past, she had responded to her MA triggers with disengagement. For example, she described disengaging after a teacher dismissed her question: "It was like, well, if you're gonna ignore me, then I'm gonna ignore you right back." However, such behavior was never witnessed during Eva's pre-observation. This was perhaps due to a lack of Eva's MA triggers in class; the course was specifically designed to emphasize the use of multiple solution strategies and the professor was attentive to students' questions.

Although Eva reported a decrease in her MA over the course of the semester, it was clear that this decrease was almost entirely instructor dependent. In her mid-interview, she indicated that the format of the course had improved how she felt about mathematics:

> The discussion element of [the course] really helps me out. And I think that is what the missing piece was when it came to my high school experience with math. So, I'm definitely feeling a lot better about it now.

Eva also stated that she felt supported by the instructor and her extensive assignment feedback: "She tells you what you're doing right and not just what you're doing wrong. Like, 'I love this thinking, can you just expand on this?' … At least I know I'm on the right track." In her post-narrative and interview, Eva suggested that her increased comfort with mathematics may not follow her into other mathematical environments. For example, she wrote, "I am not fully confident in my abilities, but I do have courage I could succeed given the right circumstances." When asked in her post-interview if her course experience changed the way she felt about mathematics more generally, she responded, "I have a better relationship with math, even though it doesn't really have to do with math, it has to do with the class."

All of Eva's observations showcased a mathematically confident student with no evidence of her identified baseline MA response of disengagement. Although this observational data appears to conflict with the MA decrease in her sMARS scores, Eva's narratives and interviews resolve this apparent incongruity. These data sources revealed that for Eva, MA was situational and dependent on the instructor's pedagogical practices. It is reasonable to assume that Eva's pre-semester sMARS responses were made referencing high school experiences, while her post semester responses likely referenced the current course environment. Eva's comfort with this environment perhaps meant that she rarely experienced MA in class during the study, fully explaining both her sMARS score decrease and the lack of observable disengagement.

Velma: "I'm just the lost person"

In her pre-narrative and interview, Velma repeatedly stated her belief that she was incapable of doing mathematics, a belief that permeated both of her primary MA triggers: *perceived mathematical inadequacy* (by self or others) and *having a wrong answer*. With respect to her first trigger, she stated in the pre-interview: "I'm not good at math. I still struggle to this day with my math. I constantly count on my fingers. … I can't do math." When asked to describe herself in her favorite class, history, she referred to her brain as a "sponge"; when asked to describe herself in math class, however, she described her brain as a "big blob."

While Velma did not self-identify a specific response to her first trigger, we observed a pattern of disengagement in class. When one of her groupmates expressed confusion, she quickly whispered "I'm confused too." When this groupmate later made progress and was praised by the teacher, Velma declared: "I'm just the lost person." After this exchange, Velma did not watch as her classmates shared their thinking in the whole group discussion and spent the rest of the class engaging in behaviors such as playing on her phone, ripping up paper, and doodling.

Her second trigger, *having a wrong answer*, emerged in the pre-interview: "I'm afraid of being wrong and being shunned for being wrong. I'm scared to be wrong. I'm embarrassed to be wrong." She then described responding to this trigger with silence: "I don't answer the questions. I don't like to answer the question in front of the class." Our pre-observation of Velma corroborated this response; she did not verbally contribute to either her group's collaborative thinking or to the whole group discussion.

Later in the semester, we found evidence of a shift with respect to Velma's second MA trigger (*having a wrong answer*), but not with respect to her first (*perceived mathematical*

inadequacy). We observed an increase in Velma's group participation, a change she identified herself in the mid-interview: "If it's not about something I like, I'm like, I can't do this. … I think [my group has] really helped get me out of that. They're like, 'Come on, you got the answer.'" Velma went as far as to say that "My group's really helped me just be less nervous and anxious." In the post-interview, Velma indicated that she had made further progress with respect to this trigger: "Now I'm like, okay, if I'm wrong, just tell me. I get it. I'm dumb sometimes, and I can't think. … Don't be afraid to tell me it's wrong. Like, teach me how I can fix it."

Despite this progress, her self-description as "dumb" and unable to think indicated that her first MA trigger, *perceived mathematical inadequacy*, remained consistent. In the mid-interview, Velma described her midterm exam experience: "I only got one and a half problems done. I got to the second page and … I couldn't get my brain around it." These self-disparaging descriptions persisted throughout the semester, with her post-interview rife with statements such as "I suck at math." Her post-narrative similarly contained statements such as "Basic math is simple but anything past that is too hard. My relationship [with math] remains the same."

Our post-observational data, however, painted a very different picture of Velma. We saw evidence of confident mathematical engagement that did not align with her own self-descriptions. As an example, consider Velma's contribution to a group discussion about the problem ⅓ divided by ⅙. Her groupmates arrived at different answers and were expressing confusion and frustration. Unprompted, Velma interjected: "It's 2. Because if you look, one sixth takes up half of the one third, and if you add another one it makes one whole, so [the answer is] 2." Velma did several things in this response that demonstrated confident mathematical engagement. First, she participated actively, sharing her thinking with her group without their encouragement. Second, she asserted her answer in a way that demonstrated certainty in her response. Finally, when sharing her thinking, she did so succinctly and in a way that made sense to her peers, starkly contrasting her representation of herself as "dumb" or unable to think.

Initially, Velma expressed confusion and frustration when working on fraction addition (e.g., "I'm just the lost person"), and rarely shared her thinking with her group mates or the whole class. By the end of the semester, and despite her expressed lack of confidence (e.g., "I'm dumb sometimes"), she participated in meaningful ways. Her contributions, both in small and whole groups, indicated that Velma was making progress with respect to *both* MA triggers, yet she only recognized improvement with respect to *having a wrong answer*. Consistent with the continued narrative that she believed herself incapable of math, she reported the highest sMARS score of the three PSTs both at the start and at the end of the semester, and her decrease was the smallest in magnitude. It was clear that Velma continued to view herself as incapable and was unaware of the shifts in confidence that we observed in her end of semester classroom observation.

Cross-Case Findings

As we looked across the three PSTs' MA journeys, the theme that emerged from comparing the data sources was that there was no single most insightful data source. In fact, the most insightful data source varied across cases. The sMARS provided insight into the existence of self-reported MA but failed to reveal how each PST's MA had changed. For example, the sMARS data showed a decrease in MA for both Mari and Eva but not the differing nature of their changes in MA; the sMARS missed that Mari's MA decrease was tied to a fundamental shift in how she viewed herself as a mathematical thinker, while Eva's was tied to a comfortable classroom context. These insights were revealed through their interviews and written narratives, through which we were able to characterize how they made sense of their MA triggers, responses, and the changes they experienced in these over the semester. Yet, the classroom

observations did not reveal any meaningful changes in behavior for either Mari or Eva. Observational data, however, was critical in understanding Velma's MA journey. The sMARS did not identify a change in Velma's MA, failing to register the shifts in how she responded to her MA triggers that were clearly evident in her interviews and second classroom observation. In fact, Velma's shifts in MA with respect to her first trigger would have gone undetected without the observational data. In conclusion, the cross-case analysis revealed that, consistent with the individualized nature of MA, each data source holds potential to reveal unique insights, though the nature of these insights will vary from person to person.

Discussion

We aimed to answer the research question: What do we learn about PSTs' MA using the sMARS, written narratives, interviews, and classroom observations? Our analysis revealed that each data source provided a unique and valuable perspective for each individual PST and presented its own affordances and limitations. For example, the validated survey scale helped us quickly identify high MA study participants and revealed some MA decreases. However, this data source could not identify the stark differences in the nature of these PSTs' MA decreases; it also did not identify the change in Velma's MA evident in other data sources.

Although our study was limited in size, each case illustrates how conclusions drawn in extant research may be enriched by analysis of additional data sources. Consider Barrett (2013), who used validated surveys (pre/post) to conclude that teaching PSTs to reason with manipulatives reduced their MA (a finding consistent with other studies, e.g., Gresham, 2018). If Eva was a participant reporting an MA decrease in this study, an interview may have yielded insight into whether this drop resulted from—or in spite of—the manipulative use. Without such data, it would be unclear if using manipulatives served as an instantiation of Eva's first MA trigger (*teachers requiring particular solution methods*) or if she viewed their use as a means to explore her own strategies.

Velma, in particular, illustrates how even the most robust data collection can fail to detect MA shifts if the study only employs self-reported data. Stoehr's (2017) study exemplified robust longitudinal MA data collection, showcasing the power of written narratives and interviews to reveal individualized aspects of PST MA. With three PSTs, Stoehr found that their unique MA coping mechanisms employed in learning mathematics carried over to teaching mathematics (e.g., working to conceal knowledge gaps). Despite Stoehr's extensive self-reported data and analysis, it is possible that a participant like Velma (who was sometimes unaware of how her MA manifested) may have presented a unique challenge to Stoehr's analysis. For participants like Velma, the absence of observational data likely translates into an absence of insight.

Beyond the value of varied data sources in MA research, the most commonly used data sources are often not utilized to their full potential, including in our own study. For example, the sMARS measures three dimensions of MA, *math test anxiety*, *numerical task anxiety*, and *math course anxiety* (Alexander & Martray, 1989). Though most validated MA surveys measure more than one MA dimension, few researchers utilizing these scales disaggregate their results by MA dimension, potentially ignoring readily available insights. Although we failed to consider disaggregated sMARS scores, our analysis suggests that, in addition to having a distinctive set of MA triggers and responses, each PST will also likely have a particular data source where changes in their MA are most evident. Because such evidence is not visible until data is analyzed, it is vital for MA researchers to utilize multiple data sources, including both self-reported and non-self-reported measures. This will honor the individualized nature of MA and ensure that MA research findings benefit a broad range of PSTs who struggle with MA.

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The Structural and External Validity of Bold Problem Solving in College Math Students

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Bold problem solving (BPS) describes an orientation towards mathematical problem-solving involving risk taking, inventiveness, and independence. Research establishing BPS is promising but requires more validity work to better position the construct. This study does so, exploring the factor structure of the BPS orientation scale, examining how BPS relates to risk taking, math identity and agency, and investigating if gendered differences in BPS orientations persist with more flexible gender measures than those used in earlier studies. Electronic survey data from 204 students enrolled in calculus or above at a single university were collected and analyzed using statistical methods. Results suggest that the BPS orientation scale has one dimension, but the measure can be improved. Risk taking and math identity have strong unique relationships to BPS. Gendered differences in BPS largely were non-existent using more flexible gender measures. Collectively, this work helps further the structural and external validity of BPS.

Keywords: bold problem-solving, mathematical risk-taking, scale validity

Participation in advanced math requires a great deal of independence, problem solving, and mathematical risk taking. A recent construct, Bold Problem Solving (BPS; Lubienski et al., 2021), is conceived as a socialized orientation towards math that involves inventiveness, independence, and risk-taking. Evidence from a different study suggests that BPS relates to advanced participation in math (Makowski, 2020). Although initial work by Lubienski et al. demonstrated that BPS differs from other orientations known to relate to math, this work left open many questions, including (1) a deeper investigation into the factor structure of the BPS orientation scale, (2) the relationship of BPS to related orientations not tested in the original work, and (3) whether observed gendered differences in self-reported BPS orientations persist when more flexible measures of gender are used. This report investigates these questions.

Theoretical Framings

BPS is a comparatively new construct and is still being validated. Flake et al. (2017) describe a three-part construct validity framework. *Substantive validity* is established by defining and linking a construct and its measure to existing literature and theories. *Structural validity* is established through statistical investigation into how the construct measure items perform together and individually. Lastly, *external validity* is established by examining how a construct measure relates to other, similar constructs and the measure's predictive power in relation to outcomes of interest. Here, we focus on the structural and external validity of the BPS orientation measure with an undergraduate students. In the sections that follow, we focus on the related constructs we test in our exploration of the external validity of the BPS measure.

Bold Problem Solving

Math problem-solving processes relate to the skills necessary for reasoning in advanced math (e.g., Furinghetti & Morselli, 2009; Savić, 2016). BPS was developed to help explore why girls underperform relative to boys in the top of the achievement distribution (Cimpian et al., 2016). Lubienski et al.'s study with high school students showed students' self-reported BPS related to, but was distinct from, both math confidence and spatial ability. The BPS measure also related to

the binary sex measure used: male-identifying students scored a full standard deviation higher on the BPS measure than did female-identifying students and it explained over half of the male advantage on a problem-solving test. Makowski (2020) showed that BPS orientation correlated with participation in advanced math: college students in non-STEM track math courses had lower BPS orientation scores than those in Calculus I or an introduction to proof class. Moreover, individuals with at least a Masters in math had higher BPS orientations than those in the more advanced math classes. Gender difference existed for students in Calculus I only.

The initial findings related to BPS are promising, but several important questions remain unexplored. First, Lubienski et al. (2021) hypothesized that their 10-item BPS measure might contain more than one dimension. Their investigation into this question concluded that the data did not support this structure, but their samples were small enough to leave the question open. Second, the relationship of BPS orientation to risk taking and independence was unexplored. Third, their use of a binary sex is misaligned with current understanding of gender.

Risk Taking

It is not uncommon to theorize that doing math involves risk. For example, Lampert (1990) noted that genuine mathematical activity is inherently risky, as it involves moving beyond what is known. Regier and Savić (2020) suggest that progression in advanced math requires comfort with risk taking. As hypothesized, BPS contains elements of risk-taking, but how it relates to BPS was not examined in the initial studies. Risk-taking might explain the gendered outcomes Lubienski et al. (2021) observed. Research from several fields suggests that females tend to be more risk-averse than are males (Byrnes et al., 1999). For example, in music, girls have been found to be less comfortable than boys with the improvisational nature of jazz (Wehr-Flowers, 2006). Similarly, in science and computing, boys were found to more often "tinker" with materials or tasks, while girls more often followed teacher directions (Jones et al., 2000).

Math Identity

Broadly, math identity captures the extent to which an individual sees themselves as someone who does math. In a recent study, Rothrock and Gay (2025) linked increased math identity in college freshmen with having high initial college math placements or having taken advanced math in high school. College students with stronger math identities are more likely to be interested in math related careers (Cribbs et al., 2020). As BPS correlates with participation in advanced math (Makowski, 2020), it seems likely that math identity may relate to BPS.

Sense of Agency/Independence

Sense of agency refers to the subjective experience of controlling one's actions and the subsequent outcomes. Students who perceive control over their learning and believe their actions directly affect outcomes are more likely to engage in exploratory problem-solving (Bandura, 1997). Agency in math is thought to serve as a gateway to creating a math identity (e.g., Boaler & Greeno, 2000) and deliberate efforts to create agency in math relate to increased interest and enjoyment of math for K-12 students (e.g., Sengupta-Irving, 2016). Thus, while BPS is hypothesized to relate to independence, a sense of agency may be an important prerequisite. We hypothesize that a positive sense of agency will relate to higher BPS orientation scale scores.

Gender

Gender identity is rooted in socialization and is not generally thought to be well-captured by the traditional binary categories used in research, including the original BPS studies. Lubienski

et al. (2021) found a binary sex difference in self-reported BPS orientations with K-12 students, which was also present for Calculus I students at the postsecondary level (Makowski, 2020). It is unclear whether more flexible measures of gender will continue to relate to BPS. On the one hand, the norms and practices of math are generally understood to replicate norms and interactions styles associated with white masculinity (Leyva, 2017). On the other hand, Alcock et al. (2014) found that personality had predictive power over and above binary sex for math achievement, suggesting that measures of personality might have a stronger relationship to BPS than binary sex while still tapping into gendered traits.

Methods

All data were collected via an electronic survey from undergraduate students enrolled at a single large southeastern university. Participants were recruited from multi-section math courses that are a standard part of a math major: Calculus I, Calculus II, Differential Equations, and Introduction to Proof. Students were eligible to participate if they were enrolled in one of these classes and their instructor allowed a researcher visit. The survey link was distributed during the research visit and again through an email distributed by the instructor.

Sample

In total, 264 students took the survey. Of these, 59 were removed for responding to an attention check incorrectly. One additional student did not provide their sex and was removed, leaving a final sample of 204. The number of students by course and binary gender are reported in Table 1. To bolster our sample size for the first research question, we supplemented the BPS orientation scale survey responses using data from 321 students enrolled in similar classes that were collected as part of a different study (Authors, in preparation).

Table 1. Sample Sizes and Percents by Binary Gender

Course	Men		Women		Total
	n	%	*n*	%	
Calculus 1	20	47.6	22	52.4	42
Calculus 2	37	67.3	18	32.7	55
Differential Equations	56	72.7	21	27.3	77
Introduction to Proof	15	50.0	15	50.0	74
	128	62.7	76	37.3	204

Survey Measures

The 15-to-20-minute online survey included questions about the student's course, their gender and racial or ethnic identity, and a battery of measures related to math attitudes and orientations. The order of the scales and the order of the items within each of the scales was randomized. We also included an attention check in the middle of the survey.

Bold problem solving. The 10-item BPS measure (Lubienski et al., 2021) was included. This measure has variety of items, some of which related to independence (e.g., "When I am stuck on a math problem, I usually play around with the problem for a while before examining worked examples or seeking help"), some to risk-taking (e.g., "I enjoy working on challenging math problems that do not have a set way of solving them"), and some to mathematical behaviors (e.g., "When solving math problems, I tend to use creative strategies"). Items used a 5-point Likert scale from "Strongly disagree" to "Strongly agree."

Risk taking. Risk taking was measured using the General Risk Propensity Scale, which is composed of eight positively worded items (e.g. "My friends would say that I'm a risk taker") and is shown to have very high internal consistency, fit a unidimensional confirmatory factor analysis model, and demonstrated predictive validity for samples of adults (Zhang et al., 2019). Items used a 5-point Likert scale from "Strongly disagree" to "Strongly agree."

Math identity. Math identity was measured using a single item asking participants to answer the question "On a scale of 1-10 (with 1 not very mathematical) how much do you identify as good at math?" While single item measures lack complexity, the large scale range and straight forward prompt provide a baseline for exploring how BPS might relate to math identity.

Sense of agency. To measure agency, which feeds independence, we used the Sense of Agency Scale, which has two subscales: Sense of Positive Agency (5 items) and Sense of Negative Agency (6 items; Tapal et al., 2017). We removed one item from the negative agency scale due to language issues and reduced the Likert scale response categories from seven to five to match the other instruments in the survey. A two-factor confirmatory factor analysis (CFA) model of the sample data suggests the reliability of the scale was still satisfactory with these modifications (RMSEA = .052, CFI = .934 with covariance of 0.308 between the factors and correlation between the subscale total of .441).

Gender. Given the relationship between gender and personality (Alcock et al., 2014), we included a measure of sex assigned at birth, two related measures of gender, and personality scale. The gender measures asked participants to rank on a typical day how masculine/feminine they felt (5 categories from "not at all" to "a great deal"). To examine personality, we used the BFI-10 (Rammstedt & John, 2007) measure, which was built based on research documenting five dimensions of personality (extraversion, agreeableness, conscientiousness, neuroticism, and openness). The 10-item scale assigns each of the five personality dimensions two items.

Analysis

Prior to the main analysis, data were cleaned and inspected for attentiveness and scale completeness. Reverse coded items were recoded to be on the same scale. Descriptive and reliability statistics were calculated for all scale items. For non-BPS measures we computed the scale score by taking the average of the responses on each item, with higher values associated with higher levels of the variable measured. The same was done for the BPS measure after the dimensionality of the 10-item scale was determined for Research Question 1.

Research Question 1: Factor Structure of the BPS Orientation Scale with Undergraduates

To assess the factor structure of the 10-item BPS orientation scale for the undergraduate student population, we used an exploratory factor analysis (EFA), examining the underlying dimensions of the construct and evaluating whether the items loaded consistently with the theoretical model of a unidimensional instrument. The analysis included tests of sampling adequacy (Kaiser–Meyer–Olkin statistic, Kaiser & Rice, 1974; Bartlett's test of sphericity, Bartlett, 1951) to ensure suitability for factor analysis. Reliability estimates, including Cronbach's alpha, were also calculated to examine internal consistency.

Research Question 2: Relationship of BPS to Other Related Constructs

To assess the relationship of BPS (using the factor structure determined in Research Question 1) to other related constructs, we performed a series of ordinary least square regressions using BPS as the dependent variable. Mimicking the original work of Lubienski et al. (2021), Model 1 assessed the baseline relationship between binary sex (entered as a binary indicator; students

who identified as female assigned a value of 1, males a value of 0) and the BPS measure. Model 2 included the other related variables (i.e., math identity, positive and negative sense of agency, risk-taking).

Research Question 3: Relationship of BPS to Flexible Gender Measures

To assess the relationship of BPS to more flexible measures of gender and personality traits, we first examined the correlations of the related measures with each other and to the BPS measure. For each of the three alternative gender-related variables (femininity, masculinity, Big 5 personality factors), we ran two models with the BPS measure as the dependent variable (6 models total). Model 1 included the gender or personality variable as the independent variable ("not at all" served as the reference category for the feminine and masculine cales). Model 2 added the other BPS-related independent variables tested in Research Question 2.

Results

All scale measures were inspected fit within standard parameters to be treated as normal in correlational analyses (±2 for skew, Tabachnick & Fidell, 2013; ±7 for kurtosis, Hair et al., 2010). These variables are thus treated as normal for the duration of the analysis. For brevity, we do not report descriptive statistics on these measures. Cronbach's alpha for the scales with more than two items ranged between .60 (sense of negative agency) and .81 (risk-taking). Cronbach's alphas were smaller for the two-item Big 5 personality measures (.26 to .75).

Research Question 1: Factor Structure of the BPS Orientation Scale with Undergraduates

Preliminary analyses indicated that the data were suitable for factor analysis. The Kaiser–Meyer–Olkin measure of sampling adequacy was .85, exceeding the recommended threshold of 0.60. Bartlett's test of sphericity was significant ($\chi^2(45) = 1489.3$, $p < .001$), indicating that correlations among items were sufficient to proceed with factor analysis. Individual MSA values for all items ranged from .75 to .90, suggesting items shared adequate variance to each other.

A parallel analysis suggested a three-factor solution was possible, with eigenvalues of 3.85, 1.34, and 1.11. EFA using principal axis factoring with a single-factor solution revealed factor loadings ranging from .26 to .76. The single factor solution accounted for 33% of the total variance (RMSEA =.12; Tucker Lewis Index [TLI] of .78). Internal consistency was strong (Cronbach's α =.81; .79 with analytic sample). Neither the two- or three-factor models were theoretically coherent. The three-factor model had strong fit indices (RMSEA = .05, TLI = .96), but a single item loaded onto the third factor and one item did not load onto any factor.

These results suggest that, consistent with the original theoretical model, the 10-item BPS orientation scale functions uni-dimensionally with this sample, with each item contributing to the overall factor. However, development of additional items and the use of cognitive interviews for response process validation are needed to create a stronger instrument.

Research Question 2: Relationship of BPS to Other Related Constructs

Table 2 displays the pairwise correlations of the various measures used. Here we only focus on the correlations between items 1 through 6. Notably, the BPS measure has a strong positive correlation with both risk taking and math identity. Consistent with prior research, the binary sex measure is also significantly correlated with the BPS measure. Neither positive nor negative sense of agency correlated with the BPS measure.

Table 3 displays the results of the three models. While female significantly negatively related to BPS (Model 1), this relation no longer held after the other covariates were added (Model 2).

Both risk taking and math identity significantly positively related to BPS (Models 2 and 3). This is interesting as risk taking and math identity were negatively correlated to each other. Neither positive nor negative sense of agency related to BPS. This makes sense as these measures were not correlated. Collectively, these results suggest risk taking and math identity together are more strongly related to BPS than binary sex.

Table 2. Pairwise Correlations of Key Analytic Variables

Scale	1	2	3	4	5	6	7	8	9	10	11	12
1. Bold problem solving	--											
2. Risk-taking	.20**	--										
3. Math identity	.19**	-.17*	--									
4. Positive agency	-.02	.16*	-.03	--								
5. Negative agency	-.06	-.04	.05	.43***	--							
6. Female	-.14*	-.06	-.04	-.05	-.03	--						
Gender scale												
7. Femininity	-.05	-.04	-.02	-.11	-.10	.81***	--					
8. Masculinity	.03	.17*	-.05	.15*	.08	-.71***	-.74***	--				
Personality												
9. Extroversion	.03	.28***	.03	.12	.10	.05	.10	.08	--			
10. Agreeableness	.09	-.11	.10	.14	.08	.03	.01	-.06	.13	--		
11. Consciousness	.02	-.04	.06	.21**	.19**	.14	.06	-.03	.11	.15*	--	
12. Neuroticism	-.13	-.32***	.03	-.19***	-.11	.38***	.34***	-.32***	-.26***	-.09	-.05	--
13. Openness	.07	.08	-.17*	.10	.05	.07	.14	.03	.15*	.13*	.14*	.00

* $p < .05$, ** $p < .01$, *** $p < .001$

Table 3. OLS Regressions Predicting BPS Orientation

	1		2	
	β	SE	β	SE
Constant	3.02***	0.06	2.12***	0.44
Female	-0.18*	0.09	-0.16	0.09
Risk taking			0.17**	0.05
Math identity			0.10**	0.03
Positive sense of agency			-0.04	0.08
Negative sense of agency			-0.06	0.08
n	204		204	
R^2	.02		.11	

* $p < .05$, ** $p < .01$, *** $p < .001$

Research Question 3: Relationship of BPS to Flexible Gender Measures

Table 2 displays the correlations between the BPS orientation scale and the flexible gender and personality measures. None of these correlations were significant. There were strong relationships between the binary sex (female) variable and the feminine and masculine 5-level categorical measures. Masculinity was positively correlated with risk taking.

Table 4 displays the regression results using the feminine and masculine gender measures in lieu of the binary sex variable. While identifying as feeling feminine "a lot" initially related to BPS, this relation was not strong and dropped once other covariates were added. Masculinity did not relate to BPS at the categorical level. We do not report the regression results using the Big 5 personality measures: the correlations between the personality measures and the BPS measure scale were not significant and the regression results reflected this pattern.

Table 4. OLS Regression Results Predicting BPS Orientation Using Flexible Gender Predictor Variables

	Femininity				Masculinity			
	1		2		1		2	
	β	SE	β	SE	β	SE	β	SE
Constant	2.96***	0.07	1.86***	0.45	2.92***	0.10	2.03***	0.45
Gender measure								
A little	0.15	0.13	0.17	0.12	-0.01	0.14	0.00	0.14
A moderate amount	0.07	0.14	0.11	0.13	0.14	0.13	0.14	0.13
A lot	-0.28*	0.14	-0.25	0.13	-0.10	0.14	-0.12	0.14
A great deal	0.03	0.14	0.01	0.14	0.15	0.15	0.10	0.15
Risk taking			0.15**	0.05			0.17**	0.05
Math identity			0.10***	0.03			0.10**	0.03
Positive sense of agency			0.00	0.08			-0.03	0.08
Negative sense of agency			-0.01	0.08			-0.04	0.08
n	204		204		204		204	
R^2	.04		.13		.02		.12	

* $p < .05$, ** $p < .01$, *** $p < .001$

Discussion and Conclusions

The analyses presented here builds on the work of Lubienski et al. (2021) and Makowski (2020), adding to the structural and external validity of the BPS construct. Research question 1 investigated the structural validity of the BPS measure, examining the dimensionality of the 10-item BPS measure. We found that for the undergraduate student population, the factor structure has the most evidence for a unidimensional structure. However, while this was the best solution, the fit statistics of the one-dimensional solution were not optimal, suggesting that developing and testing additional items is warranted.

Research question 2 examined the external validity of BPS, testing theorized relationships. Consistent with theory, BPS positively related to both math identity and risk taking. Curiously, these two variables were negatively correlated, suggesting that there might be something unique about the risk taking needed for math versus general risk taking that the BPS construct taps into. Neither a positive nor negative sense of agency related to the BPS measure, which is contrary to our hypothesis. This may be an effect of the fairly low reliabilities of the agency measures used. Alternatively, the particular measures used (which measured general agency) might be somewhat distinct from math agency. These relationships should be further probed using better, more targeted measures.

Research question 3 examined the external validity of the BPS measure, testing whether more flexible gender measures than those used in the original BPS studies still showed a relationship between BPS orientations and gender (Lubienski et al., 2021; Makowski, 2020). The results of research questions 2 and 3 together suggest that the BPS may be a "math socialization" construct that functions independent of gender. The gendered results of the initial studies may crudely measure and reflect the mathematical socialization that results from the male coded culture of math (e.g., Leyva, 2017). Measuring gender using more flexible reporting categories weakens the link between gender and BPS, while the relationship of BPS to other socialized affects remain strong. More work testing this theory should be the focus of future investigations.

Collectively, this study adds validity evidence to the theorization of BPS. However, the results are still inconclusive in important ways and additional work is needed to help understand bold problem solving in the undergraduate student population.

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Targeting Conceptions of Derivative: Introducing Derivatives in Calculus With Inquiry

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In this study, we leverage the Conceptions of Derivative (CoD) framework (Gerami et al., 2025) to analyze eight U.S. college calculus instructors' instructional tasks for introducing derivatives. During four task-oriented interviews, each instructor proposed up to eight tasks for introducing derivatives with inquiry in their calculus I courses. Based on Zandieh's (2000) process-object model, the CoD framework was used to reveal derivative conceptions that instructors were targeting in their tasks. We identified four components of derivative conceptions targeted in the tasks: contextual framing, epistemological approaches (limit-based, infinitesimal/differential-based, and instrumental/rule-based), mathematical representations, and process-object layers. While pure contextual frames and limit-based approaches dominated early instruction of derivatives for these instructors, some targeted physics and biology contexts as well as infinitesimal/differential-based and rule-based epistemologies. We discuss implications for research on calculus teaching.

Keywords: Derivatives, conception, calculus teaching, instructional tasks

While researchers have explored students' conceptions of derivatives for some time (e.g., Ferrini-Mundy & Graham, 1994; Feudel & Biehler, 2021; Jones & Watson, 2017), little research has examined how college calculus instructors teach derivatives and the conceptions they target during instruction. In this study, we investigate how college calculus instructors introduce derivatives by focusing on the conceptual opportunities they leverage in their instructional tasks. To reveal a broader range of conceptions of derivatives targeted within instructional tasks, we focused on instructors experienced with inquiry-based teaching, assuming they use a broader range of instructional tasks than those from conventional textbooks. We aim to address the following question: Which conceptions of derivative do eight college calculus instructors target in their instructional tasks when introducing the concept for the first time with inquiry?

Literature Review

Our search of the research on teaching of derivatives yielded nine studies. The first four studies were conducted with various motivations, yet they all examine non-intervention based teaching of derivatives by calculus teachers. The remaining studies (Borji et al., 2018; Hähkiöniemi, 2006; Hitt & Dufour, 2021; Kendal & Stacy, 2001) are intervention-based, as they investigate the efficacy of teaching interventions aimed at helping students make connections between multiple representations of the derivative to develop deeper conceptual understandings. Because our question revolves around teachers' natural ways of teaching derivatives, we delve deeper into the non-intervention based studies.

Park (2015) used a discursive approach to investigate how three university calculus instructors defined derivatives at a point, and on an interval (or as a function), and transitioned

[1] The first, second, and third authors contributed equally and are listed alphabetically.

from the former to the latter, given the challenges these concepts pose for students. Although the instructors used secant and tangent lines to graphically illustrate derivative at point, Park identified several disconnections. The instructors did not explicitly connect the graphical and the symbolic representations of derivatives, as they often referred to the slope of secant or tangent lines simply as 'secants' or 'tangents.' While they used graphs to illustrate the derivative at a point (via secants approaching a tangent), they did not use graphical representation to define the derivative at the function layer. Moreover, at the function layer, the instructors often showed derivative as a function with zero, positive, or negative values of derivative "rather than numerical values that change over an interval" (p. 248). Moreover, the transition from the limit to the function layer within the symbolic representation was only shown as changing the symbol for a point (x_0) to a variable (x), without addressing the reasoning behind the change of symbols. Park concluded that by not making these connections explicit, students were left to make some challenging connections on their own.

Within a study about one experienced college instructor's use of a calculus textbook in Norway, Randahl (2016) examined how the instructor introduced derivatives during a sequence of six lectures. The instructor closely followed the textbook's structure and content sequence, defining the following concepts through lecturing: 1) tangent line, 2) slope to the tangent line, 3) the difference quotient, 4) the slope of a curve at a point and the slope of the normal in this order, and 5) The derivative of a function defined using the limit definition. The definitions were presented on the board without further clarification or discussion, after which, the instructor assigned procedural exercises for group work, with some solutions discussed in front of the whole class. Because the instruction mirrored the formal introduction of derivatives in the textbook without providing additional explanations, Randahl concluded that the instruction likely did not enhance students' understanding of the abstract concepts associated with derivatives.

In a study examining teachers' views on differentiation and their knowledge of teaching it, Delos and Thomas (2003) investigated four New Zealand teachers' approaches to teaching derivatives. Although the authors found no common way of teaching derivatives across the teachers, they found that all teachers used graphical representations, particularly the approach of moving secants toward a tangent when finding derivative at a point. Despite their emphasis on graphical representation, the teachers expressed a desire for their students to develop a comprehensive conceptual understanding of derivatives, including understanding it as "a gradient function, as a rate of change, as a limit, and symbolically" (p. 56).

Lastly, in the fourth non-intervention-based study, Hitier and González-Martín (2022) identified the (in)consistencies in the use of derivatives in calculus and mechanics course, needed for solving one-dimensional kinematics tasks. As part of their broader investigation, the authors interviewed four college calculus instructors from Quebec, all of whom used Stewart's (2013) textbook to teach the course. The teachers described their ways of introducing derivatives similar to the textbook's, with interpreting derivative at a point as the slope of the tangent line. They also incorporated the symbolic limit definition both at a point and for a function and made connections with the graphical representations. Although the teachers mentioned using the four representations by Zandieh (2000) in their teaching, they expected students to heavily rely on symbolic calculations to solve derivative tasks about motion (i.e., within the physical representation of derivative).

In summary, these non-intervention-based studies on the teaching of derivatives focus on how teachers often define derivatives. They collectively suggest that instructors often utilize both graphical and symbolic representations, but that they may give less emphasis to physical

representations of derivatives. Among the intervention-based studies not reviewed here due to space, there appears to be agreement that using graphical representations for introducing derivatives is effective. However, it remains uncertain whether the physical representations, particularly those related to physics (as opposed to, for instance, population growth or decay), contribute to a better understanding of derivatives.

While these studies shed light on the importance of various representations of derivatives, they have focused exclusively on limit-based understanding of derivatives, reflecting the dominance of this approach in traditional calculus instruction. However, recent literature suggests that instructors may employ alternative epistemological approaches to derivatives that extend beyond the limit framework (Ely, 2021; Keisler, 2011; Samuels, 2017). The Conceptions of Derivatives (CoD; Gerami et al., 2025) framework, which we present next, addresses these gaps by incorporating multiple epistemological approaches and explicit attention to the physical manifestation of derivatives.

Theoretical Framework

The Conceptions of Derivatives (CoD; Gerami et al., 2025) framework extends the Zandieh's (2000) framework (Table 1) to include non-limit-based approaches to thinking about derivatives. Constructed as a cognitive model for students' understanding of the concept of derivative, the Zandieh's framework distinguishes between three derivative "process-object" layers—shown as *ratio*, *limit*, and *function* rows in the table—within four main representations/contexts of derivatives—graphical, verbal, physical, and symbolic. Relying on Sfard's (1991, 1992) notion of process-objects, Zandieh's framework uses "limiting" as the main process for deriving derivatives. The ratio layer is the average rate of change between two points: $\frac{f(x_0+h)-f(x_0)}{h}$. The limit layer is produced by *limiting* the ratio layer: $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, referred to as the derivative of the function at a point. Finally, the function layer is produced by using the same limiting process for all possible points to find derivative as a function: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Table 1. Zandieh's (2000) framework for the conceptions of derivative

		Representations/Contexts				
		Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Difference Quotient)	Other
Process-Object Layer	Ratio					
	Limit					
	Function	.	.	.	.	

Tho CoD framework, shown in Figure 1 is more comprehensive, as it includes two other ways of thinking about derivatives: infinitesimal- or differential-based, and rule- or instrumental-based. The CoD framework has four distinct but interconnected components. *Contextual Framing* is the context through which we frame or interpret derivative problems. Common contexts in calculus are pure mathematics, physics, chemistry, biology, and economics. *Conceptual Epistemology* captures fundamentally different approaches to understanding what derivatives are and how they are derived. The framework identifies three distinct epistemologies: limit-based (traditional approach using limits of difference quotients), infinitesimal/differential-based (using infinitesimal quantities or differential reasoning), and instrumental/rule-based

(procedural approach using derivative rules). *Mathematical Representations* encompass the three primary ways derivatives can be expressed mathematically: graphical (visual representations), symbolic (algebraic expressions), and numerical (calculated values). *Process-Object Layering* maintains the hierarchical structure of derivatives from Zandieh's (2000) framework but adapts it to work across different epistemologies: 1) between two points (average derivative/rate of change), 2) at a point (instantaneous derivative), and 3) everywhere (derivative function). Unlike Zandieh's framework where layers are tied to limit processes, these layers can be conceptualized through any of the three epistemological approaches.

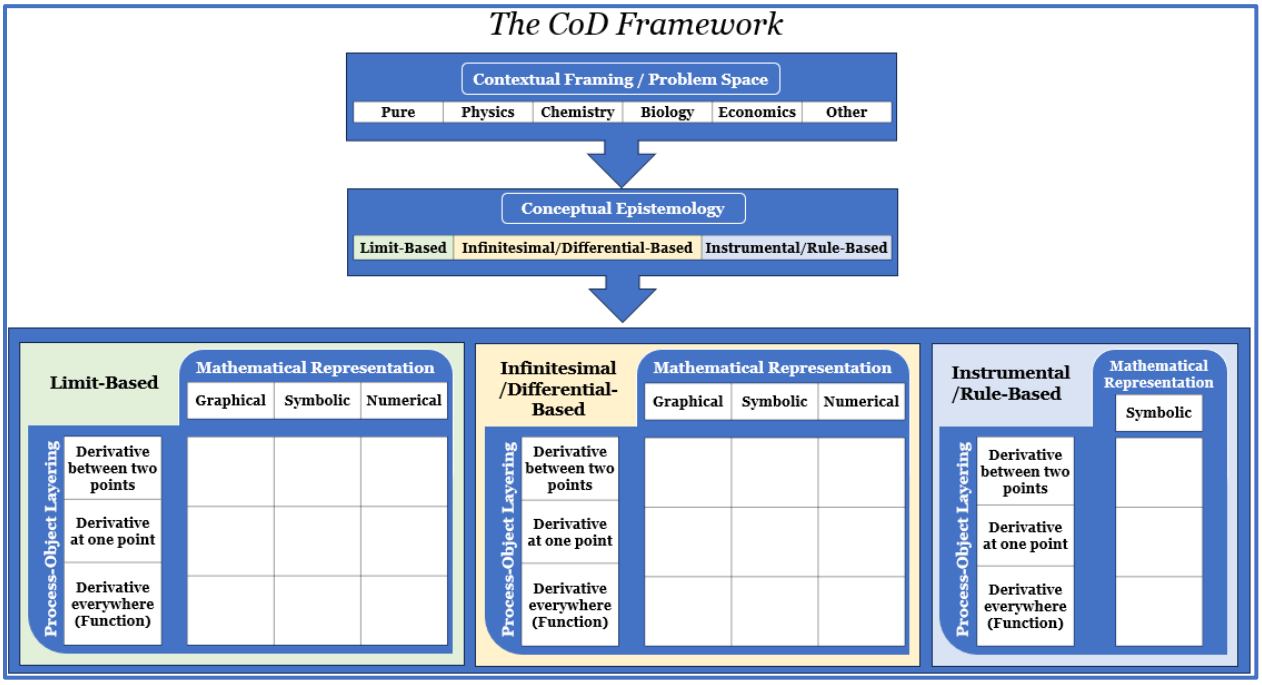

Figure 1. Conceptions of Derivative (CoD) framework

Methods

The data for this study comes from a larger study conducted to investigate calculus instructors' social and content-based framings of instructional tasks for introducing derivatives (Gerami, 2024). Eight instructors were purposefully selected via cluster analysis from a larger subset of experienced college calculus instructors who showed variety in their teaching with inquiry based on a survey (INQUIRE; Shultz, 2020). All using pseudonyms, the instructors were Adrian, Barry, Gopher, Justin, Matthew (he/him pronouns), Monica (she/her pronouns), Alex (any pronouns), and Max (they/them pronouns). The fourth author conducted four 1–2 hours long semi-structured interviews with each instructor. The interviews, organized by Zandieh's (2000) framework, asked instructors to propose up to eight tasks for introducing derivatives graphically, physically, verbally, and symbolically at a point (assuming students already knew about average rate of change), and as a function (assuming students already knew about derivative at a point).

However, the instructors often targeted multiple conceptions in their tasks, expecting students to use multiple representations of derivatives. This motivated us to analyze the conceptions of derivatives, including representations of derivatives, targeted in the tasks. Given that the proposed tasks often had multiple parts, the fourth author identified and separated the

smallest calculus-focused parts in each task to streamline the analysis; we call these calculus-specific task units (CTUs). Using the CoD framework and the qualitative software ATLAS.ti, all authors coded the four components of conceptions of derivatives targeted in each task: problem space (pure, physics, chemistry/biology), conceptual epistemologies (limit-, infinitesimal/differential-, or instrumental-based), mathematical representations (graphical, numerical, symbolic), and process-object layers (average derivative, derivative at a point, derivative everywhere/function). Problem space was coded at the task level, while the conceptual epistemologies, representations, and process-object layers were coded at the CTU level. Across the eight instructors, we analyzed 57 tasks containing 252 calculus-specific task units (CTUs). Table 2 shows the distribution of tasks and CTUs by instructor.

Table 2. Number of tasks and CTUs, as well as epistemologies used, by professor

	Instructor							
	Adrian	Alex	Barry	Gopher	Justin	Matthew	Max	Monica
Number of tasks proposed	8	8	8	8	4	6	8	7
Total number of CTUs across tasks	35	35	31	38	10	32	40	31

After coding the tasks independently, we met to compare our codes, discuss differences, and come to consensus. All final codes were agreed upon by all authors. To analyze patterns in conception targeting, we conducted chi-squared tests when possible (five or more CTUs) to determine whether individual instructors showed significant preferences for targeting specific process-object layers or mathematical representations, hypothesizing that instructors would target each layer (or representation) equally.

Findings

Contextual framing varied across instructors, as shown in Figure 2. Of the 57 tasks, 32 used pure mathematical contexts, 22 used physics contexts, and only 2 used biology/chemistry contexts. All instructors employed pure mathematical contexts, with Alex, Barry, Gopher, and Justin using them most frequently. Except for Justin, all instructors used physics contexts at least once, with Matthew, Max, and Monica using them more extensively. Adrian uniquely distributed tasks across pure and non-pure contextual frames, while being the only instructor who used biology/chemistry contexts.

Limit-based approaches dominated instruction across all instructors, appearing in 243 of the 252 CTUs (Figure 3). All eight instructors used limit-based epistemologies extensively. Infinitesimal/differential-based approaches appeared in only four CTUs, used exclusively by Gopher and Justin. Rule-based epistemologies were even less common, appearing in five CTUs across three instructors—Alex, Max, and Monica. While the instructors often targeted a single epistemological approach in their tasks, a few tasks incorporated multiple epistemologies within different CTUs.

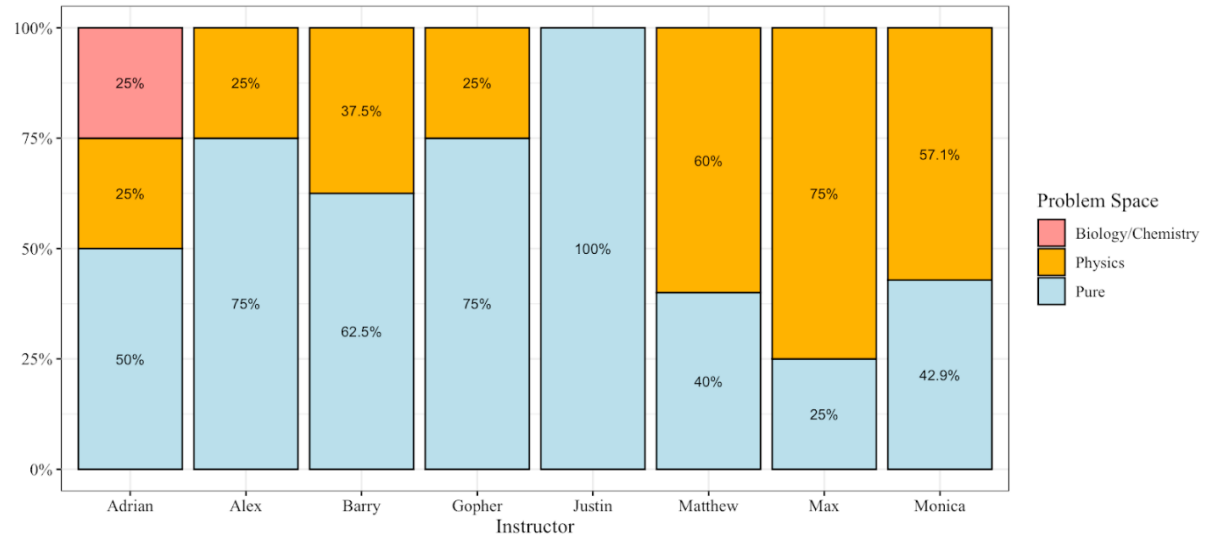

Figure 2. Distribution of contextual framing or problem space across instructors

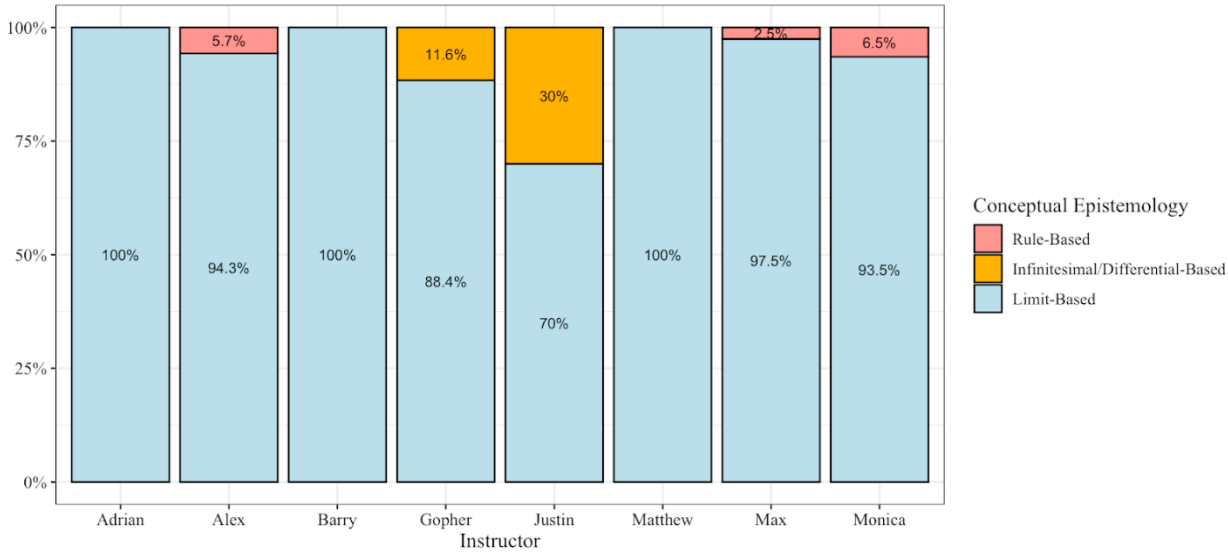

Figure 3. Distribution of conceptual epistemologies across instructors

Within the 243 limit-based CTUs, all instructors except Justin utilized the three mathematical representations and the three process-object layers, though with varying frequencies (Figure 4). Justin did not target the ratio layer or the symbolic representation. Statistical analysis of the rest of the instructors revealed that Gopher and Matthew significantly preferred targeting the limit layer over ratio and function layers ($\chi^2 > 5.99$, $p < 0.05$), while Adrian showed significant preference for symbolic over graphical and numerical representations ($\chi^2 > 5.99$, $p < 0.05$). While Justin's small number of CTUs did not allow χ^2 testing, the remaining instructors did not show significant preferences for targeting specific layers or representations.

The remaining epistemologies—Infinitesimal- or differential-based, and rule- or instrumental-based—were used infrequently. Infinitesimal/differential-based approaches appeared in only four CTUs from Gopher and Justin, exclusively at the limit layer and primarily using graphical representation (i.e., zooming in on the graph of a function to find the instantaneous derivative or the slope of tangent once the function becomes linear). Rule-based

epistemology appeared in five CTUs within tasks by Alex, Max, and Monica, all targeting the function layer with symbolic representation, reflecting the procedural nature of applying derivative rules.

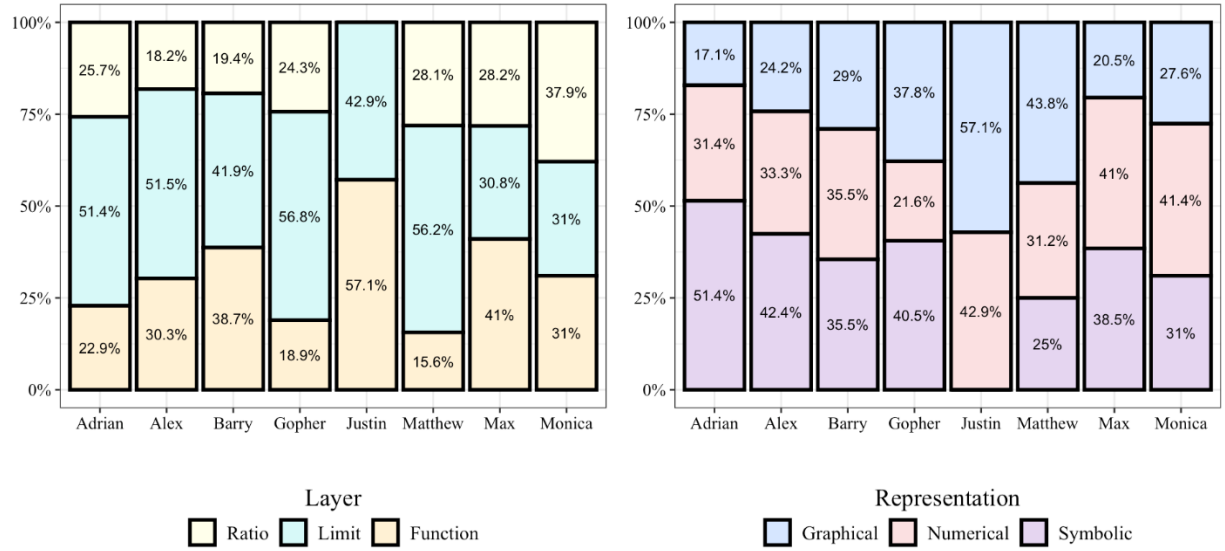

Figure 4. Distribution of process-object layers & representations within limit-based CTUs across instructors

Discussion and Conclusion

Our analysis of introductory derivative instructional tasks from eight experienced college calculus instructors showed that they predominantly targeted pure and physics contextual framing and limit-based epistemology within the conceptions of derivative laid out by the CoD framework (Gerami et al., 2025). However, the prevalence of limit-based epistemology may reflect both our sample selection and the study's original design around Zandieh's (2000) limit-based framework. Additionally, because we specifically asked instructors for tasks introducing derivatives, the focus on limit-based approaches may reflect the instructors' tendency to use this epistemology as the foundational approach when first teaching the concept.

These findings suggest several directions for future research. First, researchers could examine conception targeting across entire instructional materials, rather than just introductory instructional tasks, to capture how targeted conceptions might shift throughout the course. Second, future studies could use the CoD framework to map relationships between instructor targeting of specific conceptions and actual student conception development. Finally, given that colleges increasingly offer calculus courses designed for specific audiences such as biology and business majors, researchers might investigate whether intended student population influences instructors' choices of contextual framing and conceptual epistemologies when teaching derivatives.

Acknowledgments

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Leveraging Universal Design for Learning to Scale Inquiry-Oriented Instruction on Sequences and Series

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Although research has shown benefits of inquiry-oriented (IO) instruction for student learning, it has also been reported not to benefit all students equally. In this report, we illustrate how UDL can serve as a generative framework for (re)designing IO instruction in such a way that attends to the individual needs and strengths of students, working toward creating an equitable learning environment. We will also share UDL-aligned teaching practices for IO classrooms, including those fostering relationship-building, multimodal participation, and equitable assessment.

Keywords: Inquiry-Oriented, Universal Design for Learning, Equity

Research shows that inquiry-based (IO) instruction can support student learning, but it does not always benefit all students equally (Johnson et al., 2020). Without intentional instructional design, it may privilege certain voices or patterns of participation. For instance, Reinholz et al. (2022) showed women's (vocal) participation levels correlated with performance; when participation was equal, the gap disappeared. Other studies have similarly shown that IO classrooms can limit meaningful participation for students from groups historically marginalized in mathematics. For example, women can be excluded or dismissed during group work (Brown, 2018; Ellis & Alzaga Elizondo, 2023), and even when women offer valuable insights, their ideas may not surface in whole-class discussions (Ernest et al., 2019). Together, these findings suggest that although IO instruction holds great promise, it must be intentionally crafted to ensure that all students can participate in meaningful ways. Yet, what it means to intentionally craft this teaching remains underexplored.

One place where this issue becomes especially visible is in efforts to scale inquiry-oriented instructional materials. Research on scaling has made progress by focusing on the mathematical learning trajectory, resulting in streamlined task sequences (e.g., Larsen et al., 2013). However, this work has paid less attention to how to build in instructional supports into these materials that also address students' diverse strengths. As Armstrong (2011) notes, "classrooms are social environments, and all social environments are places where inclusivity happens—*or fails to happen*" (p. 53). Inquiry-oriented classrooms require careful attention to inclusivity, as they are structured around social interactions, which can promote inequities when not treated carefully (Battey & McMichael, 2021). Moreover, because IO instructional materials often originate from the work of a small group of students, if not intentionally designed to accommodate a broad range of learners, they risk privileging a narrow subset of the class.

Guided by Universal Design for Learning (UDL), we have been working to address these challenges to scale IO instructional materials on sequences and series in Calculus 2. In this report, we share our instructional (re)design decisions, informed by UDL principles and student interviews, as we scaled a task sequence that originated from the work of two students.

Theoretical Perspective

We engaged in design research (Cobb & Gravemeijer, 2014) to develop our IO instructional materials, following an iterative, cyclic process with three core components: (1) task (re)design, (2) implementation, and (3) data analysis. Each component informs the rest; the analysis of the

data collected during one implementation informs the redesign of the tasks that will be implemented (and subsequently analyzed) in the next. Traditionally, researchers have started with pairs of students and then scaled the tasks to a whole class setting once the tasks seem to appropriately support students in learning the content goals. However, the limited research on scaling the materials to a whole class setting has had a limited focus on streamlining the mathematical learning trajectory, rather than also attending to other social experiences that students might encounter with such learning environments. Because of this, we have adopted Universal Design for Learning (UDL) to help us also focus on other aspects of the environment. UDL complements the theoretical design frameworks that we have used to design the task sequence (e.g., Realistic Mathematics Education, Gravemeijer & Doorman, 1999; Gravemeijer & Terwel, 2000) by extending our focus from not only engaging students in meaningful mathematics but also other factors that might further support their engagement, representations, and strategic action.

We draw on UDL to support instructors to build environments that recognize and respond to the diverse identities, strengths, and needs of learners (Gargiulo & Metcalf, 2023). UDL is not a fixed set of techniques, but a generative framework for instructors that assumes from the outset that all learners are variable: in how they feel engaged in the content, how they communicate, and how they make sense of mathematics (Chardin & Novak, 2020; Tobin & Behling, 2018; Hall, 2012). Lambert (2024) outlined key elements of UDL-aligned math classrooms: (1) promoting engagement through meaningful tasks and supportive environments, and (2) supporting strategic action by helping students understand themselves as learners and offering equitable feedback and assessment, and (3) attending to mathematical representation by valuing multimodality and investing time in core ideas. (See Figure 1.)

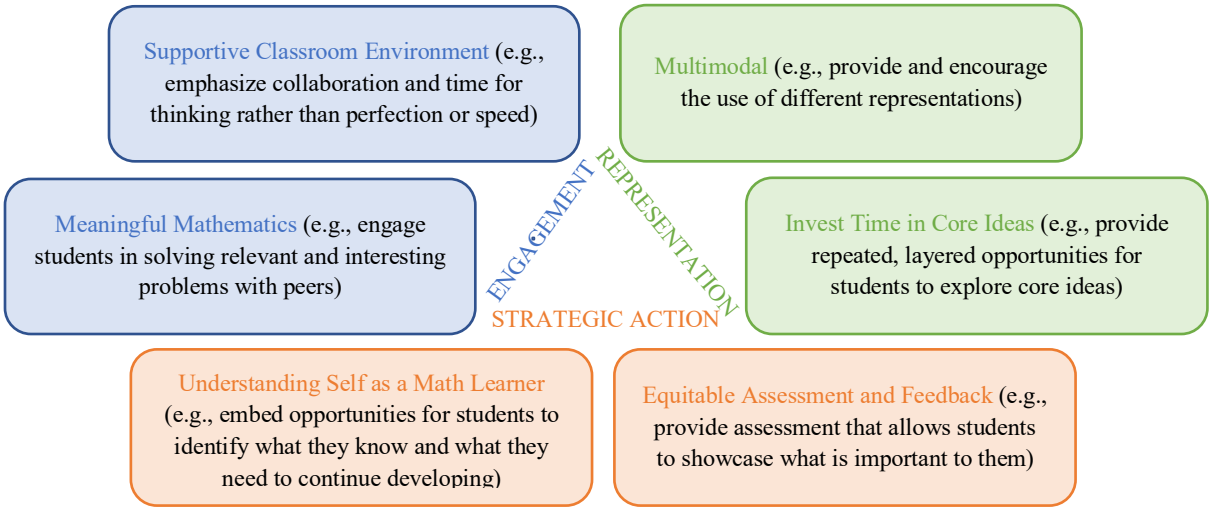

Figure 1. Key Elements of UDL-Aligned Math Classrooms.

We drew on UDL as an analytic tool during our analysis phase of the first whole-class design cycle. In particular, we wanted to (re)design teaching practices that recognize and respond to the diverse identities, strengths, and needs of learners. By teaching practices, we mean both the instructor's goal (i.e., task) and the observable actions taken toward accomplishing their goal (i.e., procedure) (Nachlieli and Elbaum-Cohen, 2021). When planning to scale the materials, we set out to test out the teaching practices that we anticipated would relate to UDL engagement,

representation, and strategic action goals. For instance, we planned to implement standard-based quizzes and portfolios instead of exams (procedure) to allow assessment to be more individualized (task related to understanding self as a math learner).

Together, we use these theories to investigate the following research questions: 1) *What are some UDL-aligned teaching practices for IO classrooms?,* and 2) *How can UDL guide instructional (re)design choices to develop such practices?* Answers to these research questions provide two main contributions. First, practical teaching practices that instructors can adapt to open up their classroom to serve a broader range of students. And, secondly, methodological suggestions for IO designers who want to scale up their materials in an inclusive way that takes into consideration the different strengths and needs of their students.

Methods

The data for this report comes from a larger project in which we are designing an IO calculus 2 unit on sequences and series. We have designed a set of tasks that, with the instructor's support, guide students to reinvent several key theorems of this unit: Monotone Convergence Theorem, all convergent sequences are bounded, and several tests for series convergence (Vroom et al., 2025; Vroom & Davis, 2024; Vroom & Alzaga Elizondo, 2025). Our design research to date has included implementing, analyzing, and revising the instructional sequence first with a pair of students in a laboratory design experiment (Spring 2022) and then in Vroom's calculus 2 class (Spring 2025). In this report, we focus on (a) some of our instructional design choices, informed by UDL and the previous paired-implementation, (b) the calculus 2 students' experiences with the instruction via interviews, and (c) redesigned instructional choices based on UDL and these experiences.

The primary source of data for this redesign relied on individual semi-structured interviews with students (13 of the enrolled 25 students in the Spring 2025 course), which were conducted a week after the course meetings ended. During these interviews, we shared our interpretation of some of the research findings on inequitable outcomes and experiences in IO classrooms (i.e., an elaborated version of the first paragraph of this paper). This was a purposeful decision to attempt to bring students into the conversation that our research aimed to address – to do research *with* the students rather than *on* them (inspired by Stinson, 2008). Following this reading, we discussed how, if at all, the research findings resonated with the students' experience during the sequences and series unit. We also asked students to share aspects (about the instruction, peer interactions, the course structure) that supported and discouraged their participation. See Table 1 for the participants self-reported identity markers (from a survey early in the semester).

Table 1. Self-reported identity markers.

Student	Gender	Race/Ethnicity	Sexuality
Aaron	Man	Chinese/White	Did not report
Alice	Female	White	Straight
Anna	Woman	White	Straight
Caleb	Male	White	Straight
Gabe	Male	Caucasian	Straight
Imran	Male	Indian	Heterosexual
Jack	Man	White	Did not report
Jason	Male	Asian	Straight
Micah	Transmasc/nonbinary	Caucasian	Asexual and Queer

Nina	Woman	White	Bisexual
Oliver	Male	White	Straight
Shaan	Male	Asian	Straight
Trevor	Male	White/Caucasian	Straight

Prior to the classroom implementation, we designed a set of teaching practices that we anticipated would align with UDL principles. For example, we planned to use Padlet, an online platform where students can post (anonymously if they choose), to provide multiple ways to participate. Following the implementation, we began analyzing these practices, initially focusing on interview transcripts from four students who identified with gender identities historically marginalized in mathematics (i.e., "woman," "female," or "Transmasc/nonbinary"). We prioritized these students' perspectives because our instructional design explicitly sought to respond to documented gender inequities in IO classrooms (Brown, 2018; Ellis & Alzaga Elizondo, 2023; Earnest et al., 2019; Johnson et al., 2020; Reinholz et al., 2022). In analyzing these transcripts, we first identified excerpts in which students referred to specific procedures associated with a teaching practice. We used these excerpts to refine our original list of practices. For instance, students commented on the practice of working in self-selected groups. Although the instructor typically uses this strategy to foster peer relationships, students' responses helped us recognize its potential alignment with UDL principles, even though we had not initially framed it that way. After identifying relevant quotes from the four transcripts, we compiled a table linking them to teaching practices. We then open coded the linked excerpts for how students perceived the practices' tasks, grouping responses to identify shared purposes. We compared these student-identified goals with our original intentions and revised (i.e., expanded, created, or maintained) practices accordingly. This process yielded a set of UDL-aligned teaching practices, which we then used to frame our analysis of the rest of the transcripts. We then read the remaining interview transcripts, coding for additional student references to these practices and incorporating their perspectives to elaborate and further refine them. Here we share a subset (eight) of these practices.

Results

Below we describe our (re)design choices. We organize this section by discussing practices that we 1) expanded, 2) created, and 3) maintained after the class implementation. In Table 2, we overview the redesigned teaching practices.

Table 2. UDL-Aligned Teaching Practices.

Revised Practice	UDL Reasoning
1. Incorporate anonymous discussion board to pose questions to provide an alternative way to participate	Representation: multimodal
2. Incorporate anonymous discussion board to create a safe space to take risk and mitigate perceived biases in discussion	Engagement: supportive classroom environment
3. Incorporate discussion board for students to increase peer engagement by providing a space for them to see or respond to each other's contributions	Engagement: supportive classroom environment

4.	Implemented standard-based quizzes and portfolios (instead of exams) to allow students to share *their* mathematics and what *they* find important	Strategic action: equitable assessment and feedback
5.	Implemented standard-based quizzes and portfolios (instead of exams) to support students to focus on their learning on their timeline rather than memorizing or their grades	Strategic action: understanding self as a math learner
6.	Allow students to choose groups that they will stay with throughout semester to build relationships	Engagement: supportive classroom environment
7.	Make sure everyone has a partner, preferable group of 2, to increase opportunities to contribute and reduce possibilities of exclusion	Engagement: supportive classroom environment
8.	Regularly share that the expectation is to share rough draft ideas (rather than perform polished ones) to normalize mistakes and emphasize growth	Engagement: supportive classroom environment

Expanding UDL-aligned teaching practices. Originally, we intended to *incorporate anonymous discussion board to pose questions to provide an alternative way to participate*. This was aligned with UDL in the sense that students would have options for how they shared their ideas so they could find a way that felt comfortable for them (*Representation: multimodal*). To do this, the instructor projected a class Padlet at the front of the room. We found evidence from the student responses that this task was realized as Anna noted:

> It's kind of an equal chance for everybody to share their ideas. Even with people, like me, that don't like talking in front of big groups of people. She kind of just let us post it on there to talk about it.

Additionally, we learned that the anonymous feature of the Padlet also supported students to feel comfortable taking risks in sharing their ideas (*Engagement: supportive classroom environment)*. For instance, Micah explained:

> I like how there's the option to be anonymous on it, and I think that takes away a lot of the fear that because of who you are, you might be perceived, or taken differently in you asking your question, and I think it gets rid of any opportunity for that, because then it's like, 'Oh. astounding chameleon asked a question'.

The students also shared that the Padlet increased peer engagement by providing a space for them to see or respond to each other's contributions (*Engagement: supportive classroom environment*). For instance, Trevor explained:

> I think the Padlets - those were helpful to me because I was kind of able to see like what other people were thinking rather than just my group members... It was helpful to see again other people's like opinions or ideas for a question.

For these reasons, we redesigned this teaching practice by expanding it to three different practices (see Table 2, lines 1-3).

In setting up the course, we decided to *implement standard-based quizzes and portfolios (instead of exams) to allow assessment to be more individualized*. This was aligned with UDL in the sense that it would give students more agency in demonstrating their understanding of what they knew and what they needed to learn (*Strategic action: understanding self as a math learner*). We found evidence that the assessment was more individualized in two different ways. First, similar to what we anticipated, it supported students to focus on their learning on their

timeline rather than memorizing or their grades (*Strategic action: understanding self as a math learner*). As Anna said:

> I'm not stressed out trying to... memorize it. Rather with this way, I feel like I could actually sit down and take my time and actually try to learn it rather than just trying to memorize everything in time for exams.

Second, it also allowed students to share *their* mathematics and what *they* find important (*Strategic action: equitable assessment and feedback*). For instance, Imran explained:

> [the portfolio was] showcasing your skills instead of doing a math test where it doesn't really show *you as a student*... [The portfolio] showed *our* understandings. It showed *our* interpretations of standards themselves. [emphasis added]

Thus, we redesigned this teaching practice by expanding it in two ways (see Table 2, lines 4-5).

Creating UDL-aligned teaching practice. One teaching practice that the instructor regularly uses in her courses was *not assigning groups in order to allow students to work with who they would like*. We were not attentive to this practice during our initial design; but, through our conversations with students, we realized that it was a practice that could align with UDL by supporting relationship building (*Engagement: supportive classroom environment*). Micah elaborated on how choosing the groups and staying with the same group was important to them:

> chosen groups versus assigned groups have it have a huge impact on what that group dynamic looks like ... there is a much, much lower chance of [exclusion] happening in chosen groups, and I think that sort of just adds another level of safety to it.

They continued to explain that staying with their chosen group helped them to "establish a consistent relationship" and that "made it so that it was a space where you guys were working together, not a space where you guys were forced together like you were actually collaborating, not just sitting next to each other doing your work." We also got insight from students about how the group size (partners vs groups of three or more) allowed (or prevented) individuals to contribute and reduced the possibility of exclusion (*Engagement: supportive classroom environment*). Students shared that it's easier to exclude yourself or others in groups of three. Shaan explained:

> in the third unit I didn't participate enough, but there was another new person who was talking to [my partner]. So, I was fine, like, I was not interacting. If ... it was just two of us, I would have like added more like to the conversation.

Trevor also explained that larger groups can lead to exclusion because he tries to find similarities with others to connect over:

> let's say we're in a group of three... And then there's two people who are more similar to each other. I think that those two people are gonna interact with each other more just because they're similar and maybe unintentionally exclude the person that's different from them.

However, when there is paired groupwork, "You gotta find, like the similarities between your partner [and] figure out how you can have... a functional group". Thus, in our redesign, we added two practices (see Table 2, lines 6-7).

Maintaining UDL-aligned teaching practice. Originally, we intended to *regularly share that the expectation is to share rough draft ideas (rather than fully formed polished ones) to normalize mistakes and emphasize growth*. This was aligned with UDL in the sense that such framing supports students to take mathematical risks by creating a safe and welcoming environment. (*Engagement: supportive classroom environment*). The students shared this

practice was realize, for instance, when Nina said, “I never felt like I had to be perfect in my understanding of everything in order to participate.” Similarly, Alice elaborated:

> I can actively give like not a fully correct answer and not feel ashamed of it. I feel like that really helps my learning. And when we worked in our small groups in class... we would all build off of each other, and we'd be... comfortable with each other not having the full answer.

Because of this feedback, we kept this practice in our redesign (See Table 2, line 9).

Discussion

Returning to our first research question, we crafted a set of UDL-aligned teaching practices for IO classrooms (see Table 2). Student feedback suggests these practices can help mitigate inequities documented in prior literature (Brown, 2018; Ellis & Alzaga Elizondo, 2023; Earnest et al., 2019; Johnson et al., 2020; Reinholz et al., 2022). We attribute this, in part, to the theoretical grounding UDL provided for scaling IO instructional materials in ways that attend to student experiences beyond the mathematical trajectory. For example, we reconsidered grouping not only in terms of mathematical productivity but also in relation to students’ comfort with taking risks and sharing emerging ideas. Similarly, UDL encouraged us to broaden what counts as participation. Although IO instruction often privileges verbal contributions, incorporating tools like Padlet helped us recognize non-verbal participation, thereby expanding whose ideas could be acknowledged and built upon. Although our insights arose from materials on sequences and series in Calculus 2, we see potential relevance across a range of IO contexts. We do not claim these practices work universally, but they may provide a starting point.

Importantly, and in response to our second research question, our work illustrates how UDL can serve as a generative framework for designing IO instruction. From the outset, UDL shaped the design of our practices by hypothesizing procedures that could accomplish a task related to UDL’s goals (engagement, representation, and strategic action). Post-implementation, we used UDL to analyze how these practices functioned, and student perspectives were crucial in this reflection. By inviting students into the research centering their perspectives, we were able to identify practices that supported them across the three UDL dimensions, practices that we at times did not even anticipate.

One notable point of divergence between our findings and the prior literature concerns group formation and size. Earlier research in mathematics education often recommends teacher-assigned groups of three or four to mitigate status differences (e.g., Cohen & Lotan, 2014). In contrast, our students emphasized that working in chosen, stable pairs reduced the risk of exclusion and fostered a sense of safety and collaboration. We interpret this difference as reflecting the distinct emphases of UDL. Whereas the groupwork literature often foregrounds managing status dynamics, our UDL framing foregrounded agency and supportive relationships. This suggests that grouping practices may be more context-dependent than the literature sometimes implies. For IO classrooms guided by UDL, chosen groups may offer advantages over assigned groups by fostering learning environments where students feel safe sharing emerging ideas, a key characteristic of IO instruction (Kuster et al., 2018).

This work takes initial steps toward scaling IO instructional materials in ways that extend beyond streamlining meaningful mathematical tasks. Future research could further examine how UDL interacts with specific IO practices across varied institutional settings, offering insight into how these practices might be refined and scaled sustainably.

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Examining Online Mathematics Instructional Videos Through a Narrative Lens: The Role of Mathematical Questions

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The increasing use of online videos in mathematics education has led to a need to understand what makes them captivating. This study applied Dietiker's mathematical story framework to online mathematics instructional videos (OMIVs), specifically analyzing several videos from a popular YouTube channel. Using a qualitative case study approach, we identified the central role of questions from a narrative lens and observed four open question types and three types of their closure. Findings also show how different question types may relate to the plot's characteristics and enrich the plot. This study is an initial exploration of mathematical stories in the context of OMIVs, providing new perspectives that would hopefully inspire further research and practical application in math education.

Keywords: mathematical story, online videos, complex Fourier series, plot, questions

The COVID-19 pandemic accelerated digital learning, making instructional videos more common in mathematics education (Alabdulaziz, 2021). Online mathematics instructional videos (henceforth called OMIVs) support engagement and understanding, and educators increasingly recognize their pedagogical value (Esparza Puga & Aguilar, 2023). Yet their quality varies, raising questions about why some videos are more engaging to the viewer (Jones & Cuthrell, 2011; Kolthof, 2021). A striking example is the YouTube channel 3Blue1Brown, whose creator Grant Sanderson argued that stories play an important role in engaging mathematical learners, with his widely spread videos deeply supporting his argument (TEDx Talks, 2020). Story, more broadly, has long been used as a pedagogical tool in mathematics for its capacity to guide emotions and foster meaning (Zazkis & Liljedahl, 2009). Building on this idea, Dietiker's (2015) mathematical story framework conceptualized mathematics lessons as unfolding events and has applied the framework to the context of lessons (e.g., Richman et al., 2019). This study adapts Dietiker's framework to analyze selected OMIVs, exploring how storytelling shapes math questions and their resolution to inform the design of more engaging instructional content.

Literature Review

Educators increasingly recognize the vital role of technology in mathematics education, as digital technologies continue to shape the current state and future of teaching (Borba et al., 2016; Darragh, 2021). With internet access and applications now available to much of the world's population, digital resources have become embedded in teaching and learning. More recently, the COVID-19 pandemic accelerated this trend, making online learning and digital products central to instruction (Ilmadi et al., 2020). The accessibility of digital devices enables learning beyond the classroom, and research highlights how such tools can enhance student attention, active engagement, and independent learning (Alabdulaziz, 2021; Fakhrunisa & Prabawanto, 2021). Moreover, compared to in-person teaching, online platforms reduce barriers of time and space and foster new forms of interaction among students and teachers (Ilmadi et al., 2020).

Whether lecture recordings or designed instructional videos, these have become a regular part of students' online mathematics learning. Researchers have examined the factors contributing to their effectiveness, such as video length, production quality, and instructor

presence, which could influence students' engagement (Acuña-Soto et al., 2018; Larkin, 2017). Equally important are pedagogical considerations: videos support students' conceptual understanding and sustained attention when they align with curricular goals, follow explicit learning objectives, and present sound mathematical explanations (Cardoso et al., 2014; Johanes & Lagerstrom, 2016). Visualization, such as animations, graphs, and text, can enhance comprehension when meaningfully aligned with learning goals (Ismail et al., 2017). In addition, recent research also highlighted that utilizing technology to transition between different realizations of mathematical objects could contribute to audience engagement in videos (Radmehr & Turgut, 2024).

While prior studies have explored technical and pedagogical aspects of OMIVs, to our knowledge, there has been less focus on their narrative and aesthetic dimensions. Researchers believe that narrative is central to mathematics learning, as it may bridge mathematics and students' experience, fostering cognitive and emotional engagement by grounding abstract mathematical ideas in familiar contexts (e.g., Toor & Mgombelo, 2015). Dietiker's (2015) mathematical story framework, developed to analyze curricula and lessons, offers a useful lens for evaluating such narrative structures. This study extends that work by applying the framework to OMIVs, aiming to illuminate how storytelling could play a role in their design and impact.

Theoretical Framework

Dietiker (2013, 2015) describes a *mathematical story* as the way mathematical content unfolds across a sequence of changes – that is, the progression of mathematical concepts. Building on this idea, Dietiker introduced four elements of a mathematical story: characters, actions, setting, and plot. *Characters* are mathematical objects, such as shapes, equations, or numbers, that may develop characteristics (e.g., a number being prime) or relationships (e.g., two functions intersecting). Character development occurs through *actions*, which mathematical actors, typically humans, perform. For example, in a video, a narrator may differentiate a function $f(x)$ – here, "differentiate" is the action on the character $f(x)$. The *setting* is the space where characters and their actions take place. Dietiker described it as the representations of the content that situate the story, which help the reader interpret the characters and their actions more clearly; for instance, $y = x$ appears as a line in two dimensions but as a plane in three. The *plot* refers to the aesthetic dimension of how readers encounter the story. Plot shifts occur when information changes, prompting readers to raise or answer questions, experience tension, and feel satisfaction or surprise.

Richman et al. (2019) further developed three characteristics of plot: density, coherence, and rhythm. *Density* concerns the number of open (i.e., unresolved) and closed (i.e., resolved or no longer under consideration) questions; and a plot becomes denser when more questions remain open. *Coherence* refers to how events connect and help learners anticipate or interpret what follows, producing responses such as pleasure or surprise. *Rhythm* is the pattern of opening and closing questions, shaping the story's pace. To interpret the overall plot, Richman et al. (2019) also introduced the idea of a *story arc*, which is the trajectory of a question and all the shifts it undergoes as it moves from being open to being resolved.

The Study

This study employs a qualitative case study to examine how OMIVs can be viewed through a narrative lens. Building on Dietiker and colleagues' work on plot analysis through opening and closing questions (2013, 2015; Richman et al., 2019), this study aims to extend this framework to video contexts. Specifically, we classify and characterize open and closed questions within the

mathematical narrative structure in OMIVs, and then examine how these question types connect with characteristics of the mathematical plot and contribute to its development. Guided by this framework, we seek to answer the following research questions:

1. What types of open mathematical questions and their closures appear in OMIVs?
2. How are these question types connected with the characteristics of the mathematical plot?

Data Collection

We collected the data from the YouTube channel 3Blue1Brown (3Blue1Brown, n.d.). As of August 2025, the channel had 7.57 million subscribers, 216 videos, and over 672 million views. Its creator, Grant Sanderson, has been recognized by the American Mathematical Society for his contributions to mathematics education (News from the AMS, n.d.). We selected this channel because of its wide reach and strong reputation in online mathematics education, and we note its pedagogical value has also been recognized in prior research (e.g., Radmehr & Turgut, 2024).

We analyzed four videos from this channel, chosen for their mathematical content, number of views, and the creator's identification of them as more narratively compelling. The videos are: V1 (3Blue1Brown, 2019c) on Complex Fourier Series within a sequence of videos on differential equations; V2 (3Blue1Brwon, 2017) on a Putnam problem and its solution; and V3 (3Blue1Brown, 2019a) and V4 (3Blue1Brown, 2019b) that constitute a two-part presentation that introduces and resolves a mathematical puzzle.

Data Analysis

We have reviewed each video multiple times and divided it into acts, defined by shifts in characters, actions, settings, or relationships (Richman et al., 2019). Acts varied in length depending on changes in focus, from under a minute to several minutes. We identified each mathematical question within each act, including explicit and implicit questions. To do so, we adapted Richman et al.'s (2019) framework by defining *explicit questions* as those posed directly by the narrator, and *implicit questions* as those that emerged from the unfolding plot in ways likely to inspire curiosity in the viewer. For example, "Perhaps you see the pattern here" could implicitly raise the question: "What is the pattern?". Also, following Richman et al. (2019), we grouped similar questions within the same act into a broader question to avoid artificially inflating question density.

For implicit questions, we included the questions we formed when watching the videos and questions we recognized that other audiences might form. Additionally, we assumed that the audience has a foundational understanding of the mathematics involved. Therefore, we focused on questions relevant to a level of knowledge that supports the understanding of the video.

Lastly, we coded the questions into story arcs and connected the story arcs to plot characteristics. Specifically, we explored the plot's density by examining the number of acquired and resolved questions across acts, analyzed its coherence by considering how arcs overlapped or linked, and traced its rhythm through the pacing of suspense and resolution. To contextualize narrative analysis with audience response, we compared the YouTube engagement graphs with plot features, offering insights into potential links between story elements and viewer engagement. The findings are presented below, with a primary focus on V1 due to word constraints.

Findings

Classification of Question Types

Types of open question. We observed four types of questions regarding how they were opened in the narrative. The first type is the *main question*, which introduces core mathematical content or ideas and serves as a central focus of the video. Progress on one main question does not necessarily contribute to a previous one. That is, while connections between main questions may exist, each relies on different knowledge. For example, in V1, the first question [Q1] was: "What is a complex Fourier series?". We classify this as a main question because it introduces one central idea without relying on earlier ones. Another example from V1 is: "How did the Complex Fourier Series create animation?". Although it seems to build on the first, we also treat it as a main question since its answer requires new knowledge rather than directly resolving the earlier one. Main questions may overlap, but each sustains its own distinct arc.

The second type is the *sub-question*, which relates to a previous question and helps make progress on it without altering its original conditions. Typically, a sub-question breaks a larger question into smaller, more manageable parts. For example, in V2, one main question is: "What is the probability that the center of the sphere is inside the tetrahedron [formed by four random points on the surface of this sphere]?". A related question is: "Which […] tetrahedra contain the sphere's center?". Because this question contributes to resolving the main question while preserving its conditions, we classified it as a sub-question.

The third type is the *helping question*. Like sub-questions, helping questions relate to a prior question, but they differ in that they modify the conditions to simplify the problem and enable progress. For example, in V2, for the same main question above, one helping question is: "What is the average size of the section that the fourth point could land if the other three points are fixed?". Fixing three points changes the conditions, making the problem more accessible. Helping questions, therefore, often serve as a bridge to solving the main question by reducing its complexity.

The last type is the *reformulated question*, which rephrases an earlier one with different wording or structure but preserves the same core idea. For instance, in V1, one question is: "Why don't we use 'vectors' instead of a 'complex number' in a complex plane?". Later in the video, the narrator revisits the same idea: "You'll notice I'm being a little loose with language, using the words vector and complex number somewhat interchangeably", which essentially asks, "Why are vectors and complex numbers used interchangeably?". Since both questions address the same underlying idea, we interpreted the latter as a reformulation of the former.

Types of question closures. We also observed three types of question closures. The first type is *normal closure*, which occurs when a question receives a clear and definitive answer, leaving no ambiguity. If a question has a normal closure, it has been fully addressed in the current context, enabling the audience to fully understand the explanation. Most of the mathematical questions in the videos fall into this category.

The second type is *inquiry closure*, in which the narrator deliberately leaves questions unanswered for further thinking or exploration. Inquiry closure occurs when the narrator poses a question and explicitly invites the audience to reflect or try solving it themselves. The purpose may be to promote active learning, deepen understanding, or develop problem-solving skills. For example, in V1, the narrator asked, "How do we find the coefficients in the step function?" and left the audience to think more about the question.

The third type is *deferred closure*, which occurs when the narrator postpones the full answer to another video or avoids giving it completely because the content is out of scope or irrelevant

to the main topic. For example, in V3, the narrator asked: "What is configuration space, or phase space?". He explicitly stated that further explanation would appear in the next video, and indeed, the following video continued with the same puzzle and expanded on this idea. Therefore, we suggest that this question exhibits a deferred closure because its scope extends beyond the current puzzle to broader areas of mathematics.

Question Types and Plot Characteristics

This subsection addresses RQ2 by examining how question types play a role in the plot's density, coherence, and rhythm.

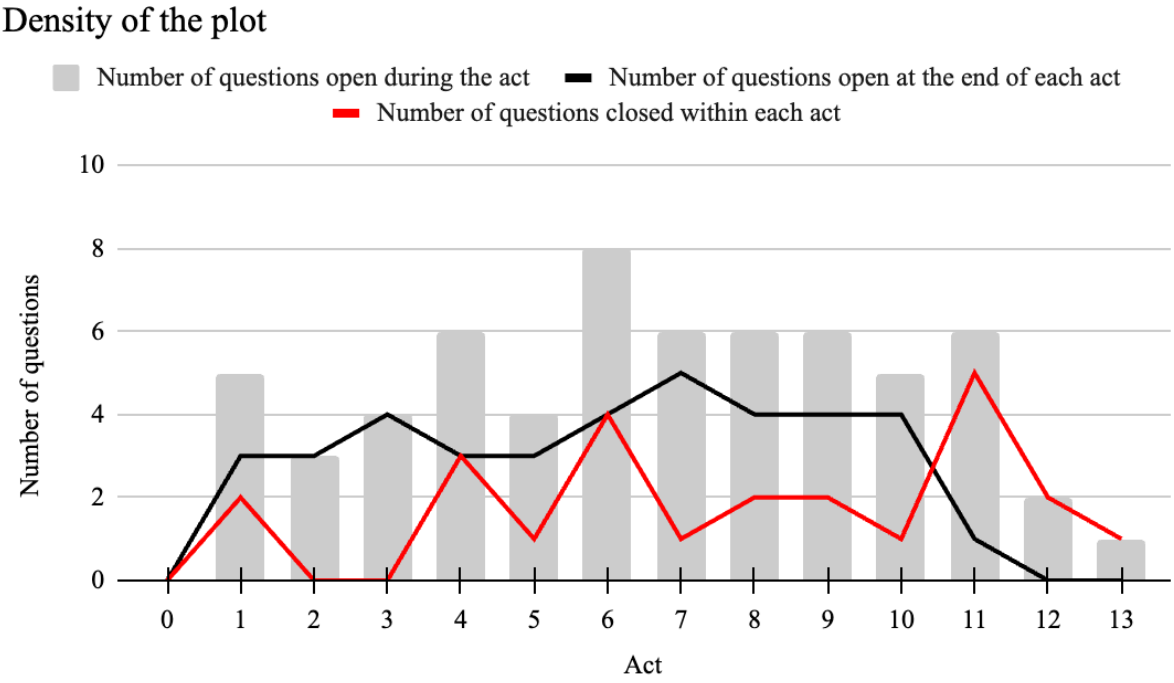

Figure 1: Density of the plot in V1

Questions and density. To recall, the density of the plot refers to how many mathematical questions remain open within a segment of the story (Richman et al., 2019). As Richman et al. (2019) explained, the plot becomes denser if more questions are left open, and a release in density occurs when the number of open questions decreases. Following the approach described in the Data Analysis section, we excluded reformulated questions when considering plot density because they do not introduce new issues, but merely re-express existing ones.

We observed a clear trend when considering how different open questions affect the density. Taking V1 as an illustrative example (see Figure 1), the density trajectory (see black line) first appears as a rising path marked by short periods of small release or sustained in between, then reaching a peak and holding steady for a time before undergoing a sudden release. Additionally, acts that open a new main question tend to generate a higher density for more than one act. The unfolding of complex mathematical concepts appears to be guided by a main question, which establishes the current topic within the narrative and where density accumulates. Supporting questions sustain and elaborate on this main question, causing the density to continue rising, reach a peak, and eventually release. This dynamic is visible in Figure 1, where peaks in density align with acts that open new main questions. For instance, Acts 1, 3, and 7 combine at least one new main question with more than one helping or unresolved sub-questions.

Figure 2: V1 engagement graph

Another perspective concerns the release of density. Figure 2 shows the engagement graph above the progress bar of V1, highlighting the most replayed moment in V1, which corresponds to Act 11. As Figure 1 illustrates, Act 11 involves a major release in density at the end of the act, with several previously open questions resolved at once. In contrast, the ends of previous acts (i.e., 8, 9, 10), show that the density remained relatively high for a while, indicating sustained tension. That is, the closures in Act 11 mark the resolution of a build-up maintained across earlier acts. The coinciding of this release and the peak of audience replays suggests that viewers are especially attentive when the accumulated tension is lifted. We therefore suggest that, rather than being drawn primarily to the opening of new questions, engagement appears concentrated at points where narrative pressure is released.

Questions and coherence. As a reminder, coherence refers to how different parts of the story connect. The classification of opening questions helps explain coherence because each type contributes a distinct way of linking ideas into the plot. In general, we suggest that main questions frame long story arcs, while sub- and helping questions sustain or clarify them, preventing the plot from fragmenting. In other words, sub- and helping questions may maintain coherence by breaking complex problems into manageable steps. For example, in V2, the main question "What is the probability that the tetrahedron formed by the four random points on a sphere's surface contains the sphere's center?" forms a long story arc that is resolved at the end of the video. Yet, it coheres with other sub- and helping questions, which develop supporting concepts and link back to the overarching narrative formed by the main question.

Closures also contribute to coherence, but in a more indirect way. We suggest that normal closures provide immediate clarity, possibly helping the audience build confidence to engage with longer arcs. Inquiry and deferred closures, in turn, may sustain coherence across videos by leaving questions open for later explanation and creating anticipation.

Questions and rhythm. To remind, rhythm refers to the pacing of the video, shaped by the pattern of how questions are posed and answered. We found that videos usually begin with more than one overarching main question in the first arc. This can create a slow rhythm, as the audience is generally unable to resolve them immediately in the same or next few acts, and they typically reach closure only in later acts. Also worth noting is that main questions are interspersed throughout the narrative but appear intermittently rather than at regular intervals. Yet most of their occurrence reinforce the video's narrative by either strengthening or shifting the focus through transitions in the main topic. This irregular pattern contributes to changes in pacing. In contrast, sub- and helping questions with normal closures maintain a steady pace by resolving smaller puzzles and advancing the narrative. The introduction of a new main question interrupts this steady rhythm, possibly breaking the audience's expectations and creating a new rhythm that may generate surprise.

For example, in V1, the rhythm begins with a consistent pace in Act 1, where two main questions have a long story arc, suggesting the story begins at a relatively slow pace. A recursive and steady pace follows, driven by shorter sub-questions that the narrators give answers to at a faster pace. However, this pace is then interrupted in Acts 6, 7, and 8, where the main question

comes into place and takes over. In Act 6, the main question, "What would happen if the heat equation solutions of the cosine waves were all added up?" initially builds up suspense. Still, the narrator answers this question at the end of Act 6, creating a sense that the next suspense would also be easily solved. However, what follows at the start of Act 7 is, "Why does the added complexity come up?", which cannot be answered within the regular pattern. After that, the pace returns to the previous in Acts 9 and 10, preparing for the changing of rhythm in Act 11, where most of the main questions get their closure. We propose that changing the rhythm, by intermittently shifting the focus at a steadier pace, can encourage students to think more, which may explain the small peaks in Figure 2.

Discussion

This study analyzed four OMIVs from the YouTube channel 3Blue1Brown to examine the role of mathematical questions in shaping mathematical narratives. We expanded Dietiker and colleagues' framework by identifying four types of open questions: main, sub, helping, and reformulated, which emerged at different parts of the story. Main questions introduced new concepts, while sub and helping questions extended them in ways that made the original idea more approachable. Reformulated questions rephrased earlier inquiries to reinforce their existence as unanswered questions. We also identified three types of question closures: normal, inquiry, and deferred. Normal closures resolved questions fully, providing clarity and closure. Inquiry closures left questions deliberately open, inviting independent exploration. Deferred closures postponed answers to the future, maintaining suspense and continuity across episodes.

The analysis of the videos showed that the type of question is related to the characteristics of the mathematical plot. It seems that main questions often bring higher plot density, slow down the development of the plot, enrich the plot, and accordingly may stimulate students' thinking. In contrast, the sub- or helping questions help the plot smoothly transition to the next stage by providing clear answers. The closure of questions, meanwhile, reinforces key mathematical concepts, providing the audience with an immediate sense of clarity and conclusion. Together, these forms of question openings and closures enrich the plot by alternating its pace, increasing and releasing density, and stimulating deeper reflection by creating connections between different parts of the story. This is in line with Richman et al.'s (2019) observation that alternating plot characteristics not only maintains students' interest but also improves the coherence and logic of the mathematics lesson.

Our classification of question types and their roles highlights the importance of a strategically structured narrative for maintaining engagement, as supported by previous research (e.g., Acuña-Soto et al., 2019). As noted above, the different question types shape the story's unfolding, contributing to the interconnected progression of the narrative. The findings suggest that an intentional use of questions in OMIVs aligns with the mathematical storytelling framework and points to the potential for future educators to apply this lens when designing OMIVs – by varying question types and strategically timing their closures.

In conclusion, this research demonstrates that questions play a central role in shaping mathematical narratives and provides a lens for digital content, which may also apply to classroom design. We suggest that by considering the type of opening questions and their closure, educators can design lessons that are logically coherent, as well as emotionally and cognitively engaging. While this study focused on a single YouTube channel and did not include audience feedback, future research could examine different OMIVs, contexts, or lesson formats, and investigate how coherence, rhythm, and density may be designed to shape learners' engagement.

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Mathematics Graduate Teaching Assistants' Discourse in College Algebra: The Case of Anthony

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Graduate Teaching Assistants (GTAs) play a critical role in undergraduate mathematics, often serving as instructors of record with limited preparation for teaching. This paper examines the case of Anthony, a College Algebra GTA participating in a semester-long professional development seminar focused on anticipating student strategies, planning purposeful questions, and reflecting on teaching. Across three observations, Anthony's lessons shifted from lecture-heavy explanations to student-centered problem solving, supported by targeted guiding questions. Monitoring charts from Anthony's planning and post-lesson debriefs revealed how his instructional decisions were shaped by prior beliefs about student-centered learning and contextual pressures such as pacing demands and coordinator feedback. This case study provides insight into the tensions GTAs face in balancing explanation and inquiry and illustrates how professional development can support incremental shifts toward more student-centered mathematics instruction.

Keywords: Graduate Teaching Assistants, College Algebra, Professional Development, Student-Centered Instruction

Graduate Teaching Assistants (GTAs) play a critical role in undergraduate mathematics, often serving as instructors of record for introductory mathematics courses (Weidert et al., 2012). Despite their instructional responsibilities, GTAs often receive little to no preparation to teach (Judson & Leingang, 2016) and limited ongoing support for effective mathematics teaching (Rogers & Yee, 2018). Because GTAs teach thousands of students nationally, their instructional practices directly influence student learning experiences and retention in STEM pathways (Blair et al., 2018).

One key challenge GTAs face is facilitating student-centered instruction. GTAs often struggle to structure lessons in balancing task setup, group work, and whole-class discussion (Speer & Wagner, 2009). Yet, student-centered teaching practices, such as posing purposeful questions, monitoring student reasoning, and facilitating whole-class discussions, can support conceptual understanding and improved student outcomes (Conner et al., 2014; Smith & Stein, 2018).

Professional development (PD) programs for GTAs seek to bridge this gap by providing structured opportunities to plan tasks, anticipate student strategies, and reflect on instructional practice (Deshler et al., 2015; Maki et al., 2019). However, less is known about how GTAs' beliefs, contextual constraints, and PD experiences interact to shape their classroom practices over time.

This paper reports on one case study from a larger project on GTA professional development in College Algebra. Specifically, it focuses on Anthony, a second-semester master's student GTA enrolled in a semester-long seminar designed to support student-centered teaching. The research question guiding this study was: *How did Anthony structure his College Algebra lessons at the beginning, middle, and end of the semester, and how were these structures influenced by contextual pressures?* By examining Anthony's case, this paper illustrates how a GTA's instructional approach shifted during the semester and highlights the role of external pressures placed on GTAs in shaping the enactment of student-centered teaching.

Literature Review and Theoretical Framing

GTAs struggle to enact student-centered instruction (Speer & Wagner, 2009), challenges that stem not only from limited preparation (Rogers & Yee, 2018), but also from a tendency to reproduce the lecture-based instruction they experienced as students (Kung & Speer, 2009; Sofronas & DeFranco, 2009). Understanding how GTAs navigate these challenges requires attention to both the knowledge they bring to teaching and the contextual factors that shape their practices. This study is framed by Hauk et al.'s (2014) knowledge of discourse, which builds on Ball et al.'s (2008) Mathematical Knowledge for Teaching (MKT) framework.

The MKT framework differentiates between instructors' subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Within PCK, knowledge of content and students (KCS) involves anticipating how learners may approach or misconceive a concept, while knowledge of content and teaching (KCT) focuses on sequencing ideas and instructional moves. Knowledge of curriculum concerns how mathematical topics fit within broader curricular progressions and instructional goals (Ball et al., 2008). This framing emphasizes the multifaceted nature of effective instruction and provides a lens for examining GTAs' instructional decisions.

Building on this model, Hauk et al. (2014) introduced knowledge of discourse to highlight the communicative dimension of teaching. Knowledge of discourse examines how instructors foster interactions among students, exchanges between students and instructors, and connections between classroom activity and broader mathematical communities. They linked this knowledge to three forms of instructor thinking: **anticipatory thinking**, which connects to KCS as teachers predict and prepare for a range of student responses; **implementation thinking**, which connects to KCT as teachers adapt instructional moves in real time; and **curricular thinking**, which connects to knowledge of the curriculum as teachers situate lessons within broader mathematical goals and trajectories. Later work reinforced that discourse is integral to both the enactment of mathematical tasks and the construction of collective understanding (Jackson et al., 2020).

Student-centered instruction in undergraduate mathematics has been consistently tied to opportunities for classroom discourse (CBMS, 2016; Yackel, 2002). Mathematics is not only a collection of procedures but also a way of reasoning, justifying, and engaging in argumentation (Conner et al., 2014; Sfard, 2001). Yet facilitating productive discussions is challenging, as instructors must guide students toward conceptual goals while maintaining space for exploration (Sherin, 2002). The five practices for productive mathematical discussion (Smith & Stein, 2018; Stein et al., 2008) provide a framework for supporting these instructional decisions: anticipating student responses, monitoring strategies during class, selecting contributions to highlight, sequencing them to build toward conceptual understanding, and connecting responses to key mathematical ideas. Questioning is central to each of these practices, both in assessing students' current thinking and in advancing their reasoning (Smith et al., 2017). However, research also shows that without targeted professional development, many GTAs continue to rely on procedural questions and extended explanation rather than orchestrating open-ended discourse, underscoring the need for sustained support (Paoletti et al., 2018; Roach et al., 2010). Thus, PD is essential in supporting GTAs from moving beyond procedural questioning toward facilitating mathematical discourse.

Although GTAs often possess strong mathematical backgrounds, content expertise alone does not ensure effective pedagogical practices (Begle, 1979; Musgrave & Carlson, 2017). PD programs vary widely and range from brief orientations to semester-long seminars (Deshler et al., 2015; Rogers & Yee, 2018). Sustained PD models emphasize iterative cycles of planning, instructing, and reflecting (Kung & Speer, 2009) as well as peer observation and mentoring

(Deshler et al., 2015). Recent recommendations highlight three priorities for GTA PD: fostering student engagement, developing knowledge of student thinking, and providing ongoing, sustained pedagogical training for GTAs (Maki et al., 2019). The seminar at the center of this study was designed with these principles in mind, engaging GTAs in planning cognitively demanding tasks, anticipating student strategies, and developing purposeful questions.

Methods

This paper reports a sub-study from a larger convergent mixed-methods investigation of GTAs teaching College Algebra. The larger project integrated qualitative (seminar recordings, reflective journals, interviews, planning monitoring charts, classroom transcripts) and quantitative (observation protocols, student surveys) data to examine GTA instruction and student perceptions. This paper focuses specifically on lesson structure across the semester using the Classroom Observation Protocol for Undergraduate STEM (COPUS; Smith et al., 2013) and a focal case analysis of one GTA, Anthony, a second-semester master's student. This was Anthony's second time teaching College Algebra. Other data sources are referenced to provide context for interpretation.

Context and Participants

This study was conducted at a large public university during one semester of a weekly professional development seminar, the Planning for Approaches to Teaching High-demand tasks (PATH) seminar, for GTAs serving as instructors of record for College Algebra. In that semester, twelve GTAs were assigned to teach College Algebra, and six volunteered to participate in this study. All participants were master's students in mathematics and were in their second semester of teaching. Each had previously completed a first-semester seminar on teaching mathematics, where the researcher was their instructor of record. Participation in the PATH seminar was voluntary, and GTAs who participated earned two credit hours and a $200 stipend provided by the mathematics department.

PATH Seminar Overview

The PATH seminar emphasized planning cognitively demanding, group-worthy tasks, anticipating student strategies, and crafting both assessing and advancing questions aligned with the five practices (Smith & Stein, 2018). Each week, GTAs engaged in a four-part cycle: (1) working through a focus task (adapted from Kuennen, 2021) as learners, (2) recording anticipated strategies and potential questions on a monitoring chart (Smith et al., 2020), (3) implementing the task the following week in one of their class sessions and (4) reflecting after implementation of the task on unanticipated student strategies. GTAs were expected to incorporate each focus task into their class the following week and share reflections during the subsequent seminar session.

Data Collection

Each GTA was observed three times during the semester, at the beginning (weeks 1–3), middle (weeks 7–8), and end (weeks 13–14), in their 80-minute College Algebra class. Observations were video recorded with the camera focused on the instructor and GTAs wore microphones to capture audio during student work. COPUS (Smith et al., 2013) was used to code observations minute-by-minute for both instructor and student actions.

In addition to the classroom observations, GTAs completed monitoring charts (Smith et al., 2020) for the seminar focus tasks. Monitoring charts asked instructors to anticipate a range of

student strategies, potential misconceptions, and plan assessing and advancing questions they might pose during instruction. The charts also included space to document which students used each approach and how to order student strategies in whole-class discussions. These documents provided a window into GTAs' planning and in-the-moment noticing of student thinking. A blank sample is provided in Table 1 to illustrate the structure of this planning tool.

Table 1. Sample Monitoring Chart

Strategy	Assessing Questions	Advancing Questions	Who and What	Order
Anticipated Strategy 1				
Anticipated Strategy 2				
…				

Finally, each GTA participated in three semi-structured interviews following their observed lessons. Interviews followed a common protocol designed to elicit GTA reflections on planning, classroom interactions, questioning, and student engagement.

Data Analysis

All classroom observations were initially coded with the full set of COPUS categories. For interpretability and comparison across lessons, codes were collapsed using the COPUS categories for instructors (Presenting, Guiding, Administration, Other) and for students (Receiving, Talking, Working, Other). Proportions were calculated for each category to generate a profile of lesson structure for each GTA across the three observations. For more details on these categories and individual codes, see Smith et al. (2013).

Monitoring charts were analyzed descriptively with attention to the strategies GTAs anticipated, the assessing and advancing questions they prepared, and the number of students noted for each approach. Particular attention was given to alignment (or misalignment) between GTA planning and classroom enactment.

Although COPUS and monitoring charts were collected for all six participants, this paper highlights Anthony's case. Anthony was selected because his trajectory illustrated both common challenges and noteworthy shifts in practice across the semester, making his case illustrative of the tensions GTAs experience in balancing instructional demands. The analysis traced changes across his three lessons, with attention to how his distribution of presenting, guiding, and student engagement shifted over time. To interpret these shifts, Anthony's interview reflections were triangulated with his monitoring charts and COPUS data, providing multiple perspectives on his planning, enactment, and instructional decisions.

Results

COPUS: Lesson Structure Across Observations

Table 2 summarizes the data for Anthony's three observations on the collapsed COPUS categories. With the COPUS data, multiple codes can occur within the same minute, so instructor and student percentages do not add to 100%.

Table 2. Proportion of Class Time for Anthony's Observations

Observation	Instructor				Student			
	Presenting	Guiding	Admin	Other	Receiving	Talking	Working	Other
1	73%	63%	5%	2%	73%	32%	27%	0%
2	71%	95%	7%	0%	73%	51%	27%	0%
3	17%	88%	5%	0%	2%	0%	98%	0%

Anthony's first two observations were dominated by presenting, with nearly three-quarters of class time coded as instructor presenting and students receiving information. Although guiding was coded relatively high (63% and 95%), this largely reflected procedural questions posed during lecture rather than student-centered activities. In contrast, Anthony's third observation reflected a shift in his instructional practices. He devoted only 17% of class time to presenting, while student work jumped from 27% to 98% of the class period. Guiding remained high (88%) in this lesson, which reflected his monitoring during group work and one-on-one questioning rather than procedural questions during lecture. Students' time receiving information dropped from 73% to 2%, highlighting the move from lecture-driven to student-centered instruction.

Monitoring Charts

Anthony's monitoring charts offered insights into how his planning aligned (or failed to align) with enactment. In his first observation, which focused on solving quadratics using multiple approaches, Anthony anticipated that "no help was needed," assuming that students would work through the task independently. However, his after-lesson notes indicated that students struggled to recall certain solution methods, and he had to provide additional support.

By the second observation, on graphing quadratics and generalizing transformations, Anthony anticipated that students would copy the x-values from $y = x^2$ (e.g., $x = -2, -1, 0, 1, 2$), and not focus on the vertex of other functions such as $y = (x-4)^2$. This did occur with some students, though in his reflections, he noted that limited time prevented many from reaching generalizations about parameters.

In his third observation, comparing exponential and linear growth models, Anthony's expectations narrowed, with his monitoring charts predicting that students would "get stuck". His reflection notes, however, documented a wider range of strategies than he had anticipated: students mistaking the exponential function for linear, attempting to compute initial population values, and experimenting with equivalent exponential functions. He also critiqued one of his planned questions, "How can we get the function?" as "terrible," noting that he revised in the moment, instead asking about patterns in the data, drawing on questioning strategies discussed in the PATH seminar the previous week. This shift illustrates both his persistent challenge in anticipating strategies and his growth in adapting questions in-the-moment.

Interview Reflections

Anthony's first interview sheds light on his rationale for lecture-dominated instruction at the beginning of the semester. Drawing on his experience, he explained:

> Last semester, the students actually were okay just reading the [ALEKS] explanation and trying to solve whatever they could. But I noticed not all of them would do that. Some of them just played on their phones or did other assignments, and that's not ideal.

As a result, Anthony saw explanations from him during lecture as "safe" structure: "Even if no one pays attention, at least they are hearing the solution… it feels like the safe side." While he

worried about being "too direct" at times, he valued the sense of security and coverage provided by lecturing.

Anthony's second interview revealed both his rationale for maintaining lecture at the start of class and reflected concerns about pacing and control: "I don't like them to start and not finish what they started… that's one of the reasons I decided to start explaining before giving them the paper." Yet, Anthony admitted that only "four or five" of the twelve students followed during lecture, while others "just ignore the lesson and try to work on something else." To address this, he focused his questions on computational struggles, like squaring negatives. He thought his questioning was a mix of single answers and explanations. Anthony tied his instructional decisions to coordinator and student feedback because he believed that this approach improved students' completion rates of the homework but constrained time for in-class exploration.

In his final interview, Anthony described a noticeable shift in his instructional approach. Having already lectured on prerequisite content in a prior class, he devoted nearly the entire session to the focus task on modeling exponential and linear population growth: "I wanted them to have the biggest chance to work on [the focus task] or figure it out." Although he was still uncomfortable with this and felt students might view his circulating as "lazy," but argued it was more demanding: "Maybe they think I'm lazy… but actually it's a lot more difficult, because you need to follow the students." He noted moments of unanticipated reasoning, such as when students connected exponential growth to the growth of world populations, or constructed different but equivalent exponential functions. He intentionally allowed them to pursue their own approaches before discussing alternatives: "Even if their ideas are really complicated, I like to let them approach the problem with the same idea they have in mind… after they are done, they will be happy because they were able to get it."

Taken together, the COPUS data, monitoring charts, and interviews illustrate Anthony's shift from lecture-dominated to student-centered instruction. Initially underestimating student needs and relying on his explanations, he gradually became more responsive, devoting more time to the focus tasks and adapting his questioning in-the-moment. At the same time, his reflections highlight ongoing tensions between content coverage and study inquiry, shaped not only by his own beliefs but also by coordinator and student feedback.

Discussion

This case study illustrates how Anthony's teaching was shaped not only by his developing knowledge of teaching but also by contextual pressures that influenced what felt possible in practice. Anthony's case demonstrates both growth and limits in his knowledge of discourse (Hauk et al., 2014). A central theme was the tension between Anthony's prior beliefs and his enacted practice. Early in the semester, he defaulted to lecture as a "safe" structure, believing that at least students would hear the solution even if they were not fully engaged. Additionally, Anthony struggled with anticipatory thinking, as seen in his first monitoring chart, where he assumed that "no help was needed." His enactment of discourse was also constrained by contextual factors. Coordinator comments about students disliking independent work and low attendance reinforced Anthony's perception that lecture was safer than leaving time for group work. These findings illustrate how contextual pressures can narrow the enactment of pedagogical content knowledge, even when instructors hold student-centered beliefs.

In the middle of the semester, Anthony attempted to integrate questioning but continued to focus on computational details rather than open-ended reasoning. His reflections about asking "mid-level" questions suggest some development in implementation thinking as he considered the role of the questions he posed. While he recognized questioning as a tool to support

engagement, orchestrating whole-class discussions remained difficult to sustain within pacing constraints. This highlights the ongoing challenge that ambitious pedagogy requires both time and tolerance for ambiguity, which Anthony felt constrained by (Stein et al., 2008).

By the final observation, Anthony's practice shifted more substantially toward student-centered instruction. Students spent most of the class working on the focus task while he circulated, adapting his questions in real time. This represented growth in implementation thinking, as he revised a planned question he later critiqued as "terrible" and instead drew students' attention to patterns in the data. His reflection that circulating was "more difficult" than lecturing reveals a developing awareness of the demands of facilitation. At the same time, his anticipatory thinking remained limited. Even in his third monitoring chart, he predicted only that students would "get stuck," failing to anticipate the diversity of strategies that later emerged. Thus, while Anthony demonstrated growth in adapting instruction in the moment, he still had room to grow in predicting and planning for a range of student strategies.

Taken together, Anthony's case demonstrates the interplay between knowledge of discourse and contextual pressures. While he developed greater responsiveness to student thinking across the semester, his ability to enact ambitious pedagogy was mediated by coordinator expectations, pacing demands, and concerns about student perceptions. This aligns with prior research showing that GTAs often struggle to align beliefs with practice when institutional forces pull in other directions (Deshler et al., 2015; Kung & Speer, 2009).

For mathematics departments, this case underscores the importance of supporting not only GTA pedagogical knowledge but also the contexts in which GTAs teach. Professional development should continue to emphasize frameworks such as the five practices (Smith & Stein, 2018) and knowledge of discourse (Hauk et al., 2014), while also explicitly addressing the institutional and cultural pressures that shape enactment. Coordinators should recognize how their feedback is interpreted and ensure that GTAs understand the broader pedagogical goals of the course. Without this alignment, GTAs may feel pressured to prioritize visible coverage over fostering mathematical discourse.

As a single case study, these findings are not generalizable to all GTAs or contexts. Anthony's reflections are shaped by his experiences, the particular coordinator he worked with, and the structure of his course. Nevertheless, the triangulation of COPUS data, monitoring charts, and interviews provides a rich account of the complex interplay between knowledge and context in GTA teaching. Future research could build on this case by examining how GTAs' practices evolve across multiple semesters of teaching, particularly on how they balance lecture with student-centered instruction. Additional studies might also investigate the pressures GTAs navigate, including departmental expectations, student feedback, pacing demands, and course structure, and how these interact with their development of anticipatory and implementation thinking. Such research could inform the design of PD and expectations that not only build knowledge of teaching but also alleviate the contextual constraints that can limit GTA growth.

Anthony's case underscores the complex reality of GTA teaching. While he entered the semester valuing student-centered learning, his practice was heavily mediated by contextual pressures. Only when schedule and content afforded greater flexibility did he more fully enact student-centered approaches. This finding highlights a broader lesson for GTA development: pedagogical knowledge cannot be understood in isolation from the institutional structures and feedback systems that shape practice. To support GTAs in enacting ambitious, student-centered teaching, PD must address both pedagogical tools and the contextual constraints that frame classroom decisions.

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Insights into Student Perceptions of Learning Assistants in Calculus I: A Mixed-Methods Study Involving Survey and NLP Analysis

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Increasingly, institutions of higher education utilize undergraduate Learning Assistants (LAs) to support students' learning in introductory calculus courses. Prior research has shown a promising impact of the LAs on students' success and retention. However, little is known about students' perceptions of LAs' roles or whether academic factors influence these perceptions. In this mixed-methods study, we analyzed 411 Calculus I students' survey responses. The results of statistical analysis indicate that, regardless of course performance or prior Calculus experience, students generally perceive LAs positively, with students who seek additional learning resources having even more favorable perceptions of LAs. Additionally, a natural language processing (NLP) technique– topic modeling—revealed two themes: Investigating Procedural Activities and Conceptually Framing Results through Collaboration. Findings support LAs' role in fostering meaningful engagement for a range of students and demonstrate the value of NLP in educational research, offering insights for LA support of active learning in Calculus courses.

Keywords: Learning Assistants, Calculus, Student Perceptions

Improving students' learning experiences in university Calculus courses has been at the forefront of many reform efforts by higher education institutions (Bressoud, 2020; Rasmussen et al., 2014). Some reform approaches involve studio course designs, group problem-solving, and personal response systems, all of which emphasize active learning. Active learning is broadly conceptualized as encompassing pedagogical approaches that center student contributions and are often viewed as the antithesis of lecturing (Freeman et al., 2014). The *Learning Assistant (LA) model* is one approach to implementing active learning that has been shown to support students' learning, engagement, and retention in the introductory STEM courses (Barrasso & Spilios, 2021). LAs are undergraduate near-peer tutors who have previously completed the course they LA for and receive pedagogical training. LAs are present in the classroom alongside an instructor and support students' learning and interactions in small groups (Otero et al., 2010).

While it has been shown that students in LA-supported courses have better content-related and affective outcomes than their peers without such support (e.g., Alzen et al., 2018), little is known about how students perceive the role of their LA within the classroom. Understanding students' perceptions of LAs' classroom roles can enhance our understanding of how to support student success in STEM courses, since students' perceptions of the course are closely linked to their persistence in STEM (Ellis et al., 2014). Recent studies have examined variation in LA-student interactions (e.g., Carlos et al., 2023) and LAs' reflections on these interactions (Butler & Buchbinder, 2023). However, students' perspectives of these interactions remain largely unexplored, including whether certain groups of students find LAs particularly helpful or not, in other words, "who benefits from interactions with LAs." Our study aims to address this gap.

Specifically, we investigate whether there are differences among Calculus students' perceptions of their interactions with LAs with respect to such variables as: final course grade, previous Calculus experience, and students' pursuit of extracurricular resources. We approach this topic through a mixed-methods analysis of Calculus I students' end-of-course survey responses about their classroom interactions with LAs. Our research questions are:

1. Do students' perceptions of their LAs differ based upon their final course grades, previous calculus experience, or utilization of resources outside of the classroom?
2. How do students describe their classroom interactions with LAs?

Prior to addressing our research questions, we conducted a psychometric analysis of the survey instrument to evaluate its validity, reliability, and overall measurement quality.

Theoretical Framing

We frame LA-student interactions within a Vygotskian sociocultural perspective on learning (Vygotsky, 1978). This perspective holds that substantial learning occurs through social interactions of individuals with a more knowledgeable other (MKO). We conceptualize LAs as MKOs since they have previously taken Calculus I at the institution of study and are tasked with aiding student learning through social engagement. This lends theoretical legitimacy to the notion that students' perspectives on their LAs' classroom roles may be distinct from how they perceive other university instructors due to differences in social and power dynamics involved.

To guide the design of survey items (described in the methods section), we utilized positioning theory and social support theory. Positioning theory attends to the ways in which individuals take up or reject broader social roles by describing the "types of rights and duties people draw on as they interact" (Herbel-Eisenmann et al., 2016, p. 103). Survey items were formulated with the aim of understanding how students perceive the "rights and duties" of their LAs within the context of a Calculus I classroom. Social support theory was utilized to further specify the specific "rights and duties" represented in these items. Social support is a well-established framework from psychology that describes one's perception that they are cared for by others (Taylor, 2011). Within the context of mathematics education, previous research has shown positive connections between levels of social support and learning-related constructs such as ability perception, self-efficacy, and grades across a variety of contexts (Monteiro, 2017).

Social support theory has been previously utilized to investigate students' perceptions of their interactions with LAs (e.g., Donis et al., 2024; Hernandez et al., 2021). These studies identified the types of social support that students perceive from LAs (emotional, informational, appraisal), and that this social support is positively related to student engagement.

Topic Modeling

To analyze open-ended students' responses, we used *topic modeling*, a natural language processing (NLP) method that utilizes unsupervised machine learning and statistical methods to categorize large corpora of text data, like open-ended survey responses (Finch et al., 2018). We utilize the Latent Dirichlet Allocation (LDA) approach to topic modeling, which generates collections of words, called topics, that frequently appear together (Chauhan & Shah, 2021). Each word within a topic has an associated probability indicating how strongly it is associated with a particular topic. From these probabilities, the proportion of each topic within each open response is determined. Thus, a single response may contain a mixture of topics, reflecting the complexity of natural language (Silge & Robinson, 2017). While the association of words with a topic is determined quantitatively by NLP, the themes underlying each topic are determined by the researcher qualitatively, considering the nature of the data (Hwang et al., 2023).

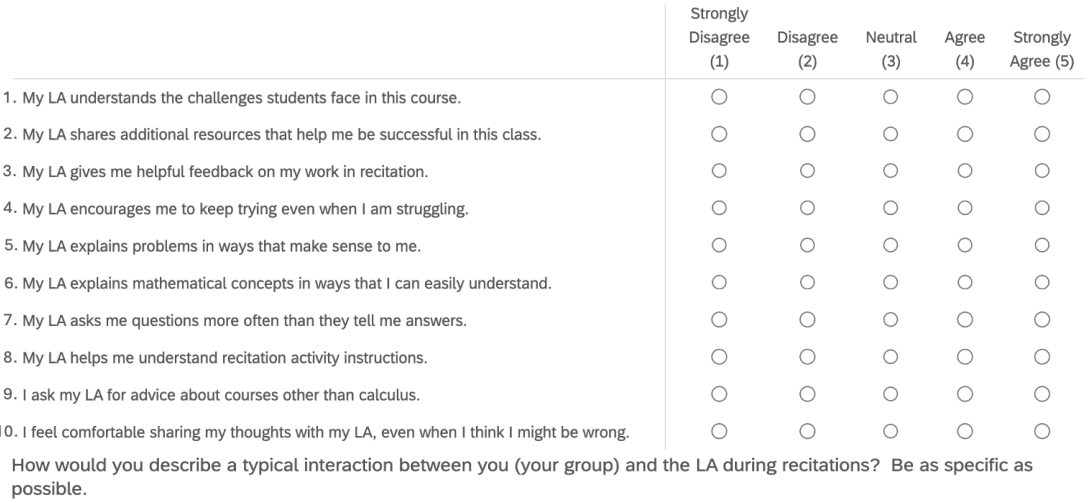
Topic modeling is a novel technique that has recently been applied in mathematics education research (Gökdağ & Özmantar, 2024; Copur-Gencturk et al., 2023). We illustrate how this novel technique can be used to analyze Calculus student survey responses.

Methods

Setting and Data Collection

This study was conducted at a large, public university in the northeastern United States. It is one component of a larger NSF-funded project that aims to transform introductory STEM courses in biology, chemistry, mathematics, and physics through active learning pedagogies. Within the mathematics department, these transformations began with the Calculus I recitations. Calculus I course follows a lecture-recitation format, where students attend a large lecture (~ 150 students) three times weekly and a smaller recitation section (~ 20 students) twice weekly. In the lecture, a faculty member presents information via traditional lecture methods. Recitations are facilitated by a graduate teaching assistant (GTA) and an undergraduate LA who jointly facilitate students' problem-solving activities in small groups. GTAs have a more authoritative role, as they are responsible for assigning groups, grading quizzes, and reviewing content.

To capture students' perceptions of the course, the research team developed a student survey and piloted it for two semesters. The items were adapted from the SEMINAL project's Student Postsecondary Instructional Practices Survey for Mathematics (Uhing et al., 2020). The survey comprises 35 Likert-scale items, scored on a 5-point scale, and six open-response items. Of those, ten Likert items and one open-response item specifically focused on students' perceptions of LAs' roles in recitations (Figure 1). Items 1-4 and 8 were sourced from Hernandez et al (2021); others were developed by the research team to capture nuances of LA-student interaction in Calculus recitations.



	Strongly Disagree (1)	Disagree (2)	Neutral (3)	Agree (4)	Strongly Agree (5)
1. My LA understands the challenges students face in this course.	○	○	○	○	○
2. My LA shares additional resources that help me be successful in this class.	○	○	○	○	○
3. My LA gives me helpful feedback on my work in recitation.	○	○	○	○	○
4. My LA encourages me to keep trying even when I am struggling.	○	○	○	○	○
5. My LA explains problems in ways that make sense to me.	○	○	○	○	○
6. My LA explains mathematical concepts in ways that I can easily understand.	○	○	○	○	○
7. My LA asks me questions more often than they tell me answers.	○	○	○	○	○
8. My LA helps me understand recitation activity instructions.	○	○	○	○	○
9. I ask my LA for advice about courses other than calculus.	○	○	○	○	○
10. I feel comfortable sharing my thoughts with my LA, even when I think I might be wrong.	○	○	○	○	○

How would you describe a typical interaction between you (your group) and the LA during recitations? Be as specific as possible.

Figure 1. LA-related Survey Items

The survey was administered during a recitation in the last two weeks of the course by researchers while the GTA and LA were out of the room. Data were collected for three semesters, across 37 different recitation sections staffed by 10 different GTAs and 7 different LAs. Of the 606 students enrolled in the course during these three semesters, 411 consented to have their responses analyzed (~68% response rate). The authors of this paper were researchers on the project and had no teaching or administrative roles in the courses studied.

Instrument Validity and Reliability

To determine the validity of these ten items as measuring the single construct of students' perceptions of LAs' classroom roles, we used exploratory factor analysis (EFA). EFA is a

multivariate statistical method that identifies latent variables in a set of directly measured variables (Finch, 2020). To ensure that this data set was suitable for the application of EFA, we conducted Bartlett's Test of Sphericity (showed significance at $p<0.001$) and the Kaiser-Meyer-Olkin Measure of Sampling Adequacy (returned values for each item above 0.89), indicating suitability of EFA (Shrestha, 2021). The EFA revealed that the instrument measures a single latent variable, with item loading factors ranging from 0.5 to 0.9, indicating moderate to strong loading (Batterton & Hale, 2017). The survey reliability was acceptable across measures, with Cronbach's alpha (0.90), split-half reliability (0.82), and McDonald's omega (0.94) all exceeding the recommended 0.7 threshold (Streiner & Norman, 2008). All psychometric analyses were conducted using the R packages *psych* and *GPArotation* (Revelle, 2017).

Analytic Procedures

A non-parametric approach was selected for this study due to the ordinal nature of the data and non-normal distribution of responses (see Table 1 in the Results section). A Bonferroni correction was applied to the Mann-Whitney p-value because multiple tests were run on the data, and to reduce the risk of Type I error (Armstrong, 2014). In the case of a significant difference ($p < 0.1$), Cliff's Delta was computed as a post-hoc test to determine effect sizes (Meissel & Yao, 2024). These statistics were computed using the R package rstatix (Kassambara, 2019).

Since some students in the sample had the same LA, both within and across the semesters, the sample could be considered as LA-dependent. However, after aggregating the data by LA, the mean difference between LAs across all ten items was 0.61, which is relatively minor for our Likert scale. This suggests that students' overall perceptions did not differ dramatically across LAs and that the sample can be considered holistically.

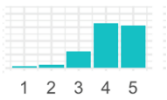
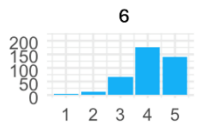
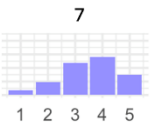
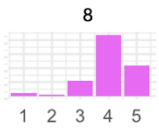
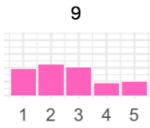
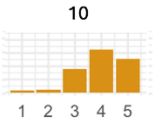
To perform topic modeling on the open-response item, we first conducted pre-processing, or text cleaning (Hvitfeldt & Silge, 2021). First, we removed stop words, which are parts of speech that typically carry little substantive meaning (e.g., a, the, did, or), using three R packages: *SpacyR*, *tidytext*, and *dplyr*. Next, we used the *SnowballC* package to reduce the remaining words to their root forms (Welbers et al., 2017), i.e., the words "explains", "explained", "explaining" are all reduced to their common stem, "explain." After text cleaning, we used the R package *ldatuning* to find the optimal number of topics in our dataset by selecting the number of topics that minimized the CaoJuan2009 metric (Nikita & Chaney, 2022). This resulted in two topics. We then used the R package *topicmodels* to construct the topic model with two topics and an alpha hyperparameter of 0.1 (Grün & Hornik, 2011). This low alpha indicates that topics are sparsely distributed across responses, aligning with our aim to capture features of LA-student interactions at the individual response level (Du, 2022).

Results

Likert Scale Item Analysis

For ease of communication, we refer to the set of 10 LA-related items in Figure 1 as the LA survey. Table 1 shows descriptive statistics for all items in the LA survey, indicating that students generally agreed with most survey items. This suggests that students' views of LAs' classroom roles were positively oriented. The exception to this trend is item 9, with a mean of 2.59 and a mode of 2, indicating relative disagreement with the statement: "I ask my LA for advice about courses other than calculus." Although this role was not explicitly required of LAs, the item was included in the survey to gauge whether LAs act as peer-mentors beyond the calculus course. Some students reported engaging with LAs in this way; most students did not.

Table 1. Descriptive Statistics of Student Responses to LA-Related Likert Scale Items

Item	1	2	3	4	5
					
Mean	4.18	3.69	4.16	4.15	4.14
Standard Deviation	0.82	0.98	0.84	0.84	0.88
Median	4	4	4	4	4
Item	6	7	8	9	10
					
Mean	4.09	3.52	4.04	2.59	3.98
Standard Deviation	0.86	1.07	0.83	1.29	0.90
Median	4	4	4	2	4

To explore whether students' perceptions of LAs differ by their final course grades, student responses to the LA survey were split into two groups: 238 high performers—students whose final course grade was B- or better, and 173 general/low performers—the remaining students. Mann-Whitney tests with a Bonferroni correction were conducted to determine whether these two groups responded differently to each of the ten survey items. No significant differences at the p=0.1 level were found, meaning that our data did not provide evidence that students' perceptions of their LAs varied by course performance.

To determine whether perceptions of LAs varied by previous experience with Calculus, student responses were divided into two groups: 174 students who self-reported having previous calculus experience and 237 students without such experience. Again, Mann-Whitney tests with a Bonferroni correction showed no significant differences between the groups at the p=0.1 level.

Next, we divided student responses into two groups based on their self-reported use of learning resources outside the course, such as the university mathematics tutoring center, online videos and other sources. There were 315 students who reported using such resources versus 91 students who did not. Mann-Whitney tests with a Bonferroni correction identified significant differences between these two groups only for items 2, 3, 4, 7, and 10 (Table 2). Since significant differences were found for these items, Cliff's Delta was computed post-hoc to determine the effect size. The results suggest that students who tend to seek outside help have more positive perceptions of LAs' roles, captured by these five items, compared to students who do not seek outside help. However, the effect sizes for all items are small (Meisel & Yao, 2024).

Table 2. Statistics for Students Who do and do not Utilize Outside Resources

Item Number	Mean Response from Students Seeking Help	Mean Response from Students not Seeking Help	Adjusted *p-value*	Cliff's Delta
2	3.77	3.42	0.00417*	0.231
3	4.21	3.97	0.0230*	0.195
4	4.2	3.96	0.0235*	0.195
7	3.61	3.22	0.00746*	0.224
10	3.82	4.03	0.0897**	0.171

Topic Modeling

Topic modeling of student responses to the open-ended question "How would you describe a typical interaction between you (your group) and the LA during recitations? Be as specific as possible" revealed two topics. Figure 2 shows the 20 words most strongly associated with each topic. For clarity, the words are displayed in full, rather than stemmed (e.g., solve instead of solv, easy instead of easi). The horizontal axis shows the probability of the word appearing within the topic; higher values indicate that a word is more strongly associated with a topic.

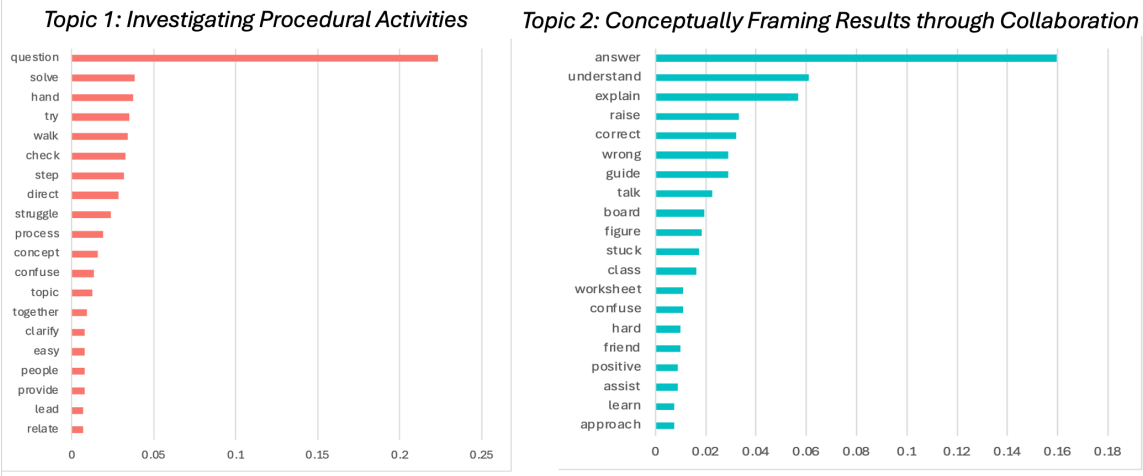


Figure 2. 20 Most Common Words in Each Topic

The first topic (Figure 2, left) includes words like *solve*, *step*, *process*, reflecting a focus on procedural aspects of Calculus. The word most strongly associated with this topic is *question*. Although it is unclear who is asking these questions (i.e., the LA or the student) or if it refers to questions on the worksheet, this word may indicate some procedural focus. The presence of words like *check*, *direct*, *provide*, *clarify*, and *lead* point to the utilization of the LAs' expertise to validate these procedural foci. The presence of the words *try* and *struggle* in this topic suggests some aspects of student difficulty or effort involved in these procedures. For example, in one response that is strongly associated with this topic, the student described "asking [LAs] specific questions when struggling or ask to have work looked over for errors and constructive criticism." Overall, the terms in this topic signal an engagement with the completion of mathematical procedures; thus, the topic was termed *Investigating Procedural Activities.*

The second topic (Figure 2, right) contains terms such as understand, explain, learn, approach, and figure (in the sense of a graph or a sketch), indicating a more conceptual focus within the topic, in contrast to the procedural focus of the previous topic. This topic also contains the terms answer, correct, and wrong, meaning that conceptual discussions were still centered on the correctness of mathematical outputs. This topic also contains words with a strong community orientation, such as guide, talk, board, class, friend, and assist, suggesting collaborative efforts during LA-student interactions. A sample response strongly associated with this topic is: "we work on a worksheet with groups and then we call [the LA] over if we are stuck and she guides us towards the right answer." We termed this topic Conceptually Framing Results through Collaboration since the frequent terms appear to represent outcome-oriented discussions between LAs and students.

Due to the nature of topic modeling, these topics are not necessarily mutually exclusive, meaning that students could describe their interactions with LAs as related to both topics. Our analysis reveals that there are two primary ways in which our students view their interactions

with LAs. Some aspects of interactions focus on dissecting specific procedural steps, while others are centered on communal engagement and understanding of outcomes.

Discussion

Statistical analysis of the LA survey revealed that across three semesters of data, *students had relatively positive perceptions of their interactions with LAs*, regardless of the LA. Also, there were no statistically significant differences between groups with high and general/low course performance, nor between groups of students with or without prior calculus experience. We consider this lack of statistical significance as *a positive result,* as it suggests that *all students viewed LAs' classroom presence as beneficial to them*: the struggling students, and the ones with a mastery of the material, those who encountered calculus for the first time and those with prior exposure to it. Statistically significant differences were only found for five Likert-scale items between students who do and do not report utilizing outside resources to aid their coursework. *Students who reported using such resources indicated slightly more positive perceptions of LAs' classroom roles*. This could be because students who seek additional learning resources tend to view LA as one such resource and were more comfortable with or proactive in obtaining LAs' assistance with their mathematical work.

The topic model indicated two primary ways in which students framed their interactions with LAs, which we termed *Investigating Procedural Activities* and *Conceptually Framing Results through Collaboration*. These two topics could provide insight into the reasons behind the often-reported student success in LA-supported courses (e.g., Alzen et al., 2018), specifically that LAs engage with both *students' procedural questions and conceptual discussion of mathematical results.* However, additional work is required to establish this connection.

Our findings align with the studies showing students' satisfaction with their interactions with LAs in large-lecture STEM classrooms (Donis et al., 2024; Hernandez et al., 2021). We extend this literature to the context of Calculus recitation sections led by GTAs. Since the instructional and power dynamics of this environment differ from those of a large lecture, establishing positive student experiences in this context is an important result. The results of this study are also relevant to the broader context of calculus reform, offering support for the LA model's impact as a transformation effort. Our results suggest that these efforts can reach students from a range of different mathematical backgrounds and achievement levels. Although focused on LAs, these findings also inform GTAs, especially those without prior pedagogical experience, by illustrating effective instructional practices. The identified themes—*Investigating Procedural Activities* and *Conceptually Framing Results through Collaboration*—offer concrete examples of valuable classroom support that can be highlighted in both LA and GTA training.

We also offer a *methodological contribution, illustrating the power of NLP tools in analyzing open-ended survey responses.* The use of topic modeling allowed us to broadly describe trends across a large volume of open-ended responses without manual open coding, but with the input and supervision of a researcher. As the use of NLP in education research continues to grow, developing best practices for the validation and interpretation of these automated analyses will be crucial to ensuring their appropriate use in understanding student experiences.

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Queer Readings of Mathematical Proof: Appropriating Norms in the Pursuit of Epistemic Justice and Queer Joy

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The enculturation of mathematics students into the mathematical norms of proof can be a source of epistemic injustice when personal, social, and cultural norms are in tension with those held by the mathematics community. This injustice can be addressed through students' conscious appropriation of these norms, and I argue that queer readings, a form of critical textual analysis originating in queer theory, offers relevant analytical tools for this appropriation. In this paper, I present findings from a study on queer readings of mathematical proof and illustrate how our queer appropriations of mathematical norms resulted in an epistemically just interaction with proof and invited queer joy into the proof reading process.

Keywords: Queer pedagogy, Epistemic justice, Proof, Proof reading

Many scholars have examined student and expert approaches to proof reading (Dawkins & Weber, 2017; Inglis & Alcock, 2012; Panse et al., 2018) in an effort to understand the "interactional accomplishment" (Weber & Mejía-Ramos, 2014, p. 90) between reader and text. However, this research often functions under an apprenticeship model where learners are expected to develop the same norms for proof as are held by professional mathematicians (Dawkins & Weber, 2017). As Tanswell and Rittberg (2020) explain, the enforcement of these norms during the learning process can create systemic disadvantages; "mathematics learners are not little research mathematicians and demanding that they operate in the same ethical order is a source of epistemic injustice" (p. 1209). Tanswell and Rittberg do not propose that the norms of proof necessarily need to shift, but that learners should "appropriate norms in order to experience the values these norms are intended to uphold" (p. 1207). In other words, mathematics students should be encouraged to explore and push back against norms of proof so that they may find when those norms are in tension with their personal background and understand why these norms are ubiquitous within mathematics. I argue that queer reading is a pedagogical method well suited to this appropriation. In this paper I investigate: [1] How do queer readings of proof encourage readers to interact with proof in epistemically just ways? [2] How does queer joy emerge during queer readings of proof?

Theoretical Perspectives

My work is grounded in the theoretical relationship between epistemic injustice and norms of proof as presented by Tanswell and Rittberg (2020), and in my conceptualization of queer readings as a proof reading strategy which encourages queer joy and epistemic justice.

Epistemic Injustice and Acontextual Norms of Proof

Epistemic injustice is a philosophical framework popularized by Fricker (2007) which exists at the intersection of epistemology, politics, and ethics. Epistemic injustice is used to discuss how systemic inequities, power dynamics and social relations negatively "affect humans…in their roles as epistemic agents" (Tanswell & Rittberg, 2020, p. 1200). For example, epistemic injustice highlights that when curricula fail to recognize Indigenous ways of knowing they discredit a culture's intellectual traditions (Pohlhaus, 2017).

Dawkins and Weber (2017) claim that norms of proof uphold four epistemological values central to mathematics. Of concern to Tanswell and Rittberg (2020), and therefore to this paper, is the second epistemological value proposed by Dawkins and Weber (2017) which states that "mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author" (p. 128). Tanswell and Rittberg (2020) argue that the norms for proof upholding this epistemic value can perpetuate epistemic injustice in classroom environments by discouraging references to the agency of either reader or author and ignoring the collaborative and less formalized contexts in which proofs are frequently read and created.

Queer Readings of Proof

Reading strategies can be described as "goal-directed actions which are undertaken by readers for planning and monitoring their efforts in order to decode a text, understand words, and construct meanings of texts" (Alexander & Judy, 1988; Pereira-Laird & Deane, 1997; as cited in Yang, 2012, p. 309). The goal of proof reading is often proof comprehension or proof validation (e.g. Dawkins & Zazkis, 2021; Hodds et al., 2014; Yang, 2012) and proof reading strategies include self-explanation (Hodds et al., 2014), summarization (Davies & Jones, 2022), and attempting to prove a theorem before reading its proof (Weber, 2015). Given proof reading strategies include any activity which readers use to "decode a text" and "construct meaning" (Yang, 2012, p. 309), they can be extended to include annotation, construction and deconstruction. For this reason, I expand queer readings into the field of mathematics as a proof reading strategy.

Queer reading is a pedagogical practice originating in queer theory which intentionally explores the "interpretive performance" (Britzman, 1995, p. 222) between reader and text while attending to a text's hidden meanings, implicit messages, and cultural, political or societal norms (Björklund, 2018; Britzman, 1995; M. Morris, 1998). Common misconceptions of queer readings is that they require queer intentions on the part of the author and/or that they must center sexuality and gender. However, many queer theorists argue that queer readings "can be just as useful in analyses of how other norms are produced in texts" (Björklund, 2018, p. 9). Consequently, queer readings of proof may invite readers to critically examine the structure and nature of a proof's argument(s), address the implicit relationship(s) between reader, author, and mathematical content, and to critique and appropriate norms of proving.

Epistemic Justice and Queer Joy

As an extension of the argument by Tanswell and Rittberg (2020), I propose that epistemically just interactions with proof are those which allow learners to appropriate norms of proof valuing acontextuality by introducing social or mathematical context, referencing author and/or reader agency, and valuing casual language and collaborative efforts. Given queer readings are strategies that explicitly work to uncover hidden meanings, consider implicit messages, and monitor norms present within a text, my work is grounded in the theoretical idea that queer readings of proof provide opportunities for readers to appropriate proof in epistemically just ways. Additionally, I consider queer joy to be a source of epistemic justice specific to members of the queer community.

Queer joy counters narratives of oppression, abuse, and isolation that are common within discussions of the queer community (Morris et al., 2022) by promoting stories of resilience, joy, and power. In educational research, Duran and Coloma (2023) invite researchers to prioritize queer joy in order to "render LGBTQ individuals as complex vibrant beings, underscore the educational and lived conditions that accept and embrace them, and prioritize our happiness,

wellbeing, and the actualization of our full potential" (p. 122). In this project, I see queer joy as a form of epistemically just interaction with proof because it embraces the identity of the reader and invites them to contextualize their gender and sexuality alongside the mathematical content so that mathematics can "meet the learners half way" (Tanswell & Rittberg, 2020, p. 1208).

Methods

Data for this paper comes from a 2025 study conducted at a large university in the American Midwest. Data was created across three phases, all of which were methodologically informed by Jeong-eun Rhee's (2020) post-qualitative writings on rememory, my personal queer research aesthetics (Author, Year), and the knowledge that "queer lives can be addressed through a plethora of methods, and all methods can be put to the task of questioning normativities" (Nash & Browne, 2010, p.12). This paper's analysis is focused on phase two, a task based group interview (Goldin, 2012) where the queer readings took place. I describe my participants and outline my methods for preparing and conducting the task based group interview below.

Participants

Four participants took part in the study. Atticus (he/him) and Jeremy (he/him) were both completing their third year of a bachelor's degree in mathematics. Audrey (she/her) was a third year graduate student working towards her PhD in nuclear physics after earning a dual bachelor's degree in physics and mathematics. The fourth participant was myself (Author, Pronouns); I was a second year graduate student working towards a PhD in mathematics education and a master's in applied mathematics. All of us had completed at least two years of mathematics coursework and identified as queer. The choice to use queer participants stems not from a belief that only queer individuals can perform queer readings, but from a belief that queer methods of doing mathematics should originate from within the queer mathematics community.

Data Creation

Prior to the task-based interview, I conducted an unstructured group interview in which I used Rhee's (2020) rememory-based methodology as an access point to the social, political, and theoretical underpinnings of queer readings. This allowed us to collectively define our intended queer readings of proof as a way to "bridge the gap" (Jeremy) between theory and "what I can actually see going on" (Atticus), make proofs "explode" (Jeremy), acknowledge the queerness of proof's "funny little symbols" (Audrey), play with "what a proof can be" (Author), and "get the arts and crafts going" (Jeremy) so that proof does not seem "so far away" (Author).

Two proofs were selected for the task-based interview on the basis that they are present in other studies on proof reading. The majority of our time was spent reading the proof taken from Mejía-Ramos et al. (2012) (see figure 1) and all findings shared are in reference to this proof.

(1) **Theorem:** There are infinitely many triadic primes.

(2) **Proof:** Consider a product of two monadic numbers: $(4j+1)(4k+1)$
(3) $= 4j \bullet 4k + 4j + 4k + 1$
(4) $= 4(4jk + j + k) + 1$

(5) which again is monadic.
(6) Similarly, the product of any number of monadic numbers is monadic.
(7) Now, assume the theorem is false, so there are only finitely many triadic primes, say $p_1, p_2, \dots, p_n$.
(8) Let $M = 4p_2 \dots p_n + 3$, where $p_1 = 3$
(9) $p_2, p_3, \dots, p_n$ do not divide M as they leave a remainder 3, and 3 does not divide M as it does not divide $4p_2 \dots p_n$.
(10) We conclude that no triadic prime divides M.

(11) Also, 2 does not divide M since M is odd.
(12) Thus all of M's prime factors are monadic, hence M itself must be monadic. But M is clearly triadic, a contradiction.
■

Figure 1. Proof of infinitely many triadic primes. Line numbers included for readability of findings.

During the interview we had access to multiple copies of this proof as well as a variety of art supplies. The interview took place in a queer community center, lasted approximately ninety minutes, and was filmed from multiple angles to ensure that facial expressions, body language, and any manipulation of the proofs would be visible. At the start of the interview we reviewed our discussion from the unstructured group interview and agreed that even if we "don't know what it's going to look like" (Jeremy) we were ready to begin our queer readings based on the desires and guidelines we had previously co-created.

Data Analysis

In order to analyze the second task-based group interview I immersed myself in the data through repeated relistening and rewatching. This strategy of data immersion was inspired by the work of post-qualitative researchers who describe data analysis not as a "recipe" or "process" but as a form of experimentation grounded in the theoretical underpinnings of the work (St. Pierre, 2018, p. 604). For me, this meant inviting the data into my life outside of the academy by consciously relistening and rewatching the data while cooking, driving, and participating in my favorite hobbies; by the end of this phase of data analysis I estimate that I had relistened/rewatched the second task-based interview upwards of fifteen times.

Having built a strong internal relationship with the data, the next step of data analysis took the form of externalizing my noticings. For the purpose of this paper, I reviewed the task-based interview again, this time pausing to take notes when I observed moments where we interacted with the proof in epistemically just ways. For example, moments in which we contributed contextual annotations (e.g. "p_1 must be 3!", "b/c primes") represent an epistemically just interaction by appropriating norms valuing acontextuality. I also noted moments when we experienced queer joy while conducting the queer reading, as evidenced by our laughter, facial expressions, and implicit or explicit connections between our identities and the proof. After collecting these notes, I watched the interview again and selected specific instances of epistemic justice and queer joy that I felt most clearly demonstrated how the queer reading process was conducive to the emergence of these moments. I then began drafting my findings with the knowledge that the act of writing is another phase of analysis which allows the writer to explore their findings through the externalization process (St. Pierre, 2018).

Findings

During our queer readings of mathematical proof, the participants and I were inspired to physically deconstruct and reconstruct part of the proof for infinitely many triadic primes by cutting apart the existing proof (figure 1) and rearranging it onto a poster while adding our own context, notes, definitions, arrows, and other forms of annotation (figure 2). Through this de/reconstruction we were able to engage in proof reading in epistemically just ways by reintroducing agency and context to the mathematical text. This de/reconstruction of proof and the appropriation of proof norms also invited moments of queer joy in which we saw our queer experiences mirrored in mathematical proof and proving. Given my understanding of queer joy as an extension of epistemic justice, these findings are shared in the second subsection. However this ordering does not imply that our experiences of queer joy are secondary in importance.

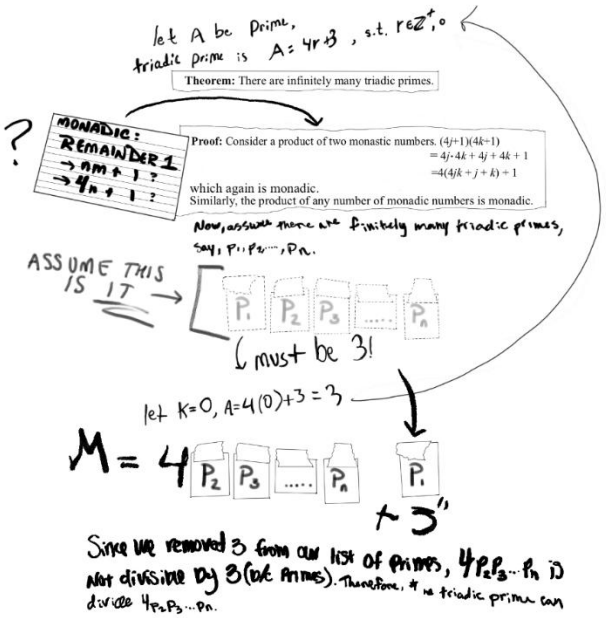

Figure 2. Partial proof de/reconstruction; digitally rendered for legibility

Moments of Epistemic Justice in the Queer Reading Process

One moment in which we experienced epistemic justice through appropriating proof norms valuing acontextuality was while discussing the assertion in line 7 and its relationship to line 8 and the overall proof by contradiction.

Audrey: So, the $p_1, p_2,$ all the way to p_n …are we assuming that *those* are the triadic primes? [she makes a chopping gesture with her hand] Are we assuming that that's all of them?

Author: Yeah, like if there was finitely many, that's it. That's our—like we could create a list and then this would be the list.

Audrey: Okay, okay. So, this is [her hands make a large sweeping gesture] all of them.

Author: All of them. [Laughing] We've done it. We've found them all.

Audrey: [She laughs and then frowns] What's the point of M though?

Author: Um…[Everyone is quiet for a moment]

Audrey: I think they're just trying to show that you can create a new triadic prime using the ones you've already created in the finite list?

Jeremy: Ohhh…

Audrey: You prove that that new product is a triadic prime…

Jeremy: Oh…okay, no I understand what you're saying.

Author: Right so it can't be finite.

Jeremy: Cause we already assumed it was finite, [He adopts a sassy tone and holds up his pencil] but actually, there's one more!

In this interaction we discuss line 7, emphasizing the proof's certainty that the list is finite while knowing that this assertion will quickly be disproven. In this way, we are acknowledging our agency as readers; we are not bound by the proof's linearity because we are able to look forward to line 12 as we read line 7. This allows us to find humor and curiosity in line 7's assertion, contextualizing the statement as it relates to the proof by contradiction. During our de/reconstruction of line 7 for our poster, Atticus, likely recalling the conversation shared above,

suggested that we emphasize the finiteness of our p's with an annotation. I wrote "ASSUME THIS IS IT" to the left of the list of p's as a playful reminder of the upcoming contradiction.

Shortly after this discussion, and as part of our de/reconstruction of line 7, I was drawn to create a physical representation of the p's, cutting out $p_1, p_2,$ etc. and taping them on our poster. This resulted in Audrey reaching out to physically move the p's from line 7 into the equation for M on line 8 as we progressed in our de/reconstruction. Her choice to physically relocate the p's to line 8 demonstrates her understanding of their equivalency and her sense of agency as a reader. Our queer reading method of de/reconstruction invited her to treat the elements from line 7 as tangible constructions which she is capable of manipulating and redefining, giving her agency over the proof in tangible ways.

Later, during our de/reconstruction of line 9, we felt the need to contextualize the conclusions being made about the terms within the equation for *M (*defined in line 8) which are central to the proof by contradiction.

Atticus: So what is it [line 8] even saying?

Audrey: Okay okay okay…so it's two [she raises her hands in a grasping gesture and then frowns, dropping her face in her hands] what's it called when things are…? [she perks up and holds her hands forward again] Terms, yes! Two terms! So we have to prove both are not—cannot—be divided by three.

Jeremy: Okay. [He uses his pinky to poke at the first term in the equation for *M*] And this one can't because we already pulled three out [he slides his pinky to the second term].

Author: And this one [I gesture to the first term for *M*] is just four and a bunch of primes.

Jeremy: Okay.

Author: So if this is four and a bunch of prime numbers…

Audrey: Yeah.

Atticus: [Slowly, while rubbing his chin] Right…then they're only divisible by four and the same prime numbers. Not three.

Jeremy: Exactly. [He leans forward and writes: Since we removed 3 from our list of primes, $4p_2p_3 \ldots p_n$ is Not divisible by 3]

Author: That makes sense to me.

Jeremy: I'm gonna say…[he adds the parenthetical (b/c primes) to the poster, and we all laugh]

In this excerpt, we discuss why the first term in the equation for M in line 8 is not divisible by three, a discussion which Jeremy summarizes by rewriting line 9 with additional mathematical context and consideration for the reader. His choice to add the phrases "since we removed 3 from our list of primes" and "(b/c primes)" demonstrate a preference for casual narrative language which guides readers towards the line's main idea and explicitly addresses the logical jumps which were implicit in the original proof. This appropriation of norms valuing acontextuality allowed us to reconstruct the proof in a manner that was more comfortable for our reading process and more closely resembled our conversations surrounding the topic.

Moments of Queer Joy in the Queer Reading Process

As our queer readings led us to de/reconstruct the proof for infinitely many triadic primes, we paused to consider the separation we felt was visible between lines 2-6 of the proof and lines 7-12 of the proof.

Author: So we have a piece of the proof that feels unrelated?

Jeremy: We need a better transition for sure

Author: [Referencing those who identified as transgender (trans) within the group] Don't we all want a better transition? [Everyone laughs]

Atticus: [Emphatically] Yes! Yes, I do!

The word transition can refer to the narrative flow between lines of text, but it is also used by the trans community to refer to the actions and experiences of modifying one's gender presentation in gender affirming ways (e.g. cutting one's hair, changing one's name). This play on words which was evoked during our queer de/reconstruction of the proof allowed us to connect to the proof on a more personal level; we were able to see our identities as relevant in a mathematical context in which acontextuality frequently disguises the existence of queer readers and queer authors of mathematics.

Other examples of queer joy were visible in our discussion of broader mathematical concepts relevant to the proof. At the start of our de/reconstruction Audrey noted that primes as a mathematical concept felt queer to her, saying "primes are queer…they're special and there are infinitely many of them." As she spoke, she smiled and wiggled side to side in her seat. I remember feeling briefly emotional about this concept; delighted that, like primes, queerness extends into an infinity of experiences within an infinity of individuals. Similarly, Jeremy pointed out that proof by contradiction is a queer way to prove a theorem, laughing and gesturing at the proof as he exclaimed "like, we said there are infinitely many of them…JK!" Again, this moment of humor represents a deeper connection between our identities as queer doers of mathematics and the mathematics itself. This identity-forward interaction with proof is evidence of an epistemically just interaction with proof since it values the context and the community that we bring with us as proof readers.

Discussion

In order to uphold epistemic values of acontextuality, norms of proof discourage reference to time, place, reader or author agency, and the realities of proof as a human creation, and these norms perpetuate epistemic injustice for students (Tanswell & Rittberg, 2020). Queer readings, a practice which celebrates reader agency and invites the deconstruction of texts for the purpose of critique, exploration, and personal meaning-making is a pedagogical tool that encourages the subversion of these acontextual proof norms. And by subverting these norms, students are able to interact with proof in epistemically just ways, which for queer students, can include epistemic justice through the cultivation and acknowledgement of queer joy.

Much of the existing literature discussing proof reading focuses on student's cognitive behavior and/or the quantifiable outcomes of specific proof reading strategies such as improved proof comprehension or proof validation (e.g. Lew et al., 2024; Mejía-Ramos et al., 2012; Panse et al., 2018; Weber et al., 2012; Yang, 2012). But in recent years researchers and educators have begun considering the humanization of proof instruction (e.g. Contreras et al., 2025; Dahlgren, 2024) and how one might incorporate equitable pedagogical practices into proof based courses (e.g. Melhuish et al., 2024). I present queer reading as a supportive critical tool in advancing these agendas, as well as an enticement for students to experience joy and justice in proof classrooms. As Atticus expressed in his reflection exercise, "this process was so, so healing for me and my math soul! I feel like I can embrace looking deeper into proofs, and even rewriting them so they make sense to me."

There is still much to conceptualize about the practice of reading proofs queerly and how it might support students of mathematics, and I plan on continuing this work with the belief that queer pedagogy can be critical, subversive, and uplifting for students regardless of their identity (Author, Year). As we continue to ask ourselves how undergraduate mathematics education might become more equitable, just, accessible, joyful, and humanizing, I offer queer pedagogies as complementary frameworks deserving of continued study.

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Designing Activities About Definition Values Through Collaboration and Attention to Professional Obligations

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Because many mathematics instructors rely on lectures as their primary mode of instruction, change efforts should engage with and build on instructors' existing values and practices rather than attempt to replace them. We report on the first year of our project, where we take a collaborative approach to improve instruction surrounding definitions through activity design and implementation. Constructing narrative cases, we explored how our collaboration aided participants in navigating alignments and tensions between their professional obligations and their activity. We found our collaboration succeeded in guiding our participants, but it required careful examination of reasons as to why instructors may not institute instructional change.

Keywords: Professional Obligations, Instructor Collaboration, Values, Definitions

Many mathematics faculty continue to rely on forms of lecture that prioritize efficiency and content coverage, both in Calculus (Kirin et al., 2017; Vroom et al., 2022) and proof-based courses (Johnson et al., 2018). Dawkins and Weber (2023) argue that such instructional choices may persist because they fulfill instructors' perceived commitments, such as ensuring mathematical precision, preparing students for advanced coursework, and managing time constraints. This suggests a need for instructional change efforts that engage with and build on instructors' existing values and practices, rather than seeking to replace them. In this study, we focus on one context where values play a central role – the teaching of mathematical definitions. We collaborated with mathematics faculty to design classroom activities that communicate the nature of mathematical definitions, hypothesizing that attending to instructors' values could support them in adopting more interactive teaching strategies while remaining aligned with their professional obligations. This focus is particularly important given evidence that mathematicians and students understand definitions differently (Edwards & Ward, 2004; Fernández-León et al., 2021; Martín-Molina et al., 2020).

Theoretical Grounding

We draw on Herbst and Chazan's (2011) theory of *practical rationality*, assuming that instructors' practices are rational within the context of their available resources and perceived professional obligations. Herbst and Chazan theorize four such obligations that instructors weigh as they make instructional decisions: (1) disciplinary, to represent mathematics authentically; (2) individual, to attend to students' needs; (3) interpersonal, to manage classroom interactions as social spaces; and (4) institutional, to meet the expectations of the schooling system. For instance, Johnson et al. (2025) exemplify such decision-making. The mathematicians they studied were hesitant to adopt active learning approaches that might limit content coverage because they wanted to avoid upsetting the department chair if their section deviated from others (institutional obligation), aimed to adequately prepare students for graduate school (individual obligation), and wanted to have time to cover all the essential topics according to their disciplinary expertise (disciplinary obligation).

Our collaborative work aimed to foreground the instructors' disciplinary obligations, as we suspected these would provide a compelling rationale for implementing an instructional

innovation. In particular, through collaboration, faculty developed activities to treat mathematical definitions in ways aligned with the mathematics community's values. By *activity*, we mean any part of an instructor's lesson that solicits some artifact of how students think about a value (e.g., a discussion, individual exercise, exit ticket). We conceptualize values, mirroring Dawkins and Weber (2017), as a community's shared goals or orientations (*values*). Rupnow and Randazzo (2023) identified two values that guide how mathematicians treat definitions: the *communication value*, emphasizing clear and unambiguous expression, and the *freedom value*, emphasizing flexibility and creativity when using or creating mathematical definitions.

As the instructors planned and enacted activities, we noticed that these activities often aligned with, but at times came into tension with, their professional obligations. For instance, a think–pair–share activity that asks students to consider multiple, conceptually distinct definitions may align with instructors' disciplinary obligation to highlight the freedom value while also supporting their individual obligation to engage students in meaningful mathematics. At the same time, such an activity might create tension with their institutional obligation to follow the textbook's chosen definition and to cover material efficiently. Recognizing and navigating these alignments and tensions proved central to the design process. Thus, we view the development of activities that either aligned with or productively responded to instructors' obligations as an advantageous goal for the activities to be realized and sustainable in the instructors' classes. In this study, we investigate: *In what ways do instructors' professional obligations align or come into tension with activity design and implementation around mathematical definitions? How can collaboration help surface and respond to these alignments and tensions?*

Methods

Our data comes from the *Supporting Mathematical Instruction around Definitions through Values-Centered Collaboration* project. We focus on the first year of the project during which participants engaged in a series of focus groups (FGs) (four total meetings) and completed surveys before or between meetings (five total surveys). We also recorded any activities that the instructors implemented in their class. We recruited ten instructors in Fall 2024 and six instructors in Spring 2025 across two PhD-granting Midwestern universities. Four FGs were held each semester.

We began the FGs by introducing instructors to the framework we used for mathematical definition values (Rupnow & Randazzo, 2023) and discussed messages that instructors may send to students about them. We (the researchers and instructors) also discussed examples from our teaching in which we aimed to share definition values with students. Author 2 shared a Real Analysis activity in which the students analyzed an informal definition for increasing sequences and used it to formulate a more formal and unambiguous version. Author 3 shared two activities: one from Abstract Algebra that engaged students in defining "prime," and, later, one in Linear Algebra where students reflected on the name "orthogonal complement" and the conceptual insight it conveys. After these discussions, instructors brainstormed activities that communicated a chosen value (and norms that uphold them) and, in the subsequent meeting, they presented these activities and exchanged feedback. After the instructors implemented their activities in their course, we debriefed and reflected on their implementations. The surveys collected demographic data, perceptions on the role of definitions in mathematics, reflections on the values discussed in FGs, descriptions of their classroom activity, and feedback from their involvement. We drew on these survey responses between FG sessions to revise our initial plan for the following session.

In this study, we highlight two cases of working with two mathematicians, who we refer to as Julian (computational mathematician) and Drew (algebraic combinatorialist). We selected these two cases because we thought both seemed like successful collaborations but in different ways. Based on our experiences with them, we thought that our collaboration with Julian supported alignment in his professional obligations and his activity whereas we thought that our collaboration with Drew supported him in responding to tensions between his obligations and his activity design.

Our data analysis involved constructing narrative cases (Auerbach & Silverstein, 2003) of our collaborations with Julian and Drew. Our narrative writing was guided by the following questions: (a) How, if at all, did [mathematician] feel obligated to share the communication value in his teaching? The freedom value?, (b) How did [mathematician's] activity align and/or come into tension with their professional obligations? (c) How, if at all, did our collaboration with [name] contribute to this alignment or tension? We developed the narratives using an iterative process. First, we re-read faculty's survey responses and answered the guiding questions, interpreting their responses through a professional obligation lens. We then discussed our interpretations until we reached an agreement. Next, we drafted what we noticed related to the guiding questions, referencing any relevant quotes from their surveys. With this first draft, we watched the FG recordings and read the corresponding meeting transcripts to test and refine our emerging claims and enrich the narrative with contextual detail. Once satisfied with the narrative, we engaged in member-checking by inviting the mathematicians to read their narratives and provide feedback, which we subsequently incorporated.

Results

In what follows, we share two abbreviated narratives that highlight our collaborations with Julian and Drew, attending to how their professional obligations aligned or came into tension with activity design, and how collaboration helped surface and respond to these obligations.

Julian's Activity: Aligning Design with Obligations

Julian's Disciplinary Obligations. Julian had an obligation to uphold the communication value in his teaching, particularly through the way he used and presented mathematical definitions in his Differential Equations class (*disciplinary obligation*). Since the course placed little emphasis on proof, he explained that the role of definitions was primarily about "communicating ideas to the students and giving something a name." In this context (and in mathematics more broadly) he explained that definitions serve key communicative purposes: they make concepts "easier to reference and talk about," help ideas "be memorable to others," and clarify "what something is and what it is not." Julian emphasized that modeling clear communication was essential because "we want our students to be clear as well when they're doing their written work- we would like them to try to emulate that."

Julian did not feel as obligated to uphold the freedom value in his teaching as compared to the communication value. While he acknowledged that definitions can also be used in the creation/discovery of concepts, he emphasized this was "mostly done in research" and not typically relevant to his classroom teaching. He explained one tension between sharing this disciplinary value and an obligation to the course to present definitions a certain way to maintain consistency for students (*individual obligation*) - "If we're using a certain textbook… I feel like I have to use whatever the textbook is using." Still, Julian shared that he exercised the freedom value in small ways in his teaching prior to the project. He shared that he sometimes replaces a traditional term with one he thinks will resonate more with students (e.g., "linear independence"

as “no redundancy”) with the caveat that he introduces the conventional term as well: “I still feel that I'm required to teach them the usual term as well, because I don't want to send them out into the world with a misunderstanding of, you know, what something is called.” That is, Julian also used traditional concept names to fulfill his obligations to students and to the discipline to promote consistency in concept names.

Activity Design. Because Julian felt obligated to uphold the communication value in his teaching (especially in comparison to the freedom value), we aimed to collaborate with Julian to design an activity that would showcase this value. At the time, he hoped to develop an activity around the Laplace Transform, one of the remaining key definitions in the course. However, he was unsure how to connect this definition to the communication (or freedom) value. In particular, he noted that in the initial teaching case studies shared, we had used the definitions of “prime” and “increasing,” which are intuitive subsets of familiar sets (primes within the natural numbers) or relate to their everyday meaning, whereas the Laplace Transform did not fit that mold. During the following focus group, we supported Julian in several ways to develop his activity so that it would better align with his desire to share the communication value. First, Author 3 shared an example from her Linear Algebra course in which she asked students to reflect on the name “orthogonal complement,” surfacing the communication value. Second, we paired Julian with another instructor who was also designing a Laplace Transform activity, to provide new perspectives during the discussion. Additionally, we encouraged both instructors to reflect on the concept name. We noted that while “Laplace” (a person’s name) may not provide much conceptual insight, the term “transform” points more directly to the heart of the concept, and what Julian described as the important part of the definition. Ultimately, Julian created an activity in which he asked students to discuss: (1) Why is it useful to give the Laplace Transform a name?, (2) Why do you think we call it a “transform”?, and (3) What are the ways that the Laplace Transform transforms something into something else? These collaborative supports helped Julian recognize ways to align his disciplinary obligation to uphold the communication value with the design of his Laplace Transform activity.

Alignment with Professional Obligations. Julian reflected on several ways that implementing the activity fulfilled his professional obligations. He emphasized that the activity elevated a potentially new obligation to himself as an instructor and to his students to make the course more interactive and fun (*interpersonal obligation*). He reflected, “it was surprising, the class kind of erupted in discussion.” He later added: “it was the best part of the class. The rest of the class felt kind of dry and boring compared to that because... I think the students had fun, I had fun.” Julian also explained how his activity supported other professional obligations. He recounted one student’s insight as he explained it fulfilled his obligation to share the communication value (*disciplinary obligation*)*:* “One student said something which I thought was interesting. He said, ‘well, if we didn't give things like this a name it would be like none of us have names and then we would just have to refer to each other, you know, without names, which would be really difficult.’ So I thought that was really a good answer.” The activity simultaneously addressed Julian’s obligation to support students in deepening their understanding of the concept itself (*individual obligation*)*.* He said, “I think the students will remember the Laplace transform and its properties more deeply now - like it'll be something that they've actually thought about. And it may help them in the future when they recall what is the Laplace transform and they'll think, ‘oh, yeah, that's that thing that transforms all sorts of things into something else that's useful.’”

Because the activity aligned with multiple obligations (*disciplinary, interpersonal, and individual*) Julian expressed a commitment to implementing similar activities in future courses. In particular, he noted that such designs would be "super easy to incorporate" when the name of a concept is chosen to convey meaning (e.g., Laplace Transform, complementary solution).

Drew's Activity: Responding to Tensions between Design and Obligations

Drew's Disciplinary Obligations. Drew desired to share different aspects of the communication value in his teaching (*disciplinary obligation*), depending on the context and audience. Across all his teaching contexts, he explained that definitions are "to get [students] to understand and reason about a concept." He distinguished between two kinds of definitions: "the short but less useful version of a definition and then the more intuitive, less precise version," using the epsilon-delta definition of a limit and its informal counterpart as examples. His choice of which version to teach depended on the course context and the needs of the students (*institutional* and *individual obligations*). In some undergraduate settings, he felt it was important to teach both: "And I try to impart to them that this is a formal thing we have to do it for these reasons, but here's maybe the useful thing that you do instead that is in itself some kind of definition, just not the usual mathematical kind, perhaps." He felt formal definitions "need to be supplemented with the informal stuff - to get students to the point where they can appreciate what the formal definition was for." In this sense, he felt a consistent disciplinary obligation to showcase how more intuitive versions of the definitions communicate conceptual meaning, and in contexts where formal definitions were expected (such as proof-based courses), he also felt obligated to the discipline to connect that meaning to formality.

While Drew felt consistently obligated to share the communication value through his teaching across different levels of courses (albeit in different ways), he sometimes felt obligated to share the freedom value, but mostly in upper-division courses. He shared an instance while teaching Abstract Algebra prior to his involvement on the project when he felt this obligation. The course textbook defined rings as not having an identity element. "And that was sort of very new to me," he reflected. "I gave them a little five-minute discussion of like, here's why we're doing it this way. These are the advantages. Not everyone does it this way. It's sort of arbitrary what the word can mean." In this instance, he felt obligated to convey to his students that definitions are choices and that there is freedom in how mathematical objects are defined.

Activity Design. Because of his professional obligation to the discipline to share the communication value, Drew wanted to develop, with our support, a classroom activity that would highlight this value. He especially wanted students to see definitions as a packaged way of expressing the main idea. He expressed interest in this after we shared a teaching scenario in which Author 2 supported her students to notice the idea of "increasing" first, and then supported the students in recognizing how precision supports clear articulation of that concept. Drew remarked, "I thought it was cool, this [activity] got the students to think about the level of precision in language that they're taking in and the way they're thinking compared to the way a textbook reads."

Drew initially began designing an activity for his current Calculus I course, centered on the definition of Riemann integration. He explained that he wanted to show his students the velocity graph for a car and then ask how far it went. He thought this would get them to "confront their intuition for like distance – time times speed equals distance – just doesn't work here because there's a curve". He then intended to support students in visually seeing how partitioning the interval could make better approximations, and then connecting this process to the definition by supporting them to see that this process "is a limit," but just one that "looks very different from

the limits we've done throughout the course." He wanted to further develop this activity to share the communication value in the sense that "this definition is a catchall uniform way to talk about the different ways of doing the same thing" and that it "[wraps] these different choices of how you might do this area approximation into one definition".

Tensions with Professional Obligations. However, Drew experienced several tensions between his task design and his professional obligations. First, he explained that he typically avoids rigorous definitions in this course, keeping things "very pictorial and heuristic-based." His students, who were primarily first-year engineering majors, wouldn't "appreciate too much rigor" (*individual obligation*). Yet, Drew thought sharing this aspect of the communication value would require linking the visual and intuitive work to a formal definition (*disciplinary obligation*). Without this connection, he felt the activity would not align with the value he was trying to promote. With this connection, he felt he would not fulfill his obligation to his students to provide meaningful coursework for them. Additionally, Drew had an obligation to stay in step with the course's tight coordination, and he did not feel free to change the pacing or topics (*institutional obligation*). From Drew's perspective, continuing to develop this activity would not fulfill his obligation to the coordinator and other instructors to offer a uniform experience for their students.

Responding to Tensions. For these reasons, we encouraged Drew to shift his focus to an activity for his upcoming Abstract Algebra class in which he felt (a) a greater sense of obligation to share the formal versions of the definitions with his students (*disciplinary obligation*) and (b) a greater sense of freedom in the content, pacing, and structure of the course (*institutional obligation*). This suggestion seemed to constructively respond to this tension as he shared, "I'm definitely more excited about not shoving something into calculus" and "I was trying to squeeze something into calculus where it maybe doesn't quite fit." In this case, collaboration did not eliminate tensions but instead supported Drew in recognizing a more fitting context where his obligations and design could be more aligned.

His abstract algebra activity focused on the concept of group isomorphism. Drew described his plan: "My tentative plan is to give them an informal definition of group isomorphism as two groups being fundamentally the same but looking different." He would begin with similar things from other math contexts such as congruent geometric shapes, different factorizations of polynomials or integers, or games like tic-tac-toe versus choosing three numbers that sum to 15. He planned to give students pairs of isomorphic finite groups and ask them to explain why they thought the groups were isomorphic. He anticipated that students would come up with the idea of pairing elements, and then he would then ask them to try write a formal definition. Finally, he would present a formal definition and discuss why it takes the form that it does. This activity appeared promising because it might allow Drew to meet several of his teaching obligations. It supports the communication value by illustrating how a formal definition can precisely capture the central idea of isomorphism – that two groups behave the same way – while also supporting the freedom value by allowing students the experience of crafting a definition (*disciplinary obligation*). It also aligned with his obligations to students, especially his desire to support their understanding: "I think it will get them to think more deeply about the concept of isomorphism and experience a line of thinking they don't usually get to" (*individual obligation*). Drew remained excited and hopeful about implementing the activity in the following semester.

Discussion

Through sustained, instructor-centered collaboration, our project aims to improve undergraduate mathematics instruction. By applying a practical rationality lens and attending to

the professional obligations that shape instructors' choices, we note our collaboration with Julian and Drew successfully supported them in navigating the alignments and tensions between their professional obligations and the design of their activities. However, this navigation required careful examination and explication of the factors that could explain why instructors are not willing to implement certain types of instructional change. In our case, this meant consideration of our participants' professional obligations and their course learning goals and contexts. Julian's case was an example of a co-designed activity where his professional obligations were aligned with his course learning goals. Further, Julian's experience with both our collaboration and his activity resulted in a commitment to continued activity implementation post-intervention. In contrast, Drew's initial activity resulted in tensions with his obligations and thus caused reluctance to implement his activity. We renegotiated the design with Drew to situate the activity in a context that aligned with – rather than put in tension – his professional obligations.

Moreover, we echo Johnson et al.'s (2025) note that professional obligations depend on the context of the particular course. Our work points to how this can be the case in two different ways. First, tight course coordination and time concerns deterred deviation from pacing and content coverage. While research on coordination has positioned it as a means to support professional development (e.g., Martinez et al., 2021), our findings show that in some cases, coordination may stifle instructional innovation. Second, as we anticipated when entering the project, the instructors did perceive the values related to definitions to be important to the discipline of mathematics. However, consistent with Rupnow and Randazzo (2023) we note that valuing something within one's discipline does not imply instructors find it necessity to share it in teaching across all levels. This raises questions about how mathematics is portrayed and how mathematicians want to portray mathematics – if instructors commonly wait until students are in upper-division courses to share notions of freedom, students who persist in the math major may expect and desire less freedom and be uncomfortable when given more, just as students who desire to think flexibly and creatively may leave mathematics before that creativity is incorporated.

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Why Pursue Graduate Studies in Mathematics? Insights Into Motivation From Personal Essays

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Deciding to pursue graduate study is a pivotal step in students' academic and professional trajectories. However, limited research exists on the motivations for applying to mathematics graduate programs. To address this gap, we analyzed 74 personal statements, in which applicants articulate their motivations, goals, and experiences, submitted as part of applications to master's and doctoral programs in mathematics and statistics at a mid-sized research-intensive university (2015–2024). Using qualitative coding with AI-assisted analysis, five key themes emerged: pursuit of knowledge, research aspirations, career advancement, growth as a teacher, and appreciation of mathematics. Most applicants expressed multiple, overlapping motivations, often linking professional goals to disciplinary growth and research. A distinctive finding was the prominence of appreciation of mathematics, where applicants described joy and fascination with the subject as central to their decision-making. These results expand understanding of mathematics graduate decision-making and carry implications for recruitment, advising, and program design.

Keywords: graduate studies in mathematics, motivation, personal essays, AI-assisted coding

The decision to pursue graduate education in mathematics is a significant milestone in an individual's academic and professional journey. While there is a growing body of literature examining motivations for graduate studies in general, much of this work has focused on broad patterns across disciplines or within professional fields, and relatively little is known about this decision-making process in the context of mathematical sciences. Previous research has identified both internal and external factors that shape an individual's decision to pursue an advanced degree. For example, individuals are often motivated intrinsically by intellectual fulfillment and self-satisfaction (Incikabi, 2013; Kowalczuk-Waledziak et al., 2017). External motivations include career advancement, prospects of financial gain, and job security (Hinkle et al., 2014; Ivankova & Stick, 2007). Mathematics-related graduate programs prepare individuals for diverse roles in academia, research, and industry, yet little research has examined the factors influencing students' decisions to apply to these programs.

This study seeks to address this gap by examining the personal essays written by applicants to graduate programs in mathematics and statistics at a mid-sized public university with a high level of research activity in the Northeastern United States. By analyzing how applicants articulate their motivations, goals, and aspirations, this research aims to provide deeper insights into the factors shaping students' decisions to pursue advanced mathematics degrees.

Literature Review and Theoretical Framing

Research on pursuing higher degrees is extensive across fields; however, fewer studies focus specifically on mathematics. This review first draws on the general literature on motivations for advanced study, then turns to factors shaping decisions to pursue graduate mathematics.

Motivations for Pursuing an Advanced Degree

Motivations for pursuing an advanced degree are multifaceted, shaped by both internal and external factors (Incikabi, 2013). Internal factors influencing the decision to pursue graduate

degrees often center on personal fulfillment, such as intellectual challenge, self-satisfaction, and a sense of achievement (Incikabi, 2013; Kowalczuk-Waledziak et al., 2017; Teowkul et al., 2009). Students are also often motivated by research interests and prior academic success (London et al., 2014; Wiegerová, 2016). Additionally, self-efficacy, influenced by both internal and external factors, is critical in this decision-making process. Positive self-efficacy—bolstered by support from faculty, peers, and family—strengthens confidence in one's ability to succeed academically and professionally (Gibbons & Schoffner, 2004; Sherman, 2021).

External motivations typically center on professional gain, career advancement, job security, and higher salaries (Hinkle et al., 2014; Ivankova & Stick, 2007; Ng et al., 2011; Teowkul et al., 2009). Advanced degrees are typically required for entry into academia and some industry fields (Basalla & Debelius, 2014). Hence, career alignment is a strong predictor of graduate enrollment (Tennant, 2004). Additionally, different social pressures, including family expectations, peer influence, and mentor encouragement, often shape the decision to pursue graduate education (Brailsford, 2010; Hinkle et al., 2014).

Factors Influencing the Decision to Apply to Graduate School in Mathematics

While the literature on pursuing graduate studies in mathematics is limited, previous research has shown that this decision is shaped by a set of academic, social, and structural influences. These influences interact with broader demographic and social dynamics—such as race, gender, and socioeconomic status—to shape who ultimately pursues graduate study in mathematics. For example, students from minoritized groups face financial barriers, such as application fees, and underrepresentation in mathematics departments, contributing to feelings of isolation (McEldowney et al., 2024; Posselt, 2016). Gender and socioeconomic status similarly shape opportunities and have been shown to encounter systemic challenges that limit access to graduate mathematics (de Brey et al., 2021; Nevill et al., 2007).

Institutional and relational factors also play a key role. For instance, students at research-intensive universities are more likely to pursue graduate study due to greater access to resources and research opportunities (Zhang, 2005), which strengthens confidence and research identity (Hathaway et al., 2002; Willis et al., 2013). Faculty mentorship, supportive advising, and inclusive teaching practices further encourage persistence and application to mathematics graduate programs (Blaney & Wofford, 2021; Mu & Fosnacht, 2019).

Methods

The primary data consist of personal statement essays submitted as a part of the application process for mathematics or statistics graduate programs at a mid-sized public research university in the Northeastern United States. These personal statements are carefully composed essays in which applicants articulate their motivations, long- and short-term goals, and relevant research or professional experiences, offering insight into the factors shaping their decision to pursue advanced study. To recruit participants, we contacted applicants from 2015 through 2024. For those who consented, the second author—who serves on the admissions committee and has access to applications from these years—extracted their statements along with background information on gender, year of application, program, and citizenship status. Race/ethnicity was an optional field on the application and could not be collected consistently for all participants.

Participants and Data Collection

The participant pool for this study comprised 74 individuals who submitted applications to graduate programs in mathematics or statistics during the academic years 2015-2024. The data

set includes applicants that fall into four groups: (1) current students at the university (N=14), (2) graduates of the university (N=11), (3) individuals whose application was rejected by the university (N=20), and (4) individuals who were accepted but chose not to attend or who started their studies but withdrew prior to graduation (N=29). The degree programs (referred here as "mathematical sciences") included both doctoral and master's programs in Applied Mathematics, Statistics, and Pure Mathematics, a doctoral program in Mathematics Education, and a Master of Science for Teachers (MST). The breakdown of the participants' data by program is specified in Figure 1 in the Results section. Out of these participants, 41 identified as male, 32 as female, and one did not specify. Additionally, 46 were U.S. citizens, 25 were not, two had dual citizenship (one of which was the U.S.), and one did not specify.

Data Analysis

We employed qualitative methods to identify applicants' motivations for pursuing graduate education in mathematics or statistics. The data analysis was completed in two stages: codebook development, followed by AI-assisted coding (Butler & Buchbinder, submitted; Dunivin, 2024), which aided in handling the large volume of qualitative data.

Codebook development. First, we manually developed a codebook of participants' reasons for pursuing advanced study in mathematics. The first author used open coding and the constant comparative method (Patton, 2002; Strauss & Corbin, 1998) to analyze a subset of 16 essays (21.6%) and generate preliminary codes for explicit and implicit motivations in the personal essays. These codes were refined into broader categories with clear definitions and illustrative quotations. This process yielded five themes: Knowledge Pursuit, Career, Research Aspirations, Growth as a Teacher, and Appreciation of Mathematics (Table 1).

Table 1: Thematic codes, definitions, and examples of applicant motivations.

Code	Description of Code	Example
Knowledge Pursuit	Applicants emphasize intellectual growth, expressing a desire to expand their mathematical understanding or develop specific skills, whether theoretical, problem-solving, or applied.	"Upon admission, my primary goals would be to understand the methods by which mathematics is applied and then utilize those methods in my study of physics."
Research Aspirations	Applicants describe motivations centered on engaging with the research process in graduate studies, such as designing studies, investigating questions, or contributing to scholarship.	"My immediate goal in continued graduate work includes gaining more research experience, as I currently have less experience than I would prefer."
Growth as a Teacher	Applicants aspire to become stronger educators, inspire students, address mathematical anxiety, or improve pedagogy. This code is more focused on instructional development rather than obtaining a teaching position.	"I want to get a PhD in Math Education to find techniques to assist a larger number of students to enjoy and appreciate the beauty of mathematics."
Career	Applicants explicitly tie graduate study to professional outcomes or a job goal, such as becoming a professor, researcher, or analyst.	"Becoming a professor has been a goal of mine for a long time, and that is the ultimate goal of getting a PhD for me."

Appreciation of Mathematics	Applicants express joy, fascination, or admiration of mathematics—highlighting its beauty, creativity, or elegance. Points to affective and aesthetic dimensions of engagement with the discipline.	"My biggest motivation for pursuing a graduate degree is my love and appreciation for mathematics. This subject has pushed me out of my comfort zone and has given me an outlet to demonstrate my creativity."

AI-assisted coding. Following the development of the initial codebook, an AI-assisted coding workflow was utilized (Butler & Buchbinder, submitted; Dunivin, 2024). For ethical considerations and in accordance with IRB approval, a temporary feature of ChatGPT was used. This means that the data was not used to train a large language model and is automatically erased from the system after 30 days. All essays were anonymized before the analysis. This use of AI was included in the participant consent form.

First, we designed a preliminary AI prompt, instructing AI to apply the codebook deductively to the 16 statements (21.6%) that the first author had already manually coded. We compared the AI-generated codes to the researcher's original codes. Based on the results, we corrected the prompt and reapplied it to the same data. This iterative process of prompt refining and applying to coded data continued until a satisfactory level of agreement between the human coder and AI was achieved, with a Cohen's Kappa of 0.83, representing a strong level of agreement (McHugh, 2012). With this level of intercoder reliability established, we applied the finalized prompt to have AI code the remaining personal statements with the five codes (Table 1). Then, the researcher blindly coded another 16 statements (21.6%) coded by AI and compared the results. There was almost perfect agreement between AI and manual coders, with a Cohen's Kappa of 0.89 (McHugh, 2012). In both manual and AI-assisted coding, each theme was counted only once per essay, regardless of how many times it appeared. This allows us to associate each applicant with the set of themes in their essay, and each theme with the number of applicants who mentioned it in their essay.

Results

Analysis of the 74 personal statements resulted in a total of 252 codes across all essays. On average, each essay was coded approximately three of the five codes, indicating that applicants typically cited multiple reasons for pursuing a graduate-level mathematics degree. All five themes were present across applicant types, degree levels, and programs, though the prevalence and emphasis varied. Results for each code in each major are represented in Figure 1. We turn now to summarizing each theme. Due to space constraints, the results section focuses on the context of each code across all essays rather than on detailed subgroup analyses.

Across the essays, the *knowledge pursuit* code was the most prevalent motivation (24.6% of total codes) across most programs, as shown in Figure 1, appearing in 62 personal essays (~84% of applicants). Some applicants with this code shared eagerness to deepen their theoretical foundations, often citing specific advanced coursework, such as abstract algebra, set theory, and analysis (e.g., "I would like to continue my education in Bayesian Statistics as well, which is one of the many reasons that I am seeking admission.") While other applicants framed their learning as open-ended exploration, curious about topics they had not yet encountered (e.g., "My long-term objectives are to expand my knowledge, likely in ways I have not yet imagined.") Overall, the knowledge theme highlights applicants' view of graduate study as an opportunity to expand and deepen their understanding of mathematics.

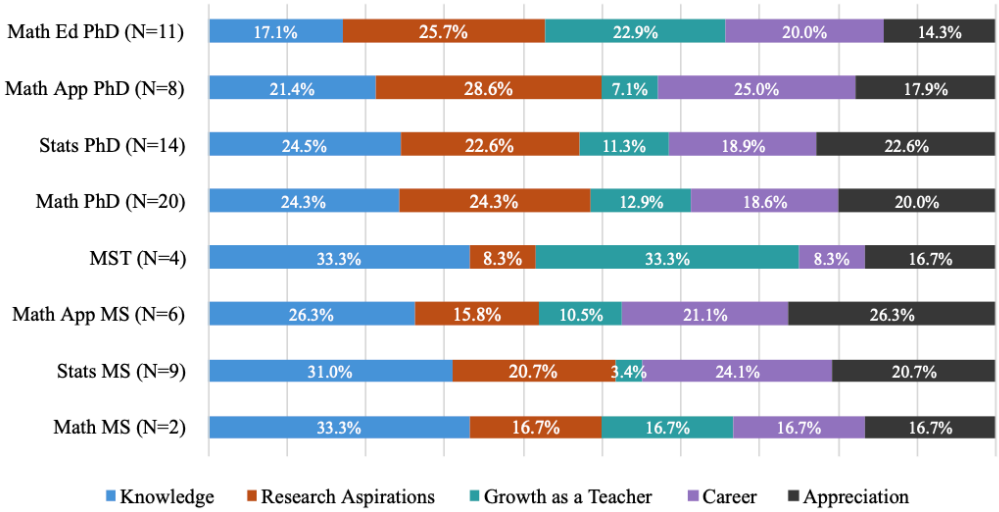

Figure 1: Distribution of codes by program (MS and PhD).

The next most common theme was *research aspirations* (22.6% of total codes), present in 77% of essays, but more prominent in doctoral applicants (Figure 1). Several applicants described a strong desire to engage in research in graduate studies, some referencing prior independent or guided projects and expressing intentions to continue such work in graduate school. For example, one applicant wrote, "I would very much like to continue research in mathematics." In contrast, others emphasized their goal of gaining new experience with research in a specific area, either emphasizing a broader ambition to contribute meaningfully to the field (e.g., "My long-term objectives are to discover and contribute to meaningful work and to do my best to offer something special to a field full of great thinkers,") or citing specific questions or area of interest. For example, one applicant stated:

> The experiences that I have from tutoring, teaching, and researching have sparked my interest in studying and seeking to improve math education. I am interested in answering questions like: Should the learning of probability and statistics be given more emphasis in high school? How can we improve students' mathematical discourse, in particular, oral communication? How much should we value our students' mathematical discourse? How can we improve students' mathematical reasoning?

Overall, this theme underscores applicants' curiosity and proactive engagement with mathematical inquiry.

In the personal essays, several applicants (~68%) also expressed a desire to pursue a graduate-level degree to achieve specific career goals. The *career* code captured applicants' descriptions of their long-term professional aspirations and represented 19.8% of the total codes. For example, one applicant wrote, "My long-term objective is to be a tenured professor of mathematics at a university so that I can conduct research alongside other mathematicians and encourage young people to engage with mathematics," implicitly highlighting how graduate study serves as a stepping stone toward this goal. Many applicants aimed for academic careers that combine research and teaching at the college or university level. Others emphasized particular fields they hope to enter and how the degree would equip them with the expertise needed to be successful (e.g., "This [MS in Statistics] will provide the foundation I need to contribute to development of soccer analytics.") Across essays, several applicants framed graduate school as integral to their broader professional goals, providing the skills, knowledge,

and experiences necessary to make meaningful contributions to their chosen field, as one applicant noted: "I firmly believe that a PhD program will enable me to delve deeper into the field, expand my knowledge, and become a successful university professor."

The *growth as a teacher* code appeared least frequently (13.1% of total codes) in applicants' personal statements, present in 44.6% of them, yet still represented in nearly half of the sample. Applicants with this code expressed a commitment to improving pedagogical practice, inspiring students, and addressing barriers such as mathematics anxiety during their graduate studies. Many reflected on past teaching experiences and the challenges they faced. For example, one applicant described the difficulty of motivating students who were held back by fear of mathematics:

> Over the years of teaching, the biggest challenge has been motivating the students to understand the lessons and see the importance of mathematics. I have seen many intelligent students who I know can be good with math but are afraid of math...I aim to find methods to reach as many students as possible. It is important to me to find teaching techniques to at least lessen the fear of math for those who are afraid.

Other applicants emphasized broader teaching goals, such as making mathematics more inclusive (e.g., "My personal teaching goal in all of my math courses is to make math accessible to all students.") Some focused on strengthening their own mathematical knowledge to better support students (e.g., "I believe [the MST program] will be beneficial for my students, in that I will have a deeper understanding of topics I present to them.") Overall, this theme was most prominent in Mathematics Education-focused programs (MST and Mathematics Education PhD), as seen in Figure 1. However, it was present across all programs, as many applicants viewed graduate study as an opportunity to develop as educators capable of creating supportive, engaging, and effective learning environments.

Similar to the career code, the *appreciation* code appeared in 67.6% of personal essays and accounted for 19.8% of total codes, across all program applicants (Figure 1). While many applicants discussed professional goals, several emphasized a deep, sustained enthusiasm for mathematics itself. They described formative experiences that sparked their interest in mathematics—from early problem-solving activities to noticing mathematical patterns in nature and art—highlighting the intellectual stimulation and challenges the discipline provides. For some applicants, this appreciation served as a core motivation for graduate study, as they explicitly linked their pursuit of graduate education to this passion (e.g., "My biggest motivation for pursuing a graduate degree is my love and appreciation for mathematics.") Others conveyed it more implicitly. As in one essay:

> I feel mathematics brings out some of the qualities I admire most in myself. I am captivated and thrilled by the constant challenge math provides and I experience a profound curiosity from within that demands the understanding of a concept, solving a problem, or writing of a satisfying proof.

These expressions of intellectual curiosity and motivation illustrate how a genuine appreciation for mathematics can drive applicants' decisions to pursue advanced study.

Discussion and Conclusion

This study provides insights into the under-researched topic of the motivations of individuals for pursuing graduate studies in mathematical sciences. Across 74 personal statements of individuals applying to master's and doctoral programs, five primary reasons for applying to graduate school emerged: Knowledge Pursuit, Research Aspirations, Career Goals, Growth as a Teacher, and Appreciation of Mathematics. These themes align with and extend prior research.

Consistent with existing literature, applicants' essays reflected both internal and external motivations. The prominence of *knowledge pursuit* and *research aspirations* suggests that mathematics applicants tend to frame their decisions around intellectual growth and scholarly engagement (e.g., Teowkul et al., 2009). *Research aspirations* as a reason for applying to graduate school was more prominent in PhD applications compared to master's, underscoring the appeal of academic research but also the differences in future professional trajectories (e.g., Wiegerová, 2016). As in other fields, *career* motivation was also frequently mentioned (Incikabi, 2013; Kowalczuk-Waledziak et al., 2017). However, in our study, the applicants often described career goals as integrally tied to knowledge pursuit and research aspirations rather than a stand-alone objective. Many applicants envisioned careers in academia, highlighting the link between mathematics graduate education and pathways into teaching and research (Tennant, 2004).

The theme of *growth as a teacher*, though less frequent, was apparent. Consistent with previous findings, applicants described a desire to improve their pedagogical practice (Kowalczuk-Waledziak et al., 2017). While this theme was most prominent among applicants in education fields, such as the Master of Science for Teachers and PhD in Mathematics Education, it appeared in applications across all disciplines, with the lowest being Statistics MS (Figure 1). Many individuals articulated a goal of addressing classroom challenges they had encountered in either formal or informal settings, and expressed a commitment to making mathematics more accessible. This suggests that for some applicants, graduate study is not only about advancing their own expertise but also about transforming their role as educators.

The novel emergent theme was a*ppreciation* of mathematics. Although prior research has identified intellectual curiosity (London et al., 2014; Teowkul et al., 2009) and personal fulfillment (Kowalczuk-Waledziak et al., 2017) as drivers of graduate study, these constructs only partially overlap with the present findings. Applicants who expressed appreciation often framed mathematics as a source of "beauty" and intrinsic value. This deep-seated passion frequently co-occurred with other motivations, indicating that research engagement and career planning are often interwoven with an enduring appreciation for the discipline itself.

Our findings help broaden understanding of applicants' decisions to pursue graduate degrees in mathematical sciences, showing that these decisions are driven by a complex interplay of reasons. Applicants view themselves as future professionals and members of a disciplinary community tied to knowledge, research, teaching, and appreciation of the field. Understanding these motivations can inform targeted recruitment, advising, and mentoring strategies. In practice, these results can guide admissions committees, advisors, and program designers by illuminating the values and aspirations students bring into their graduate studies.

Some limitations should be acknowledged. First, the dataset is drawn from a single mid-sized public research university, which may limit the generalizability of the findings across institutional contexts. Applicants' personal statements also reflect self-presentation strategies shaped by the admissions process; motivations expressed here may be idealized or selectively emphasized. Our study is ongoing, as we continue to investigate how these motivations differ across demographic groups and programs. Future research should also examine how these motivations shift as students progress through graduate school.

Additionally, we aim to understand the types of past experiences—whether positive or negative—described by the applicants in our study that shape their decision to pursue graduate studies in the mathematical sciences. Furthermore, investigating how mathematics identity interacts with these motivations may provide deeper insight into the pathways that lead students into advanced study in the discipline.

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Living as Abstract Nonsense: Ontological Tensions in Trans Experiences of Mathematics

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Ontology—the recognition or erasure of ways of being—is a justice issue in mathematics, intersecting with race, gender, sexuality, and disability. Despite growing scholarship on liberatory ontologies in mathematics education, little is known about how such paradigms might be enacted in postsecondary contexts, or how trans mathematics-doers experience ontological tensions uniquely. We address this gap by analyzing a mathematical representation of trans experience created by a transfeminine, graduate student of color in mathematics, and the ontological tensions it draws from us as trans mathematician-researchers. Our findings identify ontological erasure, disembodiment, and categorization as features of mathematics that conflict with trans ways of being. We propose ontological possibilities for postsecondary mathematics that affirm trans and other marginalized realities, and invite further exploration of postsecondary mathematics as a site for ontological justice.

Keywords: ontology, trans mathematicians, gender, equity, embodiment

Ontological Justice in Mathematics Education: Literature Review and Research Aim

Settler colonial paradigms embrace dualist framings, such as mind/body, human/nature, and man/woman (Kovach, 2021). These dichotomies may perpetuate justice issues related to ontology (how ways of being are recognized or erased). For example, settler colonial dichotomies related to sex and gender have historically framed Indigenous diversity in sexuality and gender as primitive (e.g., Morgensen, 2010) and been used to punish Black people through violations of their gendered and sexual self-definition (e.g., hooks, 1981).

Settler ontologies underpin dominant views of math, reinforcing marginalization by race, gender, sexuality, and disability (Butler, 2024; Gutiérrez, 2017; Martino & Cumming- Potvin, 2018; Westby, 2021). For example, math is framed as an intellectual pursuit, which is valorized in the hierarchical dichotomy of mind and body. Societal portrayals of Black people as primal and hypersexual rather than intellectual (McNeill & Jefferson, 2024; Strings, 2020) enact the mind/body and human/nature dichotomies, shaping dehumanizing conditions in math education, the educational outcomes of which perpetuate Black dehumanization. (McNeill & Jefferson, 2024; Joseph et al., 2019). Cultural ways of being can also be marginalized through engagement with math content. For example, since math is often construed as intellectual, its teaching can sideline physical and spiritual presence that is vital to Indigenous ways of knowing (Gutiérrez, 2017). Similarly, binary parities in math can naturalize cisheterosexuality to alienate queer students whose being is implicitly construed as mathematically illogical (Leyva et al, 2022).

A growing body of research theorizes diverse ontological possibilities in mathematics (e.g., de Freitas & Sinclair, 2013; Gutiérrez, 2017, 2022). For example, seeking to honor materiality and embodied knowledge in school mathematics, de Freitas and Sinclair propose teaching the number line as an elastic line that can be grabbed and stretched "so that it brings forth an infinitude of numbers that were imperceptible a moment earlier" (p. 465-466). At the postsecondary level, emerging scholarship in mathematics (e.g., Ardila-Mantilla, 2016; Paton,

2025; Cheng, 2025) integrates mathematical and relational ways of being and knowing to advance equity. For instance, Paton challenges both the mathematical and social value of equality, framing category theory as a paradigm for equivalence that embodies equity. These emerging disruptions of dominant mathematical ontologies reflect a demand for liberatory ontological possibilities in postsecondary mathematics education research.

Ontological tensions can be particularly salient for trans[1] people, as we navigate gendered societal assumptions and expectations that are often incongruous with our internal realities. However, research knowledge about trans ontologies in mathematics is limited. Two studies of trans experiences in mathematics exhibit ontological themes, but report on social dimensions of undergraduate contexts (e.g., gendered assumptions of mathematics ability, positive experiences in collaborative learning environments; Cassiday, 2019; Kersey & Voigt, 2021). For example, Kersey and Voigt asked students to create graphs of their trans mathematics experiences with axes representing masculinity/femininity and cisgender non/conformity. However, they did not examine how students' trans being shaped their engagement in the mathematical task.

Ontological tensions can be a generative and powerful informant for liberatory ontological possibilities in mathematics education and beyond (Anzaldúa, 2002; Gutiérrez, 2012, 2016; Myers et al. 2023). Towards theorizing ontological possibilities, we draw from a study of trans mathematicians' experiences, centering AT, a transfeminine graduate student of color. We analyze a mathematical representation of her intersectional experiences as a trans mathematician, queering the researcher–participant boundary by examining resonances with our own ontological experiences (Heckert, 2016; Nash, 2016; Rooke, 2016) as trans mathematicians. We pursue the research question: How does AT's representation illuminate, and invite us to collectively inhabit, generative ontological tensions between mathematics and transness?

Theoretical Perspectives

We ground our work in gender theory (Butler, 2011), queer theory (e.g., Muñoz, 1999) and new materialist ontologies (deFreitas & Sinclair, 2013). Queer theory critiques and destabilizes sex and gender binaries in culture, society, and politics to "offer alternative ways of thinking and performing in the world" (Voigt, 2022, p. 4). However, alternative, liberatory ways of performing gender are not free from embodied and political realities. Addressing common misinterpretations of "gender performance" (Butler, 2002) as independent from corporality, Butler (2011) prompts us to consider how the sexed body is socially constructed. Bodies "only appear, only endure, only live within the productive constraints of certain highly gendered schemas" (Butler, 2011, p. x) and become intelligible only under certain conditions. The act of imposing intelligibility to the body as either male/man or female/woman creates a category of being which is unintelligible by dominant society, fueling (trans) oppression through limited social and political recognition (Butler, 2011; The White House, 2025). This societal exclusion from intelligibility defines "foreclosures and repressions that constitute the discontinuity and incompletion of the [trans] subject" (Butler, 2011, p. 141), framing some bodies as "unthinkable, abject, unlivable bodies" (p. x). Relationality between the body and society also extends past gender: its interconnections with race, in particular, structure which bodies (often white ones) are rendered legible in their rejection of cisnormativity, and which (often non-white bodies) might *disidentify* with a politicization of gender that presumes whiteness as normative (Muñoz, 1999). We leverage these ideas of gendered, raced, bodily intelligibility to consider how trans ontology may resist governance by political-mathematical cisgender binaries, especially as experienced by

[1] We use *trans*, abbreviated from transgender, to reflect language used within trans communities.

AT, a trans mathematician of color. Drawing from Muñoz's (1999) disidentification as "productive interventions in which politics are destabilized" (p. 102), we name this ontology as a gendered experience in a perpetual state of flux between (dis)embodiment and (dis)identification.

We engage new materialism to deepen our exploration of mathematics as a dis/embodied, dis/identified gendered experience. New materialism frames matter and agency as intertwined and cocreated (deFrietas & Sinclair, 2013). Providing both non-human and human examples, deFrietas and Sinclair contend that the uncertainty principle in quantum physics demonstrates that "indeterminacy is intrinsic to matter" (p. 460) and similarly "the [human] body is…[not] passive matter upon which culture is inscribed... it is centrally implicated in the constitution of itself and society" (p. 460). So too is matter's relationship to mathematics. The body can be viewed as "an assemblage of human and non-human mathematical concepts" (p. 453) that, in turn, resonates with and extends mathematical ideas. Recognizing how the body both creates and is constituted from mathematics provides a bridge between queer theory, mathematics, and ontology, creating mathematics and trans being as together fluid, ambiguous, and queer.

Methods

In alignment with the study's aim of legitimizing trans ways of knowing and being in mathematics education, our inquiry drew from queer and embodied methodological paradigms.

Context, Researchers, and Participants

This analysis comes from a larger study of trans mathematicians' experiences, inclusive of those who completed at least one year of a U.S.-based doctoral program in the mathematical sciences. In early 2025, we recruited participants through professional mathematics organizations that centered marginalized identities (e.g., Spectra, Lathisms, Sines of Disability). We selected 20 participants, prioritizing those with multiple marginalized identities. Our research team consists of five white trans people (two faculty, two doctoral students, and an undergraduate).

We selected our focal participant, AT (she/they/it), because their mathematical representation was uniquely evocative in its interplay between mathematical and trans ways of being. AT is a transfeminine graduate student of color who holds additional marginalized identities. AT was consulted about identifiability risks, and a draft was shared for her review.

While data were collected by the full team, analysis and writing were completed primarily by the first three authors. These authors identify as mathematicians, spanning graduate to faculty experience in mathematics. Queer methods embrace researcher subjectivity and blur boundaries between researchers and participants, establishing credibility through reflexivity (Heckert, 2016; Nash, 2016; Rooke, 2016). We leveraged our lived experiences as mathematicians to occupy this researcher-participant boundary. McNeill (they/them) is a femme trans man with a history of mental illness and disability. Davis (they/them) is a neurodivergent, nonbinary transmasculine graduate student. Black (she/they) is a queer, transfeminine, neurodivergent, able-bodied graduate student. The team acknowledges racial power dynamics inherent in white scholars drawing analytic conclusions from narratives shared by a participant of color. Informed by analytic autoethnographic methods (Anderson, 2006), we make our positionalities transparent throughout the study to acknowledge our subjective influence on the knowledge it produces.

Data Collection

Participant Data Collection. We conducted, recorded, and transcribed two 90-minute interviews over Zoom. The first interview explored participants' trans and mathematics histories. Afterwards, participants engaged in an open-ended, reflective exercise—to create a mathematical

representation of their intersectional experiences as trans mathematicians. The second interview asked participants to narrate their representation, and probed further into connections between mathematics, transness, and intersecting identities. We present AT's representation in Figure 1.

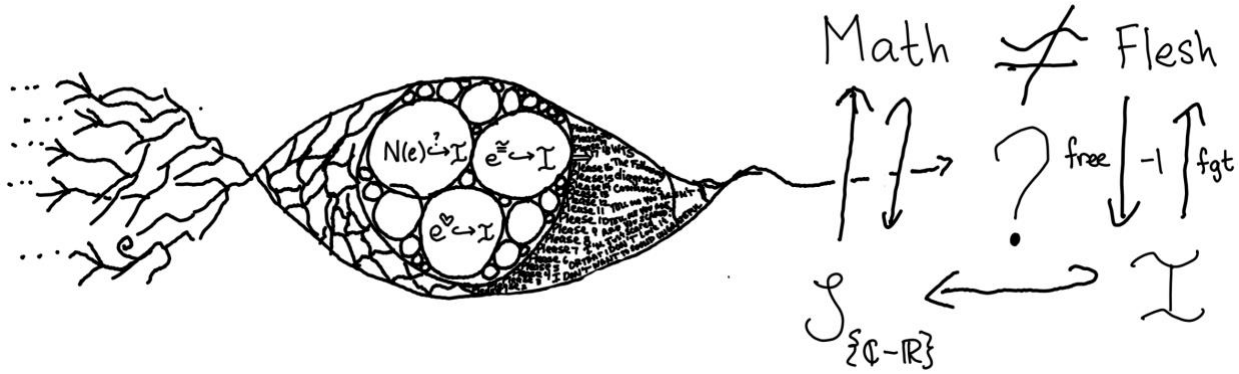

Figure 1. AT's mathematical representation of their intersectional experiences as a trans mathematician. AT's handwriting has been replaced by McNeill's to retain authenticity while protecting AT's anonymity.

We briefly explain AT's representation to contextualize later findings. The right side is a (potentially) commutative diagram. AT uses *I* to represent the self and conceptualizes the self as flesh with extra structure. There is a "free" map that includes the flesh into the self, and the forgetful functor maps the self to the flesh. The self is mapped injectively into an imaginary (the complex numbers without the real line) realm. Math is also included in the imaginary. On the left side is an eye, whose iris contains circle packings. The large circles pose possible injections of semi-mathematical objects (e.g., $e^{\heartsuit}$) into the self. Lines branching from the eye on the left represent both nerves and eyeliner. The right side of the eye contains the words "WTS [wish to show] The Following diagram commutes. TELL ME YOU AREN'T. ARE YOU FLESH? TELL ME YOU ARE. ARE YOU SCARED? I'M Just Scared. OR THAT I DON'T Love it. I DIDN'T WANT TO SOUND UNGRATEFUL" (capitalization in original). The right side of the iris is lined by 20 numbered instances of "Please."

Researcher Data Collection/Analytic Note-Taking. The last stage of data collection and first stage of analysis consisted of (embodied) analytic note-taking. The first three authors each read AT's transcripts alongside its representation. In alignment with our research question about ontological tensions we collectively inhabit, we each recorded our embodied experiences of engaging with AT's data (e.g., emotions, physical sensations), and the personal experiences and stories to which these embodied responses were attached, through in-line analytic note-taking. We intentionally brought awareness to our racial experiences and observations to mitigate race-evasive interpretations of AT's experiences. As empirical inquiry (Richardson, 1994; Richardson & St. Pierre, 2005), this writing externalized our thinking, fostering "adventure and experimentation" and bringing forward 'the unintelligible' (St. Pierre, 2018, p. 605) that conventional forms of data analysis often miss. We combined our notes into one document, AT's transcript augmented with researchers' reflections placed in their original locations. We color-coded each person's contributions to preserve the distinctness of our perspectives.

Data Analysis

Guided by queer and embodied methodologies' attention to researcher subjectivity, we conducted a reflective thematic analysis (Braun et al., 2023). To address the research question about how AT's narratives illuminate, and invite us to collectively inhabit, ontological tensions

between mathematics and transness, we first coded the augmented transcript for connections to ontology. We then identified collective thematic tensions. Guided by our positionality, we selected themes that were salient in AT's narrative rather than those driven mainly by researchers' reflections, preserving the intersectional complexity of AT's ontological tensions even when intersectional dimensions were not mirrored in the researchers' realities.

Findings

Analysis revealed three collective ontological tensions between mathematics and transness: mathematics is construed as real while trans people are construed as imaginary, mathematics is positioned as disembodied while trans people are made especially aware of their bodies, and mathematics is categorical in nature while trans self-understandings are often anti-categorical.

Imaginariness: "Weird Nonsense that has Implications… with a Capital 'I'"

AT's maps of the self and mathematics into non-real, complex numbers suggests a parity between the imaginariness of her being and mathematical truth. However, she witnessed a social validation of mathematics as real that she did not experience in relation to their own existence. Its sense of "not being real…, being imaginary, just that sense of unreality or being invisible" was sustained by the marginalization it experienced as a woman, trans person, and religious minority in a largely cisgender, male, secular field. They described frequently being "talked over" or having their bodily needs neglected through professional mathematics structures (e.g., conferences where they could not safely use the restroom, speaker dinners during religious observances). Interactions reinforcing this ontological erasure were largely unintelligible through a dominant mathematical lens, which failed to connect such experiences to AT's identities: "you can't repeat the social interaction in a double-blind exam or anything." Our resonances with AT's invisibility communicated implicit pressures to legitimize transness in math, questioning "What aspects of [my gender presentation] do I need to erase" (Davis) and noting that the lack of recognition for non-binary identities "positions me as non-real on both sides" (McNeill).

This gendered invisibility contrasts the invisibility of mathematics. AT recognized much of mathematics as "abstract nonsense" that was widely legitimized. Black noted, "to be not real in a mathematical sense carries a sort of intriguing depth,... vision, and excitement; meanwhile, to feel not real as a person… carries a mournful and sad heaviness,... disassociation or… disconnection." This represented a double standard, as McNeill commented:

> A lot [of] mathematics… prov[es] the existence of… things [that] are… not tangible… Things that are tangible, like the body, paradoxically, do not really exist in mathematics. The supposed objectivity of mathematics renders the subjectivity of the body invisible.

This double standard materialized through experiences of misogyny. AT described a traditional view that "mathematicians are discovering secrets of the universe," which evokes a sense that "math [i]s this higher power or perfect form of knowledge and logic" (Davis). AT characterized mathematicians holding this view as "super misogynistic… they're like, 'Yes, I'm better than you.'" This power imbalance renders mathematics as real and trans existence as imaginary. Yet, she described the field as "moving away" from this view: "the gays" in mathematics often said, "look at this weird little thing I have made… Can I show you my arrows?"—a framing that presents mathematics as a human endeavor and may open more ontological possibility.

(Dis)embodiment: "Does Everyone… Think about Flesh… as Much as… a Trans Person?"

AT's ontological tensions with the disembodied framing of mathematics stemmed from the interplay of three factors: her need for positive connection to her racialized, transfeminine body,

the abstract nature of her subfield, and her recognition that mathematical language often borrows flesh-related terms. Category theory uses many flesh-related words, which tempted her to force a mathematical understanding of her own body, as evidenced in AT's mathematization of their embodiment, considering the self as "flesh with extra structure in a monoidal category".

AT's reconciliation of (dis)embodiment in mathematics was both playful and poignant. For example, in their representation they denoted the "heart of a category" with the symbol ♡. They also described playing a game with non-category-theorists: "Is this a math word or am I bullshitting you?" using "body-related things" like "hearts", "skeletons", "cores", and "nerves." Black felt especially "tickled" to learn of category theory's "Theorem of the Heart". Despite these linguistic connections, AT's embodied experience remained mathematically unintelligible. Considering the heart of a category, she questioned, "[Are] these structures inside of me?" and lamented how "math isn't helping [her]… in terms of understanding [her] relationship to the flesh." Resonating with AT's sentiments, the authors noted that "I can touch a knitted sweater in a way that I can't touch my math homework" (Davis), which contributed to ontological disconnection as mathematicians who "experience the body only through metaphor" (McNeill).

AT's sentiments illuminate liberatory ontological possibilities of welcoming the body's subjectivity in mathematics. AT perceived that mathematicians' appeal to the body is neither a coincidence nor an accident. For example, they argued that ♡ is "much more indicative" of the essence of the heart of a category than the "abstract instructions" of the formal definition ("...the truncation above by zero and the truncation below by zero…"). Furthermore, AT noted that "people in math… lov[e] going on hikes and rock climbing… they're also connecting to their bodies even if they're cis[gender]," which she conjectured might represent a universal need among mathematicians to be more connected to their physicality. Moreover, McNeill noted that embodied subjectivity is inherent to mathematical thought: "I think with a piece of meat. We all do. How objective can meat be?" We collectively considered the political possibilities of recognizing such subjectivity. For example, AT noted that mathematics is "heavily funded by the NSA," an organization that placed their community under heightened surveillance. A more embodied mathematics might resist such forms of dehumanization.

Categorical (Un)Intelligibility: "This Doesn't Make Any God Damn Sense"

AT's visual references to math, flesh, and mathematical "unreality" illustrate its desire to categorize its world, identity, and mathematical experiences in logical ways. By representing itself in a commutative diagram, its intersectional trans experiences are described through sources and targets. This presents an ontological tension between what can be categorized and her intersectional trans experiences which defy logical structure and categorization.

For example, while discussing the non-equivalent relationship between mathematics and the flesh represented in her commutative diagram, AT explained "math is not flesh, it is sort of different… [I'm] trying to think about how I'm structuring my life." AT's attempt to "structure" her life category theoretically led to her questioning whether her diagram (and therefore herself) "commute up to a higher homotopy." Such sentiments represent AT's impulse to mathematize and categorize her body, self, and mind, which were shared by the authors. For example, Davis described a desire for gendered coherence, expressing frustration at not being categorized correctly in social interactions. They noted that their androgyny "doesn't read for people," which led them to wonder, "what cues am I missing?" In our trans experiences changing names, appearance, or pronouns also reflected the pursuit of intelligibility by cisgender society.

AT's attempts to categorize their existence, however, were in tension with the fluidity they embodied as a creative and gender-transgressive person. Later in the interview, she abandoned attempts to categorize herself, explaining that the difficulties of trying to represent herself mathematically resolved when she chose to treat the commutative diagram as poetry and to "not [be] super worried about the math." However, she recognized this approach opposed accepted approaches to mathematics: "if an actual category theorist saw this, they'd be like, what the fuck have you written?" AT's playfulness and creativity with conventional categorization was also reflected in her nonbinary gender expression where "ambiguity is some part of the point for me." For example, she described dressing in a more feminine way in mathematics contexts where "girliness is not allowed." Similarly, when discussing her pronouns, she described picking different pronouns to use on different days, or inviting fellow trans people to use he/him/his pronouns to refer to her exclusively when "it's funny", saying "If I'm wearing a ball gown, that's a 'he.'" The authors shared AT's desire for incoherency. McNeill expressed a desire to "live apart from the cisgender world," a nod to unintelligibility as freedom from dominant binary categorization (in mathematics). AT's abandonment of the rigor of her commutative diagram, thus, represents our collective search for mathematical freedom and unintelligibility as a tool for self-expression. Overall, AT's attempt to describe itself through the systems of category theory illuminates ontological tensions between its embrace of life as an unintelligible system and its desire for a life that can be described systematically through their field of mathematics.

AT's desires to use mathematics in ways that defied standard mathematical coherency illuminate ontological possibilities for trans affirmation. McNeill noted that "there is something delightfully metaphorical about questioning the existence or coherence of the self in relation to mathematics at higher dimensions" and compared this to aspects of physiology, such as the placebo effect, that "escape understanding." Black also highlighted how, as trans people and mathematicians, there is a "persistent push to interrogate and challenge and wonder about whether or not our existing models of the world are working for us." When existing models fail, we, as mathematicians and trans people, ask "what perceived limitations or assumptions do we have that we need to forget?" These desires explore the possibility of unintelligibility as a mathematical value, resolving ontological tensions between conceptualizations of mathematics as structure and trans realities that often revel in nonconformity and unintelligibility.

Discussion & Conclusion

AT's representation offers a pathway toward trans ontological justice in mathematics, extending theories of liberatory ontologies to reveal new possibilities in postsecondary contexts. Scholarship on ontological justice in mathematics education often seeks to disrupt settler colonial paradigms by contextualizing mathematical ideas within ecological and social imperatives (e.g., Gutiérrez, 2017). Yet, abstract mathematics, such as category theory, defies contextualization. AT's representation offers new directions for disrupting settler colonial, binary paradigms, reminding us that the materiality of the body—and the political contestations that shape its social signification—are ever present in mathematical inquiry. By recognizing "pure" mathematics as a social construction, AT positions mathematics as a flexible human creation whose ways of knowing and being can expand through embodied and social experience, revealing situated and partial truths; mathematics need not be an absolute truth to which our humanities must yield. AT's experiences also highlight that transness marks avenues for *dis*identification from settler knowledge production, illustrating ways of doing and teaching mathematics with decolonial intent. We call on mathematicians and researchers to explore how attending to students' ontological affirmation in mathematics can generate new and diverse educational possibilities.

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Shore to Shore: Students' Progressive Mathematization of Null Spaces

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This study explores how undergraduate students develop reasoning about the relationship between a matrix and its null space through progressive mathematization within an inquiry-oriented linear algebra context. Using a playful hallway map task, students engaged in a paired teaching experiment to investigate how modifying the number of rooms and hallways affects the dimension of the null space. We drew on a river journey metaphor to examine students' movement along and between shores of contextual or formal symbolic reasoning. Findings reveal a dynamic interplay between contextual reasoning and formal symbolic representations, as students moved fluidly between the realistic hallway context and matrix-based interpretations. Through successive horizontal and vertical mathematization, students gradually developed reasoning that approached a reinvention of the rank-nullity theorem. The results highlight the complexity of this journey and suggest further research into providing supports for instruction.

Keywords: Progressive Mathematization, Student Reasoning, Linear Algebra, Null Space

Inquiry-oriented instruction (IOI) is a student-centered teaching approach where the instructor investigates students' reasoning (Kuster et al., 2017). One source of materials for IOI is materials designed using Realistic Mathematics Education (RME) because it leverages realistic contexts, promotes the instructor acting as a guide, and is designed to take students from reasoning about a context to formal mathematical activity (Gravemeijer, 1999). Sometimes this transition might be seamless, where students might have developed a robust enough understanding within a context to begin reasoning outside of the context to general cases. Other times, students might bring in prior mathematical experiences on the topic and this transition might be messier. In these cases, students go back and forth between contextual and formal symbolic ways of reasoning to consider more general cases (Rasmussen et al., 2005). The result of discussing these different paths is a rich interplay where students can build off each other's knowledge taking them on a complex journey that traverses between developing understandings of realistic contexts and formal symbolic perspectives.

In this paper, we detail one such journey in a playful mathematics task designed to supplement the Inquiry-oriented linear algebra (IOLA) task sequence about subspaces (Andrews-Larson et al., 2023). The goal of the activity was to help students develop ways of reasoning about the relationship between a matrix and its null space by drawing on intuition developed in the context of school hallway maps. We present the results of a paired teaching experiment with students and highlight their complex journey of going between contextual and formal symbolic reasoning which ends with students approaching a reinvention of the rank-nullity theorem.

Theoretical Framework and Related Literature

The task sequence used in this study was developed drawing on RME design heuristics. Of particular interest are heuristics related to students' reinvention of concepts and the purposeful use of realistic contexts (Gravemeijer & Doorman, 1999). In such tasks, students move between a realistic context and formal symbolic reasonings through progressive mathematization. Treffers' (2012) progressive mathematization consists of two types of activity: horizontal mathematization and vertical mathematization. For Treffers, *horizontal mathematization*

involves "transforming a problem field into a mathematical problem" (p. 247), where *vertical mathematization* involves making progress within symbolizing more general notions of a concept. In this case, the ultimate goal is to move "from the world of life to the world of symbols" (Freudenthal, 1991, p. 41).

Oftentimes, however, students' progressive mathematization is more complex than simply developing intuition within a realistic context, transferring that to mathematical symbols, then formalizing concepts through those symbols. Unlike Treffers' conception, horizontal mathematization can also present itself as students going from formal symbolic ways of reasoning back to reasoning within a realistic context (Rasmussen et al., 2005). For instance, students could learn row reduction and how to interpret its results and then use this method to reconsider a previously examined context. Similarly, vertical mathematization can take place within a realistic context and is not unique to reasoning within formal symbolic mathematics.

Considering horizontal and vertical mathematization in these ways instead highlights how students can move between horizontal and vertical mathematization within and out of a realistic context. Students do not have to solely work within a given context then shift to and stay within formal symbolic reasoning for progressive mathematization to occur. Rasmussen and Keene (2019) used a river metaphor to highlight "places (islets) where students pause to engage various ways of reasoning…as they explore the river and progress in their mathematical journey." We expand on this metaphor by adding shores to the river which represent the realistic context and formal symbolic mathematics. As students progress down the river, students may choose to work within the context or formal symbols. Perhaps a student is trying to vertically mathematize on the formal symbolic shore but hit an obstacle stalling progress. That student can go back to the other side of the river, or back to reasoning within the realistic context, to anchor what they were trying to formalize within the context. These stopping points along or close to one shore act as opportunities for students to cross the river and horizontally mathematize a concept, where vertical mathematization occurs along one shore. This gradual movement down the river between and along the shores represents students' progressive mathematization.

In this paper, we focus on students' progressive mathematization of the relationship between a matrix and its null space. Research regarding null space specifically is limited, especially research focusing on students' concepts or understandings of null space. Two studies touch on student reasoning about null space, but mostly for other purposes. Wawro (2015) details one student's interpretation of the Invertible Matrix Theorem. This study is not about null space alone but does involve the student describing relationships between null space and other attributes of matrices or collections of vectors. Caglayan (2019) examines undergraduate students' classifications of subsets of various vector spaces as either subspaces or not. One of those subsets was the null space of some given vector space. There has been recent work exploring students' conceptual resources while reasoning about null space (Andrews-Larson et al., 2022) and the ways students are positioned while reasoning about null space in the context of the hallways task (Andrews-Larson et al., In Progress; Smith et al., 2025). Because student reasoning about null space is understudied, we use this topic as the subject of this research study. More specifically, we are interested in the ways students move between reasoning about null space within the realistic hallway map context and formal symbolic reasoning about null space. Our research question is: how do students develop their reasoning about the relationship between a matrix and its null space through progressive mathematization?

Methods and the Hallways Task

Methods of Data Collection

During Summer of 2024, our research team conducted a five-day paired teaching experiment (PTE; Steffe & Thompson, 2000) over Zoom with three participants enrolled in a differential equations course. The third author acted as the teacher-researcher for all five interviews, while the first and second authors took turns acting as the observer-researchers. While the teacher-researcher was engaged in the in-the-moment interactions with all three participants, the observer-researcher was taking field notes, or asking clarifying questions when needed.

When initially designing the study, our research team was curious if participants would position each other differently based on perceived experience with the content (Smith et al., 2025). This was relevant because two of the three participants (Henry and Tyler) previously completed linear algebra before the first interview. We focus our attention on discussions between Henry and Tyler because they represent two different journeys down the river of progressive mathematization. By the start of the fifth day of the PTE, all participants had completed the subspaces unit of the IOLA curriculum, which we refer to as the hallways task.

The Hallways Task

The hallways task has three parts (See Wawro et al., 2013). The first involves the notion of closure under linear combinations of camera data vectors, where 5D vectors represent paths from classrooms in the hallway map pictured in Figure 1. Participants reasoned about the set of camera data vectors that emerge from sending participants from room C back to room C as a set closed under linear combination. Participants referred to these paths as "loops", which became a common contextual term for this set of vectors. In Task 2, participants began to associate camera data vectors with room population change vectors by considering how many people move past certain cameras between each period. This association often led to a discussion of how a system of equations can be represented by a coefficient matrix. The span of the "loop" vectors resulted in a set of vectors that resulted in no change in population because they corresponded to a path that started and ended in the same room. Finding all the "loops" became a way for participants to conceptualize the null space. Finally, in Task 3, participants were provided with a 5 by 7 matrix and asked to create a new one-way hallway map that would correspond to this new coefficient matrix.

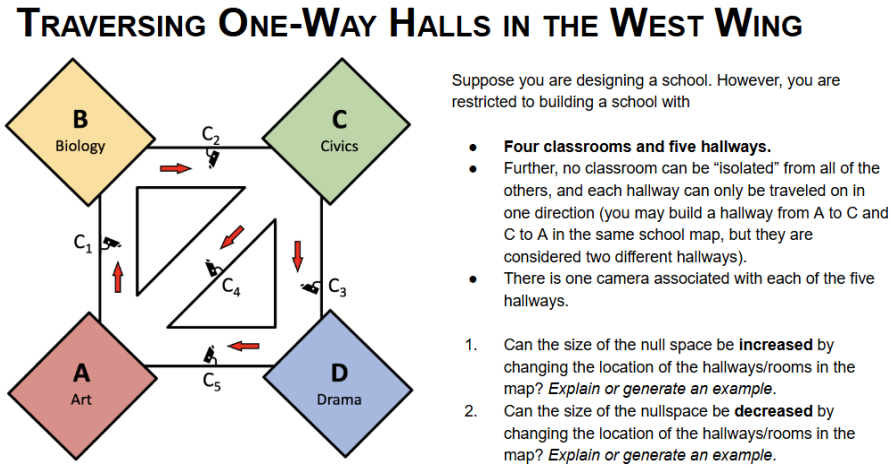

Figure 1. The hallway map from the hallways task sequence with the first Day 5 task statement.

Our team designed an additional task as a follow-up to the hallways task sequence stated at the right of Figure 1. Our initial design was related to the development of a playful math task (Ellis et al., 2024), where participants first attempt to change the dimension of the null space of the original coefficient matrix associated with the hallway map. Then, participants became "structural engineers" tasked with developing a new map where they could change the number of rooms and hallways to determine the impact this had on the dimension of the null space of the associated matrix.

Methods of Data Analysis

Our initial analysis of the data involved reviewing the Zoom recordings as a whole team. We began by noticing instances of collaborative reasoning between the pairs of the three participants. We then highlighted these specific excerpts as examples of collaborative reasoning individually, then coded these excerpts in pairs to develop our collaborative reasoning actions (Smith et al., 2025). We examined the ways participants made space for, acknowledged, or leveraged each other's ideas. For this paper, we reanalyzed the video data from the fifth day, focusing more on participants' mathematical reasoning. We noticed Henry and Tyler were both horizontally mathematizing (i.e., going between reasoning within the realistic hallways context and the more formal matrix representations) and looked to further explore what started this movement and what happened because of this movement. Note that we use the term "realistic context" to represent the context chosen by the designers of the task sequence. We examined two more salient shifts between realistic context and formal mathematical symbols, adding context by noting the catalyst for and results of these shifts. By doing this, we studied Henry's and Tyler's progressive mathematization.

Results

Part 1: Keeping the Number of Rooms and Hallways the Same

At the beginning of Day 5, participants were trying to change the dimension of the null space by generating a different hallway map with four classrooms and five hallways. Tyler was the last to share his hallway maps after some individual working time. Tyler's first map demonstrated his attempt to increase the dimension of the null space by increasing the number of loops:

> Unless I violated a rule I'm unaware of, I think I was able to decrease it by just decreasing loops, but I was not able to increase it. I thought I did, but then I noticed two of the loops just added together to be the third one. So…it's not actually a third independent vector…so I haven't really increased the span at all.

Tyler realized that he had not "increased the span" since the camera data vector that represented the "new loop" was a linear combination of the original two (e.g., <1, 1, 0, 0, 0> + <0, 0, 1, 1, 1> = <1, 1, 1, 1, 1>) as pictured in Figure 2. Next, Tyler described his success in decreasing the dimension of the null space by decreasing the number of hallway loops from two to zero:

> There's no loops here on the right. I just want to make sure this didn't violate any things…does there have to be one main loop? Do you have to be able to go throughout the school? Cuz obviously here you kind of get stuck at D, which that's no good. So, I don't know if that's against what we're doing here. If I changed any of them, I would get a loop…. I'm more confident in my inability to increase the null space.

Tyler's continual attention to the "rules" of the task, and his concern with violating them was an indication of his care in contextualizing his mathematical reasoning. It is interesting to note that the task states that students would be able to get from one room to any other room, but as Tyler said, "you're gonna get stuck" in room D. If one were to row reduce the coefficient matrix

associated with Tyler's map, there would be two free variables associated with loops that travel backwards on hallways, which is also "against the rules" of the task.

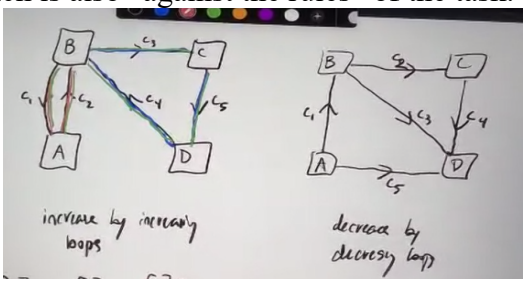

Figure 2. Tyler's maps with two loops (left - 2D null space) and no loops (right - 0D null space).

Tyler's contribution led to Henry discussing the number of rows and columns of the coefficient matrix, as represented by the number of classrooms and hallways in the map:

> I think I agree with Tyler, because if you think about the matrix representation then the most amount of linearly independent variables you can have is the number of rows that you have which corresponds to the room distributions. Since you have four rooms, then that means that you can only have four linearly independent variables. Then the amount of free variables would be the number of weights minus the number of independent variables. If we have five weights, then the most linearly independent variables you could have is four, then that would have at least one loop or something like that…I think that assumes though that it has to be onto, in which case, if it doesn't have to be then you could add more and more zeros.

Henry's approach of using the greatest number of pivots by comparing the number of rows to columns was accurate. However, when re-presented with Tyler's hallway map with no loops (see Figure 2), Henry struggled to reconcile the initial statement with his context-based interpretation of the dimension of the null space as a number of hallway loops. Henry went on to say, "I don't have a solution to my original problem, but the largest possible free space you can have is five, so you would have a 5D null space." This led to a discussion of a contextual interpretation of a matrix of all zeros, with Henry stating, "I'm not sure how to reconcile the matrix representation with the physical layout." Tyler goes on to respond:

> I think this is all just connecting to my sort of lingering interest in how we're actually making these things from matrices. Because it seems sort of convenient whenever there's like a -1 and a 1, like those pairings to just draw it like that. But I don't see what's like absolutely making us do it like that. Like I'm sort of thinking. Based on how we've been doing that would be no rooms and no, well not even. It would be like…a bunch of hallways to nothing.

Tyler appeared to be leveraging a zoomed-out perspective of coefficient matrices to question the impact of the restrictions placed on the entries of the coefficient matrix by the task sequence (e.g., the -1 1 pairs within each column). The group continued to consider what a matrix of all zeroes would represent in the context of a hallway map before working on the second task.

During this part of the task, we saw Tyler start his journey by reasoning within the hallways context, then horizontally mathematized by crossing the river to formal symbolic notation. This horizontal movement took the form of Tyler trying to generalize what he was noticing within the hallways context. Henry, however, started on the formal shore but could not make sense of Tyler's findings in relation to his more formal reasoning

Part 2: Changing the Number of Rooms and Hallways

For the second task, participants were positioned as structural engineers and allowed to make edits to the map (e.g., adding rooms or moving and removing hallways) and determine the impact this alteration had on the dimension of the null space. After Henry presented his initial map, he presented two more maps that had five classrooms instead of four:

> Tyler said something about adding rooms and I thought was interesting. So, thinking back to the matrix, I thought that would increase the amount of rows that you have…so then I drew the pentagon because that was the simplest geometry in my head to draw. So, then I was thinking about what would happen if you had no null space. So then according to this, you have five hallways and five rooms so then you have a square matrix, so then you could make something with no null space, so then I made the thing on the right.

Henry appeared to be saying that to have a 0D null space you need to have the same number of rows and columns to ensure there is a pivot in every column, which corresponds to the same number of hallways and classrooms, as indicated in Figure 3. However, it should be noted that Henry's second map emerged from a discussion about a contextual interpretation of the zero matrix. A hallway map with no null space would have no loops, meaning that there is no way to keep the population of each room fixed between class changes. A coefficient matrix with all zeros, would have a 5D null space since it would collapse all camera data vectors to the zero vector. So, even though Henry said the right-hand map in Figure 3 would have no null space, we interpret him to mean this was his attempt at constructing a map that had an associated coefficient matrix of all zeros since, "you get stuck in the room you start in."

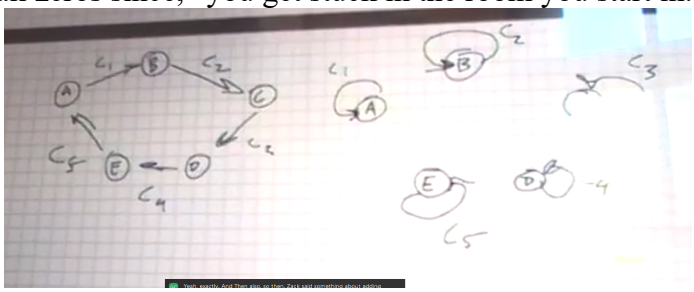

Figure 3. Henry's second set of maps demonstrating 1D null space (left) a no null space map (right).

Similarly to Henry attempting to determine the limits on the possible values of the dimension of the null space, Tyler also attempted to generalize the relationship between the number of rows and columns to determine the dimension of the null space:

> [I] added an additional pathway between C and D going back up. I think that acts as an additional linearly independent vector, so increasing the null space to three. But…I looked at my old one, and sort of realized, I mean yeah if this one has already reduced the null space, then I can just get rid of any of the classrooms or any of the pathways and I can't ever make more loops by reducing, by getting rid of things I already have, so I just said hey I'll start where I'm at and I can remove things to show…you can do either and I've reduced the null space…I'm trying to find out like a rule of thumb for this. But it was sort of confusing because I got there was six columns, and four rows equals two…[I'm] trying to sort of figure out how can we, say we wanted to like minimize how many hallways and things that we're using, how could we do that.

As seen in Figure 4, Tyler was attempting to generalize the relationship between the number of rows and columns and what this meant for the dimension of the null space. However, the difference in the number of rows and columns was not equivalent to the number of loops Tyler

generated with his new map (pictured on the left side of Figure 4). The participants ran out of time to completely finalize their thoughts at the end. However, we view Tyler and Henry's progress in investigating the relationship between the number of rooms, hallways, and loops and the number of rows, columns, and dimension of the null space of the associated coefficient matrix as progress towards reinventing the relationship between the dimension of the null space and the number of rows and columns of the that coefficient matrix.

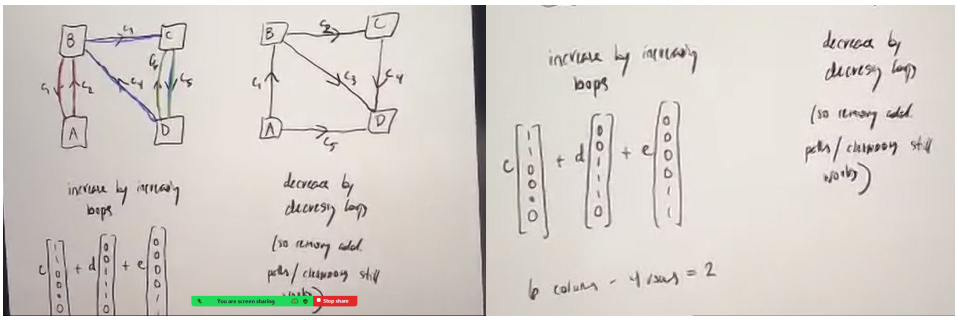

Figure 4. (Left) Tyler's second set of maps with three loops and no loops. (Right) Tyler's attempt at formalizing the relationship between the number of rows, columns, and loops.

Here we see Tyler horizontally mathematize to again cross the river to the formal symbolic side to generalize the relationship between the number of columns, rows, and loops. Henry, on the other hand, horizontally mathematized back to the hallway maps to demonstrate a map that corresponded to his largest value of the null space being five dimensions.

Discussion and Conclusion

In this work, we found that Tyler and Henry had two different approaches to the playful math problem. They were each vertically mathematizing, but on different "shores". The rich discussion between the two resulted in horizontal mathematization, generating connections between shores. For example, Tyler and Henry linked adding columns to adding hallways and adding loops to increasing the dimension of the null space. At the end of the conversation, they were close to understanding how altering the map or matrix impacted the dimension of the null space. For Tyler and Henry, their discussions resulted in successive horizontal mathematization which led to vertical mathematization along both shores of the concept of null spaces.

As both designers and IOI practitioners, we believe that students should be encouraged to revisit and create connections between their formal mathematical experiences and a variety of contextual experiences. This approach paves the way for two new avenues of research. From an IOI instructor's perspective, what are the asset-based approaches to navigate conversations between students who are approaching tasks from opposing shores of the river? From an RME designer's perspective, attention should not only be paid to appropriate entry points but also how frequently the context can be revisited throughout the course and how well these ways of reasoning support the mathematization of later concepts. For example, the hallways context can used to support the guided reinvention of null space as well the rank-nullity theorem.

While this study was limited in number of participants, it charts the start of a journey to a guided reinvention of the rank-nullity theorem. It also highlights the complexity of student reasoning and progressive mathematization down the river. The result of this journey is not only a deep understanding of formal mathematics, but also a rich context from which students can refer to considering new ways to reason about related ideas. Future research will focus on developing this additional activity to supplement the IOLA curriculum.

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Utilizing Discourse Analysis to Explore the Merits of Nonverbal Communication in Higher Education Mathematics Classrooms

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Discursive interactions have been of great interest to researchers studying students' sense of belonging in the classroom. The study of discourse helps us understand positions of authority and agency, as well as creating a common language to describe expected social roles in student-teacher and student-student interactions (storylines). Positioning theory provides the groundwork for understanding how our beliefs about social roles influence how we express ourselves. In each of these areas, verbal communication is typically studied directly, with nonverbal communication used as confirmatory or add-on illustrations. This study centers on instructor nonverbal communication, using empirical data from an observation of a higher-education mathematics classroom across two class sessions. We seek to identify how the instructor's use of classroom space (proxemics) and body movement (kinesics) demonstrates the assignment and shifting of power structures, as well as the enactment of storylines.

Keywords: Positioning theory, discourse analysis, nonverbal communication

Mathematics, as a discipline, is notorious for creating the unfortunate dichotomy of "math people" versus "non-math people" in the classroom. Identifying oneself as a "non-math person" engenders a lack of confidence, a high incidence of math anxiety, and a significant impact on the rate at which students pursue STEM careers (Beilock, 2019; McDermott-Murphy, 2022). According to a study by Bressoud (2015), Calculus 1, a course required for most STEM and some non-STEM majors, had significantly higher drop/fail rates and lower levels of student confidence. One possible finding Bressoud identified was in the connection between student confidence, enjoyment, and a desire to continue mathematics as related to "good teaching" (Bressoud, 2015).

"Good teaching" and "ambitious teaching," according to Bressoud (2015), include qualities such as eliciting student thinking, holding frequent whole-class discussions, helping students become better problem-solvers, and providing time for student thinking. Each of these qualities positions students as co-contributors to classroom knowledge, sharing authority between teacher and student, creating a narrative of "teacher as facilitator of learning" and "teacher as participant in learning" (Drageset & Ell, 2024).

In this study, we follow one instructor, Kim (pseudonym), to examine how he positioned students and the messages he conveyed through verbal and nonverbal cues. After determining the importance of Kim's nonverbal communication through an initial analysis of verbal communication, the research question we sought to answer was: How does nonverbal communication demonstrate the assignment and shifting of authority and the enactment of storylines?

Few studies exist in mathematics education discourse research that describe communication acts beyond gestures. Therefore, this study lays the groundwork for understanding the relationship between verbal and nonverbal communication and examines the merits of nonverbal communication in mathematics education. We first consider the theoretical frameworks employed to understand the verbal and nonverbal communication in Kim's classroom discussions. We then describe our process for analyzing instances of verbal communication,

which led to the discovery of the central role of nonverbal communication. Lastly, we will discuss the implied messages conveyed by Kim's nonverbal communication acts and their impact on student engagement.

Theoretical Framework

Communication is a collection of socially constructed verbal (words, tone, pauses, and inflection) and nonverbal (gestures, gazes, movements) cues that deliver messaging consistent with our contextual understanding. Researchers in mathematics education have studied discourse extensively as common phrases used in the classroom (Biber & Barbieri, 2007; B. Herbel-Eisenmann et al., 2010), as constructors of relationships of power and authority (Langer-Osuna & McKinney de Royston, 2017; Wagner & Herbel-Eisenmann, 2014), or as indicators of social roles and inherent responsibilities (Davies & Harré, 1990; Drageset & Ell, 2024; B. A. Herbel-Eisenmann et al., 2015).

Discursive analysis, using positioning theory, can be applied to both verbal and nonverbal communication to ascertain the subject's beliefs and attitudes. Most discursive analyses have focused on verbal communication (words, utterances, intonation), while using nonverbal communication (gestures, facial expressions, movement) as an add-on to provide further depth to their findings. In this analysis, verbal communication analysis led the research team to a deeper conversation about nonverbal communication. Therefore, we will begin with a discussion of the theoretical basis for our verbal analysis, followed by a discussion of the nonverbal frameworks.

Analysis of Verbal Communication

Categories. Teachers' utterances can be classified into seven distinct statement types based on their intended purpose. For example, bookend statements summarize a previous activity and serve as a springboard to the following activity. Norm-setting and “math talk” center mathematics within the social and contextual setting. Other categories include affective discourse, technical tools, discursive moves, organization, and information. Identifying the purpose of teacher statements allows for comparison across disciplines, instructors, or class periods (Knott et al., 2008).

Positioning Theory. Discourse comprises a collection of communication acts that convey messaging about our perceived roles and the roles we attribute to others. In a classroom setting, for example, a teacher may see themselves as a facilitator of discussion, with students as primary consultants who provide the necessary dialogue to engage with and interact with other students. Students may expect to be the recipients of the knowledge that the teacher provides. In this case, they may perceive themselves as passive observers, whereas the teacher is the subject-matter expert. Both sets of expectations *position* students and teachers into roles based on their individual understandings of the storyline of student/teacher interactions. These roles are negotiated constantly and can change from moment to moment (Davies & Harré, 1990; Drageset & Ell, 2024; B. A. Herbel-Eisenmann et al., 2015).

Analysis of Nonverbal Communication

Several types of nonverbal communication are identified in disciplines such as linguistics, sociology, psychology, and social anthropology. For this study, we focus on two: proxemics and kinesics.

Proxemics. The distance in a conversation varies with the relationship between the speakers. For example, if two strangers are talking, they tend to naturally stand further apart than siblings. Farsani and Rodrigues (2021) highlighted four distances commonly used in discourse: intimate

space (up to 45 cm), personal space (up to 120 cm), professional space (up to 370 cm), and public space (more than 370 cm). In a classroom setting, for example, teachers speak at the front of the class using public space. When speaking with small groups of students, teachers may step closer, into the professional space, or into personal space in more relaxed settings (Farsani & Rodrigues, 2021).

Kinesics. Body position, movement, and facial expressions are all examples of kinesics (Khan et al., 2025). This includes movements such as folding the arms, leaning forward or back, shrugging the shoulders, or using a rolling hand gesture to indicate that the topic continues (or should continue).

Methodology

As part of a larger study, one algebra instructor, Kim (pseudonym), at a Midwestern university was observed twice within two weeks at the end of the Spring 2025 semester. The class was being conducted using the Building Thinking Classrooms methodology (Liljedahl, 2021). In the first observation, the tables were arranged in rows, with some rows near the side walls, leaving little room for movement. In the second observation, tables had been moved into clusters and away from the walls. In both class periods, vertical whiteboards were affixed to the walls, with students clustered in groups around each board. Kim was selected for analysis because of his approach to teaching in higher education, changes in furniture arrangement, and his use of technology to decenter the classroom.

At the front of the room was a table with a technology station, next to which was the screen for the overhead projector. Kim had his laptop set up and alternated the overhead display between a Desmos (now known as Amplify) activity on the laptop and an Apple TV setup connected to his iPad. In this way, he could walk around the classroom, taking notes on his iPad visible to the students, and, during large-group discussions, he would sometimes hand the iPad to a student to operate the technology without walking to the front of the room.

Swivl Robot technology used a recording device on a lanyard to capture audio/video recordings that followed Kim's movement around the room. Four additional recorders were placed at tables around the room to pick up student voices during small-group discussions. These additional recorders could be enabled and disabled during analysis to improve transcript accuracy.

Data Analysis

Following the observation of each class, the resulting audio/video recording was used to create a transcript. During the cleaning process, special attention was paid to nonverbal communication. The transcripts were independently coded into categories and positioning acts by each member of the research team.

The research team then convened to compare their individual coding of the transcripts. When interrater discrepancies occurred, the researchers relied on the context of the instructor's statement to reach an agreement. This involved listening to the instructor's statement in the recording while attending to tone, phrasing, nonverbal cues, and student reactions.

During these deliberations, the research team realized their reliance on nonverbal cues to more accurately categorize the instructor statements in question. While the team had previously noted common hand gestures, the researchers began using more subtle physical movements to gain a deeper understanding of the instructor's statements. This included how the instructor positioned himself among the groups, how he leaned toward or away from a speaking student, and shifts in posture that drew attention to other groups.

To supplement their understanding of these nonverbal communication acts, the research team created a map of teacher movement in 20-minute segments at the beginning, middle, and end of the two observed classes. To do so, a two-dimensional scale of the classroom was created for each observation because the furniture was moved between the two observed class sessions. Researchers then watched the recordings and manually traced the instructor's physical movements through the classroom using different colors to represent the "trips" that occurred during the 20-minute segments. A "trip" was considered to be the path that Kim took before returning to a given starting point, typically the front of the classroom or the podium. Researchers then noted specific patterns that emerged in Kim's movements.

Results

The research team identified four primary categories of nonverbal communication consistently used by Kim: position in physical space, gestures, movement, and pacing. We will discuss the patterns identified for each of these categories before addressing their implications.

Position in Physical Space

Across both observations, Kim used the classroom space differently during small-group and large-group discussions. During the first observation of large-group discussions, Kim spent most of the teaching time at the front of the classroom, asking students in their groups questions and using the vertical whiteboard. In the second observation, Kim used technology to teach from the center and the back of the room, with only a brief period of instruction from the front.

During small-group discussions, Kim adopted several physical positions in relation to the groups. When providing students time to brainstorm and collaborate on ideas, Kim primarily positioned himself in the circle created by students around the vertical whiteboards, as in Figure 1. However, as time progressed and Kim shifted to more direct teaching, he tended to stand behind the groups, either speaking directly to them or over their shoulders as they continued working on their boards.

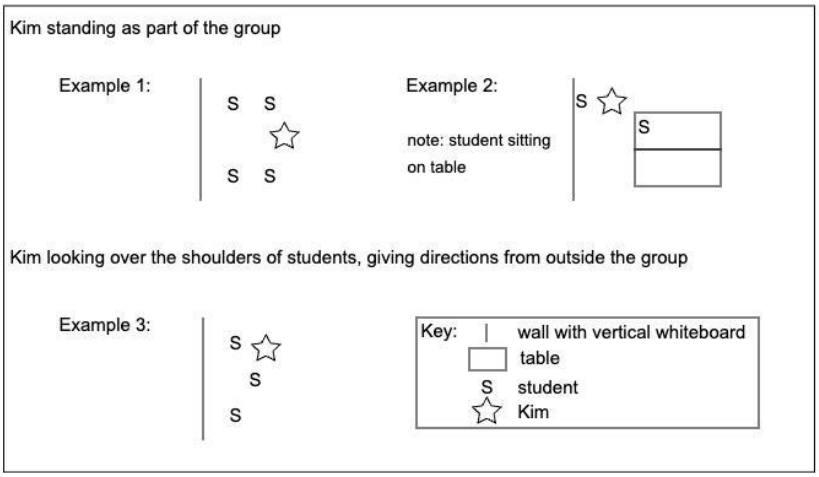

Figure 1. Kim's physical positioning around small groups.

Gestures

Physical movements were used to help students differentiate among ideas and to aid memory recall. When discussing the concept of "balancing," Kim alternated raised hands to indicate the rising and falling of opposite sides of a balance beam, as opposed to the leveling of a bar chart,

demonstrated by a line drawn in the air. Kim repeated these movements with each group toward the end of the small-group discussion, connecting the gestures to the two ideas.

Pointing was consistently used to convey three distinct ideas. Kim used pointing to indicate the person he was directly addressing, often accompanied by a lean inward as if sharing a secret. He also used pointing to signal when he wanted to capture a statement, thereby validating the response. Lastly, he used pointing to direct the students' attention to either another group, a computer, a vertical whiteboard, or the class notes projected on the screen.

One notable use of gestures to direct student attention occurred when Kim crossed his arms, set his shoulders beside a student, and leaned back as if they were both reclining on a couch. The student then followed his gaze and shifted again when Kim moved. This movement had a more lasting impact than a simple point to shift focus.

Another method Kim employed to direct attention was general arm sweeps, such as when he asked a speaking student to "tell them," motioning to the other students in the class. The student looked up and followed Kim's arm around the room before speaking again, a little louder than before.

Movement

A class norm that had clearly been established was to look at other whiteboards when a group became stuck. After conversing with one group, Kim suggested they ask another group, then turned and walked toward the group. The students followed, and when Kim stepped back, the group continued in the desired direction.

In large-group discussions, Kim would turn his body to face the projection screen while students discussed their work, thereby providing a model posture for students to emulate. Student gaze would then shift from Kim to the board, following his gaze and posture.

This happened again when in a large-group discussion, shortly after Kim had been teaching from the middle of the classroom. Kim had asked a student to show the class their ideas on the iPad. The student took the iPad and moved to the center of the classroom, mimicking Kim's movement.

Pacing

During shorter conversations, Kim moved and talked very quickly, setting the tone for the classroom discussion. In small groups, Kim quickly provided validation or correction after an idea was shared. He moved away briskly after a group began discussing his answers to their questions. When releasing the class to their groups, he gave directions and then issued a curt "Let's go" to signal that the students should act quickly. Occasionally, this fast pace meant that he spoke over students as they tried to voice their thoughts. An interesting consequence occurred during a large-group discussion, when a student spoke over him, picked up the iPad, and provided an answer before Kim was ready to relinquish authority. In this case, a student appeared to take on the role Kim demonstrated.

However, this was not always the case. When students were working in small groups, Kim would saunter around the room along a set path, such as a circle in the middle of the room, listening to conversations, reading the students' vertical whiteboards, and then checking online for their responses in the Desmos activity. At these times, he provided opportunities for students to think, collaborate, and formulate their own responses. Maps of these moments showed dark paths along which Kim repeatedly moved in the same direction.

Each of the nonverbal cues discussed above (position in physical space, gestures, movement, and pacing) conveyed messages directly to students, sometimes overriding verbal messages. In

the following section, we discuss the implications of this messaging on the classroom environment and the direct impact on students.

Discussion

In response to our research question, two primary areas of analysis were examined directly: the assignment and shifting of authority and the enactment of storylines. One unintended result became apparent as well: agreement (or lack thereof) between verbal and nonverbal messages.

Assignment and Shifting of Authority

Authority figures are participants who direct attention, whose ideas are taken up, and who have control over the situation (Bishop, 2012). In analyzing nonverbal communication, we can observe the student and teacher posture, physical position, and classroom/group attention to identify authority at each moment. In this study, the shifting or sharing of authority was most clearly identified with shifts in student focus and with students mimicking Kim's behavior.

The researchers found that Kim shifted students' attention solely through nonverbal cues, altering his posture and gaze toward their work. The student mimicked these movements, thus accepting the directional shift without any verbal acknowledgment from Kim. Their change in physical position followed his direction without explicit verbal direction.

On another occasion, he taught from the center of the classroom. When a student offered to answer a question, he handed the iPad to the student and stepped back. Without verbal prompting, the student took Kim's place in the center of the classroom to speak. The authority shifted to the student as Kim stepped aside, de-centering himself.

Storylines

In verbal communication, we talk about storylines of "teachers as providers of insight," "teachers as facilitators of learning," or "teachers as participants in learning" (Drageset & Ell, 2024). When examining Kim's observation recordings, we observe elements of the first two at various points during the class periods.

At times, Kim joined the student groups, standing alongside them in a circle. In these moments, he leaned in when asking questions, gesturing to the student or to the board. As he listened to students, he turned his whole body toward them, frequently with his head tilted to the side. Between groups during this time, Kim circled the classroom at a distance and observed students' conversations. These movements suggested that he participated in the discussion, in which students were the providers of collaboratively constructed knowledge. Our interpretation was validated by his verbal communication acts, which included soliciting students' thinking, highlighting key concepts articulated by students, and encouraging collaboration within and between groups.

At other times during the two observations, Kim took a more direct approach to teaching, standing behind groups to talk over their shoulders and sometimes pointing through students to the board. Listening to his language use, he was providing more direct instruction in these moments, enacting a storyline of "teacher as provider of insight."

While the two storylines were both present, they do not necessarily indicate conflicting messages. Teachers frequently move between storylines during a single class period, taking on different roles and shifting their expectations of student behavior (Drageset & Ell, 2024).

Agreement (or Lack of) in Messaging

In analyzing nonverbal communication through verbal communication, the research team hypothesized that the two would align. Before the observations, Kim stated that he intended to use students' thinking to create a productive learning environment. Classroom norms were established that students would work together, with Kim facilitating their discussion and occasionally providing input. Students were encouraged to try new things, take chances, and participate collaboratively.

During student group work, when Kim followed a set path around the room, observing conversations and reviewing students' work on boards, he created space (both physical and metaphorical) for students to collaborate. At times when students asked for help or when he noticed students were stuck, he facilitated discussion by asking questions and requesting that students provide feedback to one another. These are the moments he moved to a physical position directly in line with the students in the group. When responding to questions, he validated correct responses by pointing to the student who provided the proper response and leaning in toward them. Leaning toward a student in this manner reduces the distance between Kim and the student, shifting from personal to intimate space. All of these verbal and nonverbal cues indicate a shared space between the student and teacher. Students are encouraged to provide information, and the teacher serves as a facilitator.

However, there was a significant period during which the nonverbal cues conflicted with the established classroom norms and occasionally with the verbal cues. During the observations, Kim began to speak over students, moving the conversation along at a quicker pace. He began walking briskly and visiting groups for shorter periods. When he visited groups, he spoke to them over their shoulders and, occasionally, pointed at their board between their shoulders. The faster pace and Kim's removal from the groups led to a shift in the storyline and a new expectation that students respond quickly and accurately, which conflicted with the class's explicit and implicit norms. Although student perceptions were not measured in this study, some students were hesitant to speak in the groups at this time, and many looked to one another while he was speaking, which seemed to indicate some unease.

Identifying conflicting messages provided a mentoring opportunity for Kim, who was unaware of the change in his behavior and its impact on student engagement.

Conclusion

Discourse analysis is an effective method for gaining insight into classroom interactions. While it is tempting to focus primarily on verbal communication, nonverbal communication provides a wealth of information that can highlight hidden messaging or infer interpretations of messaging.

The limitations of a study such as this, however, are that perceptions of nonverbal communication acts differ across cultures, making international classrooms difficult to analyze. Future research may include the cultural contexts of the ethnicities represented by the teacher and the students present. Also, obtaining the teacher's intended messages and the students' perceptions of these communication acts could be helpful.

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A Social Network Analysis of Race/Ethnicity and Gender in Inquiry-Oriented Linear Algebra

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This study investigates mechanisms of equity in active learning by examining student networks in an Inquiry-Oriented Linear Algebra (IOLA) classroom through social network analysis (SNA). Drawing on the sociopolitical perspective, particularly the dimension of power in equity, the research analyzes how race/ethnicity and gender shape students' peer nominations. Data were collected from an IOLA class of 64 students at a Southeastern U.S. university, using discussion board "shout-outs" and replies to track social networks. Findings reveal strong patterns of racial and gender homophily and highlight that women of color occupied disproportionately central positions within the network. While these positions suggest potential empowerment, they may also reflect additional labor on the part of women of color to sustain visibility and influence. The study underscores the need to frame equity in active learning not as inherent but as contingent on power dynamics, solidarity, and segregation within mathematics learning communities.

Keywords: Inquiry-oriented instruction, equity, social network analysis, women of color

Equity in postsecondary Science, Technology, Engineering, and Mathematics (STEM) education remains an urgent issue due to systemic barriers. Interventions often focus on "helping" or "fixing" students, yet such approaches rarely address the structural issues of racism, sexism, and microaggressions that persist in STEM classrooms (McGee, 2020). Mathematics is widely recognized as a gatekeeper in STEM (Battey et al., 2022; Leyva, 2021), where narratives of innate ability and deficit views of minoritized students contribute to inequity. These conditions call for research that investigates how equity is enacted or constrained.

Active learning has emerged as a reform strategy by shifting the focus from lecture to participation, collaboration, and inquiry. Evidence shows that active learning can narrow achievement gaps for historically underrepresented students in STEM (Theobald et al., 2020). However, equity is not automatically guaranteed in active learning. This highlights the need to examine not just whether active learning works but also how equity operates within it.

In light of this necessity, Social Network Analysis (SNA) offers a promising quantitative methodology. By mapping classroom relationships through nominations and interactions, SNA makes visible the patterns of influence that shape student participation (Grunspan et al., 2016). When tied with the critical perspective and the quantitative approach on equity, SNA allows researchers to interrogate how race, gender, and power intersect in active learning classrooms. This study fills the research gap in the intersection between equity, active learning, and SNA.

This study focuses in particular on women of color. Black and Latina women remain significantly underrepresented in STEM fields and often face the "double-bind" of racism and sexism (Ong et al., 2011). Asian women occupy a paradoxical position: while Asian students are frequently stereotyped as overrepresented, Asian women encounter gendered racism that renders them invisible or devalued in STEM spaces (Castro & Collins, 2021). These complexities make the experiences of women of color crucial to understanding equity in mathematics classrooms.

Accordingly, this study investigates "What is the student social network in an inquiry-oriented linear algebra class, and how do race and gender relate to these networks?"

Literature Review and Theoretical Perspective

Equity in Active Learning: Not Automatically Equitable

Active learning refers to a pedagogical approach that promotes students' active involvement. Active learning commonly entails student-centered learning, in-class activities, small group discussions, and challenges to lecture-based and instructor-focused teaching (Freeman et al., 2014; Theobald et al., 2020).

Several studies state that active learning can aid equitable learning. The meta-analysis result by Theobald et al. (2020) shows that active learning courses in undergraduate STEM have lower achievement gaps on historically marginalized groups of students than do traditional lecturing STEM courses. Similarly, Laursen et al. (2014) found that Inquiry-Based Learning (IBL) eliminates a gender gap, based on their quasi-experimental design that compares student outcomes across four universities with over a hundred IBL and non-IBL courses.

However, equity is not inherent in active learning. According to Theobald et al. (2020), one-third of active learning courses in the meta-analysis data had wider achievement gaps in active learning versus passive learning. Also, Johnson et al. (2020) reported a gender gap in inquiry-oriented abstract algebra, with men outperforming women. Additionally, undergraduate mathematics classrooms that implemented small-group work and whole-class discussions involve inequitable experiences reported by women and/or racial minority groups of students, such as microaggressions or sexist and racist events (Ernest et al., 2019; Leyva et al., 2022).

The Sociopolitical Perspective: How to Frame Equity

Given the issues of underrepresentation of historically marginalized groups in STEM education, sociopolitical concepts and theories highlight the critical perspective on underrepresentation and marginalization in mathematics education (Gutiérrez, 2013; Gutiérrez, 2009; Adiredja & Andrews-Larson, 2017). From a sociopolitical perspective, mathematics and mathematics education are political because they involve issues of power and have the potential to challenge or reproduce existing deficit narratives about underrepresented groups of students (Gutiérrez, 2017; Adiredja, 2019).

Specifically, Gutiérrez (2009) suggests framing equity in four dimensions: *achievement*, *access*, *identity*, and *power,* and this study focuses on the ***power*** dimension, one of the dimensions of the critical perspective on equity, beyond gap-gazing. Because of the political attributes of mathematics and mathematics education, power can be measured across various components of mathematics learning, such as voices in the classroom—who gets to talk, who decides the curriculum, and opportunities for students to engage in mathematics learning (Gutiérrez, 2009). Given the need to investigate the mechanisms of equity in active learning, this study suggests a focus on students' power in such settings.

The Social Capital Perspective: How to Interpret SNA

The social capital perspective is adopted in this study to interpret SNA and students' power measures. According to Knaub et al. (2018), social ties are resources such as knowledge, material, and social actors like humans, and there is no free will, so social actors strategically and purposefully seek out others and generate ties. Thus, knowing people is more than just knowing people; it is a matter of privileges and marginalization among students. Particularly, students from historically marginalized groups may have fewer social ties than they want, or may not have them, because they strategically access social ties as coping strategies to engage more actively in class.

Methods

This study was conducted in a 4-credit-hour Linear Algebra course, which used the IOLA setting; 64 students were enrolled, taught in Fall 2023 at Gauss University (the pseudonym), a 4-year public Southeastern predominantly White institution in the United States. The instructor is a tenured White cisgender woman at Gauss University, an experienced IOLA instructor who understands the nature of inquiry-oriented instruction and equity in mathematics education.

Except for two students under 18 and one who opted out, 61 students consented to the use of their written work for research purposes. Then, among 61 students, 38 consented to self-identify their race/ethnicity and gender on the consent form. These 38 students are referred to as "network participants" in this study. Figure 1 shows the race/ethnicity and gender of the network participants. Note that there is no Black woman among the network participants, even though there seem to be some Black women in the classroom, from the classroom observation.

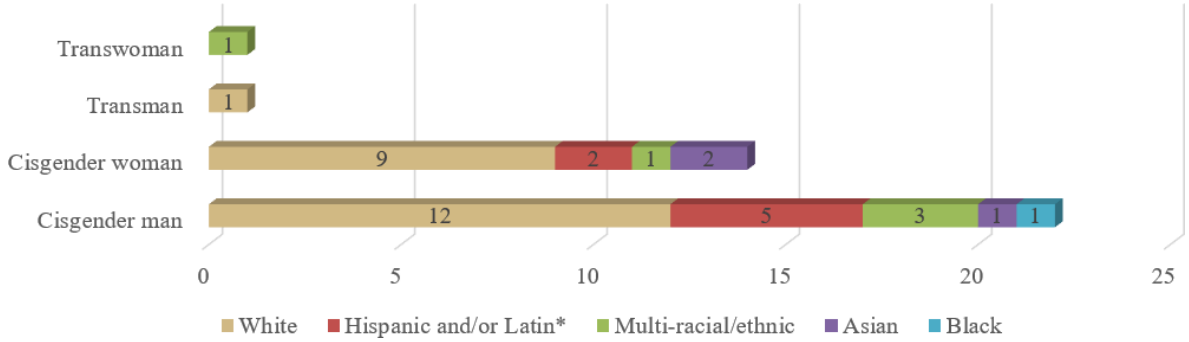

Figure 1. Demographic Information of the Network Participants
Note that students "of color" refer to "non-White," that is, Hispanic and/or Latin, multi-racial/ethnic, Asian, and Black. Latin is an inclusive term intended to recognize the presence and identities of people across the gender spectrum, including gender-nonbinary people of Latin American descent.*

The collected data sources are, first, discussion boards, where students submit online as part of their graded assignments; a survey that asks about students' prior experiences in mathematics; and self-reports of race/ethnicity and gender. The author also attended the IOLA class to observe and take field notes regarding notable patterns of social interactions.

In this study, "nomination" refers to the student nomination assignments on the discussion boards by either "shout-out" or "reply." "Shout-out" means that students nominate other students' names for those most influential in their learning, corresponding to IOLA content topics. Also, students had the option to "reply" to classmates who were helpful with their understanding, and in that case, those replies were considered the same as a "shout-out." There were 14 discussion boards assigned in the semester; 11 of them asked students to "shout-out," and three did not require a "shout-out" but allowed "reply".

SNA was conducted with "Gephi," an open-source graph and network analysis software. The first step of SNA is converting student nominations to matrices. Each matrix represents one discussion board, and students in the network are represented as numbers in the row and column matrix. For instance, for 64 students, each is represented as S01, S02, …, S64, and one discussion board is represented as a 64×64 square matrix. For one discussion board per day, entries in the matrix are either 0 or 1, where 0 indicates no nomination between two students and 1 indicates a nomination. Then, by summing the matrices across days, it shows accumulated networks throughout the semester and allows visualization of the entire network in Gephi.

Then, the number of nominations is analyzed using mean values and standard deviations, box-and-whisker plots of degrees in race/ethnicity and gender, and the Mann-Whitney U-test. Hypothesis tests were also conducted in "DATAtab," an online statistics tester using artificial intelligence. Also, ***homophily*** —the measure of patterns of who relates to whom in terms of similarity in attributes—race/ethnicity and gender —was tested using chi-square.

This study measures four types of centralities to find who is the most central in the network. The first measurement on the social network is ***degree centrality***, which identifies the most nominated students, as it represents the total number of connections. Next, ***Betweenness centrality*** measures how often students bridge paths between two other students. So, a student with higher betweenness centrality is a broker in student groups. ***Closeness centrality*** measures how students connect to others in the entire network, unlike degree centrality, which focuses on direct connections. Therefore, if a student has a high closeness centrality, that student is indirectly connected to other students through nominations, potentially influencing the whole class. Lastly, the ***eigenvector centrality*** measures how much students are networked to well-connected neighbors. So, if a student has a high eigenvector centrality, then that student has access to well-connected student neighbors.

Findings

This part addresses the social networks in an IOLA class and how race/ethnicity and gender relate to those networks generated from student nominations. The social network diagram (See Figure 2) illustrates how students nominate one another.

Who Received and Sent More Nominations? Descriptive Results of Mean Values and Standard Deviation

Note that this part uses the binary groups in race/ethnicity and gender. The racial/ethnic groups are non-White and White, and the gender groups are women, both cisgender and trans, and men, both cisgender and trans. On average, IOLA students received nominations from 3.86 (SD = 2.26) students 7.64 (SD = 5.32) times and sent nominations to 3.86 (SD = 2.66) different students 7.83 (SD = 4.13) times. Regarding gender, on average, women received 10 nominations from around 5 people, which is about 1 more than men did. Women and men sent nominations to around four people, but women sent nominations around ten times, more than men did. This comparison between non-White and White students indicates that non-White students received nominations from around one more person. Still, both groups received slightly more than nine nominations. White students sent nominations to 0.36 different people. However, according to the Mann-Whitney U-test, there is no statistically significant difference between the Non-White and White groups, and between women and men, in how often or how many different people they receive and send nominations to.

Does the Network Have Homophily? Chi-Square Test Results in Race/Ethnicity and Gender

From here, "target" refers to nodes that send nominations, and "source" refers to nodes that receive nominations. Figure 3 shows the Sankey diagrams of the homophily in race/ethnicity and gender. A Chi-square test was performed between the Source Race and the Target Race. All expected cell frequencies were greater than 5; thus, the assumptions of the Chi-square test were met. There is a statistically significant relationship between the Source Race and the Target Race ($\chi^2(1) = 42.39$, $p = <.001$, Cramér's $V = 0.4$). The calculated p-value of <.001 was lower than the defined significance level of 5%. Hence, the Chi-square test was significant. This means that the tendency for White students to nominate White students and non-White students to nominate non-

White students is statistically significant. Also, according to the Chi-square test, a statistically significant relationship exists between the Source Gender and the Target Gender. ($\chi^2(1) = 61.78$, $p = <.001$, Cramér's $V = 0.48$) The calculated p-value of <.001 was lower than the defined significance level of 5%. Therefore, the Chi-square test was significant. This means that the tendency for men to nominate men and for women to nominate women is statistically significant.

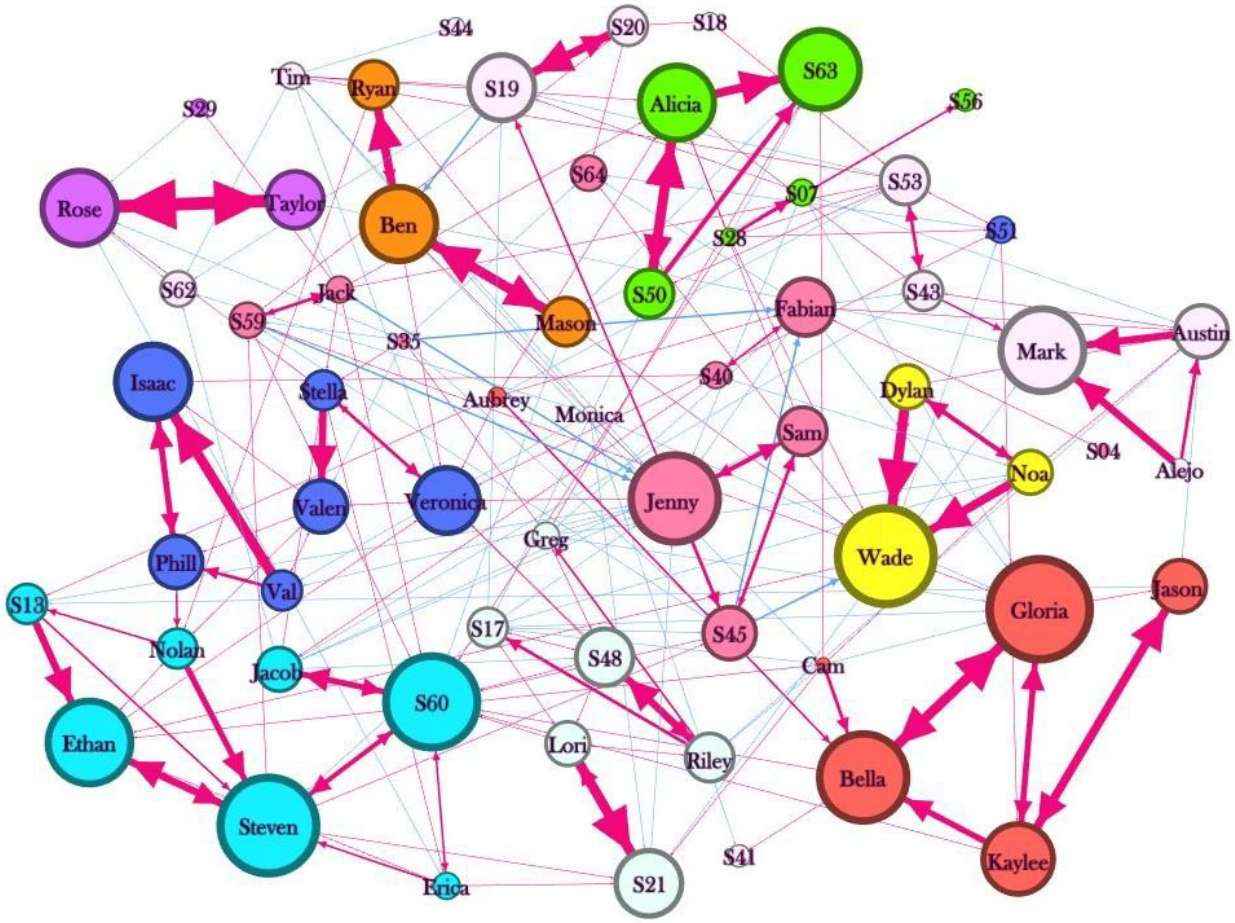

Figure 2. Social Network of the Number of Student Nominations in the IOLA Class at Gauss University.
Note: Circles with pseudonyms are network participants, and circles with codes are students who did not identify their race/ethnicity and gender. Node size represents weighted in-degree—how often the student receives nominations, regardless of redundant people. Node colors represent the small communities of the network determined by nominations. Pink arrows mean nominations by shout-outs, and blue arrows mean nominations by replies.

Who Were the Most Central? Results on Centralities Regarding Race/Ethnicity and Gender

With a comprehensive analysis of the four different centralities, the students of higher centralities include two women of color, who are disproportionately more represented in terms of race/ethnicity and gender in the class. Considering the intersections of the top 6 students (10% of the classroom population) of in-degree, betweenness, closeness, and eigenvector centralities, there are three students among the highest centralities: Jenny, Veronica, and S60. Jenny, a Chinese American cisgender woman, and Veronica, a Hispanic and/or Latina cisgender woman, are women of color. Hence, S60, Jenny, and Veronica not only received nominations from a greater variety of students (in-degree centrality) but also were able to reach other students more

closely (closeness centrality), build bridges with different classmates (betweenness centrality), and connect to other well-nominated students (eigenvector centrality).

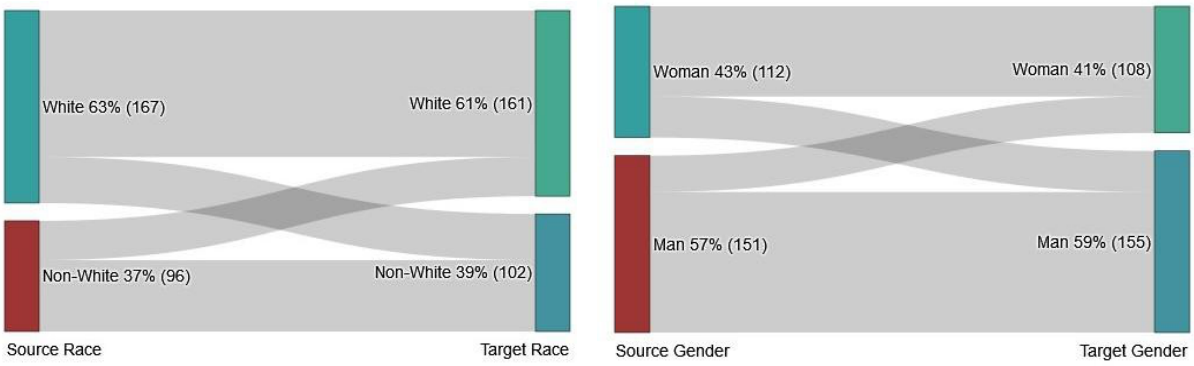

Figure 3. Sankey Diagram in Race/Ethnicity and Gender

Discussion

Limitations of This Study

This study has the following limitations. First, this study could not include Black women as network or focal participants, though Black women were observed attending the Applied Linear Algebra course at Gauss University. So, this study could not document how Black women would have social networks and experience the IOLA setting class. Second, the written works from discussion boards, including student nominations, are collected from only some classrooms because, of 64 students, two are minors and one opts out. Thus, the social network generated by nominations might miss some information. Also, since the network participants are 38 students out of 64, the SNA results related to race/ethnicity and gender might not represent the entire groups of some racial/ethnic or gender in the class.

The IOLA Class Could Imply Equitability

Considering the social capital perspective on the SNA, the results indicate that women of color could construct more empowerment. The rationale for this is that, first, women of color were disproportionately represented among the students of the highest centralities. From a social capital perspective, higher centrality indicates greater social capital, which means greater empowerment. Also, these high centralities mean that women of color could be exposed as influential students in others' learning. This implies that the linear algebra learning community could provide women of color more opportunities to be identified as influential. Additionally, the results showing no significant difference in the number of nominations between men and women, and between White and non-White students, indicate that the IOLA classroom may be where classmates perceive other students as influential. This perception would be reasonably similar across race/ethnicity and gender.

Then, was the IOLA class inherently equitable? Was the IOLA class disproportionally supportive of women of color? The response to this study remains "no." The next part addresses the reason the IOLA class was not necessarily equitable for women of color.

Potential Interpretation of Additional Labor of Women of Color

This study also interprets this potential equity in the IOLA class as resulting from additional ***labor*** by women of color. ***Labor*** refers to how students from historically marginalized groups process and respond to discouraging experiences (Battey et al., 2022). This study found that

Jenny and Veronica had the highest in-degree, betweenness, closeness, and eigenvector centralities in the class, indicating that they may need to put in additional effort to participate in linear algebra. For instance, regarding in-degree centrality, Jenny and Veronica were nominated by the most diverse peer groups in class. In contrast, White students with the highest weighted in-degree centrality were usually nominated by redundant classmates. Additionally, Jenny's and Veronica's higher degrees mean that they had quicker accessibility to the entire class (closeness centrality), made bridges between different classmates (betweenness centrality), and possessed connections to other well-nominated students (eigenvector centrality) and these measures can be interpreted as Jenny and Veronica being exposed to relatively more peers, implying they may need to work on additional labor.

How to Interpret Racial/Ethnic and Gender Homophily: Solidarity and Segregation

There is strong homophily in student nominations regarding race/ethnicity and gender in the social networks in the IOLA class. This study suggests two contrasting interpretations of this homophily: ***solidarity*** and ***segregation***. The first view is to see homophily as the solidarity of minoritized groups of STEM students. That is, the tendency for women to nominate other women and for students of color to nominate other students of color can be interpreted as solidarity within the linear algebra classroom community. The second view is to interpret homophily as segregation between minoritized groups and non-minoritized groups in STEM. More specifically, the tendency of White and/or cisgender male students to nominate each other segregates female, non-cisgender, and/or students of color in a linear algebra learning community. These two perspectives contrast yet possibly coexist. That is, solidarity within minoritized groups of students and segregation between them and non-minoritized groups can occur simultaneously in a mathematics class. For instance, a group of racially diverse cisgender men would be racially inclusive but also might be exclusive from women and women of color, thus segregating them. Also, a group of White women and White non-cisgender students can be interpreted as a community where students stand in solidarity regarding gender inclusivity.

Conclusion and Suggestion for Future Work

This study demonstrates that active learning is not inherently equitable but is shaped by classroom power dynamics. In an IOLA course, women of color held central positions in peer networks, reflecting both empowerment and additional labor. Strong racial and gender homophily suggested overlapping patterns of solidarity and segregation. These findings emphasize the need for intentional design to ensure equity in mathematics. Future research should extend SNA across broader contexts and include groups not represented here, particularly Black women, to better understand how equity is enacted and sustained in active learning.

Acknowledgments

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Exploring Attentive Behavior While Reading Mathematics Textbooks

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Textbooks remain central in undergraduate STEM education, yet little is known about how readers distribute attention across the diverse elements of extended mathematical texts. This study investigates how mathematics majors, non-mathematics STEM faculty, and non-math majors read a four-page excerpt from a standard calculus textbook. Using eye-tracking, we compared participants' attention to expository prose, worked examples, figures, and a highlighted formula. Exploratory analyses suggested differences between the groups' distributions of attention. However, a compositional data analysis did not reveal statistically significant group-level differences. These findings point to the importance of disciplinary literacy frameworks in interpreting attention patterns and raise questions about how readers' identities shape engagement with mathematics texts.

Keywords: Mathematics Textbooks, Reading, Didactical Disciplinary Literacy, Eye-Tracking

In STEM education, pedagogical texts, such as textbooks, are central curricular resources; they provide a basis for content and sequencing as well as serving as learning resources for students. In conjunction with increased interest in pedagogical formats such as "flipped" classes (e.g., Maxson & Szaniszlo, 2015), educators and researchers have highlighted the "need for greater attention to reading in mathematics instruction" (Doerr & Temple, 2016, p. 6).

Despite the central role played by textbooks, detailed evidence about how readers distribute attention across extended expository passages in pedagogical texts remains limited. Eye-tracking has been used extensively to study reading and learning across educational contexts (Holmqvist et al., 2011; Rayner, 1998), and mathematics education researchers have begun to apply it to understand students' attentive practices (e.g., MacKay et al., 2025; Zhang et al., 2025). Still, there are relatively few studies that examine how readers engage with longer instructional texts.

The goal of the present study is to extend this line of work by investigating how different groups of readers allocate their visual attention when reading an excerpt from a calculus textbook. Building on prior research contrasting students' and experts' engagement with calculus texts (e.g., Shepherd & van de Sande, 2014; Weinberg et al., 2023; Wiesner et al., 2020), we examine how mathematics majors, non-math majors with calculus experience, and non-mathematics STEM faculty attend to prose, worked examples, figures, and highlighted equations in a common calculus text (Stewart, 2016). In particular, we ask how attention patterns vary across these groups, and how these patterns can be interpreted in relation to readers' disciplinary and didactical literacy practices.

Literature Review

Reading in Educational Contexts

Pedagogical texts are situated within disciplinary genres with conventions that require readers to engage in particular reading practices (e.g., Rezat, 2006; Rezat & Rezat, 2017). Mathematics textbooks are highly multimodal: they integrate prose, formal definitions, theorems, examples, and diagrams (Fan, et al., 2013; Lemke, 1998; O'Halloran, 2005; Pepin & Haggarty, 2001). Prior research has shown that novices tend to read superficially, focus on worked

examples, and under-utilize expository text (e.g., Mesa, 2010; Randahl, 2012; Shepherd et al., 2012; Shepherd & van de Sande, 2014; Weinberg et al., 2012). While design elements like boxed formulas or diagrams can scaffold learning (Lai et al., 2012), they may also constrain attention to salient but less integrative features.

Some of these results have been investigated using eye-tracking (ET) methods. ET measures like fixation times and scan paths have provided insight into how readers select and integrate multimodal information (e.g., Coskun & Cagiltay, 2022; Lai et al., 2013; Liu & Cui, 2025), although most of this work has been descriptive rather than theoretically grounded. Studies suggest that readers selectively attend to various text components (e.g., diagrams or highlighted equations), and that this varies by reader characteristics (knowledge, motivation, and disciplinary identity) and task context (e.g., Drexler et al., 2019; Hochpöchler et al., 2013; Ögren et al., 2017; Panse et al., 2018; Schnotz & Wagner, 2018; Schnotz et al., 2014; Tóthová & Rusek, 2022). Such findings underscore that reading mathematics texts is not uniform between readers, and that readers' attention is shaped by knowledge, task demands, and textual design.

Disciplinary and Didactical Disciplinary Literacy

The perspective of disciplinary literacy was introduced to examine how reading and writing are tied to disciplinary values, norms, and reasoning practices (Moje, 2007; Shanahan & Shanahan, 2008; 2012). Disciplinary literacy is socially situated within intellectual communities (Boaler, 2002; Bishop, 2012; Moje & Luke, 2009). Thus, knowing mathematical content cannot be separated from knowing how to engage with the texts where that content resides.

Mathematics texts encode conciseness, formal justification, and multimodal coordination (Morgan, 1998; Doerr & Temple, 2016; Lee & Spratley, 2010); and mathematicians' disciplinary literacy complements these features. In a comparison between mathematicians, chemists, and historians, Shanahan et al. (2011) found that mathematicians use the structure of the text to support understanding; reconcile the text with their own knowledge to focus attention; emphasize accuracy; view graphics as an integral component of the prose itself; and engage in intensive rereading. In contrast, historians viewed graphics as potentially ancillary to the text; chemists demonstrated more subtle differences, such as viewing graphics as alternative forms of overlapping information.

Textbooks, which are "pedagogically prepared for classroom use" (Kling et al., 2017, p. 2), differ from research texts in both style and function (Fauvel, 1988; Richards, 2002). Weinberg et al. (2023) extended disciplinary literacy to account for texts specifically written for instructional purposes by introducing the concept of didactical disciplinary literacy (DDL), which involves practices such as framing the text as a product of authors; referring to instructional purpose; attending to teaching-related structures in the text; and expressions of authority in teaching and learning (Weinberg et al., 2023). Readers' meaning making depends on both disciplinary background and sensitivity to pedagogical design (Wiesner et al., 2020; Fulmer et al., 2022).

Gaps and Research Question

Comparative ET research in mathematics rarely juxtaposes novices and experts, despite evidence from other fields (Harsh et al., 2019) that expertise mediates attention. Furthermore, ET work often lacks integration with disciplinary literacy frameworks, limiting interpretation of gaze measures (Andrä et al., 2020). In addition, the intersection of disciplinary literacy, DDL, and attention allocation remains underexplored, especially in the context of mathematics texts. The present study addresses these gaps by examining how STEM faculty, mathematics majors, and non-majors allocate attention to expository text, figures, examples, and highlighted equations.

Our research question is: How is a reader's distribution of attention across different types of text (examples, text, figures, and equations) in a standard calculus textbook excerpt related to their positioning in the mathematics and school community?

Theoretical Framework

The core of our framework is the idea that readers actively select components of the text to focus on as they read, that this selection is guided by disciplinary norms and their own content knowledge, and that their eye fixations reflect their attention.

Reading Attention and Eye Movements

Researchers in mathematics education have increasingly begun to use eye-tracking to study how readers attend to texts, solve problems, and engage with multimodal representations (Schindler & Lilienthal, 2019). As Schindler et al. (2025) point out, most eye-tracking work operates based on the eye–mind hypothesis (Just & Carpenter, 1980; Rayner, 1998), which proposes a positive connection between visual fixations and cognitive attention, and in which, as König et al. (2016) note, gaze patterns are seen as indicators of higher mental processes such as perception, memory, problem solving, and abstract reasoning.

Gaze data are not inherently self-explanatory. For example, several scholars (e.g., Andrá et al., 2015; Hartmann et al., 2015; Holmqvist et al., 2011; Jacob & Karn, 2003) have proposed that readers' cognitive behavior might not match what their eyes are fixated upon, and some research (e.g., Schindler & Lilienthal, 2019; Thomaneck et al., 2022) has demonstrated evidence of such mismatches. Despite these limitations, numerous researchers (e.g., Alemdag & Cagiltay, 2018; Deubel & Schneider, 1996; Remington, 1980; Scheiter & Eitel, 2017; Schindler & Lilienthal, 2019; Thomaneck et al., 2022; van Gog et al., 2014; Wang et al., 2019) have found that eye movements are generally linked to attentional foci.

Disciplinary and Didactical Disciplinary Literacy

As Barton and Hamilton (2000) described, literacy practices are "the link between the activities of reading and writing and the social structures in which they are embedded" (p. 7). Disciplinary literacy refers to the specialized ways of reading, writing, and reasoning that are characteristic of each academic field (Fang & Schleppegrell, 2010; Rainey & Moje, 2012). In mathematics, this entails precision with notation, logical structure, coordination of representations (symbolic, graphical, verbal), and orientation toward proof and generalization (Morgan, 1998; O'Halloran, 2005; Rezat & Rezat, 2017). Readers engage not just with words and symbols but with the epistemological norms and values of mathematics itself.

Didactical disciplinary literacy (DDL) extends disciplinary literacy to the context of instructional texts (Wiesner et al., 2020; Weinberg et al., 2023). It involves readers' ability to recognize and interpret instructional scaffolds such as boxed definitions and worked examples; critique texts as instructional artifacts; and navigate their positionality within institutional settings and norms, among other skills and habits. DDL highlights how readers' practices are simultaneously disciplinary, didactical, and situated within relations of power.

Methods

Participants and Materials

The study is structured as a comparison between three groups expected to bring a range of literacy practices and content knowledge to the reading experience. The first group consisted of

five students who had declared either a major or minor in mathematics and had completed at least the first two semesters of college-level calculus and a semester of linear algebra with a grade of at least C-; we hypothesized that students in this group had developed and refined literacy practices for reading didactical mathematics texts and would also have sufficient knowledge to learn new calculus concepts by reading a textbook excerpt.

The second group consisted of five students who had completed a single college-level calculus course with a grade of at least C- in the previous semester. We hypothesized that these students had developed sufficient calculus knowledge to learn a new concept but were less likely to have developed sophisticated DDL practices for mathematics.

The third group comprised five faculty members (biology (2), chemistry, environmental science, and physics), all with PhDs, at least four years of undergraduate teaching experience, and active research agendas. As a result, they were skilled in interpreting disciplinary texts and using textbooks pedagogically. All had taken calculus; but, except for the physicist, none had used calculus in over a decade. We expected the group to have less calculus knowledge than the student groups, but disciplinary literacy practices similar to those of mathematicians (as supported by the findings of Shanahan et al., 2011).

Participants were recruited via email from lists of all students and faculty who met the above criteria. Participants were selected from among the respondents based on whether they continued to respond to scheduling emails and their availability during scheduled interview time slots.

We sought to have all participants read for the same purpose—namely, to learn a new concept. We selected the section "Newton's Method" from a widely used calculus textbook (Stewart, 2016). This four-page excerpt included expository prose, a formula (presented inside a blue box) for obtaining the n+1st approximation from the n^{th} approximation of the root of a polynomial, three worked examples, and seven figures. At the beginning of the reading session, all participants were asked about their familiarity with Newton's Method, and all indicated they were not familiar with the concept.

Data Collection

During each interview, the textbook excerpt was displayed on a computer monitor, and the participant was able to navigate between the pages using the computer keyboard. Participants were asked to read the excerpt "to learn about the concept of Newton's method" at their own pace, while the interviewer worked independently at a separate desk. They were provided with a pen and paper and told they could take notes as they read. The participant's eye movement was recorded with a Tobii X2-60 eye tracker mounted below the monitor and calibrated according to manufacturer guidelines (Tobiipro, 2019).

AOI Construction and Data Processing

We created a rectangular Area of Interest (AOI) around each section of expository prose, figure, example, and the formula in the blue box. The AOIs were grouped by type: blue box formula, example, figure, or text. This choice of grouping was made to reflect the ways in which mathematics disciplinary literacy practices differ from those in other disciplines by tightly integrating the interpretation of expository prose, graphics, and formulas, with the goal of enabling us to (1) explore ways in which these elements might be approached by the different participant groups and (2) the extent to which the didactical nature of the text might lead to different attentive patterns.

We restricted the set of eye movements to fixations, created a variable to indicate which AOI each fixation was contained in, and then computing the total duration (in seconds) for which each

participant had fixations within each AOI. We then computed the total fixation duration for each type of AOI for each participant. One participant (one of the biology faculty members) had long stretches with no valid fixation data, so we omitted their data from our analysis.

Results

Participants distributed their attention across the different types of text in a wide variety of ways. For example, one non-math major spent approximately 60% of their time looking at the examples and 20% in each of text and figures, while one math major spent approximately 55% of their time looking at the figures, with 20% looking at the examples and 25% at the text.

The percentages for the blue box formula were uniformly small (approximately 2% among all participants). Thus, we examined the distributions of the other AOI types using the ternary plot shown in Figure 1. In this plot, the three colors (red, green, and blue) indicate data from the three groups (MM, NMF, and NMM). Each group's dots are connected by a convex hull, and the large black dots are the hull centroids, which represent group averages. From the centroids, we can see that the math majors tended to spend a larger proportion of their time reading the expository text; the non-math majors tended to spend a larger proportion of their time looking at the examples; and the non-math faculty tended to spend a larger proportion of their time examining the figures. Although there is some overlap between the hulls, visual inspection suggests that the three groups exhibit relatively distinct distributions of attention.

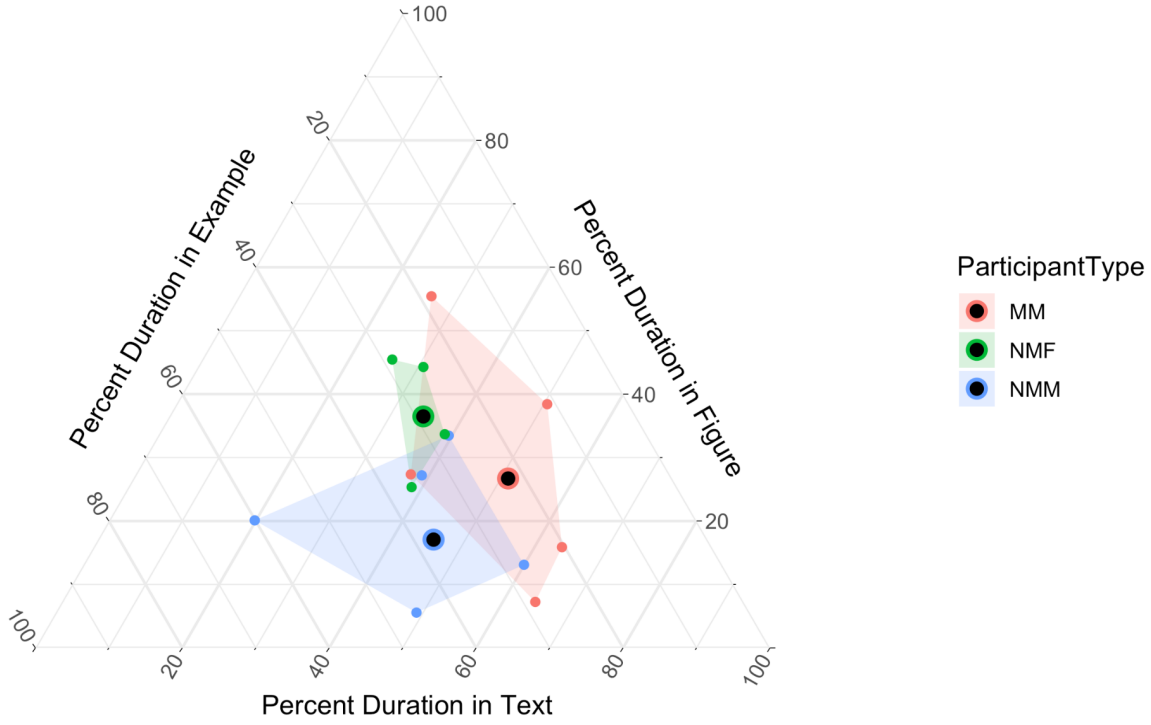

Figure 1. Ternary plot of participants' proportional durations in the example, figure, and text

Within these global patterns, however, there was a relatively considerable amount of variation among the participants within the MM and NMM groups compared to the NMM group, as demonstrated by the large areas of the hulls. In addition, the separation between the three hulls does not occur along a single axis, suggesting that between-group differences might not align with the three AOI types.

Compositional Data Analysis

Our data are non-negative and multivariate, with values that only carry relative meaning and, within each individual and group, must sum to a fixed value (in this case, 100%). This constant-sum constraint violates assumptions of independence in standard statistical methods. Thus, compositional data analysis, which uses logarithms of ratios of values, is the most appropriate statistical tool.

To conduct this analysis, we began by transforming our data using an orthonormal basis of log ratios of parts, which maps the data into a set of orthogonal components and eliminates collinearity. This isometric log-ratio (ILR) transformation is: $ilr(x) \coloneqq V^t clr(x)$, where $clr(x)$ is the centered log ratio that expresses each component relative to the geometric mean of all parts, and V is an orthonormal basis of the clr plane. We then compared the resulting three (now independent) ILR coordinates across groups (MM, NMM, NMF) using a permutational multivariate analysis of variance (PERMANOVA; Anderson, 2001). The analysis, based on Euclidean distances with 999 permutations, indicated that participant type did not significantly predict variation in the duration data ($F(2, 11) = 0.89$, $R^2 = 0.14$, $p = 0.50$). Thus, when considering all three ILR coordinates jointly, our data do not provide evidence of a significant overall difference in the distribution of attention across the stimulus types between groups.

To follow up this non-significant result, we conducted a principal component analysis (PCA) on the CLR-transformed duration data to investigate the potential for more complex relationships between factors. Four components were extracted, with PC1 accounting for 65.1% of the variance and PC2 for 18.9% of the variance. Loadings indicated that blue box (0.61) and figure (−0.79) duration loaded strongly on PC1, whereas example (0.53) and text (0.45) loaded on PC2. A plot of the first two components is shown in Figure 2.

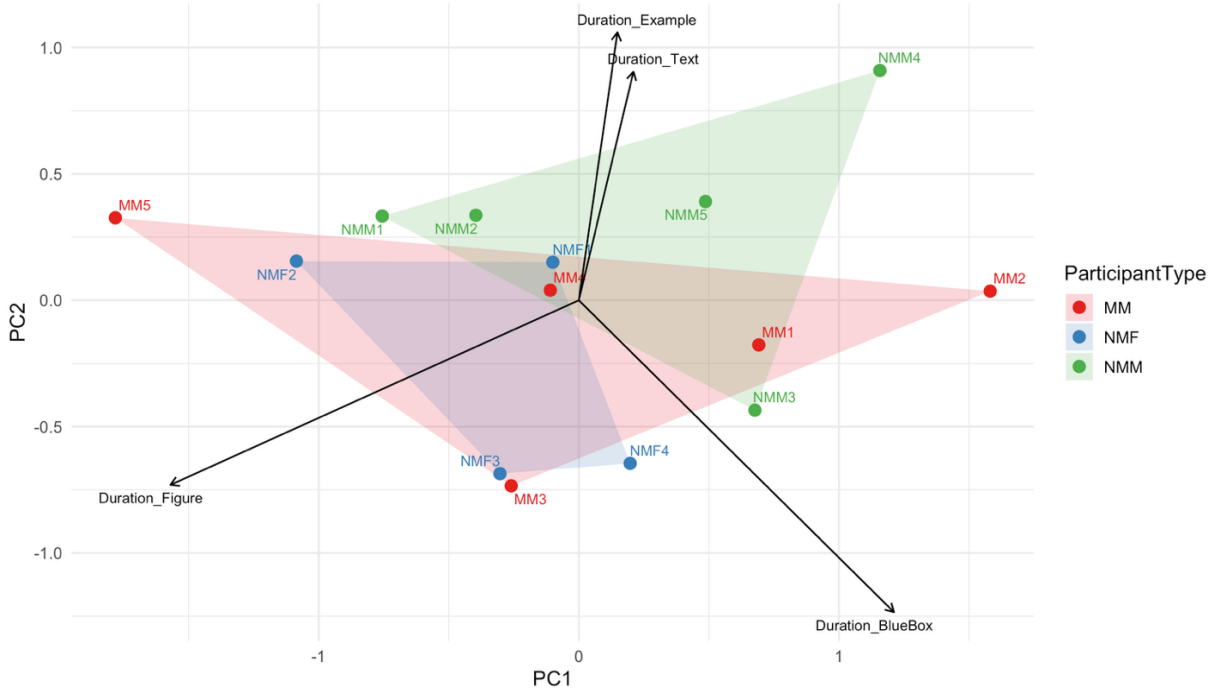

Figure 2. PCA biplot of duration variables.

In this plot, the arrows represent the contribution of each stimulus type to the principal components; these arrows suggest that the figure and blue box components most strongly influence PC1, while example and text primarily play a role in PC2.

Each point in the graph represents one participant's score along the PC1 and PC2 axes. The points are connected by colored convex hulls to highlight similarities and differences within and between groups of participants. In the green hull (NMM), four of five points had relatively higher PCA scores along PC2. The blue (NMF) and red (MM) hulls occupy roughly the same range along the PC2 axis, suggesting a similar balance between text and example durations. However, the MM hull suggests that some MM participants spent relatively more or less time on blue box and figure compared to others, whereas the clustering of NMF along PC1 indicates more uniformity in their distribution of blue box and figure attention. However, these group-level interpretations are exploratory and not inferential.

Discussion

The initial EDA revealed differences in how the participant groups allocated their attention across types of text. The faculty members distributed their attention roughly evenly among the example, figure, and text, while the math majors tended to focus on the expository prose and the non-math majors focused on the examples. One explanation for these patterns could be the participants' disciplinary literacy practices. The faculty members were all scientists, and one disciplinary literacy practice in science is to view different text elements as alternative forms of overlapping information (Shanahan et al, 2011); this might have led faculty members to attend equally to the different types of text. Similarly, the math majors might have focused on the expository text to closely read the derivation of Newton's Method because understanding derivations is an essential mathematical practice. In contrast, the non-math majors' focus on the examples could be interpreted from a perspective of what it means to know mathematics, in which understanding a concept is viewed as equivalent to being able to perform a procedure.

The perspective of didactical disciplinary literacy provides a slightly different set of explanations for the participants' distribution of attention. One DDL practice involves attending to the textbook authors' instructional purpose. In this textbook excerpt, the figures played the role of providing an intuitive explanation of more formal aspects of the text. The faculty members might have recognized this purpose and, consequently, focused their attention on the figures, while the math majors' attention was still guided by the disciplinary focus on the formal symbolic derivation. In contrast, the non-math majors' focus on the examples could be explained by their identity as students and their anticipation of being asked to solve similar problems.

The permutation MANOVA did not reveal a statistically significant difference between the groups' distribution of attention. One explanation for this is that there are no significant differences between the groups. Another possible explanation is that there is significant variation within each group, and the small sample size didn't provide sufficient statistical power to identify the differences. The PCA biplot suggests yet another explanation: There might be differences between the groups, but these differences exist on more complex combinations of groups of variables, such as contrasting figure + formula with text + examples.

By examining how different reader populations allocate attention across types of text, the present study offers one of the first attempts to link measurable patterns of attention with theoretically grounded models of literacy. Future work could include combining AOIs along other dimensions, such as mathematical content or pedagogical purpose, to further examine relationships between DL, DDL, and participants' attentional distribution.

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Investigating Students' Ways of Reasoning When Estimating an Unknown Population Parameter

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To examine the forms of reasoning novice statistics students employed when reasoning about sampling distributions, I used Charles S. Peirce's three classic forms of inferential reasoning—deduction, induction, and abduction—defined by case, rule, and result. In this paper, I report on a subset of these findings by describing and comparing the reasoning of seven undergraduate students. I engaged each student in a statistical task that I designed to investigate their reasoning in a repeated sampling environment when given a population with an unknown parameter. Findings indicate that abductive reasoning was powerful in making inferences from sample data to the unknown population.

Keywords: statistics education, reasoning, sampling distribution

Data are everywhere. Data are often reported in the form of polls, medical studies, and advertisement information. Introducing students to the basic language and fundamental ideas of statistics will help them develop statistical reasoning skills to evaluate data-based claims, enabling them to become better educated, well-informed members of society (Bargagliotti et al., 2020; Ben-Zvi & Garfield, 2004; GAISE College Report ASA Revision Committee, 2016). A major component of statistical reasoning is inferential reasoning, or drawing conclusions from data. Noll and Hancock (2015) noted that inferential reasoning is "a fundamental skill for participation in a global, technological, and data-driven society" (p. 362). Inferential reasoning includes many statistical ideas, such as repeated sampling, variability, and randomness. Understanding these ideas is important for improving students' statistical literacy and productive citizenship (Franklin et al., 2007; Saldanha & Thompson, 2007).

Despite the importance of understanding these statistical ideas and being able to reason inferentially, research suggests that students struggle with these ideas (Saldanha & Thompson, 2002, 2014; Sotos et al., 2007). For example, many students may be able to carry out the calculations involved in formal inference procedures, such as confidence intervals and hypothesis tests, but they often struggle to understand the underlying process and logic behind statistical inference (Chance et al., 2004). One reason for this difficulty stems from the complex and abstract concept of sampling distribution, which requires students to coordinate multiple ideas such as sample, population, distribution, variability, and repeated sampling (Chance et al., 2004; Kadijevich et al., 2008; Noll & Shaughnessy, 2012; Saldanha & Thompson, 2002, 2007). Another source of difficulty is that the forms of reasoning that are expected and highlighted in statistics differ from those that are standard in mathematics (Groth, 2015).

Theoretical Framework

Definitions of reasoning include some process of transforming given information to develop a conclusion or make an inference (e.g., Galotti, 1989; National Council of Teachers of Mathematics, 2009). Deduction, induction, and abduction are three classic forms of inferential reasoning. I adopt the definitions of deductive, inductive, and abductive reasoning from Peirce (1878) and others (e.g., Reid & Knipping, 2010). *Deductive reasoning* is the process of drawing logically valid conclusions based on stated assumptions or rules and is thought to be the only form of reasoning that establishes a conclusion with certainty. *Inductive reasoning* involves

generalizing a rule from several observations. *Abductive reasoning* starts with an observation and involves *hypothesizing* what rule might lead to that particular observation.

Peirce (1878) defined deduction, induction, and abduction in terms of a triadic structure involving a *case*, *rule*, and *result* that are linked in a particular order. Reid & Knipping (2010) interpreted Peirce's definitions and defined a *case* as a specific observation that a condition holds. A condition describes an attribute of something, or a relation between things. A *rule* is a general proposition that states that if one condition occurs then another one will also occur. A *result* is a specific observation, similar to a case, but referring to a condition that depends on another condition linked to it by a rule. The order in which one links a case, rule, and result determines the kind of inference needed to gain more knowledge about the situation. In deductive reasoning, a rule and a case conclude a result. In inductive reasoning, a case and a result (or often, many cases and many results) lead to a rule. In abductive reasoning, a result and a rule infer a case. Consider the different ways case, rule, and result can be linked using Peirce's (1878) original example about a pile of beans found on a table: (1) If I know that all of the beans in the bag on the floor are white (the rule) and that a pile of beans on the table is from the bag on the floor (the case), then I can *deduce* (with certainty) that all of the beans in the pile on the table are white (the result); (2) If I know that a pile of beans on the table is from the bag of beans on the floor (case) and that all of the beans in the pile on the table are white (result), then I can *induce* that (probably) all of the beans in the bag on the floor are white (rule); and (3) If I know that all of the beans in the pile on the table are white (result) and all of the beans in the bag on the floor are white (rule), then I may *abduce* that (possibly) the pile of beans on the table is from the bag of beans on the floor (case).

In both mathematics and statistics, we see all three forms of reasoning. Conjectures and generalizations through inductive reasoning occur in both mathematics and statistics. Abduction occurs in both mathematics and statistics when hypotheses are constructed to explain some observation or event. Although mathematics and statistics share similar forms of reasoning, the emphases in each field differ. In mathematics there is an emphasis on deduction and proof, arriving at conclusions with certainty. The emphasis in statistics is on reasoning with uncertainty, a feature of both inductive and abductive reasoning. This paper reports on a subset of my findings from a larger project in which I investigated how novice statistics students understand sampling distributions. For this paper, I studied the research question: When given a population with an unknown parameter, what forms of reasoning (i.e., deduction, induction, and abduction) do novice statistics students employ when making inferences from one sample outcome to the population from which it was drawn, and what do these forms of reasoning reveal about their understanding of sampling distributions?

Methods

Participants for the larger study were 11 undergraduate students from a large university who had recently completed an introductory statistics course. I engaged each student in two clinical interviews, each 60-75 minutes in length. Across both interviews, the context was about the true population proportion of intended business majors at two large universities. North University's website states that 20% of their undergraduate students are intended business majors while South University's website does not have this information posted. In the first interview, students reasoned about sample outcomes drawn from North University, a population with a known parameter. They made predictions for sample outcomes, physically drew samples from a large box of beads with proportions corresponding to the population of interest, and simulated drawing many samples using a web-based applet (Rossman & Chance, 2021). Throughout the interview,

they updated their predictions for sample outcomes as they continued to draw more samples. In the second interview, I tasked students with estimating the unknown population parameter at South University based on the result of one sample outcome. They drew one sample from a box of beads that modeled the population of interest, made an initial prediction for the unknown parameter, tested and refined their prediction using the web-based applet, and ultimately provided a range of plausible values they believed captured the true population proportion. This paper reports on data collected from the second clinical interview.

Primary data for this paper include the interview transcripts. After checking and timestamping interview transcripts against the audio and video recordings of the interviews, I adapted Simon's (2019) three-level approach to analyzing qualitative data. In the first level, I analyzed the interview transcripts line-by-line and identified segments of data in which students provided an explanation or a justification. Taking a convergent approach (Clement, 2000) in the second level of analysis, I applied Peirce's (1878) case-rule-result framework to each segment to infer the form of reasoning students used. This involved identifying what the student used as a rule and how they either applied the rule to a particular case (deductive reasoning), developed the rule from multiple cases (inductive reasoning), or hypothesized a rule to explain a particular case (abductive reasoning). In the third level of analysis, I identified emergent themes that provided additional insight into students' understanding of sampling distributions beyond the three broader forms of reasoning.

Results

This paper reports on a subset of the findings of the larger study. By describing and comparing the reasoning of seven students, I highlight a critical difference in two forms of reasoning. All seven students used the applet to continually make and test predictions for the unknown population parameter. In addition, all seven students used the likelihood of their sample outcome to assess their predictions for the unknown population parameter. However, only three students related one or more sampling distributions that they constructed using the applet to a hypothetical population. These three students reasoned abductively by hypothesizing multiple populations that could have produced their one sample outcome. In contrast, students who did not consider a hypothetical population developed a rule—using the likelihood of their sample outcome—to deduce a range of plausible values for the unknown population parameter. I identified a subtle yet critical distinction in students' reasoning across these two groups; those who reasoned abductively coordinated two levels of distribution—population distribution and sampling distribution—and those who reasoned deductively did not show evidence of this coordination.

Abducing Plausible Values for the Unknown Population Parameter

Three students related one or more sampling distributions that they constructed using the applet to a hypothetical population that could have produced the collection of sample outcomes. These students abduced a range of values based on the likelihood of the outcome of their one random sample that they drew from the box of beads that modeled the South University context. They repeatedly predicted and tested multiple values for the unknown parameter and assessed the plausibility of each value by considering how often their sample occurred in the many samples they drew using the applet (see Figure 1). Each time they determined that a hypothesized value was plausible, they abduced a case from a result and a rule. Starting with their one sample outcome (result), they hypothesized a value for the unknown population parameter, then constructed a sampling distribution using the applet with the probability of

success equal to the hypothesized value and drew many samples. Because their sample outcome was likely to have occurred under the hypothesized value (rule), they abduced that their sample could have been drawn from a population whose parameter was equal to the hypothesized value (case).

For example, when testing her prediction of 60% for the unknown population parameter, Lorraine drew 500 random samples with the applet using a probability of success of 0.6 (see Figure 1). Using the results of her simulation, Lorraine assessed the likelihood of her sample outcome of 52% when the probability of success was 0.6. She said,

> It looks like 52 would be reasonable…it looks like there's a pretty significant number of results that are…52. And then past that is where it tends to kind of taper out. So maybe if the true proportion were 60%, the lower bound would be like 48 or so.

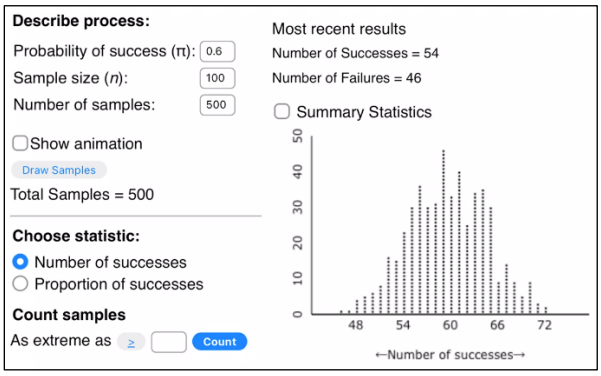

Figure 1. The Result of Lorraine's Simulation When Testing 60%

Given her sample outcome of 52% (result), Lorraine assessed the likelihood of her sample outcome, determining that her sample outcome "would be reasonable" "if the true proportion were 60%" (rule). She abduced that her sample outcome could have come from a population whose true parameter was 60% (case) and included 60% in her range of values for South University's true parameter. Lorraine related the sampling distribution she constructed to a hypothetical population with a "true proportion" of 60%, indicating she understood that the sampling distribution represented the outcomes of 500 random samples drawn from a population whose parameter was 60%. Lorraine continued to use the applet in this way to abduce multiple values that were plausible for the true percentage of undergraduates at South University who were intended business majors, ultimately providing a range of 42% to 62% for the unknown population parameter.

Like Lorraine, Eric and Lyla repeatedly predicted and tested multiple values for the unknown population parameter, assessing the plausibility of each value based on how often their sample outcome occurred. Furthermore, both showed evidence of relating one or more sampling distributions they constructed using the applet to a hypothetical population that could have produced that collection of sample outcomes. When Eric tested his prediction of 40%, he explained that the 40% represented "the assumption that if South University had their proportion online…that proportion is 40%. It's like the 20% from North University," indicating that he related the sampling distribution he constructed with a probability of success of 0.40 to a hypothetical population whose true parameter was equal to 40%. To assess the plausibility of his prediction, he considered the likelihood of his sample outcome under this assumption. He said, "If you say it's 40, then 50 could be a possibility," pointed to a sample outcome of 50 on the dot plot, and said, "So we could say, oh, that one [sample] that I drew was that one." When Lyla

tested her prediction of 50%, she explained that she "made a distribution based on this 0.5" and looked for "the probability of drawing 46 students in a sample of 100 from this larger population." She related the sampling distribution she constructed with a probability of success to a "larger population" and considered the likelihood of her sample outcome of 46. Like Lorraine, both Eric and Lyla repeatedly reasoned in this way, abducing a case from a result and a rule. They abduced that their sample outcome (result) could have been drawn from a population (case) whose parameter was equal to their prediction (rule).

The conclusion of this abductive inference is that the one random sample these students drew could have been drawn from a population with some parameter, *p*. Without connecting a collection of sample outcomes to the population from which they were drawn, this conclusion is not possible. Without this connection, considering the likelihood of the outcome of the one sample can be used as a procedure to deduce a range of values for the unknown population parameter. This was the case for Tami, Tyra, Mindy, and Corrina.

Deducing Plausible Values for the Unknown Population Parameter

Four students used the likelihood of their sample outcome to assess the plausibility of multiple values for the unknown parameter, without relating each sampling distribution they constructed to a hypothetical population that could have produced those outcomes. Rather than hypothesizing what population could have produced their sample outcome, they used the likelihood of their sample outcome as a rule to determine whether they should include their prediction in the range of values for the unknown population parameter. These students reasoned that if their sample outcome occurred often when the probability of success was equal to *p*, then *p* should be included in their range of values, without connecting the collection of sample outcomes to a population whose parameter is equal to *p*. This connection is the subtle yet critical distinction between the abductive inference I described in the previous section and these four students' deductive inference. By considering the likelihood of their sample outcome each time they constructed a new distribution using the applet, they understood that the distribution they constructed represented a collection of sample outcomes to which they could compare their one sample outcome. However, they did not show evidence of understanding that each sampling distribution they constructed with some fixed parameter, *p*, represented a collection of possible outcomes from a population whose true parameter was equal to *p*.

Corrina's reasoning is representative of the group. When testing her prediction of 40%, Corrina considered the likelihood of her sample outcome of 51 (see Figure 2a). She said, "It's kind of on the higher end. It's not as likely, but it totally could happen" and included 40% in her range of values for South University's true parameter. When she tested 60%, she said, "Yeah, so that 51, again, is kind of on the lower end, but still likely, so I think it could be 60" (see Figure 2b) and, again, included 60% in her final range of 38% to 62% at the end of the task.

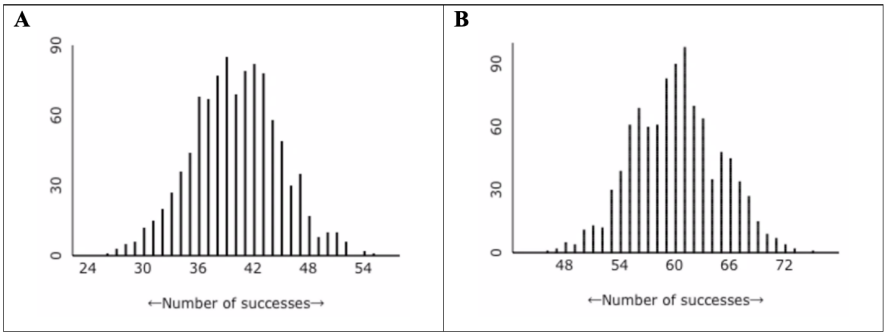

Figure 2. The Results of Corrina's Simulations When Testing 40% and 60%

Corrina applied a general rule to each of her predictions to deduce whether she should include the prediction in her range of values for the true parameter at South University: If my sample outcome occurred in a large collection of outcomes when the probability of success is equal to *p*, then *p* should be included in my range of values. She applied her rule to her prediction of 40% and reasoned that because her sample outcome of 51 "could totally happen" when the probability of success was equal to 40% (case), then 40% was a plausible value for the unknown parameter (result). Similarly, because her sample outcome of 51 was "on the lower end, but still likely" when the probability of success was 60% (case), then 60% was a plausible value for the unknown parameter (result).

Students' Summaries

At the end of the task, students summarized how they determined their range of plausible values for the true percentage of intended business majors at South University. These summaries provided additional evidence of the difference in students' understanding across the two groups. Table 1 shows the summaries of a subset of students who are representative of each group.

Table 1. Students' Summaries

Form of Reasoning	Summary
Abductive	Lorraine: If I put in the probability of success as 42, or if I put 0.42 in the center, that would be the proportion of business students. Then it'll tell me, or it won't tell me, but I can guess how likely it is that I would pull the result that I pulled (motions to her sample of 100 beads on the scooper) from that [box of beads]…If it was likely enough that I could pull a sample of 100 students where 0.52 was 52 is the percent that were business majors, then it would be likely that that number could be within the range of the actual proportion…It's kind of a messy way to explain it, but since we put in 0.42 and 52 was a reasonable, or it wasn't an uncommon result, yeah, it wasn't significantly high or significantly low, then it's likely that this [0.42] could be the actual proportion. So, it should be included in the range.
	Lyla: We started by drawing a sample of undergraduates without knowing the true proportion of students that are intended business majors, which my value was 0.46. And then I just made a prediction what the true population may look like based on this sample. I didn't stray much from the 0.46, I did 0.5 and then made a distribution based on this 0.5 and saw where the probability of drawing 46 students in a sample of 100 from this larger population fell on that success rate. And then based on that, tried to find values where drawing 46 was the very furthest extremity and then the lowest extremity to try to find cases in which it seems nearly impossible to get 0.46, both it being the smallest number and the largest number. And from that, knowing the proportions in which it was basically impossible to get the number 46, we know where it is possible to have 46.
Deductive	Corrina: So, when looking at the dot plot, I would put in our prediction of maybe it's 50% and then look for that 51 on the dot plot. And then just obviously we looked at the other numbers, but if you were really looking for, is it this or not, you could just kind of look at 51 and see how likely that happened. So, normally I would just kind of go straight for that 51 and see on this one it's this teeny tiny dot, but on one of the other ones it was the most probable thing on there…Just kind of where 51 was sitting on the scale. The number that we put in here (points to the probability of success on the applet) kind of is usually towards the middle, just kind of how close it's getting to this sample [outcome of 51] was kind of deciding whether or not it ends up in the scale.
	Mindy: So, this [first] one was just kind of a random prediction, because it's super close to 49. And then we plugged in different ranges, and I determined whether or not 49 would fall on the graph. And because of that you can kind of determine whether or not you think 20 or 30 would actually be in the range…When we saw 20, 49 wasn't super likely to show up.

There are two important differences between the summaries of students who reasoned abductively and those who reasoned deductively. First, those who reasoned abductively mentioned the context of the problem multiple times when they referred to "business students," "business majors," "a sample of undergraduates," and "a sample of 100 students." Second, they related the context to a hypothetical population when they referred to "the proportion of business students," "the true proportion of students that are intended business majors," and "students in a sample of 100 from this larger population." In addition to the explicit connections these students made to a hypothetical population in their explanations when testing multiple predictions, these key differences in their summaries provide additional evidence that this group of students coordinated two levels of distribution: population distribution and sampling distribution. In contrast, students who reasoned deductively referred to "the dot plot," "this teeny tiny dot," and "the graph," without ever referring to the context of the problem or using words or phrases that suggested they were thinking about a hypothetical population in relation to the samples they drew using the applet.

Discussion and Implications

This study examined the forms of inferential reasoning seven novice statistics students employed when reasoning about sampling distributions. When given a population with an unknown parameter, students used abductive or deductive reasoning when making inferences from one sample outcome to the population from which it was drawn. Those who reasoned abductively hypothesized multiple populations (and corresponding parameters) that could have produced their sample outcome. In doing so, they reasoned about the multi-tiered repeated sampling process (Saldanha & Thompson, 2002) and considered multiple levels of distribution in their reasoning—a critical understanding to reason productively from sample data to the population from which it was drawn. In contrast, students who reasoned deductively used the likelihood of their sample outcome in their rule for determining whether to include a particular prediction in their range of values for the unknown parameter. These distinct ways of reasoning have important implications for research and teaching.

Using Peirce's (1878) framework illustrates how these three broad forms of inferential reasoning are useful in distinguishing between the critical aspects of statistical reasoning and mathematical reasoning. In particular, I defined statistical inference in terms of abductive reasoning, which enabled me to understand how students coordinated multiple levels of distribution to draw inferences from sample data to a larger population. This approach of using Peirce's (1878) forms of inferential reasoning could be adopted by other researchers to investigate students' understanding of other important statistical concepts and statistical reasoning more broadly.

The focus of many introductory statistics courses at the undergraduate level is on statistical inference, whether informal or formal. The results of this study showed that abductive reasoning was powerful in making inferences from sample data to a larger population. Providing students with opportunities to reason abductively can support them in understanding the underlying logic and processes in estimating an unknown population parameter. For example, designing tasks that provide students with opportunities to continually hypothesize multiple populations from which a sample may have been drawn can promote abductive reasoning. Engaging with simulation tools can promote these critical forms of reasoning, but more research is needed to understand the meanings students construct when using these tools.

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Interpreting Components as Coordinates in Two Coordinate Systems

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This study explores how linear algebra students reason with a novel coordinate system, the COSLA system. As part of a larger study, individual clinical interviews were conducted with three students, and their work was analyzed through the ReNaming and ReLocating framework (Lee; 2023, 2024). Findings include various approaches in interpreting n-tuples and vector notations and different ways to reason about a displacement vector. This paper also presents challenges students may face with non-Cartesian systems and suggests future implications for teaching change of basis.

Keywords: student thinking, change of basis, n-tuples, vectors, coordinates

Introduction and Literature

Coordinate systems are widely used in undergraduate mathematics. In precalculus and calculus, students work with various coordinate systems, such as the Cartesian and Polar coordinate systems. They use n-tuples to denote point locations on plane or to express the results of component-wise operations. In linear algebra, students also work with non-traditional linear coordinate systems generated by basis vectors distinct from the Cartesian unit vectors, where vector notation is typically employed.

Several studies have reported on student thinking about vectors in n-tuple form. Students often interpret vectors as a particular string of numbers, independent of any coordinate systems (Hillel, 2000). Similarly, students may conceive the image of a vector under a linear transformation as a string of numbers rather than as a vector representation relative to a basis. (Hillel & Sierpinska; 1994, Sierpinska et al; 1999) More recently, Selby (2016) found that linear algebra students could answer change of bases questions by memorizing formulas, however, often neglected to reason about what is accomplished through the process. To support students in learning change of basis, he proposed a change of basis project emphasizing physical interpretations.

A few other studies have focused on developing instructional materials in the context of change of basis. Wawro et al. (2013) developed a task sequence to support students in reinventing matrix diagonalization, $A = PDP^{-1}$ by working with a non-traditional coordinate system which uses y=x and y=-3x as the new axes. Zandieh, Wawro, & Rasmussen (2017) identified four types (S1-S4) of linear algebra students' symbolizing evolution found in their class the reinvented matrix diagonalization curriculum was applied to. Levenberg (2015) developed a task for pre-service teachers that involved reading values from axes labeled with non-numerical quantities. He found that such activities prompted logical thinking because the relationship among the non-numerical values were directly tied to their associated meanings.

Conceptual Framing

ReNaming and ReLocating

I developed a framework that describes how students reason when identifying locations within a single coordinate system and across multiple coordinate systems. Also, the framework details how they represent numerical values as locations within one or more coordinate planes

(Lee; 2023, 2024). The framework, referred to as the coordinate system framework, includes four distinct components: Naming, Locating, ReNaming, and ReLocating.

Naming describes how locations or shapes in a coordinate plane are assigned names, such as ordered pairs and equations, while Locating describes the reverse process, how ordered pairs or equations are placed into a coordinate plane.

ReNaming refers to when a location or shape, initially measured in one coordinate system, is assigned a new name in a second coordinate system laid atop the first. For example, a circle centered at (1,0) with radius 1 can be named as $(x-1)^2+y^2=1$ in the Cartesian coordinate system and renamed as $r=2cos\theta$ in the polar system (Figure 1. Left).

ReLocating, on the other hand, occurs when an existing name creates two different locations or shapes in space, depending on the coordinate systems being used. The new one may appear different from the first, but they share the same name. For instance, the polar curve, $r=2cos\theta$, can be sketched in the Cartesian coordinate system (Figure 1, Right).

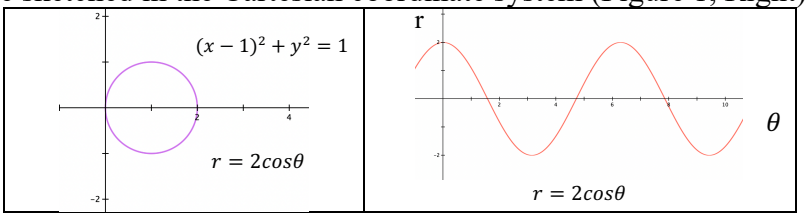

Figure 1. ReNaming Circle with Two Systems (Left), ReLocating $r=2cos\theta$ into Cartesian System (Right)

Coordinates Not the Same as Components

Coordinates are not always identical to the components in n-tuples (Lee, 2024). Coordinates represent relative sizes with respect to the units employed by a coordinate system, whereas components are the number values listed in an n-tuple or a vector notation. This distinction may not be apparent when considering a point's coordinates within the Cartesian coordinate system, however it becomes salient when reasoning with non-Cartesian coordinate systems. For example, when a student attends to the components of a point's ordered pair (1,2) in the Cartesian coordinate plane, s/he may think of it as "over 1, up 2". His/her interpretation may not have much problem to view (1,2) in terms of the Cartesian basis vectors as such "1 of $\begin{bmatrix}1\\0\end{bmatrix}$ and *2* of $\begin{bmatrix}0\\1\end{bmatrix}$" because the calculation $1\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}$ becomes the same as $\begin{bmatrix}1\\2\end{bmatrix}$. However, when thinking of a point viewed in another coordinate system, things get different. For example, there is a point named as (2,1) relative to the basis $\boldsymbol{B}=\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix}\}$, which would mean "2 of $\begin{bmatrix}1\\-1\end{bmatrix}$ and 1 of $\begin{bmatrix}0\\2\end{bmatrix}$". The location of (2,1) is being represented as a linear combination of the two vectors, $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\2\end{bmatrix}$, that is $2\begin{bmatrix}1\\-1\end{bmatrix}+1\begin{bmatrix}0\\2\end{bmatrix}$ in the new system. Figure 2 illustrates the location of coordinates (2,1) in the new black system.

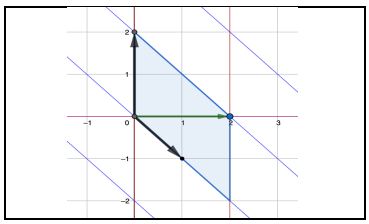

Figure 2. (2,1) in the New Black Coordinate System

While (2,1) is represented as the green vector in the plane, it is also represented as (2,0) relative to the Cartesian basis. In other words, 2 and 1 in (2,1) are the *coordinates* of the location because they are defined relative to the basis $\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\}$, while 2 and 0 in (2,0) are the *components* if it is the calculated result from $2\begin{bmatrix}1\\-1\end{bmatrix} + 1\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ with no attention to the basis.

This proposal reports on student reasoning with a novel coordinate system, the COSLA system, to address the research question: How do students interact with the novel coordinate system and how they interpret components and coordinates- are these interpretations distinct?

Methods

As part of a larger dissertation study, this proposal draws on data from individual face-to-face clinical interviews (Clement, 2000). At the time of interviews, participants were STEM majors who had completed Calculus 1 or 2 as prerequisites and had taken an inquiry-oriented linear algebra course at a large public university in the United States. Both their written work and interview conversations were video recorded. Written work was archived as pdfs, and conversations were transcribed into spreadsheet. Student reasoning was coded line by line on the spreadsheet using the ReNaming and ReLocating framework. All the names used in this report (Hann, Wilson, Jeraldo) are pseudonyms.

Description of the Task

I designed my dissertation task protocols in three packets each called clinical interview, teaching experiment 1, and teaching experiment 2. This proposal focuses on a task administered at the end of clinical interview (Figure 3, Left). The task was designed to explore how students interpret and use a novel coordinate system, referred to as the COSLA system, in which the unit vectors, C_1 and C_2, do not have length 1 relative to the rectangular system, and the origin is not located at (0,0). Prior to this task, students participated in clinical interview tasks themed around Gulliver's Travels, set in an experientially real place, Cocos Island, including a treasure hunt task. Students worked with multiple coordinate systems, such as the Gulliver (slanted) system (Figure 3, Middle) and the Rock (rectangular) system (Figure 3, Right). This series of tasks was developed according to the central idea of Realistic Mathematics Education (RME) informed by Freudenthal (1991). During the clinical interviews, students approximated the location of the Treasure on the Rock's coordinate system as roughly (10, 6.6) or (9.8, 6.6).

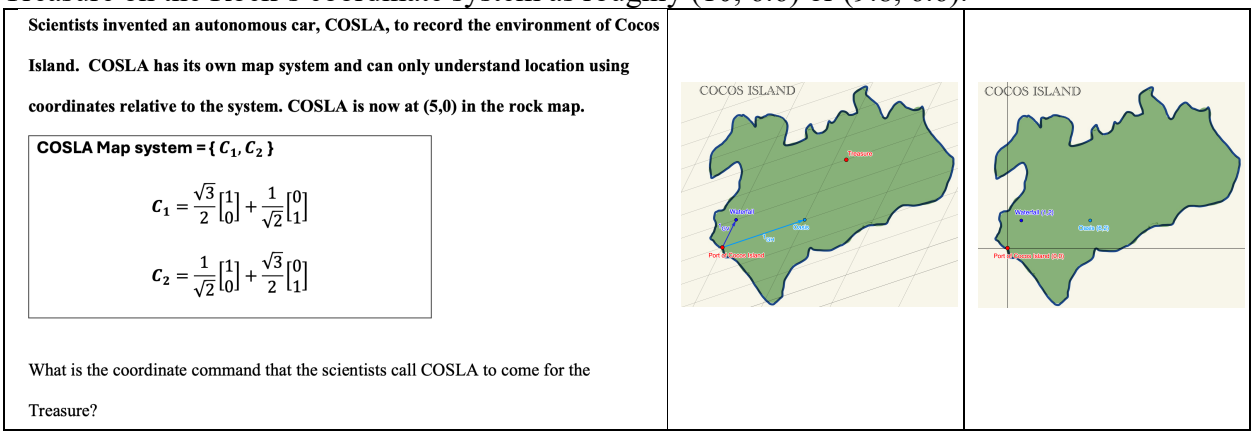

Figure 3. The COSLA Task (Left), Gulliver System (Middle), Rock System (Right)

In the COSLA Task, two coordinate systems are used: the COSLA system, which employs C_1 and C_2 as unit vectors, and the Rock system, which is identical to the rectangular system. Scientists, as inventors of the car, have access to both systems and represent components in C_1 and C_2 in terms of the rectangular coordinate system. That is, $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are the standard unit vectors, and $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$ are relative sizes to the '1'in the standard rectangular system. The car, COSLA, can only interpret commands expressed relative to its own unit vectors, C_1 and C_2.

Results

This section reports how two participants progressed in their search for the coordinate command in the COSLA task. It also examines how components in an ordered pair or a two-dimensional vector came to be understood as coordinates in their reasoning.

Interpreting Components as Coordinates in ReLocating Unit Vector

A participant, Hann, began by interpreting vectors as relative to a coordinate system.

> "We have this definition of our map system[COSLA system]. Okay, so I'm going to interpret these vectors [pointing to the $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ in C_1 and C_2] as being expressed in relation to the rectangular coordinates. So, this one zero, zero one is in rectangular."

He clarified $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ inside C_1 and C_2 as vectors within the rectangular system, and then continued: "What does that [pointing to C_1 and C_2] tell me? …Okay, so we have C_1 which is one zero, C_2 which is zero one (Figure 4)."

He went on to conceptualize C_1 and C_2 as unit vectors relative to the COSLA system and represented them again as $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. The $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ appearing inside C_1 and C_2, and the $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ representing the exact C_1 and C_2, although numerically identical, were distinguished by Hann through coordination with their respective coordinate systems. The two pairs of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ thus came to indicate two distinct locations, which I categorize as ReLocating. To Hann, the components of ones and zeros in each vector were interpreted as coordinates of ones and zeros relative to their respective coordinate systems.

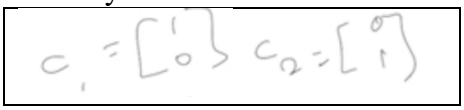

Figure 4. ReLocating Unit Vector to COSLA system

Interpreting Components as Coordinates in ReNaming Unit Vector

Hann next simplified C_1 and C_2 component-wise into vectors and related them to the COSLA unit vectors, saying: " that [COSLA unit vectors] should equal square root three over two, one over square root two in rectangular coordinates…(Figure 5)"

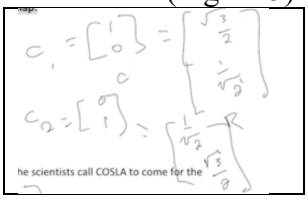

Figure 5. ReNaming COSLA Unit Vector

He placed equal signs even though the entries in the vector notation do not equal their respective components. Here, the equal sign was used to indicate the same physical unit length, despite the unequal components. These are two names for the COSLA unit vector C_1 and C_2, which I categorize as ReNaming. Hann interpreted components in $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{\sqrt{2}}\end{bmatrix}$, $\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{\sqrt{3}}{2}\end{bmatrix}$, as coordinates relative to their respective coordinate systems.

Interpreting Components as Coordinates in ReNaming Displacement Vector

Hann continued his search for the coordinate command by identifying the displacement vector from (5,0) to the Treasure location in the rectangular system. Drawing his earlier work in the clinical interview, he used the Treasure's location (10, 6.6), and again clarified both points, (5,0) and (10, 6.6), as well as the displacement vector from one point to the other, all expressed in rectangular coordinates.

> "What I think I'm looking for is a vector from five zero to the Treasure which is 10 and 6.6. Again, these are rectangular coordinates, and I need this vector to be expressed eventually in terms of the car [COSLA]…We can just do like the head minus the tail, so it's going to be rectangular coordinates. Now, I have the rectangular vector for this command (Figure 6)."

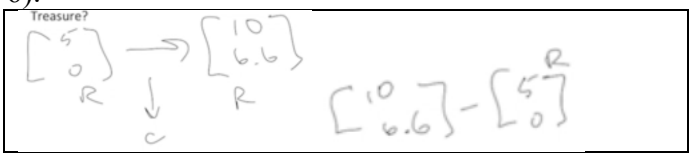

Figure 6. Naming Displacement Vector in Rectangular System

To rename the displacement vector relative to the COSLA system, Hann set up a matrix equation.

> " I need to convert it into the COSLA system, which I think I will need another matrix to do that… You want to multiply it [the matrix with A B C D in it (Figure 7)] by the COSLA coordinates that's one zero, and that will give me my square root numbers [wrote 'C→R' under M (Figure 7) to clarify COSLA to Rectangular]."

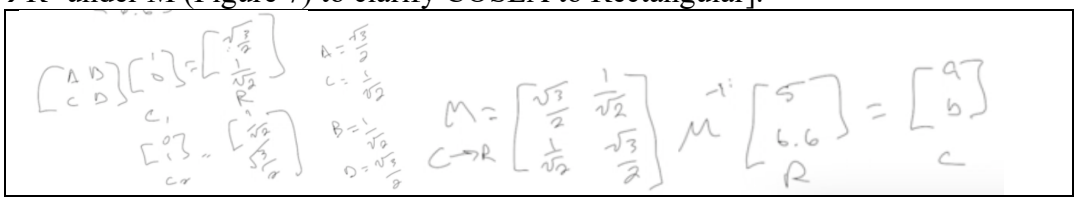

Figure 7. Matrix that Renames Vector

He continued: "to get the coordinate command we're looking for, we need to do M^{-1} times our coordinates we found before…that'll give us our car coordinates. Inverse because we're doing the opposite of what this matrix does. So, M goes from the car to the rectangular, and I want to go from rectangular to the car so taking the inverse."

Hann indicated that the coordinate command is $\begin{bmatrix}a\\b\end{bmatrix}$ (Figure 7) as a result of inverse matrix multiplication, without performing the operation by hand. The vector, $\begin{bmatrix}5\\6.6\end{bmatrix}$, is renamed as $\begin{bmatrix}a\\b\end{bmatrix}$, representing the same displacement vector, which I categorize as ReNaming. Hann interpreted components in these vectors as coordinates relative to their respective coordinate systems.

Interpreting Components as Coordinates in Rectangular System

Wilson, another participant in the interview, approached the same task in a different way. His primary reasoning was to visualize a path that connects the initial location of COSLA to the Treasure as a linear combination of vectors (Figure 8. Left), rather than as a directed displacement vector. In this approach, he did not coordinate between two coordinate systems, but instead interpreted all vector components within the rectangular system. He simplified C_1 and C_2 component-wise into vectors (Figure 8. Middle top), approximated them with decimals (Figure 8. Middle bottom), and used them as a pair of travel vectors. All components were thus expressed relative to the rectangular system. Wilson searched for the COSLA coordinate command as the coefficients a and b attached to each travel vector (Figure 8. Right).

> "The treasure in the rocks map [rectangular map] is 9.8 and 6.6… [he found the Treasure coordinate earlier as (9.8, 6.6)] what I'm trying to find …is how the commands for COSLA would put the treasure to that span… coordinate command… that's what my initial thought is the command for the coordinates to travel this path to the treasure."

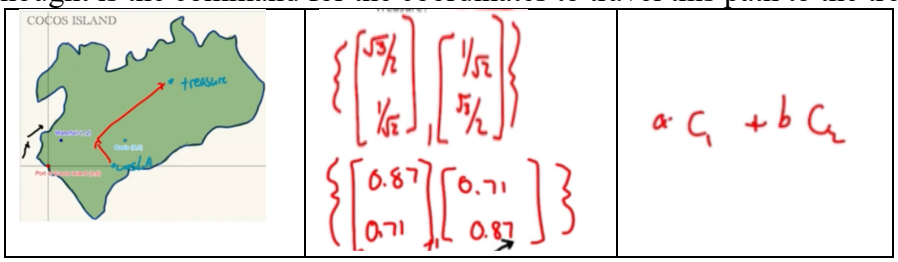

Figure 8. Path from COSLA (5,0) to Treasure (9.8, 6.6)

He continued: "I'm going to expand them [C1 and C2] out… Now that those are C1 and C2 fleshed out (Figure 8. Middle), I guess that's how those set of vectors can get to the point 9.8 6.6."

Wilson then marked travel vectors on the map as multiples of each vector from COSLA to the Treasure (Figure 9, Left) and expressed the path as a vector equation from the initial point (5,0) to (9.8, 6.6) (Figure 9. Right). His drawing and equation reflect an interpretation of the displacement vector as a journey from the initial to the final location. He then spent considerable time plugging whole numbers into a and b in the equation to approximate the Treasure's location as closely as possible.

> "Creating a path from the origin or from a point within a coordinate system to a point that isn't outright just noticeable …How many of these combination of vectors would be used as the coordinate command within this problem."

Rather than ReLocating and ReNaming the units, Wilson expanded every component of the new units and interpreted them relative to the rectangular system. In other words, he treated the components in the vectors C_1 and C_2 as coordinates within the rectangular system.

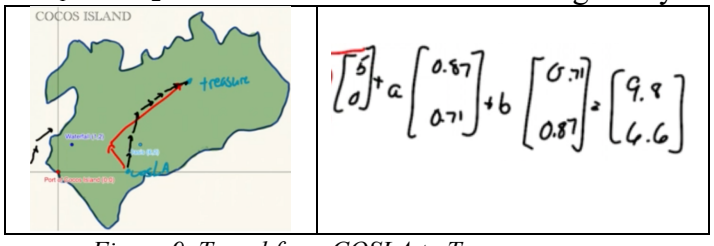

Figure 9. Travel from COSLA to Treasure

Another participant in the interview, Jeraldo, approached the task another way. Like the other two, he began by simplifying C_1 and C_2 component-wise into vectors. He then voiced the need of a transformation matrix to rename coordinates from one system to the COSLA system. Rather than relating the rectangular units to the COSLA units directly, he decided to use the landmarks on the map, the Oasis and the Waterfall, to set up a matrix equation (Figure 10).

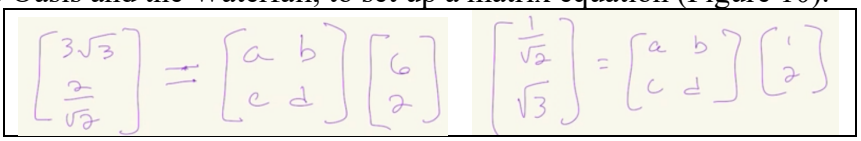

Figure 10. ReNaming Oasis (Left) and Waterfall (Right)

The Oasis is at (6,2) and the Waterfall at (1,2) relative to the rectangular system. From his matrix equations, Jeraldo appeared to assume that the two systems, rectangular and COSLA, shared the same origin at (0,0). This suggests either that he have not yet had the opportunity to distinguish the two, coordinates and components, or that he did not consider such a distinction necessary, or that he simply did not fully understand the problem.

Discussion

Similar to what Moore et al. (2013) and Zazkis (2008) noted, students express locations or equations within the system in which they have developed thinking- that is, the Cartesian coordinate system. Consequently, when they encounter another system, they tend to think of the new system in terms of an old familiar system. In many curricula and textbooks, components in an n-tuple and vector notation are by default interpreted relative to the Cartesian system, making it compelling for students to view numerical values as corresponding to the Cartesian units. To support students' flexibility in moving and expressing between coordinate systems, it is helpful to include instructions on conversions between non-Cartesian coordinate systems, independent of the Cartesian system.

As noted in the literature section, many of linear algebra studies have emphasized the importance of incorporating more realistic tasks and physical interpretations into instruction. The COSLA task was developed as part of my larger study, in line with these recommendations. I have also created a GeoGebra Applet (**https://www.geogebra.org/classic/vz2mcxqr**) designed to support students' experiences with vectors and coordinate systems, helping move instruction beyond purely abstract and computational approaches.

Lastly, although I was fascinated by Wilson's reasoning, in which he envisioned reaching the Treasure from the initial location as a kind of journey using linear combination, I came across a study reporting that students often employ similar "journey" imagery when adding vectors (Watson et al., 2003). For example, students may justify the equality in $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ by reasoning that it represents moving from A to B and then from B to C. But this imagery can become problematic when interpreting the commutative property for addition such as $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{BC}+\overrightarrow{AB}$ since the latter expression does not give a satisfying meaning to them with the journey reasoning.

Acknowledgements

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The Persistence of Motion Graph Reasoning Amid Quantitative Conceptions of Coordinate Systems: The Case of Ava

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We report on a qualitative case study of a college algebra student, Ava, who participated in a series of online task-based interviews involving graphs of different dynamic situations. Our analysis focused on the nature of Ava's graph reasoning and coordinate system conceptions within and across these tasks. During Ava's interviews, she primarily demonstrated Variation and Covariation graph reasoning while conceiving of the coordinate system quantitatively. Yet, she also had an expectation that her graphs would represent the motion of the objects in the dynamic situations or the motion of the traces in the graphs themselves. Ava's Motion graph reasoning amid her quantitative conceptions of coordinate systems demonstrates overlap between quantitative and physical conceptions. Her case speaks to the richness of students' conceptions of graphs; even when they conceive of quantitative coordinate systems, they can also expect a graph trace to resemble the motion of an object.

Keywords: undergraduate education, technology, quantitative reasoning, graphing conceptions

Having students engage in tasks where they represent attributes of objects in dynamic situations graphically can be a productive way to foster students' quantitative and covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). However, even when the task is designed in a way that supports their graphical representations of quantitative relationships, students may or may not actually conceive of their graphs in this way. Instead, they may conceptualize them as literal pictures of physical objects (Clement, 1989; Lai et al., 2016; Leinhardt et al., 1990), depictions of motion (Kerslake, 1977), or chronological records of events (Janvier, 1998). Furthermore, students may or may not conceive of a given representational space, such as a Cartesian plane, as a coordinate system that can be used to represent quantitative relationships. Thus, whether or not a particular graphing task is effective depends, in part, on students' graph reasoning (Johnson et al., 2020) and underlying conceptions of coordinate systems (Lee et al., 2020).

Design considerations for digital graphing tasks revealed relationships between students' graph reasoning and their conceptions of coordinate systems (Johnson et al., 2024). We report on a case study combining two theoretical frameworks related to graph reasoning (Johnson et al., 2020) and coordinate system conceptions (Lee et al., 2020) to explain aspects of students' graphing activity that may seem counterintuitive or contradictory. We ask the following research question: What is the nature of students' conceptions of coordinate systems and graph reasoning within and across digital graphing tasks involving different dynamic situations?

Background

Graph Reasoning

Johnson et al. (2020) developed a graph reasoning framework to characterize students' conceptions of graphs, identifying four broad types: Covariation, Variation, Motion, and Iconic. *Covariation* and *Variation* graph reasoning involve students thinking about quantities and their relationships. For example, suppose a student is shown a dynamic situation in which a "Cannon Man" is shot vertically out of a cannon and descends with a parachute (Figure 1) and is tasked

with sketching a graph relating his height from the ground and his total distance traveled. A student demonstrating Covariation graph reasoning may sketch a graph and describe how it shows Cannon Man's height increasing and decreasing while simultaneously showing how his total distance traveled continues increasing. If the student exclusively describes their graph in terms of one of these changing attributes, they would be demonstrating Variation graph reasoning (e.g., only describing Cannon Man's height increasing and decreasing).

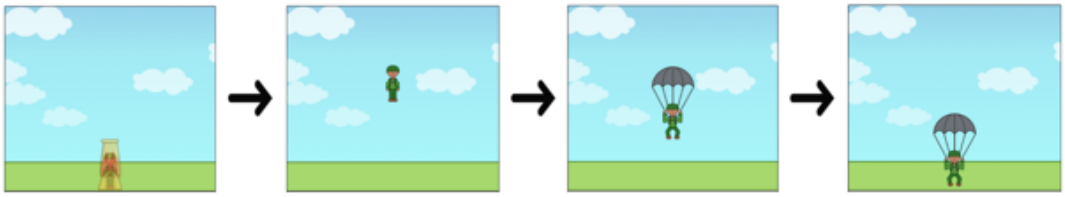

Figure 1. Cannon Man Dynamic Situation

Motion graph reasoning refers to students' reasoning about the physical movement within a dynamic situation. For example, a student demonstrating Motion graph reasoning may consider the Cannon Man dynamic situation (Figure 1) and describe their graph as representing Cannon Man's changing speed. *Iconic* graph reasoning, on the other hand, involves reasoning about physical features of the situation itself; a student demonstrating Iconic graph reasoning may describe their graph as representing Cannon Man's literal path. Johnson et al.'s (2020) framework highlights the diverse ways students may reason about and represent dynamic situations graphically.

Conceptions of Coordinate Systems

Lee et al. (2020) identified two types of coordinate systems students may conceive of when graphically representing dynamic situations: spatial and quantitative. A coordinate system is considered to be *spatial* when it is "mentally overlaid onto the space being represented" (p. 32). When students conceive of spatial coordinate systems, they determine relative locations of objects in the dynamic situations within that space. For example, given the Cannon Man dynamic situation (Figure 1), a student may conceive of a spatial coordinate system and plot points on a graph to represent the literal positions of Cannon Man in the air. Alternatively, students conceive of *quantitative* coordinate systems when they conceive of attributes in a situation as quantities and "project them onto some new space, which is different from the space in which the quantities were originally conceived" (p. 34). For example, in the Cannon Man situation (Figure 1), a student may conceive of a quantitative coordinate system and plot points on a graph to represent the relationship between Cannon Man's height from the ground and his total distance traveled. Notably, even when students are given an implied quantitative coordinate system, they may or may not conceive of the coordinate system quantitatively (e.g., using a coordinate system to represent physical locations even if neither axis entails a distance). Hence a student's meanings for the coordinate system can shape their graphing activity.

Methods

We report on a qualitative case study (Stake, 2005) of a college algebra student who participated in a series of four task-based clinical interviews (Clement, 2000; Goldin, 2000). This case stems from the first author's dissertation (Knurek, 2025a). In a preliminary report, we explored connections between a student's graph reasoning and conceptions of coordinate systems (Knurek et al., 2025b). Ava's case advances the prior work by demonstrating how

Motion graph reasoning can persist amidst a student's quantitative conceptions of coordinate systems. Ava completed interviews online via video conference, during which she interacted with a sequence of dynamic graphing tasks housed in the Desmos classroom (www.desmos.com) known as techtivities (Olson & Johnson, 2022).

During a techtivity, students are shown an animation of a dynamic situation and are asked to consider two specified attributes. Next, students are to explore changes in these attributes by dragging sliders (i.e., "Axis Sliders") along the axes of a Cartesian plane and interacting with a dynamic graph illustrating their slider movements. Then, students are to sketch a graph representing the relationship between attributes. Figure 1 (Cannon Man) and Figure 2 (Dynamic Tent, Toy Car, and 3D Printer) illustrate the four dynamic situations used for the interviews. In the Dynamic Tent techtivity (Figure 2a), students are asked to consider the height and width of a tent. In the Toy Car techtivity (Figure 2b), students are asked to consider a toy car's total distance traveled and distance from a cone. In the 3D Printer techtivity (Figure 2c), students are asked to consider the total amount printed and the height of the printed object.

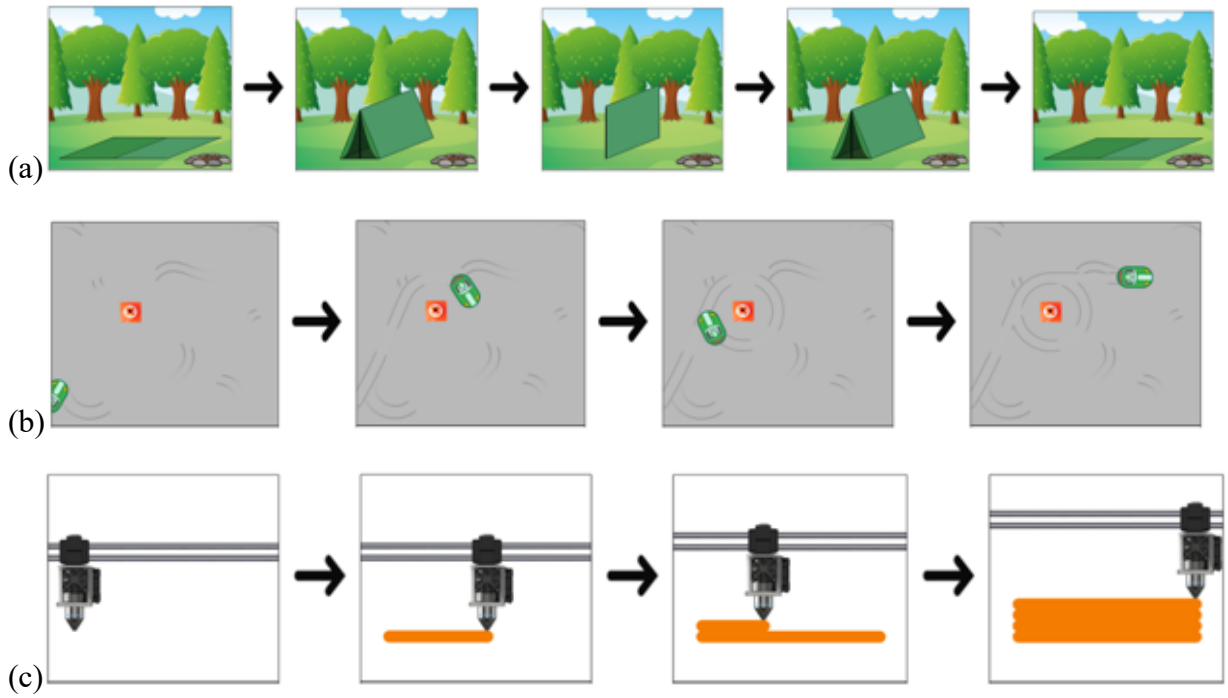

Figure 2. Dynamic Tent, Toy Car, and 3D Printer Dynamic Situations

Data Collection and Analysis

To collect data, the researcher conducted task-based clinical interviews (Clement, 2000; Goldin, 2000) to elicit students' current ways of reasoning by using interview protocols that support flexible and responsive questioning. As such, the four clinical interviews were akin to exploratory teaching (Steffe & Thompson, 2000), which is part of teaching experiment methodology.

Ava's interviews lasted approximately 20-30 minutes and were video-recorded. During the interviews, Ava shared her screen and completed techtivities while the researcher asked questions from the interview protocols, which were designed and refined during an earlier pilot study. The researcher took jot notes (Erickson, 1986) immediately after each interview to document reflections and takeaways to inform minor adjustments to interview questions for

subsequent interviews, making the approach iterative and responsive to Ava's emerging reasoning.

To systematically analyze the data, we used Wolcott's (1994) three-part framework: description, analysis, and interpretation. The first part, description, entailed reviewing both the transcripts and screen recordings of Ava's interviews and generating a detailed account of all her words, gestures, and cursor movements. For the second part, analysis, we engaged in conceptual analysis (Steffe & Thompson, 2000) to make inferences about how Ava conceived of quantities, coordinate systems, and graphs based on the descriptions. For the final part, interpretation, we drew on established theories and frameworks to make sense of Ava's thinking and answer the research question. This allowed us to take a step back and consider the broader meaning of our findings, building on the descriptions and analyses we had already conducted to make claims about Ava's reasoning and conceptions within and across techtivities.

Results

Our analysis of Ava's interviews revealed the persistence of her Motion graph reasoning (Johnson et al., 2020) amid her quantitative conceptions of the coordinate system (Lee et al., 2020). Across three interviews (i.e., Cannon Man, Dynamic Tent, and 3D Printer), Ava consistently conceived of the coordinate system quantitatively while demonstrating Variation and Covariation graph reasoning. Yet, she also conceived of some aspects of her graph as representing motion. During her other interview (i.e., Toy Car), Ava conceived of parts of the coordinate system spatially while demonstrating Motion graph reasoning.

In the following sections, we describe Ava's reasoning about each individual attribute while interacting with the Axis Sliders. Then, we present Ava's graph sketches and annotations while highlighting evidence of her graph reasoning and conceptions of coordinate systems.

Variation in a Quantitative Coordinate System

While interacting with the Axis Sliders, Ava consistently conceived of attributes in the dynamic situations as quantities and represented her conceptions of change for each attribute in a new space (i.e., the axes of the given Cartesian plane). For example, during the Dynamic Tent techtivity, Ava dragged the horizontal slider (Width) to the left as the tent's width decreased, then dragged the slider to the right as the width increased. She said, "So it starts at the widest it gets to in this video, and then it slowly like decreases until it gets to zero, and then it comes right back to its initial point." Next, Ava dragged the vertical slider (Height) up as the tent's height increased, then dragged the slider back down as the height decreased. She said, "The height starts at zero and it gradually increases and then like goes back to being zero again." Similarly, during the other three techtivities, Ava used the sliders along the axes to represent her conceptions of change for each attribute. Across all her interviews, Ava demonstrated Variation graph reasoning while conceiving of each axis in the coordinate system quantitatively.

A Coordinate System Involving Spatial and Quantitative Elements

During her Toy Car interview, Ava sketched a graph representing the toy car's proximity to the cone, rather than the relationship between the specified attributes (i.e., the toy car's total distance traveled and the toy car's distance from the cone). She started her graph by plotting a single point on the horizontal axis to represent the maximum distance the toy car traveled (Figure 3). Then, she plotted three points along the vertical axis, labeled "0ft," "5ft," and "10ft." Next, she labeled one of the points on the vertical axis as "cone" and said, "This is where the cone is at. So the cone is five feet from where [the car] started." Finally, Ava plotted points diagonally from

left to right in the plane while describing the toy car's proximity to the cone. We numbered each point in Figure 3 to indicate the order in which she plotted these points. Ava described the motion of the toy car while plotting the points:

> When it's further from its journey, it's closer to the cone [plots points labeled 1-4 from left to right on a diagonal line, Figure 3] and then further in its journey, approaching the cone, blah, blah, blah. And then we'll eventually get to the cone at some point in its journey [plots point labeled "5," Figure 3]. And then as it goes down, or as it continues its journey, it's getting further away from the cone [plots points labeled 6-10, Figure 3].

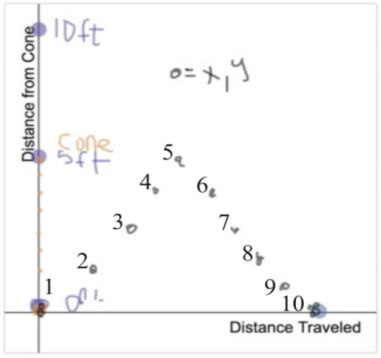

Figure 3. Ava's Graph Sketch during the Toy Car Techtivity

Ava sketched a graph to represent the toy car going to and from the cone. She labeled "cone" on the vertical axis to show the vertical location of the cone in the animation, then plotted points to show how the toy car approached the cone and continued away from the cone. We infer that Ava plotted the points from left to right either because the toy car traveled across the screen from left to right, or because she was focused on the temporal progression of the motion, accounting for experiential time (Thompson & Carlson, 2017). Ava's sketch and response suggests that she conceived of a coordinate plane involving spatial and quantitative elements, and provides evidence of Motion graph reasoning.

Covariation in a Quantitative Coordinate System

During her other three interviews (i.e., Cannon Man, Dynamic Tent, and 3D Printer), Ava sketched graphs representing the relationships between attributes in the dynamic situations. Figure 4 shows Ava's graph sketches for the Cannon Man, Dynamic Tent, and 3D Printer techtivities, respectively. Ava started each graph sketch by plotting points along the axes and in the region bounded by the axes. Then, she connected her points with line segments while describing the relationship between attributes. For example, during the Dynamic Tent techtivity (Figure 4b), Ava said, "When the height is lower at four, we see the width gradually become higher, gets to six, and when the height is even lower, the width is even higher." Ava described the relationships between attributes similarly during the Cannon Man and 3D Printer techtivities. In all three interviews, Ava demonstrated Covariation graph reasoning while conceiving of the coordinate system quantitatively.

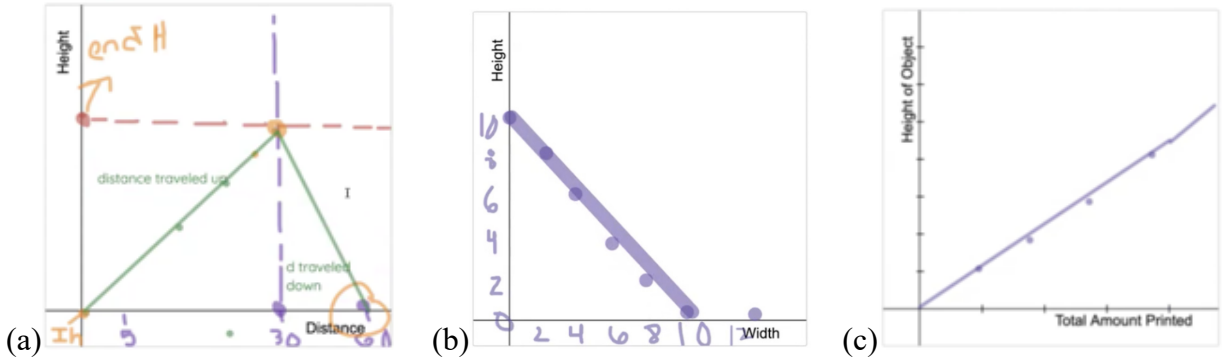

Figure 4. Ava's Initial Graph Sketches

Motion in a Quantitative Coordinate System

After Ava sketched each graph, her subsequent annotations and responses suggested that she conceived of her graphs not only as representing the relationship between attributes, but also as representing visible motion. For example, during the Cannon Man techtivity, the researcher asked Ava, "If Cannon Man just went up at a constant speed and down at a constant speed, would that change anything?" Ava responded, "Yes, because instead of my graph looking like this, it would look more symmetrical like this." She drew a picture in the top-right section of the plane that resembled an upside-down V with a vertical line through it (Figure 5a). She said, "The differences in the slopes would tell you like how fast it's going." Ava described the asymmetrical look of her graph sketch as representing Cannon Man's changing speed, with the different slopes representing different speeds (i.e., how fast it's going). During the Dynamic Tent, after finishing her graph sketch, Ava drew an arrow pointing downwards above her line segment (Figure 5b) and said, "The line goes this way." Then she drew an arrow pointing upwards below the diagonal line, crossed it out, and said, "It doesn't go this way." Similarly, during the 3D Printer techtivity, Ava drew an arrow along her line segment (Figure 5c) and said, "I think it goes in this direction." She proceeded to talk about how her graph showed the motion of the object getting "higher." Although Ava conceived of her graphs in terms of quantities and their relationships across all three interviews, her responses provide evidence that she also had expectations of her graphs representing motion; this included either the motion of the object in the dynamic situation or the motion of the trace in the graph itself. Thus, even though Ava was still conceiving of a quantitative coordinate system and evidencing Variation and Covariation graph reasoning for the majority of her interviews, her Motion graph reasoning persisted. To Ava, a graph should represent more than just the relationship between attributes; it should also incorporate some kind of motion that is reflected in the situation itself.

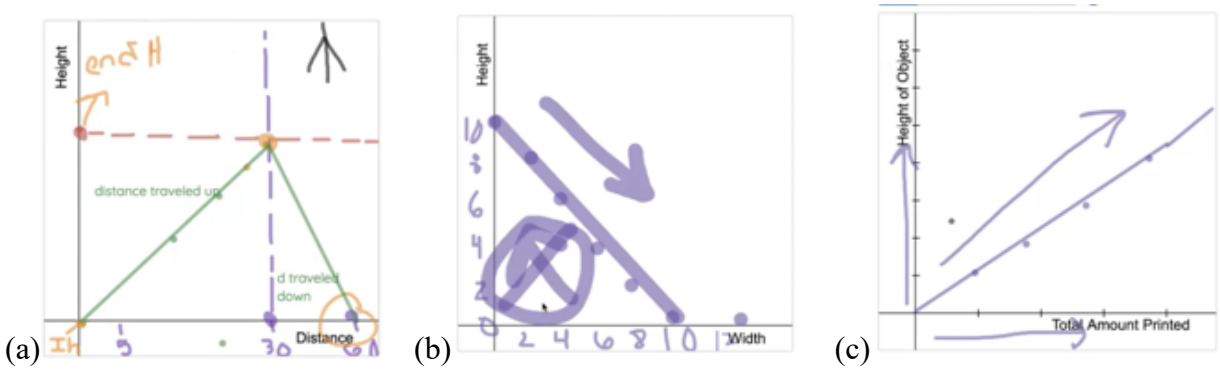

Figure 5. Ava's Graph Sketches with Subsequent Annotations

Discussion and Conclusion

This study provides empirical evidence demonstrating interrelationships between students' graph reasoning (Johnson et al., 2020) and their conceptions of coordinate systems (Lee et al., 2020). Ava's graph reasoning often aligned with her conceptions of coordinate systems in ways to be expected. Across interviews, Ava's Covariation and Variation graph reasoning emerged when she conceived of the coordinate system quantitatively. In conjunction, her Motion graph reasoning persisted amid her quantitative conceptions of coordinate systems. Ava's case shows how seemingly contradictory elements can hold together for students when they have multiple goals for graphing. Ava expected her graphs to represent both the relationships between attributes and the physical motion in the dynamic situations. This speaks to her conception of what a graph *should* represent (Johnson et al., 2020); to Ava, a graph should incorporate both quantitative and physical aspects of whatever situation she is representing.

Our findings have implications for researchers investigating students' graphing activity through tasks involving dynamic situations. Ava's case suggests that even when students incorporate the physical motion of objects into their graphical representations, they also can hold quantitative conceptions of coordinate systems. Thus, there is not a simple one-to-one correspondence between how students reason about graphs and how they conceive of coordinate systems. Researchers can use Johnson et al.'s (2020) and Lee et al.'s (2020) frameworks in tandem in future studies to gain deeper insights into students' graphing activity.

Lastly, we drew on two theoretical frameworks (Johnson et al., 2020; Lee et al., 2020) related to students' conceptions of graphs and coordinate systems, respectively. We acknowledge that other theoretical lenses, such as Moore & Thompson's (2015) constructs of static and emergent graphical shape thinking, could provide additional insight into some of Ava's sketches and responses (e.g., her use of arrows to describe the motion of a graph). However, we intentionally focused our examination of students' graphing activity to Johnson et al.'s (2020) and Lee et al.'s (2020) frameworks to investigate the interconnections of graph reasoning and coordinate system conceptions. In doing so, we were able to uncover some of the nuances involved in these constructs. Future research can continue to investigate students' graph reasoning alongside their conceptions of coordinate systems to gain deeper insights that may support the refinement of these frameworks to better reflect the complexity of students' conceptions and reasoning processes. Studies with larger and more diverse student populations could help validate and extend these findings, providing a clearer picture of how these constructs manifest across different mathematical contexts.

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A Narrative Inquiry Into Students' Proof Journeys and Mathematical Master Narratives

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Tenchita Alzaga Elizondo, University of Texas Rio Grande Valley
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This study uses narrative inquiry to explore undergraduate students' experiences learning mathematical proof and engaging with proof culture. Drawing on interviews and journey plots from five students at Hispanic-serving institutions, we constructed interpretive narratives to examine how their stories reinforced or resisted mathematical master narratives, such as mathematics being done in isolation or requiring innate ability. Students' journeys revealed moments of joy and triumph, as well as loneliness and frustration. While elements of their stories reinforced master narratives—such as feeling isolated—others emphasized the effort required to develop proof skills, countering the narrative that mathematical ability is innate. Our findings highlight how learning environments and instructional practices influence students' engagement with proof and their perceptions of who belongs in mathematics. By analyzing how students navigate proof culture, this work contributes to efforts to make mathematics more inclusive and to better understand the cultural narratives that impact students' mathematical development.

Keywords: Proof Culture, Narrative Inquiry, Student Experiences, Master Narratives

Students' experiences learning mathematical proof—both inside and outside the classroom—offer valuable insights for the mathematics education community, yet remain underexplored in undergraduate research. Existing studies have examined what students in proof-based courses take away from lectures (Krupnick et al., 2018; Lew et al., 2016) and their beliefs about various aspects of proof, including the reading, writing, and understanding of proof (Guajardo & Lew, 2024; Lew & Mejia-Ramos, 2019; Weber & Mejia-Ramos, 2014). While these studies shed light on students' perceptions and learning outcomes related to proof, further research is needed to explore how students' experiences evolve over time as they navigate proof-based courses and engage with proof culture. Their experiences with this culture shape not only their beliefs about proof itself but also their identity and sense of belonging within the proof community.

Recent studies have begun to explore students' experiences through their own mathematical journeys (Melhuish et al., 2024; Wrightsman, 2025). Melhuish et al. (2024), analyzing student reflections on mathematicians' autobiographies, found that students often challenged dominant narratives within the mathematics community—such as the belief that mathematical ability is innate—especially when reflecting on their own paths, including the idea that one can struggle with math and still pursue a mathematics major. Similarly, Wrightsman (2025) used narrative inquiry to examine how students' perceptions of and engagement with norms in inquiry-oriented introduction to proof courses shaped their participation and mathematical identities. She found that while some students adapted to the change in expectations (i.e., collaboration, public reasoning), others struggled as their prior identities related to being "good at math" were unsettled. These studies highlight the importance of understanding how students' beliefs and identities related to mathematical proof and proof culture are shaped by the content, courses, and broader cultural narratives they encounter.

Building on the need to understand students' mathematical journeys, we adopted a narrative inquiry approach to explore students' evolving experiences with learning proof and engaging with proof culture. Broadly, we view proof culture as the commonly shared values, goals, norms and practices within the mathematics community related to proof and proving (e.g., Dawkins &

Weber, 2017). Students' stories about their experiences learning proof over time and engaging with proof culture are powerful because they have the potential to open a window into elements of master narratives that are being reinforced or challenged (e.g., Melhuish et al., 2024). When students' identities and stories diverge from the master narratives, they may feel excluded from the broader mathematical culture (Melhuish et al., 2024). Therefore, identifying how master narratives are reproduced or resisted in students' personal narratives is a necessary step toward making proof culture more inclusive. Without this understanding, we cannot determine which aspects of the master narratives need to change. To address this, our study investigated the following research question: How did students' retelling of their journey through proof-based courses and experiences with proof culture reinforce or counter master narratives in mathematics?

Theoretical Framing

The stories that people tell (and retell) about themselves in relation to the world around them are inextricably linked with their identities (Sfard & Prusak, 2005). Students' stories about their journeys learning proof are powerful because these personal narratives have the potential to challenge existing master narratives within mathematics and proof culture (McLean & Seyd, 2015; Melhuish et al., 2024). Master narratives are common cultural stories that influence how people behave and view themselves within a culture, and shape perceptions of who is considered 'good' at being part of the culture (McLean & Seyd, 2015). Unlike personal narratives—stories of individuals' lives—master narratives are "culture-specific stories" that individuals may "internalize or resist, consciously or unconsciously" (p. 323). Master narratives become particularly important in mathematics and proof culture as they often link 'doing mathematics' with 'doing masculinity' (Jaremus et al., 2020; Leyva et al., 2017; Mendick, 2006), while also perpetuating the dominance of whiteness (Battey & Leyva, 2016; Martin, 2019) and Eurocentric viewpoints (Rowlands & Carson, 2002). Notably, Melhuish et al. (2024) identified several mathematical master narratives: Mathematics is done by privileged white men; Mathematics is done by those who are brilliant, but socially inept; Mathematics is done in isolation; Mathematics is dry and not fun; Mathematics ability is innate, and easy for those with high ability; and Mathematics is individualistic and a meritocracy.

Following McLean and Seyd (2015), we adopted a narrative approach to theorize the relationship between individual identity and societal structures. Given their shared adaptability and subjectivity, narratives offer a fitting lens for examining the complex interplay between identity and culture. McLean and Seyd's model connects individuals' personal narratives with master narratives and potential alternative narratives. When individuals conform to master narratives, the dominant social structures are reinforced, whereas resisting them constructs an alternative narrative. Interestingly, McLean and Seyd argued that master narratives can be strengthened in both cases—"directly through unconscious internalization and indirectly through conscious opposition and negotiation" (p. 325). Although internalization is usually an unconscious process, master narratives can be evident in individuals' personal narratives.

Within a narrative inquiry framework, stories are used not to uncover objective truths, but to understand the meanings embedded in lived experiences and how those meanings connect (Coulter & Smith, 2009; McAlpine, 2016; Polkinghorne, 1995). Rather than aiming to "describe the world as it is," narrative inquiry seeks to retell events retrospectively, acknowledging that within constructivist or postmodern paradigms, a singular reality does not exist (Coulter & Smith, 2009, p. 578). Narratives are structured through a temporal and sequential ordering of events (McLean & Seyd, 2015) and typically include elements such as "social context,

complicating events, and an evaluative conclusion" (McAlpine, 2016, p. 33). Narrative inquiry typically takes two forms: *analysis of narratives*, which involves collecting participants' stories through interviews or written reflections (e.g., journaling) and identifying patterns or themes (Coulter & Smith, 2009; McAlpine, 2016) and *narrative analysis*, which begins with data that may not be in storied form and constructs narratives through analytic interpretation (Coulter & Smith, 2009; Polkinghorne, 1995). We leveraged both forms of narrative inquiry to examine students' accounts of learning proof and engaging with proof culture.

Methods

Data Sources and Participants

Data for this study was collected at two southern Hispanic serving research universities in the U.S. We conducted interviews with students who completed a survey and responded "yes" to the interview invitation. All interviews were conducted over Zoom. Students were asked to reflect on their journey through proof-based courses and their experiences with proof culture. We asked them to create a journey plot (McAlpine, 2016) by graphing their experiences using time as the horizontal axis unit and any unit they wanted for the vertical axis (see Figure 1). While we provided some possible units (e.g., level of confidence in writing/reading proofs, sense of belonging in the proof community), we emphasized that they could use *any* unit they wanted to visually represent their experiences with proof culture. Participants used the annotation feature in Zoom to draw their graphs. Two participants created multiple graphs on the same coordinate plane, meaning they chose several different vertical units to describe their varied experiences. We prompted them to walk us through their thought processes while creating their graphs, asking follow-up questions to elicit any specific experiences or memories they had related to pivotal moments. Table 1 contains self-identified background information about the five study participants and the vertical unit(s) they chose to represent with their graphs.

Table 1. Demographics and Background of Interview Participants

Name	Race/Ethnicity	Gender	Proof Courses	Vertical Unit(s)
Aiden	White Hispanic	Man	Intro to Proof, Modern Geometry	Support from professors, outside support, relationship with professors, "proofing ability"
Arthur	Hispanic	Man	Discrete Math, Algorithms (at elite university); Intro to Proof, Modern Algebra, Real Analysis	Understanding of proof, difficulty with proof
Ethan	Hispanic	Man (Cisgender)	Intro to Proof, Topology, Real Analysis, Modern Algebra I	Confidence reading/writing proof, understanding of proof, sense of belonging
Liam	White	Non-binary	Intro to Proof, Real Analysis I and II, Modern Algebra	Comfortability in reading and writing proof

Rollin	White Hispanic	Man	Intro to Proof, Modern Geometry, Modern Algebra, Real Analysis Boundary Value Problems	Understanding of what a proof is and how to read/write proofs

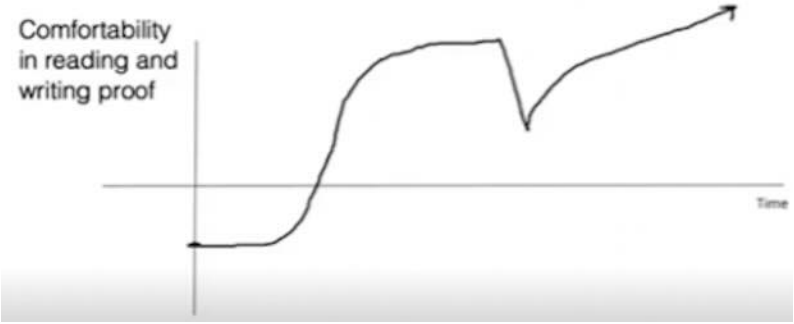

Figure 1. Liam's Journey Plot with vertical unit "Comfortability in reading and writing proof."

Data Analysis

We followed McCormack's (2000a, 2000b) guidance for transforming interview data into interpretive narrative accounts. First, we watched each interview and recorded active listening notes to re-engage with the storyteller after time had passed (McCormack, 2000a). As part of the note-taking process, we reflected on our positions as researchers in relation to the participants. Due to the subjective nature of this work, it was important for us to think about how our own subjectivity and positioning might influence the analysis process. For each interview, two authors independently drafted initial narratives. Although participants created journey plots with respect to time, their descriptions of the events within the plot were not always linear. Since narratives unfold over time and have a sequential structure, we focused on piecing together the participants' stories based on temporal aspects and sequences of events as we drafted initial narratives. Then, we synthesized the two narratives for each participant, paying attention to features one narrative captured that the other didn't, as well as the tone, flow, tense and structure. On average, each synthesized narrative was about four paragraphs long.

To strengthen credibility, we sent the synthesized narratives to each participant, asking if our interpretation was accurate and whether they would add or change anything. Two of the five participants responded affirmatively without suggesting changes. We then compared and contrasted the syntheses to identify shared or stand-alone experiences. For example, comparing Arthur and Liam's stories revealed a common experience of having an "aha" moment while independently studying proofs after hitting a low point with feeling comfortable understanding proof. Then, we used the master narratives identified by Melhuish et al. (2024) to indicate elements of the participants' stories that countered or endorsed the master narratives.

Results

The participants' stories reflected different challenges, involving individual reflection, reorientation towards proof, and striving to find community and support. Several journeys were marked by extreme highs and lows (Liam, Arthur) while others showcased a positive upward trend over time (Rollin, Ethan). We first present summarized versions of each participant's narrative, then highlight how certain aspects of their stories either reinforced or countered the following master narratives: *mathematics is done in isolation*, *mathematics is individualistic and*

a meritocracy, *mathematics ability is innate and easy for those with high ability*, and *mathematics is dry and not fun.*

Students' Summarized Personal Narratives

Aiden's Story: "It's us and the professor. If the professor can't help us, no one can." Aiden's journey into proof-based math courses was represented by an experience of loneliness and yearning for support. He was "surprised and very joyful" to discover the rigor in proof classes, expecting his "niche little hobbies" would help him succeed. But when asked to "translate Hilbert's axioms into symbols," he struggled to find help—tutors and other professors couldn't assist, leaving only the course instructor. The intro to proof class consisted mostly of lectures, and students "struggled" equally, offering little peer support. He realized proof is a "very lonely subject compared to other math subjects," and over time, support "dropped" due to its rigor. Still, he appreciated engaging with the professor's "hobby." Despite the loneliness, his love for math and proof reached a high point, and he believes his ability to engage with proof has improved.

Arthur's Story: "Cleared the fog a bit, wiped the mirror...it [was] a moment of clarity." Arthur's journey was marked by a battle between understanding and difficulty, and it was a readjustment of understanding what proof was that ultimately supported a positive outcome. Before college, he felt fairly confident with proof, but at a prestigious university, instructors expected him to know the basics without support, making him feel like a "single cog" and "the weakest link." After transferring to a local university, he struggled to balance life, work, and school, using math and proof as an escape. Working independently—sometimes at 3:00am—he had a "real moment of clarity," recalibrating his view of proofs to focus on how they relate to other exercises. The fog lifted, like he had "finally taken off his glasses to clean them." His experience shows that learning proof isn't nurtured by prestige, but by support, clarity, and connection, and through readjusting his perspective, he rebuilt his relationship with proof by finding the right environment to feel supported.

Ethan's Story: "I'm learning…getting there, I just don't believe I'm quite at that level yet." Ethan's journey through proof-based courses was one of growth and self-awareness as he strived to find his place within the proof community. Initially unsure of his skills in reading and writing proof, his confidence grew through courses like Modern Geometry and especially Topology, whose abstract nature "really forces you to hone your skills." Abstract Algebra further improved his understanding, though he still felt like he hadn't written proofs "nearly long enough" to take himself seriously among experts. Until he could answer questions on Math Stack Exchange, he couldn't confidently see himself as part of the proof community. He was "learning and getting there," even if not quite at that level yet. In contrast, his classroom community felt more supportive—he and his classmates were "in the same boat," helping each other through topics. Ethan's journey shows that mastering proof is not just about skill but about finding belonging, and while being part of the broader proof community still feels distant, he's found support and connection among peers.

Liam's Story: "Proof can be whatever it needs to be." Liam's story is one of turbulence, related to their experiences with "comfortability reading and writing proof." Starting college math in Calculus 3, they felt "thrown in" to advanced concepts like limits and epsilons, making proof feel like a "big beast" to conquer. An introduction to proof course with a norm-breaking instructor changed everything—accessible proofs increased their comfortability, and they began to love proofs, unlike their friend in a more "rigid" program. However, independently studying category theory challenged them again with its rigor and conciseness, forcing them to reckon

with their understanding. Things shifted when they eventually realized that each subfield has its own proof conventions. After a year of analysis, they needed to "get [themselves] out of this little box" and reframed proof as "whatever it needs to be…it's just a way to explain our reasoning for why this thing is true." This turning point allowed them to flourish, raising their comfortability with proof as they continued their studies in mathematics.

Rollin's Story: It is one thing to "read a recipe" and another to create a delicious cookie. Over the course of his education, Rollin encountered proof in various forms and contexts, gradually developing his proof knowledge and writing skills. In high school, he learned that a proof was a series of logical steps leading to "a correct answer," but this basic understanding couldn't handle the advanced proofs in research papers. At university, he learned to "make" proofs—logical steps and axioms—but realized that "writing" proofs required creativity and deeper knowledge, like the difference between "reading a recipe" and creating a delicious cookie. His skills grew through coursework and an opportunity he had to participate in a Research Experience for Undergraduates (REU), where he authored a proof of an optimal solution to Conway's game. Though he viewed the proof as "simple," it marked a turning point, showing he could "comfortably make them for research," even if he's "not quite there yet." Rollin's journey evolved from rigid steps to crafting original arguments, with research as the "ultimate goal" of proof writing.

Elements that Reinforced or Countered Master Narratives

Mathematics is done in isolation. Parts of Aiden's story both reinforced and countered the mathematics is done in isolation master narrative. On one hand, his experiences with math before proof classes countered this narrative because he actively sought help, not wanting to do math in isolation. On the other hand, when he started taking proof classes, his realization that only the professor can help reinforced this narrative because the situation forced him to learn proof mostly alone (only with the professor's help). Ultimately, his experiences in proof courses reinforced this narrative's strength. Arthur's experience in proof courses at the elite university—feeling like a "single cog" and "the weakest link"—also reinforced this narrative. Ethan's experience, however, countered this narrative because he felt like he and his classmates were all in "the same boat" and they helped each other when they could.

Mathematics is individualistic and a meritocracy. Aspects of the student stories reinforced and countered the master narrative mathematics is individualistic and a meritocracy. Liam's and Arthur's experiences reinforced it because they both arrived at a "moment of clarity" through individual reflection and effort, and felt personal satisfaction making it through a low point with continued persistence and hard work. Contrastingly, Ethan's experiences reflected valuing peer support and how people in the broader community helped each other on Math Stack Exchange, countering the narrative that one works for one's own personal gain (individualism).

Mathematics ability is innate, and mathematics is easy for those with high ability. The language Ethan and Rollin used throughout their stories reflected building proof knowledge and skills along the way, implying mathematics ability is not innate—countering this master narrative. For example, Rollin recognized that he was "not quite there yet" and Ethan was "learning and getting there," both articulating that even though they had learned a lot and their confidence with proof had grown, there was still more to learn. In comparison, Arthur and Liam both experienced struggling with proof at the beginning of their journeys because of the classroom environments they were in. While they did not directly endorse this narrative, the commonalities between their experiences—feeling like they were expected to have prior knowledge and grasp new concepts quickly—reinforced the master narrative to some extent.

Mathematics is dry and not fun. None of the participants' stories directly reinforced the master narrative mathematics is dry and not fun. Aiden's experience of entering proof courses and noticing how his "niche little hobbies" (which were logic related) were connected to proofs made him feel "surprised and very joyful," countering this master narrative. Even though he eventually found learning proof to be a lonely experience, he continued to enjoy proofs through the experience of interacting with his professors' hobbies (which were presumably exciting and fun to them). Logan similarly experienced joy and love for math when they took the intro to proof course with the professor who made proofs more accessible, and their enjoyment continued even after they hit a low point when studying category theory.

Discussion

In this paper, we shared five students' journeys through proof-based mathematics, each offering a distinct perspective shaped by personal experiences with proof culture. These narratives included moments of joy and accomplishment, as well as periods of isolation and struggle. While each story was unique, common elements emerged that connected students' experiences to mathematical master narratives previously identified in the literature (i.e., Melhuish et al., 2024). As discussed earlier, mathematical master narratives are stories tied to mathematical culture that shape perceptions of who belongs in mathematics and what it means to do mathematics. By examining how these narratives surfaced in students' accounts, we gained insight into how proof culture is experienced on a personal level. While elements of students' stories reinforced master narratives—such as Aiden's experience of isolation—others, like Ethan and Rollin, emphasized the sustained effort required to develop proof skills, which countered the master narrative that mathematical ability is innate.

A key takeaway from our results is the influence that learning environments have in shaping students' experiences with proof and their sense of support. Both Arthur's and Liam's stories contrast welcoming environments with isolating experiences at more "prestigious" universities. Arthur began his journey at an elite university where he felt unsupported and like the "weakest link" due to a lack of prior knowledge. After transferring, he found an environment that allowed him to take time for himself and succeed in proof. Similarly, Liam described a "norm-breaking" instructor who made proofs accessible and brought joy to learning proof—an experience that stood in stark contrast to their friend's more "traditional" and rigid program, which led to a strong dislike of proofs. Their redemption stories also revealed the "good side" of meritocracy (Weber & Melhuish, 2020), in which striving for improvement was motivating without being rooted in competition or comparison (i.e., the "bad side").

Taken together, these student narratives offer a meaningful lens into how proof culture is experienced and interpreted by students. Understanding which master narratives are endorsed or resisted in students' personal accounts of learning proof and engaging with proof culture provides a starting point for shifting these narratives. While master narratives are deeply embedded and unlikely to disappear entirely, elements within them can evolve to become more inclusive—especially when we recognize which narratives are most salient and how students are already challenging them through their lived experiences.

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Resources, Orientations, and Goals Informing Calculus Instructors' Enactment of Equitable Active Learning Teaching Practices Learned Through Approximations of Practice

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We implemented a semester-long professional development workshop series, training Calculus I and II instructors to implement active learning and equitable teaching practices. The workshops actively engaged instructors in approximations of practice (Grossman et al., 2009) in which they learned to enact various teaching practices like designing tasks; eliciting, noticing, and responding to evidence of student thinking; orchestrating class discussions; fostering inclusive social norms; and equitably managing student participation. After they participated in the workshops, we interviewed the instructors about their decisions to implement the teaching practices from the approximations of practice in their teaching. We analyzed the instructors' resources, orientations, and goals that informed their decision to enact those practices.

Keywords: Approximations of Practice, Professional Development, Calculus, Active Learning, Equitable Teaching Practices

Despite the benefits of active learning and equitable teaching practices (Bartell et al., 2017; Freeman et al, 2014; Theobald et al., 2020; Thiry et al., 2019), university faculty have been slow to adopt them (Rasmussen et al., 2019; Stains et al., 2018). There is a need for more professional development (PD) on these teaching practices. At the K-12 level, PD and teacher preparation programs use Approximations of Practice (AoPs; Grossman et al., 2009) to give instructors opportunities to develop and enact specific teaching practices. *AoPs* are activities with a hypothetical teaching scenario in which instructors can simulate a teaching practice without the complexity of being in a real-time classroom. AoPs have been shown to be effective in helping K-12 mathematics teachers develop active learning practices (e.g., Amador et al., 2021; Howell & Mikeska, 2021; Krause et al., 2020; Lai et al., 2021; Roberts & Olarte, 2023; Schack et al., 2013). Their use in PD programs for university faculty has not been commonly documented, however. We propose that such implementation of AoPs in PDs holds promise for the training and professionalization of university mathematics instructors.

In this paper, we investigate how engaging in AoPs can support university instructors in implementing active learning and equitable teaching practices. We report on a study where Calculus I and II instructors participated in a PD program focused on equitable teaching practices and active learning through using AoPs. The instructors' decisions to implement the teaching practices from the AoPs could be informed by their resources, orientations, and goals (Schoenfeld, 2010), so we address the research question: *What are the resources, orientations, and goals that inform instructors' implementation of the teaching practices from the AoPs?*

Literature Review

Active learning is a large umbrella term that captures various pedagogies (e.g., group work, student presentations, class discussions) related to engaging students in class activities beyond passively listening to lectures or taking notes. Although there are numerous benefits associated with active learning in STEM courses (e.g., Freeman et al., 2014; Kogan & Laursen, 2014; Theobald et al., 2020), active learning is not necessarily equitable for students (White et al., 2025; Brown, 2018; Johnson et al., 2020; Reinholz et al., 2022). Thus, it is essential for

instructors who use active learning to attend to equity and inclusion in their courses. There are several teaching practices instructors can use to equitably implement active learning that have been suggested by researchers (e.g., Bartell et al., 2017; Freeman et al., 2014; Gutiérrez, 2010; Kuster et al., 2018; Laursen & Rasmussen, 2019; Reinholz, 2023) and professional organizations (e.g., Mathematical Association of America, MAA, 2018; National Council of Teachers of Mathematics, NCTM, 2014). Serbin et al. (2025) suggested that these practices of varied grain size align with these broader teaching practices: "(i) use meaningful engaging math tasks, (ii) facilitate student collaboration, (iii) elicit, notice, and use evidence of students' mathematical thinking, (iv) orchestrate class discussion on students' mathematical ideas, (v) attend to participatory equity and access in the classroom, (vi) foster an inclusive class community, and (vii) equitably assess student work" (p. 1568). Given the success of active learning and equitable teaching practices in calculus courses (Bressoud & Rasmussen, 2015; Hagman, 2021), mathematics departments in the US have made strides toward institutionally changing their mode of instruction in calculus to active learning (Smith & Rasmussen, 2025; Williams et al., 2022).

It is difficult for instructors to change their instruction, so there is a need for PD on active learning. PD programs for mathematics instructors can be implemented in a variety of formats, like pre-semester workshop days, workshop series, faculty professional learning communities (Stoll et al., 2006), and lesson study (Fernandez & Yoshida, 2012). PD programs can also be informed by various design principles. Soto and Marzocchi (2021) suggested such programs should emphasize building community among faculty participants, create opportunities for faculty to collaborate, encourage faculty reflection and persistence in improving their practices, build faculty's confidence and motivation, and provide a variety of PD formats. These and other researchers have emphasized the importance of actively engaging participants in PD programs to model active learning approaches (e.g., Akin et al., 2023). One such way to actively engage instructors in teaching practices during PDs is to provide opportunities to participate in AoPs, which we describe as follows.

Theoretical Framework

Theoretical Perspective Informing the PD Design

AoPs are productive activities for teachers to engage in during teacher education and PD programs (e.g., González, 2017; Kavanagh et al., 2020; Sapkota & Max, 2024). AoPs are "opportunities to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p. 2056). These activities simulate a teaching scenario or practice, which provides an opportunity for teachers to rehearse and develop teaching practices without the complexity of a real-time class setting or risk of affecting students (Howell & Mikeska, 2021). AoPs can simulate various teaching practices, such as noticing students' mathematical thinking (Jacobs et al., 2010; van Es & Sherin, 2021) or responding to students' ideas in class discussions (Campbell & Elliott, 2015; Howell & Mikeska, 2021; Jackson et al., 2024; Kaka et al., 2021). The design of AoPs can vary by the type of teaching practice approximated, the medium of the scenario (e.g., classroom videos or written artifacts of student work), the degree of authenticity of the approximated teaching practice, and the scenario content.

Theoretical Perspective Informing the Data Analysis

To analyze our data, we drew on Schoenfeld's (2010) decision-making framework, which suggests a teacher's decision-making can be explained by their resources, orientations, and goals.

Resources. In Schoenfeld's (2010) framework, *resources* encompass an instructor's

knowledge inventory (including content and pedagogical knowledge), as well as the material resources available to teachers (e.g., technology, physical classroom spaces). These resources determine what is possible in a classroom at any moment. Resources also include interpersonal skills and practical knowledge that teachers bring into the classroom (Schoenfeld, 2011). What an instructor knows and has access to influences the decisions they make in real-time teaching.

Orientations. *Orientations* refer to a teacher's beliefs, values, preferences, and tastes (Schoenfeld, 2010). These orientations influence how teachers perceive and engage with classroom situations and what goals they consider to be important. Orientations are often implicit and develop over time with experience. They significantly shape teachers' real-time decisions.

Goals. *Goals* are the objectives, either explicit or implicit, that guide a teacher's decisions and actions (Schoenfeld, 2010). They can range in granularity, from broad aims, such as helping students to foster conceptual understanding of a subject, to more specific aims, like maintaining the class pace. Schoenfeld (2011) posited "goals recruit resources" (p. 459); teachers activate and apply their resources in service of the goals they prioritize. Instructors constantly re-evaluate their goals based on evolving classroom dynamics and shift their actions accordingly.

Methods

The 11 participants included part-time lecturers, full-time lecturers, and tenured professors, who taught Calculus I or II. Eight were faculty at a large public university, and three were faculty at a nearby community college in the southern US. Most faculty primarily lectured in their courses but were interested in implementing active learning.

The first author led a PD workshop series on using active learning while attending to equity and inclusion. The program consisted of two full-day sessions before the spring semester, with six two-hour sessions during the semester. The PDs drew on K-12 and undergraduate math education literature and focused on the following evidence-based teaching strategies: fostering inclusive classroom social norms; designing group-worthy mathematical tasks (Lotan, 2003) that require high levels of cognitive demand (Stein & Smith, 1998); eliciting, noticing, and responding to evidence of student thinking (NCTM, 2014); orchestrating class discussions using the 5 practices: anticipating, monitoring, selecting, sequencing, and connecting (Smith & Stein, 2018); and fostering equitable student participation (MAA, 2018). Instructors practiced these strategies through 15 AoPs that used videos of student thinking, scripts of classroom discussions, student quotes, and hypothetical student work to simulate classroom scenarios.

We conducted hour-long semi-structured interviews with each instructor after the PD series. Questions focused on participants' enactment of the teaching practices from the AoPs in their calculus classes. We provided participants a list of topics covered in the PDs and asked them to identify practices they tried implementing and the result. We asked about their experiences in the workshops and what they found helpful about the AoPs. Interviews were conducted over Zoom by the second author and an outside research team member who were not involved in facilitating the AoP workshops. Interviews were transcribed through Zoom and cleaned for analysis.

Our data analysis began by extracting excerpts where participants discussed their experience or decision to enact active learning pedagogies that were highlighted in the PDs. We deductively coded (Miles et al., 2013) these excerpts using the codes, resources, orientations, and goals, while considering Schoenfeld's (2010) descriptions of those. We characterized the types of resources, orientations, and goals discussed via inductive coding (Miles et al., 2013). For each participant, we compared excerpts and created a summary of how their resources, orientations, and goals seemed to influence their decision to enact teaching practices they learned in the AoPs.

Results

For this paper, we present case studies for two participants and describe the resources, orientations, and goals that informed their implementation of the teaching practices from the PD.

Case 1: Sumaya

Overview and What Practices She Implemented from the AoPs. Sumaya had taught university mathematics for 17 years. She primarily taught Calculus I, and during the semester that she participated in the PD, she taught Calculus II. During her interview, Sumaya claimed she implemented these teaching practices from the AoPs: fostering inclusive social norms, designing group-worthy mathematical tasks, eliciting evidence of student thinking, orchestrating class discussions, and fostering equitable student participation. When asked about which of the AoPs impacted her teaching the most, Sumaya brought up the steps of orchestrating class discussions. She remarked, "I wasn't aware that some of the practices that I had would fall into this definition of anticipating, monitoring, selecting, sequencing, and connecting." She went on to say, "After I attended the workshop…I learned that these are actually a sequence of different things that we can do for the class discussions, and so I became more mindful of doing them." Sumaya talked about how, before the workshop, she would anticipate student responses and monitor students during class, but because of "time constraints," she would often just take "the first answer that I would get. I would just say either yes or no and continue with that." But after attending the workshop, Sumaya "was more patient" and would "wait for more responses and then build upon those responses." Sumaya discussed how her participation in the AoPs helped her reflect on her teaching and find areas to improve upon. She said,

> I participated in this program because I wanted to better myself. But then some of the things, for example, the anticipation and selecting and sequencing, when I heard it, my first reaction was, "Oh, yeah, I'm doing this." But then when…we talked about it, and then when I went to the class, and I reflected on it, I realized that there is a lot of improvement that I can make...I became more mindful of everything.

Participating in the AoP helped Sumaya become more aware of the steps in orchestrating class discussions, which led her to be more intentional about selecting, sequencing, and connecting student responses. Below, we discuss our claim that Sumaya's resources, orientations, and goals informed her implementation of the teaching practices from the AoPs.

Goals and Orientations. A central focus of Sumaya's interview was her goals to *grow as an educator* and *engage students in their learning*. These goals were closely linked to Sumaya's orientation that *active learning is important*. Sumaya discussed how she had been committed to using active learning teaching strategies even before participating in the AoPs. She explained that at the beginning of her teaching career, she was very "lecture-based" because that is how she had been taught. However, over time, she "became interested in the teaching practices that best involve students and engage students in their own learning," attending conferences and workshops to learn more about active learning. Sumaya discussed the value of PD, saying they help with "becoming very mindful of our practices, reflecting on what we're doing, so that we can make adjustments." She went on to say that "if the focus goes into learning and what we can do for the students to engage them through active learning, how to be a truly equitable educator, then we can reach more students and help them with learning." These goals and orientation seemed to support Sumaya's decision to use active teaching practices from the AoPs in her class.

Another one of Sumaya's goals was to *provide students with appropriate scaffolding* so that they could engage with material and continue to work in small groups without needing constant assistance. Sumaya linked this to the teaching practice of designing group-worthy mathematical

tasks, which had been the primary focus of one of AoPs. She discussed how facilitating group work in a large class of 56 students in an auditorium classroom without the help of a TA was challenging. However, by carefully scaffolding group activities, Sumaya felt that the students "had some guidance to go through it [the worksheet]" and that providing "questions that would guide them to go to the next step, not necessarily making it too easy...would be one of the better ways to manage the large classroom." This goal seemed to influence Sumaya's decision to implement group-worthy tasks in her class, despite the challenges hindering her implementation.

Sumaya also exhibited the orientation of the *importance of valuing student contributions* and discussed her goal to *have students feel comfortable sharing their ideas* in class. She talked about how Calculus II "has difficult topics, so not all students are comfortable sharing their answers." Sumaya went on to say, "It's really important for them to engage without putting them in a position that would embarrass them. So, that was something that I tried to give value to all answers that I was receiving from the students." She concluded by saying, "I would always try to bring positive remarks to all sorts of answers, so that they feel more comfortable. Building trust with 56 students wasn't easy...But I think I was successful in that." Helping students feel comfortable directly connects to the teaching practices of fostering an inclusive classroom social norms and equitable participation from the AoPs. Sumaya commented that her participation in the AoPs "validated...how I want to approach my students, making sure everyone is included, making sure everyone's opinion does matter, that they have ownership over their learning, that I give credit for their effort, and not necessarily correctness of their answers." Her orientation and goal here supported her enactment of productively responding to student thinking and fostering an inclusive classroom social norms and equitable participation, which she practiced in the AoPs.

Resources. In terms of resources, Sumaya discussed how her *awareness of student backgrounds* and *knowledge of common student struggles* informed the way she structured class and designed tasks. Sumaya discussed the steps in orchestrating her class discussions, saying,

> Anticipating students' answers, that's always been in my mind the way that I make the worksheets or the way that I create those review sessions at the beginning of each class...So that I knew that, okay, this part seems a little bit difficult,...students are going to get caught...[or] make a mistake in this part.

She also described the challenge of supporting students with various mathematical backgrounds and trying to "balance" meeting the needs of different students. Sumaya remarked how she had "math majors in the class who are ready to learn more, and they want to go more in depth with the proofs," while also teaching "students who are really struggling with derivative, anti-derivative, very basic rules. So, how to balance that and be still equitable and inclusive. That was a challenge." Sumaya's resources here seemed to influence her implementation of equitable and inclusive teaching practices, which were addressed in the AoPs.

When discussing how she had implemented the practices of fostering equitable participation and inclusive social norms in her class, Sumaya brought up the cultural context of the institution. She talked about how during group work time, she would "make sure that everything is clear to the students, especially because we are a Hispanic Serving Institution, and some of the students, English is their second language." She went on: "I was very mindful of that...if the language is clear for the students...I let the students know that as I'm walking around, if something is unclear, make sure that you ask me, so I can clarify for you." By drawing on the resource of her *knowledge of students' backgrounds*, Sumaya could attend to equity and inclusion in her class.

Case 2: Richard

Overview and What Practices He Implemented from the AoPs. Richard taught university mathematics for 35 years. He taught a variety of courses, primarily Calculus II, which he taught during his participation in the PD. Richard claimed he implemented the practices of: eliciting evidence of student thinking during class, noticing and responding to evidence of student thinking, fostering inclusive social norms, and some of the five practices for orchestrating class discussions. Richard discussed how the AoP on responding to student thinking "made me more aware of how I need to be interacting with the students in certain areas." The AoP prompted participants to discuss what an instructor should have done differently. Richard recalled the scenario where a student asked about a computation, "From an instructor point of view, it was trivial, but the student was not understanding… the instructor had simply replied, 'You're supposed to already know that.' Of course, that puts the students down." He remarked,

> That scenario reinforced how, when a student asks a question, it's important to take it seriously…Not get frustrated that they didn't understand something trivial…That's important in teaching and interacting with students…to answer them as if they're serious mathematical questions and dig in to see where their misunderstanding was coming from.

Richard exemplified how he implemented the approximated practice while simplifying the expression x^{n+1}/x^n: "The first time that an expression comes up, I usually do it in slow motion and say, from now on, we'll just go to this." When a student asked about the simplification,

> Instead of saying, 'we talked about this earlier'…I tried to explain it in a different way... That's when that connection with the scenario in the PD came in…Just be patient, treat the question as a serious question, and let's see if we can figure out where the misunderstanding is and how to get him over that."

The AoP helped Richard intentionally respond to students in non-dismissive ways and elicit further evidence about their thinking. We posit that Richard's goals, orientations, and resources informed his implementation of these teaching practices that he learned in the AoP.

Goals and Orientations. Richard provided evidence of the goal of *helping students feel comfortable sharing answers and asking questions* and the related orientation that *it is important for students to feel comfortable and confident in class*. Richard described an instance of a student asking a question in class and being "more interested in just learning why, rather than is somebody gonna think I'm dumb." Richard claimed the student "felt comfortable in asking, and so to me, that meant that I maybe was partially successful in having an environment where they could ask questions like that…I found students were very open in offering up their mistakes." Richard perceived value in having a positive class environment where students felt comfortable and confident sharing answers, making mistakes, and asking questions during class. This goal and orientation seemed to support Richard's decision to enact teaching practices of fostering inclusion and productively responding to student thinking, which he learned about in the AoPs.

Another goal and orientation salient in Richard's interview were to *respond to students in productive, non-disparaging ways* and *it is important to value student contributions*. Richard claimed to "try to create a classroom environment where questions are taken seriously and with enthusiasm, and…that the failures are celebrated. So that we can learn a lot more by making mistakes and learning how to fix something than just by getting something right the first time." He valued students' questions, answering them with enthusiasm, and responded to students' mistakes in positive ways. This goal and orientation seemed to support Richard's implementation of responding to evidence of student thinking because he was intentional in responding to students' questions and contributions in ways that did not disparage the students. He described trying to productively respond to students in class by celebrating their mistakes, eliciting further

evidence of their thinking, and providing more explanations without embarrassing the student.

Resources. Richard's knowledge and experiences served as resources that seemed to influence his decision to enact the teaching practices from the AoP, like responding to student thinking in productive ways. He discussed how the AoPs contributed to his *knowledge and awareness of student interactions*, a resource that helped him consider different ways to respond to students. He stated, "That activity stuck with me. There were some discussions that made me more aware of how I need to be interacting with students." This *experience engaging in the AoP* and *learning about specific teaching practices regarding student interactions* served as resources he could draw on while responding to student thinking. His prior *knowledge of how to respond to students* was also a resource informing his decision to implement the teaching practices from the AoPs. He referred to an AoP scenario where an instructor "put a student down," explaining, "I think I'm pretty good about not responding to students that way. That particular scenario... reinforced how when a student asks a question, it's important to take it seriously." Richard had prior knowledge of how to interact with students based on his years of experience teaching, and the AoP helped reinforce that knowledge to use as a resource in his teaching.

Discussion and Conclusion

Given the calls to implement active learning and equitable teaching practices in calculus courses, we led PDs, training Calculus I and II instructors to use these pedagogies. The PDs were designed to actively engage instructors in AoPs to enact various active learning and equitable teaching practices. After participating in the PD, we interviewed instructors about their implementation of the teaching practices they learned in the AoPs. We focused on two case studies, Sumaya and Richard, for this investigation. Both instructors were able to describe how their experience engaging with the AoPs informed their implementation of the approximated teaching practices in their own classrooms. We investigated how their resources, orientations, and goals informed their decision-making while enacting teaching practices from AoPs.

Both instructors had goals and orientations related to engaging students, helping students feel comfortable in class, and valuing student contributions. We posit that having these goals and orientations influenced the instructors' decisions to enact active learning and equitable teaching practices like eliciting and responding to evidence of student thinking, fostering inclusive social norms, and fostering equitable student participation. Furthermore, an instructor's resources (e.g., knowledge, experiences) influence the decisions they make in their teaching (Schoenfeld, 2010), particularly whether to enact the practices they learned in the PD. For Richard, the AoPs served as a resource, reinforcing his goals and prior knowledge of how to respond to students, which seemed to influence how he responded to student thinking in class. For Sumaya, her knowledge and awareness of student backgrounds informed how she implemented practices from the AoPs, such as fostering inclusive social norms and equitable participation. Overall, our case studies exhibit how instructors' resources, orientations, and goals could influence the instructors' decisions to enact the teaching practices learned in the AoPs.

Our findings contribute to the literature on PD for university mathematics faculty and provide implications for PD designers. The AoPs in our PD served as a resource that supported instructors in deciding to implement active learning and equitable teaching practices. AoPs have great potential to train university instructors to implement active learning. They provide a low-stakes environment for instructors to develop new teaching practices. Our case studies showed how engaging in AoPs allowed instructors to reflect on their teaching practice, be more mindful of how they interact with students, and reinforce their pedagogical knowledge. Thus, we suggest that designers of PD programs for university faculty actively engage their participants in AoPs.

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Can Self-Regulated Learning Practices Shift Without Intervention? An Exploratory Study of Calculus I and II Students

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This study examines changes in self-regulated learning (SRL) skills among undergraduate Calculus I and II students, without explicit interventions. Using pre- and post-surveys aligned with Zimmerman's SRL model, we identified both improvement and decline across items assessing various SRL practices. Calculus II students showed significant improvements in planning, self-control, and information management, which appear to be supported by student maturity, conceptual course content, and collaborative instruction. In contrast, Calculus I students exhibited a decline in performance-phase strategies, reflecting challenges from large lectures, formulaic content, and first-year adjustment. The researchers' hypotheses on potential reasons for the observed results were corroborated by the course instructor and teaching assistants. However, more research is needed to understand these patterns and their underlying causes. Our findings suggest that while some SRL skills may develop naturally in more advanced contexts, first-year students may require targeted support to become more effective learners.

Keywords: Calculus, Self-Regulated Learning (SRL), Study Skills

Calculus courses play a critical role in university education, particularly for STEM majors, as they often serve as gateway or gatekeeper courses. Success in these courses can significantly influence students' trajectories in STEM fields, while difficulties can create barriers to persistence. Due to their central role, improving student learning and success in Calculus has been a priority for both instructors and education researchers. Understanding not only what is taught but also how students approach learning is essential for designing effective instructional reforms. In particular, it is important to understand students' use of self-regulated learning (SRL) skills, which encompass cognitive, metacognitive, behavioral, motivational, and affective aspects of learning (Panadero, 2017).

This study was conducted within the context of a larger initiative aimed at improving teaching and learning in the Calculus sequence at a large public university in the northeast of the USA. The larger reform effort at this university encompassed Precalculus, Calculus I, and Calculus II. While all courses use a lecture–recitation format, the primary focus of the reform was on recitations. In these sessions, students engaged in conceptually rich activities and collaborative group work, supported by Learning Assistants (LAs) alongside Graduate Teaching Assistants (GTAs). After several years of applying these reforms, students often continued to use unproductive study methods despite instructional changes. Motivated by this, we sought to investigate the learning strategies of students in Calculus I and II and examine how these strategies "naturally" change over the semester without any particular intervention. To this end, we administered a 64-item Likert survey at both the beginning and the end of the semester.

In this paper, we report on survey items that showed statistically significant changes, whether positive (improvement) or negative (decline), in specific SRL skills. We focus on items where the direction of change differed between Calculus I and Calculus II students. To explore possible explanations for the observed changes, we enlisted interpretations from the course instructors, TAs, and LAs, and included them in this report. Our research question is: *How do undergraduate*

Calculus students' SRL skills change over a semester without formal intervention, and how do these changes compare between Calculus I and Calculus II students?

Literature Review and Theoretical Background

Calculus Reform

Undergraduate calculus courses function as critical gateways, affecting students' success and persistence in STEM. National data show that while many students enter calculus with prior high-school exposure, fewer than 60% earn an A or B, and confidence and enjoyment often decline (Bressoud, 2015). Across institutions, about 11–20% of STEM-intending students do not continue, with women's attrition roughly 1.5 times that of men, even controlling for prior achievement (Ellis et al., 2016; Rasmussen & Ellis, 2013).

To address these issues, many institutions reform introductory courses by integrating active learning and student-centered pedagogies to reduce attrition and promote deeper learning. Courses emphasizing collaborative problem solving, conceptual tasks, and real-world applications yield higher concept-inventory gains, improved exams, and reduced equity gaps (Kramer et al., 2023; Rasmussen & Ellis, 2013). Scaling these practices remains challenging: only 23% of Precalculus–Calculus II courses use active-learning strategies, with usage declining in higher-level courses (Rasmussen et al., 2019).

Metacognitive and study-skill development is another critical factor for student success or lack thereof. Structured study routines, including problem sets, review tasks, and guided reflection, improve performance and engagement (Boelkins & Pfaff, 1998; Khan, 2020). These practices support consistent study habits, conceptual understanding, and monitoring of learning – core self-regulated learning (SRL) skills. Despite their importance, SRL practices are rarely made explicit in traditional courses, leaving students to develop them independently.

Self-Regulated Learning (SRL)

Self-regulated learning (SRL) includes cognitive, metacognitive, behavioral, motivational, and affective aspects of learning (Panadero, 2017). Students with strong SRL skills set goals and actively select, monitor, and adjust strategies to achieve them. They also demonstrate more expert-like thinking individually and in groups (Stanton et al., 2021; Van De Bogart et al., 2017), which correlates with higher achievement, motivation, and lifelong learning (de Boer et al., 2018; Donker et al., 2014; Richardson et al., 2012; Rosário et al., 2019; Theobald, 2021).

Metacognition, a core SRL component, refers to awareness and control of one's thinking and learning (Stanton et al., 2021). Students with strong metacognitive skills assess understanding, identify gaps, select strategies, and evaluate outcomes. In mathematics, metacognition supports problem-solving and abstraction (Desoete & De Craene, 2019; Pólya, 1945/2014; Schoenfeld, 1992). Research shows SRL skills can be taught through explicit interventions (de Boer et al., 2018; Donker et al., 2014; Richardson et al., 2012). Yet, college instructors often prioritize content over SRL scaffolding (Moos & Ringdal, 2012), and secondary-level opportunities are often implicit (Dignath & Büttner, 2008). As a result, undergraduates enter with limited metacognitive knowledge and SRL skills (Pintrich, 2002). This gap is especially consequential in mathematics, where first-year calculus requires shifts in thinking, strategies, and time management. Research on SRL at the undergraduate math level remains sparse, even though its discipline-specific nature underscores its relevance for early STEM courses (Panadero, 2017).

Cyclical Phases SRL Model

This project uses Zimmerman's (2000) Cyclical Phases Model of SRL, involving three phases: forethought, performance, and self-reflection. In forethought, students analyze tasks, set goals, and plan strategies guided by motivation. In performance, they execute tasks and monitor progress. In self-reflection, they evaluate outcomes and attributes, shaping future learning. Zimmerman's model, grounded in social-cognitive theory, is widely used in STEM SRL research (Panadero, 2017; Gamby & Bauer, 2022; Jin et al., 2023; Layco, 2020; Marchis, 2012).

SRL is closely linked to motivation influencing learning performance (Zimmerman & Schunk, 2011). Expectancy-Value Theory frames motivation as learners' expectations for success and task value (Wigfield & Eccles, 2000). Within the SRL model, motivational beliefs affect goal setting, strategy selection, and engagement during forethought, while reflection shapes future expectations. Effective SRL enhances control, mastery, autonomy, and confidence, supporting intrinsic motivation (McCombs, 2013; Reeve et al., 2012). Framing this project with Zimmerman's SRL model allows analysis of how calculus students plan, enact, and reflect on their learning. The framework was used to both develop the survey and to interpret survey data on strategy use, motivation, and metacognitive engagement.

Methods

Setting and Participants

This study is part of a broader NSF-funded initiative to transform STEM gateway courses, focusing on the calculus sequence at a large public university in the northeastern United States. Calculus recitations were redesigned with conceptually rich activities and active learning, including collaborative problem-solving in small groups (CBMS, 2016). Calculus courses follow a lecture-recitation format: large lectures (~ 120 students) led by a faculty instructor three times a week, and recitation sections (~ 20 students), led by GTAs supported by LAs twice a week.

Data came from Spring 2025 students in Calculus I and II, who are predominantly STEM majors (89% and 99%, respectively), and share similar demographics, reflecting the university's predominantly white population. In Spring 2025, the same experienced instructor taught both Calculus I and II courses. There are minor but meaningful distinctions in the prior preparation of the two groups. Spring Calculus I students had not reached the mathematics proficiency threshold and were placed in Precalculus in the Fall. Spring Calculus II students usually completed Calculus I in the Fall, indicating their initial stronger mathematical preparation. Additionally, the instructor of both courses in Spring 2025 had taught Calculus I in the Fall. Thus, students in Calculus II experienced continuity of instructional style, whereas Spring Calculus I students had a different instructor for their prerequisite Precalculus course.

Instrument and Data Collection

Students' SRL skills were assessed with a 64-item survey. Items are measured on a 4-point Likert scale ranging from "1 (disagree)" to " 4 (agree)." The survey is theoretically grounded with items aligned with Zimmerman's (2000) SRL phases – forethought, performance, and self-reflection – and capture a range of skills (e.g., planning in forethought, task analysis in performance, and evaluation in self-reflection). Due to space constraints, examples of survey items will appear in the results section. Many items were drawn from well-established SRL instruments, such as Schraw and Dennison's (1994) Metacognitive Awareness Inventory, and Marchis' (2012) SRL inventory, and modified to the context of college-level calculus. The initial survey was piloted in Fall 2024 with 299 calculus students and revised following statistical and

qualitative analyses. The revised instrument used in Spring 2025 had good to excellent reliability (Cronbach's Alpha values ranging from 0.87 to 0.94) of the three SRL subscales.

The same survey was administered at the beginning and at the end of the semester. Students completed the survey online via Qualtrics, which took approximately 15 minutes. Only data from consenting students were included in the analysis. The final sample comprised of 74 (out of 181) students in the Calculus I course and 78 (out of 193) students in the Calculus II course, who completed both pre- and post-surveys.

Data Analysis

Quantitative analysis involved descriptive statistics for all survey items and assessing internal consistency using Cronbach's alpha. Although Likert-scale survey data are ordinal, parametric methods, such as paired *t*-tests, are widely accepted in educational research when sample sizes are adequate and data approximate normality (Norman, 2010; Chen & Liu, 2020). Given the size of the current study and alignment with these established practices, paired two-tailed *t*-tests were conducted on pre–post differences within each course, and effect sizes were calculated. The survey was not anonymous, allowing for matching pre- and post-responses. Although we initially did not intend to compare across the courses, course-specific analyses revealed notable differences between the Calculus I and Calculus II cohorts, which were further explored.

To contextualize the quantitative results, particularly the statistically significant changes observed on certain SRL items, we reached out to the course instructional staff, shared with them the statistical analysis, and asked them to comment on these findings and suggest possible explanations based on their classroom experiences. Responses were received from one instructor, two GTAs, and three LAs. We incorporated the background information and the insights they provided into possible interpretations and explanations for the observed findings.

Results

Across all three SRL phases, the average student response was "somewhat agree," showing moderate use of SRL strategies in both Calculus I and II. This suggests that students in both courses engage in SRL strategies frequently, though not consistently at the highest level. Considering pre–post course changes, on average, Calculus I students' SRL scores decreased from 3.034 (SD = 0.688) to 2.935 (SD = 0.782), which is statistically significant ($t = -2.141$, $p < .036$, Cohen's $d = -0.315$). Calculus II students' scores increased slightly from 2.948 (SD = 0.741) to 2.990 (SD = 0.778). Furthermore, when examining pre-to-post semester shifts in overall SRL scores, a distinct pattern emerged: Calculus I students' responses declined on most SRL items, whereas Calculus II students' responses increased on most items (Table 1).

Table 1: Counts of Items by Average Change in Calculus I and II

	Declined	No Change	Improved
Calculus I	47	1	16
Calculus II	18	2	44

At the item level, 20 survey items showed a statistically significant change over the semester. Of these, several items reflected significant declines in Calculus I students' responses, while many indicated significant gains for Calculus II students. These differences between Calculus I and II students warranted further examination of how their SRL habits change.

Items with Statistically Significant Changes in Both Calculus I and Calculus II

Three survey items, seen in Table 2, demonstrated statistically significant change for both Calculus I and II. Importantly, these items followed opposite trends: Calculus I students reported declines, whereas Calculus II students reported improvements.

Table 2: Items with Statistically Significant Changes for Calculus I and II

Item	Class	Pre-Mean	Post-Mean	p-value	Effect Size
18: I try to understand the reasons behind mathematical formulas and concepts.	Calc I	2.946	2.703	0.0407*	-0.405
	Calc II	2.782	2.974	0.0348*	0.323
31: I ask questions during lectures or recitation sessions when I don't understand something.	Calc I	2.919	2.649	0.0101*	-0.401
	Calc II	2.641	3.026	0.0010**	0.611
57: When I need to solve a mathematical problem, I think of several ways to solve it and choose one that makes the most sense for me.	Calc I	3.054	2.824	0.0428*	-0.38
	Calc II	2.756	3.000	0.0109*	0.403

Item 18: "I try to understand the reasons behind mathematical formulas and concepts." Table 2 shows that Item 18 was statistically significant for both Calculus I and II, but in opposite directions. Calculus I students showed a significant decrease in this SRL skill, with mean scores declining from 2.946 on the pre- to 2.703 on the post-survey ($p < .0407$). Conversely, Calculus II students showed a significant increase, with mean scores rising from 2.782 to 2.974 ($p < .0348$). This item falls within the Performance Phase of Zimmerman's (2000) SRL model and is categorized under Information Management Strategies. One possible explanation for the difference in the direction of change in responses between Calculus I and II is the course content. Calculus I emphasizes limits and derivatives, often in procedural, formula-driven ways, potentially discouraging deep engagement with mathematical reasoning. By contrast, Calculus II introduces integrals, sequences, and series, which are conceptually complex and allow multiple solution strategies, fostering motivation to understand the reasons behind computational strategies. Reflections from the instructor, GTAs, and LAs provide additional perspective. The instructor noted many Calculus I students were "pretty focused on just getting through," contributing to reliance on procedural strategies. In Calculus II, instructional staff emphasized the "why" behind concepts, theory, applications, and deeper understanding, discouraging "plug and chug" approaches. Taken together, these results suggest both course content and instructional practices shape students' development of information management strategies.

Item 31: "I ask questions during lectures or recitation sessions when I don't understand something." As seen in Table 2, Item 31 revealed notable differences in help-seeking approaches between Calculus I and Calculus II. In Calculus I, mean scores significantly declined from 2.919 on the pre- to 2.649 on the post-survey ($p < .0101$). In contrast, Calculus II students reported significant growth in this area, with mean scores rising from 2.641 to 3.026 ($p < .0010$). This item is in the Performance Phase of Zimmerman's (2000) SRL model and is categorized under Help-Seeking. The contrasting outcomes between Calculus I and II may be partially explained by the differences in classroom climate and familiarity with instructional staff. The course instructor for both Calculus I and II noted that "Calc I was a huge lecture," which may have discouraged students from asking questions. On the other hand, many of the Calculus II students had the same instructor throughout the year in both courses, which may have fostered a stronger rapport and comfort. A Calculus II GTA and LA in different recitations described

deliberate efforts to encourage questions and create a supportive environment. One GTA reflected, "I was very happy with the learning environment my LA and I created along with all of our students." Collectively, these perspectives suggest that differences in classroom climate, continuity of instructional practices, and explicit encouragement of questioning likely contributed to the increase in those practices among Calculus II students.

Item 57: "When I need to solve a mathematical problem, I think of several ways to solve it and choose one that makes the most sense for me." Comparisons of planning skills in Calculus I and II, measured by Item 57, revealed opposite trends in student development over the course of the semester. As shown in Table 2, Item 57 was statistically significant for both Calculus I and Calculus II. Calculus I students' mean scores significantly decreased from 3.054 on the pre- to 2.824 on the post-survey ($p < .0428$), whereas Calculus II students' scores significantly increased from 2.756 to 3.000 ($p < .0109$). Within Zimmerman's (2000) SRL model, Item 57 belongs to the Forethought Phase and is categorized under Planning. Similar types of differences in course content and instructional practices, as described for Item 18, likely influenced students' planning skills. Calculus II's focus on integration techniques, and sequences and series, where there are multiple solution strategies, may be more conducive to seeking various solution paths, compared to Calculus I, which focused more on procedures for finding limits and derivatives. The instructor noted that planning skills may have improved in Calculus II because students must evaluate "all the paths we can go down to solve" a problem, whereas Calculus I problems follow a straightforward path. A GTA emphasized recitations highlighting multiple solution strategies, and an LA noted that peer collaboration exposed students to alternative approaches. Together, these factors support the development of planning skills in Calculus II, while the more linear nature of Calculus I may limit growth.

Course Specific Significant Changes

In addition to the items discussed above, several other survey items showed statistically significant changes for either Calculus I or Calculus II, though not both. Due to space constraints, we summarize the results here and focus on a select few items.

Calculus I students demonstrated significant improvement in two areas of self-regulated learning: Item 42, reflecting careful reading of instructions before starting a task (Forethought Phase/Planning Category), and Item 52, highlighting self-control in exam preparation (Performance Phase/Self-Control Category). These gains suggest that students were increasingly attentive and deliberate in their problem-solving approaches and exam strategies. Eleven other items showed decline, including Item 49 (Performance Phase/Help-Seeking Category), which measures help-seeking from instructors or GTAs. This decrease aligns with findings for Item 31, suggesting that Calculus I students were generally less comfortable seeking support from the instructional team. These results reflect areas for growth and provide valuable insight into where additional encouragement or support could further strengthen students' SRL skills.

For Calculus II, four items showed statistically significant changes. Students improved on two items, which reflected enhanced skills across all three SRL phases, but declined on Item 8: "I can motivate myself to learn mathematics when I need to" (Forethought Phase/Conditional Knowledge). Interestingly, this item had a slight, non-significant increase for Calculus I, but a significant decrease for Calculus II students ($p < .019$), making it the only item with a reversed direction of change between the two courses. The instructional team was unsure why motivation decreased for Calculus II students, indicating a need for further research to better understand the factors contributing to this decline.

These results indicate that both courses can foster growth in students' SRL skills. Calculus I students showed improvements in self-control, while Calculus II students generally improved across several categories. These patterns highlight the potential for targeted instructional strategies to support continued development of self-regulated learning in both courses.

Discussion and Scientific Significance of the Study

This study examined changes in self-regulated learning (SRL) skills among Calculus I and II students that occurred without SRL interventions. The analysis revealed contrasting patterns across the two courses, despite their similar pedagogical approach and being taught by the same instructor, which we found surprising and tried to explore further. Calculus II students showed measurable gains in SRL skills, including planning, self-control, and information management, whereas Calculus I students exhibited declines across most SRL skills. Prior research on SRL demonstrated that students often struggle to develop self-regulatory skills independently and that explicit scaffolding or intervention is typically required for meaningful growth (Panadero, 2017; de Boer et al., 2018; Dignath & Büttner, 2008). Our findings regarding Calculus I students align with these previously observed results, but not for Calculus II students.

Calculus II students showed notable increases in many of their SRL skills. Several factors may explain this pattern. Improved instructional practices through a larger reform effort fostered opportunities for students to practice SRL skills. One GTA noted that her students "were very active participants in recitation and also seemed to develop good relations amongst each other," illustrating how a supportive, interactive environment can encourage self-regulated behaviors. Student maturity and prior experience with college-level mathematics also likely contributed to increased capacity for planning, monitoring, and reflecting on learning. Finally, course content that offered multiple solution strategies and encouraged deeper engagement may have supported intrinsic motivation and skill development.

In contrast, Calculus I students showed declines in several SRL skills, consistent with both SRL research and literature on challenges in introductory calculus. Students were less likely to seek help and engaged less in performance-phase strategies, reflecting the difficulties of large, formula-focused lectures and the cognitive and emotional demands of first-year calculus (Bressoud, 2015; Ellis et al., 2016). Instructional team members noted that many students were still adjusting to college-level study habits and heavier course loads, suggesting that SRL skills may not develop automatically (cf. de Boer et al., 2018). These findings underscore the importance of supporting SRL skill development, particularly in university STEM gateway courses where students may lack experience and confidence.

The results have several implications for calculus instruction and undergraduate mathematics education. They highlight the potential for instructional practices and course design to promote SRL even without formal interventions, while reinforcing the need to target first-year students with explicit scaffolding. SRL remains under-researched in undergraduate mathematics despite its role in promoting persistence, motivation, and conceptual understanding (Panadero, 2017). There is a clear need for research-based interventions, and we are currently designing one tailored for calculus students. Overall, supporting SRL must complement curriculum reform, as improving teaching alone cannot equip students with the skills to regulate their own learning.

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Descriptive Interpretations of Equivalence in Measurement Contexts: The Case of Emily

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Proportional reasoning is a central concept for students to learn, yet they often struggle when solving proportionality-type problems. While prior work has examined students' solution strategies, little attention has been given to the meanings of equivalence that support proportional and multiplicative reasoning. This study investigates how students reason about equivalence in measurement contexts with tasks adapted from Cook et al. (2021). Using task-based clinical interviews with four preservice secondary teachers, we analyzed responses to tasks involving relationships such as "1 lunar cycle = 28 days." Focusing on one participant, we identified the use of two ways of reasoning about equivalence which proved productive for her. Findings highlight the critical role of descriptive reasoning for connecting quantities and magnitudes in proportional situations. We argue for expanding research on equivalence in measurement-based proportional contexts.

Keywords: proportional reasoning, multiplicative reasoning, equivalence, measurement problems

Introduction & Literature Review

Studies using the students-and-professors problem (S&P problem), as well as others like it, highlight the nuance and complexity that can be involved when using proportional reasoning to successfully reason about quantities involved in such problems (Christianson et al., 2012; Clement et al., 1981; Cohen & Kanim, 2005; Fisher, 1988; Van Dooren et al., 2009). Proportional (and multiplicative) reasoning has been well-studied (e.g. Clark et al., 2003; Lesh et al., 1988; Modestou & Gagatis, 2010; Vergnaud, 1983, 1988) and for good reason- such reasoning has been referred to as a critical connector between elementary-level math and higher-level math (Lesh et al, 1988). Despite the importance of proportional reasoning in the mathematics curriculum, students' experiences have not been sufficient in supporting them in developing such reasoning (Tjoe & de la Torre, 2014); this highlights the need for research continuing to explore students' proportional reasoning.

Although much scholarship has been done on students' solution processes in S&P-type problems, little attention has been paid to the meanings of equality and equivalence that support students in reasoning productively about these scenarios. More broadly, we observe that, while interpretations of the equals sign have received a large amount of attention (Jones, 2008; Jones & Pratt, 2006; Kieran, 1981), very few studies have examined the interpretations that support productive reasoning about equivalence in proportional, quantitative scenarios. Additionally, while much of the research into the S&P problem (and those like it) has focused on producing the correct algebraic equation (e.g., $S = 6P$), very little attention has been paid to how students might reason about the magnitudes involved in such problems. This leads us to pose the following research question: In what ways might students productively reason with equivalence as they engage in tasks involving proportional quantities?

Theoretical Perspective

Equivalence

For a context in which to discuss students' ways of reasoning about equivalence in proportional contexts, consider the following tasks originally described by Cook et al. (2021):

> Task A: In Glenn Junior High school, there are 28 students for every 1 teacher. Determine the number of students if there are 37 teachers. Then write an equation that generally relates the number of teachers, *T*, to the number of students, *S*, in Glenn Junior High.
>
> Task B: There are 28 days in a lunar cycle. Determine the number of days that pass in 37 lunar cycles. Then write an equation that generally relates a number of lunar cycles, *L*, and the number of days that pass during those lunar cycles, *D*. (p. 36)

Both tasks are variations of the S&P problem described by Clement et al. (1981). In the spirit of inquiring about solutions to these tasks, Cook et al. (2021) developed a conceptual analysis to answer questions a student might ask when approaching Task A and Task B. The solutions to and interpretation of both tasks require flexible use of equivalence. The authors hypothesized three ways of reasoning that could support students' productive engagement with these measurement/proportionality tasks: *numerical*, *transformational*, and *descriptive*.

A *numerical* way of reasoning identifies two expressions as "equivalent when they have the same numerical value" (Cook et al., 2021, p. 36). In the context of expressions such as $2(x + 1) = 2x + 2$, a numerical way of reasoning is a natural default because, for all values of the variable x, the left-hand and right-hand sides evaluate to the same number. However, numerical ways of reasoning can fail when referring to a statement such as '28 days = 1 lunar cycle' (as 28 is not equal to 1). These kinds of measurement equations require a different kind of reasoning. To that end, Cook et al. (2021) identified two other ways of reasoning about equivalence, a *transformational* way of reasoning and a *descriptive* way of reasoning. The transformational way of reasoning involves some manner of algebraically appropriate modification of the expression or equation (e.g., distributing the 2 in $2(x +1)$ to obtain $2x +2$). A descriptive way of reasoning, adapted from the work of Prediger & Zwetzschler (2013), "treats two objects as equivalent if they describe, represent, or model the same quantity or situation" (Cook et al., 2021, p. 37). For example, a student might engage in descriptive reasoning when referring to a clock to grapple with modular arithmetic or referring to area under a curve to justify why two integral expressions might be the same. Descriptive ways of reasoning are particularly important for proportionality tasks, as attention to measurement and magnitude can help students reason through the task.

Quantities and Magnitudes

Undergirding the solution to proportionality problems is attention to the quantities involved. Thompson (1994) defined *quantity* as a scheme consisting of three components: quality of object, unit or dimension, and the ability to quantify the object. An individual, for Thompson, engages in quantitative reasoning when working with all three components of the scheme. Thompson et al. (2014) establish that *magnitude* can be thought of as a scheme at several levels. Of relevance to this report, one of Thompson's levels refers to Wildi's (1991) definition of magnitude of a quantity *Q* as "the measure of *Q* with respect to a unit *u*, times the magnitude of *u*" (Thompson et al., 2014, p. 5). The invariance of magnitude with respect to change in measure is highlighted by using this definition; this creates a powerful mental image for magnitude which can be used by the student in proportionality contexts. In the context of Task B, 28 days and 1 lunar cycle are equal because 28 and 1 both describe measurement of the same duration of time, albeit in different units (the length of one day, |day|, and the length of one lunar cycle, |lunar

cycle|, respectively). In the language of Thompson and Wildi, 28*|day| = 1*|lunar cycle|; this setup helps lead to the algebraic solution to the task, *28L = D*. The solution equation treats the variables *L* and *D* as measurements, while |lunar cycle| and |day| are units of measure. We associate this approach with a descriptive way of reasoning. We hypothesize that attending to these features can be productive for reasoning about proportional quantities; it is unclear, however, whether and how these ideas might emerge when working with students.

Methods

We conducted task-based clinical interviews (Clement, 2000) with four students at a large public research university in the United States. All four participants were pre-service secondary teachers, completing the third year of their coursework in the Spring semester of 2025. Individual interviews (one participant per interview) centering on Tasks A and B from the previous section were completed in Spring 2025 and were approximately 40 minutes in length. We recorded student words and utterances, as well as their written work, via iPad and the ExplainEverything app. We presented participants first with Task A before presenting Task B. As the participants worked on both tasks, the interviewer (third author) asked them to explain their reasoning via clarifying follow-up questions, focusing on key proportional and quantitative relationships in the two tasks (e.g., does 28 students = 1 teacher, and why? Does 28 days = 1 lunar cycle, and why?). We also asked participants to describe similarities or differences between the two tasks to elicit more details from their line of reasoning. Our analysis in this proposal focuses primarily on one participant's responses to Task B. We selected Emily (pseudonym) because her responses were particularly vivid and illustrative of the types of reasoning conjectured by Cook et al. (2021).

Transcripts were produced for each interview session. The first three authors analyzed the transcripts and video recordings of the participants' work, employing Clement's (2000) cycles of interpretive analysis with the ways of reasoning hypothesized by Cook et al. (2021) serving as initial codes. Our goal was to develop models of the student's reasoning that provided a rational description of her language and actions; models were discussed and negotiated until all three researchers were in agreement. The transcripts were coded for instances of each way of reasoning about equivalence in Cook et al. (2021)— numeric, transformational, and descriptive—and screen captures of relevant portions of the video recording were used to augment the transcripts.

Results

For Task A, Emily began by finding the answer and using numbers to work backward in an attempt to find the relationship that made sense ("if there's one teacher, then there's 28 students, there's two teachers, then there's 56 students"), leading her to the correct algebraic solution of *S = 28T*. Further, she justified her choice by using a numerical way of reasoning to eliminate the reversal error as a possible answer ("If you did it the other way…you have five students, you do not have 140 teachers"). Emily used this line of reasoning for Task B as well:

> *Emily*: So we have 37 cycles that have 28 days in each of them [*writes 37*28*]... So that would be [1036] days that pass in 37…days. Um, then write an equation that generally relates the number of lunar cycles, L, the number of days that passed during those lunar cycles. Um, so if we have one lunar cycle, it's gonna be the same way that it was before [referring to her work in Task A]. If we have, I think it's gonna be 28 L equals D [*writes 28L = D*]. 'cause if you had it the other way, if you had 28 D, would still be like after 28 days it should equal one lunar cycle [*writes 28D = L*]. But if you have, if you put in 28

for D, you get 28 squared, which is 784 [*writes 784 L*]. And that is not one lunar cycle…So I think that this would have to be the general equation for L and D [*boxes 28L = D*].

We infer Emily relied on numerical ways of reasoning to justify her equation by arguing that the rule written the other way would provide an incorrect numeric relationship between the number of days and lunar cycles.

The interviewer continued by asking how Emily might respond to an assertion that one does not equal 28, so therefore one lunar cycle cannot equal 28 days. Her response pivoted to using a descriptive way of reasoning about equivalence:

Emily: Well, I would ask them if one foot equals 12 inches [*writes 1 ft = 12 in*]. So I think that's a pretty common…conversion… so one and 12 are not equal, but our units are different [*underlines 'ft' on the left and 'in' on the right*]...your units are different, but these two things [*referring to 1 ft and 12 inches*] are equal. But just because the units are different does not mean that in this case [*scrolls up to her work on Task A, which has 1 teacher ≠ 28 student*] I can say one teacher [*underlines 'teacher'*] equals 28 students [*underlines 'student'*]. Yeah, our units are different, but the value of those units is not the same.

The interviewer then asked her to elaborate about what she meant by the displayed equation, 1 LC = 28 days. Emily responded by drawing the number lines in Figure 1 and partitioning it into pieces which she claimed represented the 28 days ("28 is the number of subsections contained inside of one lunar cycle"). She articulated:

Emily: We can't cut one teacher up into 28 even pieces and say that those are the students… each of these [tick marks] is not equal to one student…one teacher's not made up of 28 students, but one lunar cycle is made up of 28 days.

Emily argued that one lunar cycle is composed of 28 days (28*|day| = |lunar cycle|). Hence, she leveraged a meaning for magnitude consistent with Thompson et al.'s (2014) description of a Wildi magnitude to justify the former equation; Emily stated that the same magnitude of time could be measured in terms of 28 times the magnitude of a day or 1 times the magnitude of one lunar cycle. Further, Emily contended it cannot be said that 28*|student| = |teacher| as they did not represent the same magnitude.

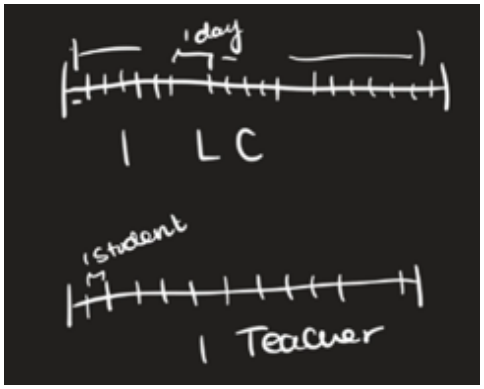

Figure 1. Emily's number lines comparing a partition of one lunar cycle and one teacher.

We infer that here Emily attended specifically to the idea that there is a measurement comparison that can be made between days and lunar cycles, while the same cannot be said for students and teachers; as a result, the latter magnitudes are incommensurate.

She continued to focus on the measure, consistent with understanding that 28*|day| = |lunar cycle| when asked to identify what is true for Task B which is not true for Task A:

Emily: So here we're saying the amount of time that has passed is the same. So if you're talking about, um, the length of one day multiplied by 28 [*writes length (day) * 28 = 1 LC*]...that will give you one lunar cycle. And you can say the same thing the other way…you can divide a lunar cycle up into 1/28 of a lunar cycle, and that's one day [*writes 1/28 LC = 1 day*]. But if we come down here and we say, "Okay, well one student times 28, does that equal one teacher? How many teachers does 28 students make up?"... 1/28 of a teacher that does not pop out one student. Like I can very confidently say that that is not one student.

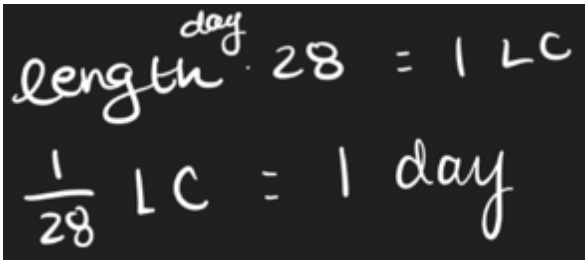

Figure 2. Emily's comparison of units for Task B.

In this excerpt, Emily explicitly sets off a distinction between days and length of days, indicating an attention to the magnitudes at play. Importantly, she identifies that (1/28)*|lunar cycle| = 1*|day|, which is a feature distinguishing the Wildi magnitude scheme from less sophisticated magnitude schemes (Thompson et al., 2014).

The interviewer continued by later asking Emily to respond to a student who wrote both $D = 28L$ and $28D = L$. First, Emily says "so I guess that's the difference. I'm talking about length here [*refers to L in 28D = L*], but here I'm talking, if we're talking number of days [*refers to D in 28D = L*], which is what the problem states and the number of days that pass…". Here, she clearly distinguishes the measurement of the number of days and lunar cycles ("if we're talking number of days") versus the magnitude of each unit ("I'm talking about length here"); this is again consistent with the descriptive way of reasoning of the equation as proffered by Cook et al. (2021) (i.e.., "they describe, represent, or model the same quantity or situation" (p. 37)).

As she continued to reason why $D = 28L$ and $L = 28D$ could not both be true, she paralleled the initial reasoning she gave for why $D = 28L$ was the correct answer; Emily used numbers to justify why four days would not equal 112 lunar cycles, as illustrated in Figure 3 below.

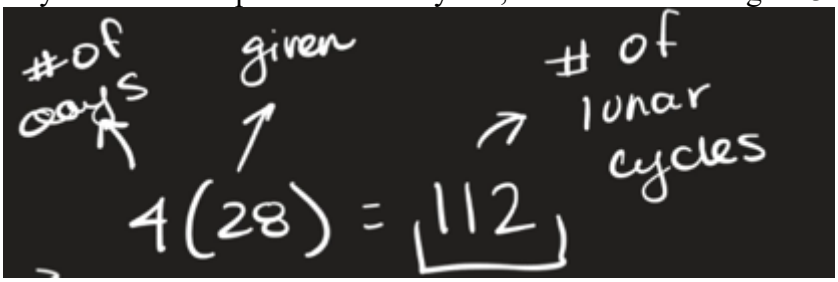

Figure 3.Emily's use of numerical ways of reasoning justifying why D = 28L.

She immediately referred to her initial line of reasoning ("the way I reasoned with it to begin with was with 37 times 28 gives me [1036] days…so then either that [*referring to her original reasoning*] would be wrong or this reasoning [*referring to the claim that 28D = L*] would be wrong"). Immediately after reestablishing the falsehood of the equation $28D = L$, Emily used both numerical and descriptive ways of reasoning by saying "if we're gonna think about it in

practical terms, [one] lunar cycle, we can say that's a month…we have four days. That's not 112 months". We infer that Emily used a descriptive way of reasoning to support the numerical way of reasoning she had established earlier.

The distinction between measurement and units of measure became more evident as the interview continued:

Emily: This [*referring to D = 28L*] would be the number of lunar cycles . But here [*referring to her work in Figure 2b*], this is the number of days and this is the number of lunar cycles that have passed, which is not equivalent. These two things are not equivalent. Length of a lunar cycle and number of lunar cycles that have passed.

Int: Okay. And so those are, those are, those are different things, but clearly they're related in some way. [*Emily agrees.*] Um, so how…are they related?

Emily: Mm. Well, if you have three lunar cycles that have passed, right? So we're talking about the length of three lunar cycles… that is the length of those three lunar cycles. Each one of these is a lunar cycle. [*draws the number line in Figure 4a*] The length is not three…or I guess, um, 'cause if we're talking about like, each of these have 28 days [*produces Figure 4b*]...So I would say yellow is the length of one lunar cycle, and then blue is saying that three lunar cycles have passed [*produces Figure 4c*]….Because this is three lunar cycles, this is one, the length of one lunar cycle. Within here you have 28 [*produces Figure 4d*].

Int: So are you thinking about, about the, say the space between the blue brackets?...Is that the number of, are you thinking about that as the number of lunar cycles or the length of three lunar cycles? Or both, or neither?

Emily: Well, I guess the length, this is, this would be saying three lunar cycles have passed….We have three lunar cy- three full lunar, lunar cycles that have gone by, but the length of each one is the length of a day, times 28. So that's like each of these have 28 little, little markings in them [*produces Figure 4e*].

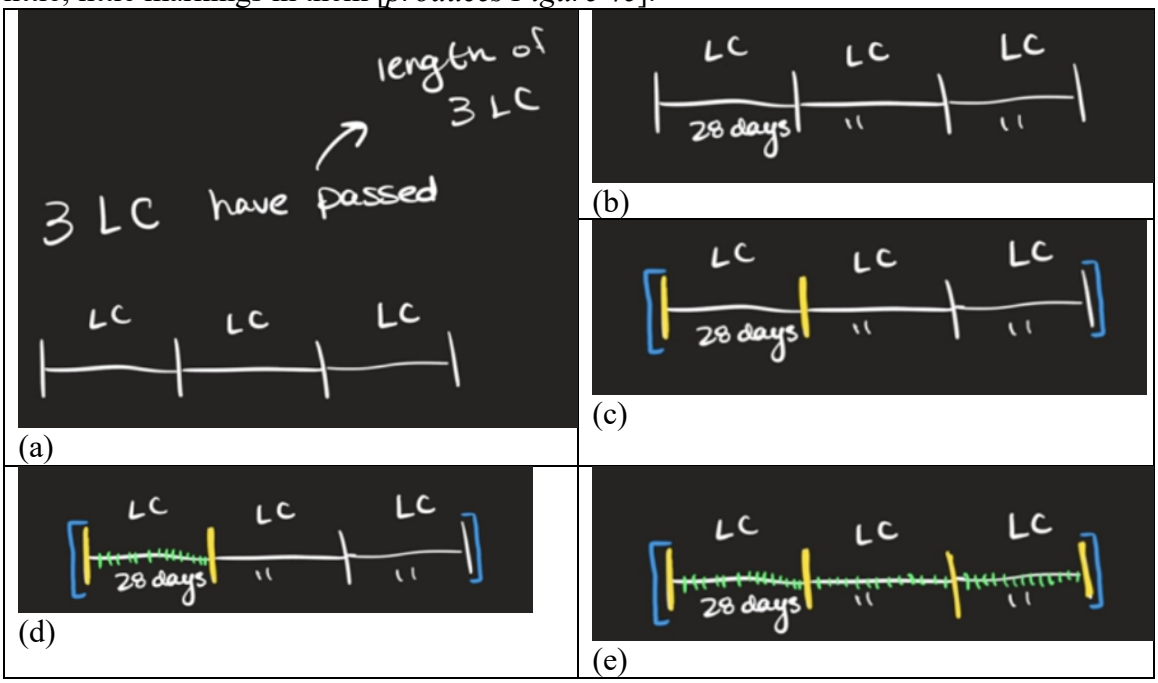

Figure 4. The evolution of Emily's number line diagram for 3 lunar cycles.

From this interaction, we infer that Emily argued 3*28*|day| = 3*|lunar cycle|; we particularly highlight Emily's careful attention to the distinction between the length of a lunar cycle (in yellow) and the accumulated duration of three lunar cycles (in blue). She then described how each lunar cycle consists of 28 days to draw a final representation. The final argument is an extension of her prior use of the Wildi magnitude accompanied by a clear representation of her reasoning. Namely, a single duration of time, as represented by the total line segment in Figure 4a, could be measured using two different units. With lunar cycles as her unit, the duration comprised 3*|lunar cycle|. With days as her unit, the duration comprised 3(28*|day|), which was an extension of her earlier reasoning with Wildi magnitudes. Combining these two arguments Emily concluded 3*28*|day| = 3*|lunar cycle|.

Discussion

Emily successfully used both numerical and descriptive ways of reasoning about equivalence when solving Tasks A and B. We claim that both the numerical and descriptive ways of reasoning about equality were essential to her productive approaches to these tasks. Whether she used one or the other, we observe, depended on the nature of the objects she was equating. When attempting to derive and verify the correct algebraic equation to Task B (relating, for example, D and L via $D = 28L$), Emily relied on numerical way of reasoning. On the other hand, when equating magnitudes of time (e.g., 1 lunar cycle and 28 days), Emily relied primarily on descriptive ways of reasoning. Put another way, Emily demonstrated numerical ways of reasoning to relate the *measurements*, and a descriptive way of reasoning to relate *magnitudes* (durations of time). Moreover, we note that equating durations of time (such as 1 lunar cycle and 28 days) do not lend themselves to a numerical way of reasoning because, indeed, the numerical values (measurements) are different. Taken together, these insights underscore that a descriptive way of reasoning can not only be useful but *critical* for students' reasoning about proportionality and measurement, especially considering that a numerical way of reasoning is predominant when working with most other types of algebraic expressions.

This paper contributes to the field by establishing that a descriptive way of reasoning about equivalence can be critical to students' thinking and provide a detailed example. We highlight that her work provides a clear and explicit empirical example of a student reasoning with Wildi magnitudes as described by Thompson et al. (2014). Specifically, by viewing the equality of 28 days and one lunar cycle in terms of being different measurements of the same magnitude, Emily was able to productively reason through the apparent contradictions of the solution process and extend the equality to larger periods of time than the one initially proposed by the problem. We note that while the students-and-professors problem has been studied extensively, we are unaware of research examining measurement conversion proportional situations such as Task B. We call for additional research exploring such scenarios and how students may rely on different ways of reasoning about equivalence (and magnitudes) when reasoning about such situations.

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Choice and Circumstance: Sources of Faculty Momentum Toward Active Teaching Practices

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To study what shapes instructors' use of student-centered active teaching methods, we examined multiple influences across the faculty career span, using survey data from over 600 mathematics faculty. We present a path model that incorporates factors from graduate teaching experience and job choice to professional development, departmental norms, and involvement in student-oriented scholarly and professional activities. We adapt Adelman's concept of momentum to interpret results and highlight the role of early decision-making and professional environments that reinforce teaching-focused choices in shaping faculty teaching practices.

Keywords: active learning, student-centered teaching, professional development, institutions

Improving instruction has been called the "best bet" for strengthening undergraduate STEM education (Singer, 2013, p. 768). Yet many students do not experience research-based instruction in STEM classes, where 60%-90% of instructors rely on lecturing (Hurtado et al., 2012; Larsen, Glover & Melhuish, 2015; Rasmussen et al., 2016). While minority, first-generation and poor students benefit most from these high-impact practices, these groups least often encounter them (Kuh, 2008). As Fairweather (2008) asserted, "The key… lies in getting the majority of STEM faculty members to use more effective pedagogical techniques than is now the norm" (p. 13).

High-quality, teaching-focused professional development (TPD) can be powerful in shaping instructors' teaching practices (Lattuca, Bergom & Knight, 2014; Laursen et al., 2019). For example, Archie et al. (2022) showed that TPD on inquiry-based learning (IBL) fostered gains in mathematics' instructors knowledge, skills and availing beliefs, that were in turn positively associated with later classroom uptake of IBL. Most future faculty have had little prior learning about teaching (Austin et al., 2009; Golde & Dore, 2001); thus, early career may be an optimal time to offer TPD, when faculty are learning to balance teaching with other duties (Turner & Boice, 1987) and before they settle into habits that serve students less well. Many STEM disciplines have thus offered professional development to early-career faculty (CSSP, 2012), focused on teaching and how to navigate research, service, and collegial relationships. However, the evidence is slim about the impact of this early-career positioning (Connolly & Millar, 2006).

We set out to understand whether and how TPD targeting early-career mathematics instructors influenced their teaching practices. In this paper, we focus on findings about alumni who had previously completed the program. We wanted to learn how alumni are teaching now, and what influenced their teaching—including early-career TPD, other professional learning, and their work environments. As part of a holistic take on the influences on teaching, we also sought to understand alumni careers more fully, considering their other roles such as scholars, leaders and participants in professional societies, and we compared alumni with a matched comparison group to learn whether and how the groups differ. Our research questions and hypotheses are:

RQ1. What are the overall differences in the teaching, scholarly and leadership activities, and institutional environments of alumni and comparison group faculty?
Hypothesis: Alumni program status (vs. Comparison) will predict more active

teaching when other independent variables in the model are held constant.

RQ2. What social, professional, and institutional factors predict differences in the use of active learning strategies for teaching? *Hypotheses: Departmental expectations and instructor beliefs that support active teaching are associated with more active teaching in practice. Instructors who engage in career activities related to active teaching, equity and outreach also will teach more actively.*

RQ3. How do early career experiences around teaching influence the role and nature of teaching in instructors' later careers? *Hypothesis: Indicators of early-career interest in and pursuit of teaching-oriented jobs will predict the emphasis on teaching in instructors' careers, as expressed by the number of courses they teach.*

Prior Work and Conceptual Frameworks

Many studies have examined barriers and supports to instructors' uptake of active teaching approaches (e.g., Dancy & Henderson, 2010; Henderson & Dancy, 2007, 2011; Lund & Stains, 2015; Walczyk, Ramsay & Zha, 2007; Yik et al., 2022). Psychological and personal factors, such as beliefs about learning and teaching and prior experience with active teaching, are often highlighted in these studies, as are classroom factors such as the layout of classroom space, class scheduling, and the nature of the course and students. The institutional context where faculty work also has a sizable influence on their ability and motivation to change their teaching, including teaching loads, course coordination, and institutional reward systems (Austin, 2011).

Removing such barriers can therefore serve as leverage to improve teaching. Professional development can increase knowledge and skills and foster facilitating beliefs. Other strategies operate on parts of an institutional system, such as structures that increase accountability for teaching quality, and cultures that support teaching change (Borrego & Henderson, 2014). Manduca (2017) centers the individual learner in an even broader landscape, "as an active agent who learns over time and is responsible for making changes" (p. 416). In addition to formal professional development and institutional context, she highlights "the community of learners that surrounds and supports an individual as they learn and change" (p. 418). Peer communities may support teaching change both during TPD and through disciplinary or departmental groups. Together these ideas cast growth in teaching as a complex, incremental and long-term process, not a simple result of one-off professional development or binary decisions to change.

Moreover, teaching is only one part of a faculty career, so a full picture of faculty teaching must attend to academic careers and how they develop. Calling out insights from the sociology of science, Hermanowicz (2012) noted that the concept of a career is inherently sociological, with its dependence on "advisors, mentors, college origins, doctoral programs, social class backgrounds, colleagues, collaborative networks, postdoctoral appointments, employing departments, employing universities, disciplines, fields, professions, and reward systems… that make careers possible and which differentiate them" (p. 210). This view leads us to attend to educational history and career activities in our study, and to the norms of the overlapping social communities in which faculty careers develop (Ebert-May et al., 2025).

As theoretical framing, we use Adelman's notion of "momentum" (1999), developed in his study of bachelor's degree attainment and racial inequities in this outcome. Dissatisfied with standard accounts of academic achievement, he explored how early elements of students' academic paths, such as the level of their high school mathematics courses, had shaped their college outcomes many years later. This approach highlights that many factors influence individuals' paths and may play out over decades, and draws attention to influences that are not pre-set but can be changed by institutions and individuals themselves. Wang (2017) uses the

concept of momentum to clarify antecedents of community college student success, highlighting learning experiences and motivation as well as academic factors that may combine to overcome inevitable friction and help students toward their academic goals. While we focus on a different sphere, our study likewise addresses multiple factors—preparation, motivation, environment, choice—that help to move people toward or away from specific outcomes over long times.

As a metaphor, the term "momentum" highlights forward impetus or motion toward an outcome, what a person needs to build and maintain in order to progress. Momentum may vary in pace or direction over time, depending on diverse individual and environmental factors that may boost momentum or break it by adding friction. It recognizes individuals as "active, responsible participants" in their trajectories, neither laying blame on them for disruptions in momentum nor treating them as "passive creatures whose fate is wholly molded" by institutions (Adelman, 2006, p. xxiv). In this project, we are interested in sources of momentum that impel faculty toward use of active approaches to teaching. The notion entails a developmental perspective on faculty careers (Baldwin & Blackburn, 1981) and is well suited to our retrospective, multi-factor study that considers instructor change as a trajectory, rather than an event, and to our use of path analysis as an approach to modeling a complex data set.

Context for the Study

Particular challenges face studies of TPD that focus on outcomes that are important but take time to develop, such as use of research-based teaching practices. The program must be large, long-lived, and stable over time in design, so that multiple cohorts can be combined to develop a sample with sufficient statistical power. Our study gathered data from MAA Project NExT (PN), which had several features favored by research (e.g., Giersch & McMartin, 2014; Manduca, 2017), related to duration and frequency, mentoring, content, format and facilitation. The year-long TPD program includes three multi-day gatherings and online interactions. Participants interact with peers and seasoned faculty, face-to-face and online, using a variety of mechanisms to foster interaction. Program topics focus on the needs, concerns and interests of new faculty, e.g. teaching methods, curriculum choices, inclusive teaching and advising, outreach, undergraduate research, time management, and balancing multiple faculty duties. The format of the first summer workshop typically includes 1-2 hour interactive blocks for plenary talks, table discussion, mini-workshops, a networking fair, and group meals. Later meetings emphasize participant-planned sessions, plenaries, and social time. Finally, the program is overseen by a committed leadership team with expertise in professional development, bolstered by a diverse set of session leaders, speakers and mentors who model diverse careers, lives, and specialties.

Study Methods

Study Participants

Current analyses of the survey dataset used 634 responses from 492 instructors who participated in Project NExT ("alumni") and 142 from a comparison group constructed by matching alumni survey respondents to non-alumni peers. We surveyed all program participants from 1994-2018. We used the Mathematics Genealogy Project (Jackson, 2007) to match alumni to non-alumni peers by graduate institution, year of Ph.D., and advisor. The peer group is broadly matched in general experience of doctoral education, mathematical subfield, and prestige of doctoral degree, and faced comparable job markets as new PhDs. The final sample reflected a 50% survey response rate. All responses contained complete information about instructors'

teaching and described in-person classes. The IRB-approved survey was administered in winter-spring 2022.

Survey Instrument

The survey contained 47 items (some with long lists of choices) asking about topics related to careers as math instructors. These included: 1) the benefits of Project NExT or an alternate professional development project, 2) academic career activities, 3) participation in professional development, 4) participation in professional societies, 5) involvement in research and grants, and 6) their department's expectations in evaluating their work, with a parallel section asking about activities that brought respondents personal career fulfillment. To assess the teaching component, we used items similar to the TAMI-IS survey (Hayward et al., 2018). It asked instructors to choose a course they had taught recently and estimate the amount of class time spent in a range of teaching activities (e.g., Lecture or Group Work). We also probed basic demographics, years of experience teaching at the college level, and institutional characteristics.

Data Analysis

Based on our hypotheses, we tested a three-part model with the outcome variables of Active Teaching, Amount of Teaching and Program Status. We used the statistical software AMOS to examine model fit and regression weights of the independent variables entered into the model using a predetermined set of variables. Of the original set of 28 variables, 16 were significant predictors. To save space, only significant predictors are presented in Table 1 and Figure 1.

We followed best practices in confirmatory path modeling by checking assumptions, reporting multiple model fit statistics and specifying the model *a priori* to analysis. The model was calculated using Maximum Likelihood Estimation and model fit was evaluated using two indices: the Comparative Fit Index (CFI) and the Root Mean Square Error of Approximation (RMSEA). CFI values range from 0 to 1, with values < 0.95, and RMSEA values ≤ 0.09 interpreted as evidence of good fit (Rigdon, 1996).

The path model had three parts, all using items from the alumni survey. Hypothesized relationships between independent variables and *Active Teaching* asked if departmental and personal expectations and career activities for active learning instruction translated to practice. We also wanted to know the predictive association between early career activities and the amount of teaching instructors reported. The third part of the model examined selection into Project NExT, seeking to hold possible confounding variables constant to gauge the relationship between attending PN and later active teaching.

Results

Table 1 describes the variables used in the path analysis, showing where they appear in the path model (denoted by →), and the value of the path coefficient. Figure 1 shows the variables in the model and their path coefficients. We report variables with significant standardized regression coefficients, where * denotes probability $p<0.05$, and ** denotes $p<0.01$. Model fit was "good" with a Chi-Square/Df = 5.8 ($p < 0.001$), CFI = 0.94 and RMSEA = 0.088 (Hu & Bentler, 1999). Correlations (not shown in Figure 1) between independent variables in the model were lower than r=0.2, with no collinearity of predictors. While 16 predictors are shown, another 12 were used in the model but were not significant predictors and are not included in Table 1 or Figure 1.

The model describes three interlocking regression analyses. As hypothesized, departmental expectations and personal beliefs were strong predictors of Active Teaching, our primary

outcome measure (Outcome Model, Fig. 1 right side). The R-squared value for this model component was 0.27. Departmental expectations to use methods other than lecture had the strongest independent association with Active Teaching (b=0.28), and the personal importance of active teaching also predicted the outcome variable (b=0.19). The factor variable TDO, an aggregate of six items, gauged participation in career activities around teaching, equity and educational outreach. TDO was positively associated with teaching actively (b=0.21). Instructors who taught a greater number of courses in the past two years also incorporated methods of active learning (b=0.14).

Table 1. Variables in Path Analysis, Description, Path Model Association and Path Coefficient

Variable	Description	Path & coefficient
Active Teaching	Factor variable derived from survey questions asking for estimates of class time spent in teaching activities such as Lecture, Group Work, Discussion, Presentation and Short Explanations. (Higher Score is more active, -3 to +3)	Main outcome variable
Expectation_Variety	How does YOUR ACADEMIC DEPARTMENT rate the importance of the following for evaluating your career progress? - Use a variety of teaching methods (1-4)	→ Active Teaching -0.12**
Expectation_Not Lecture	(as above) -Departmental expectation to use methods other than lecture (1-4)	→ Active Teaching 0.28**
TDO	Factor variable asking about teaching, diversity and outreach career activities derived from checklist of survey items.	→ Active Teaching 0.21**
Important_Active	How do YOU rate the importance of the following in terms of what fulfills you personally?- Teaching in more active and engaging ways (1–4)	→ Active Teaching 0.19**
Important_Equity	(as above) - Promoting equity and diversity in your department and institution (1-4)	→ Active Teaching 0.09*
Teach_Freq	How many course sections have you taught in the last two academic years? Integer)	→ Active Teaching 0.14**
Program	Has participated in Project NExT or comparison (0 = Comparison, 1 = NExT)	→ Active Teaching 0.13**
Female	Gender Female (1= Female, 0= Male) (Other was listed as option but no one chose)	→ Program 0.07*
Years_Taught	Number of years taught (Integer)	→ Program -0.11**
PD_Inst	Factor variable of checklist items asking about participation in professional development at institution, higher score indicates more participation (-3 to +3)	→ Program 0.12**
Belongs_AMS	Belongs to American Mathematical Society (0, 1 = belongs)	→ Program -0.08*
Sought_Teaching	Please agree or disagree with the following statements about your early teaching career. - When I started my career I sought out a faculty position that emphasized teaching (1-4)	→ Program 0.20** → Teach_Freq 0.12**
Highest_Degree	Highest mathematics degree offered at institution (1-4, Associates to Ph.D.)	→ Program -0.14** → Teach_Freq -0.23**
Early Teach	Which of the following was (or is) true about yourself during your early career? (1-4 years after you received your Ph.D). - I taught more than one class each semester	→ Teach_Freq 0.10*
Early_Active	Which of the following was (or is) true about yourself during your early career? (1-4 years after you received your Ph.D). - I taught using active or inquiry-based methods	→ Teach_Freq 0.08*
Tenure_Elig	Is your position tenure eligible? (0, 1 = Yes)	→ Teach_Freq 0.14**
EarlyTeachwanted	Which of the following was (or is) true about yourself during your early career? (1-4 years after you received your Ph.D) - I designed my own courses	→Teach_Freq -0.11**

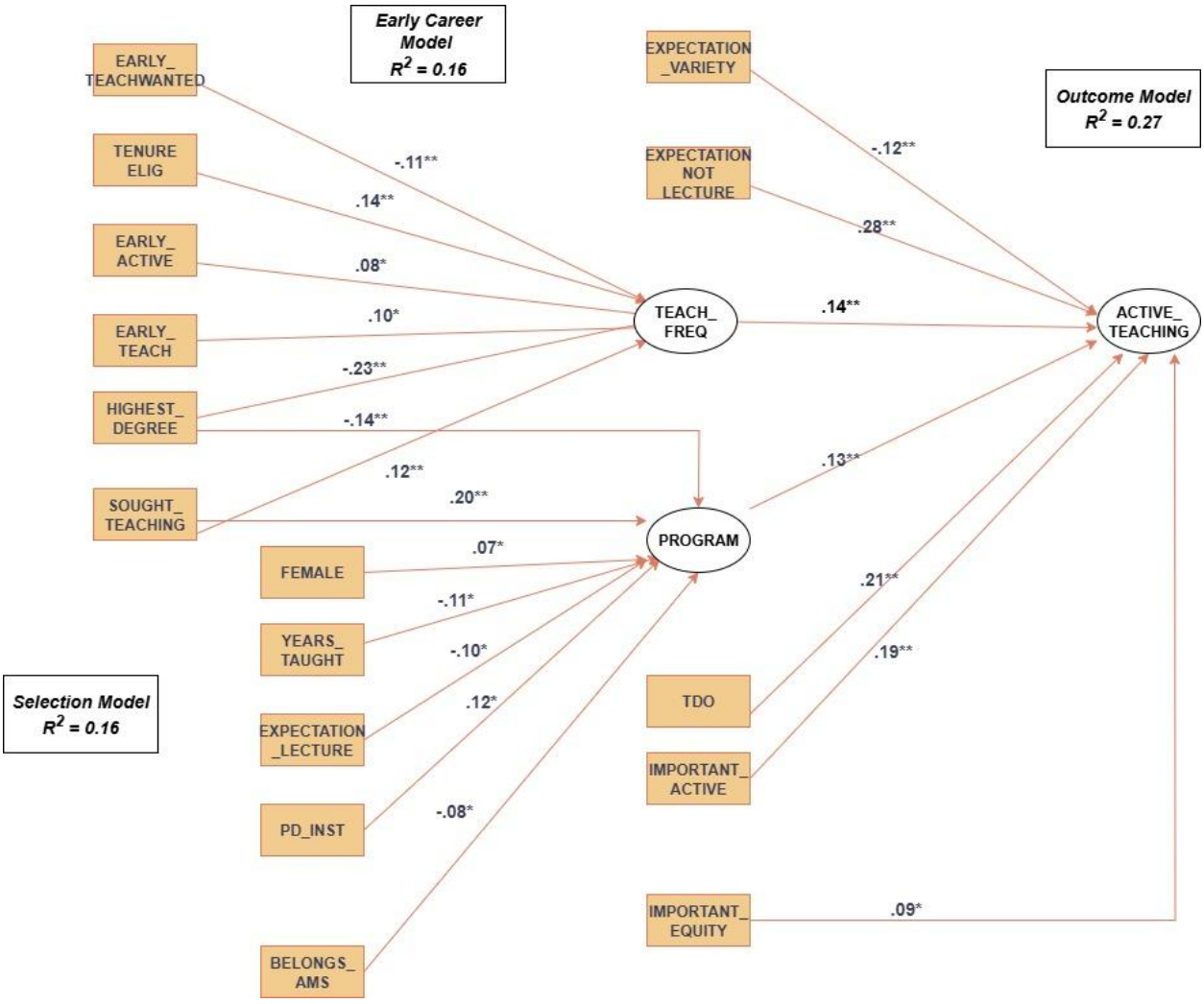

Figure 1. Three-Part Path Model of Relationships between Active Learning, Program Status & Teaching Frequency

Early career practices and decisions were associated with the amount instructors taught (Early Career Model predicting Teach_Freq variable, Fig. 1 upper left). This component of the model had an R-squared value of 0.16. Teaching at institutions offering Masters or Bachelors as the highest degree was highly associated with the amount instructors taught in the past two years (b=-0.23), as was having a tenure-eligible position (b=0.14). Initial career choice, reflected by the statement, "When I started my career I sought out a faculty position that emphasized teaching," was associated with more teaching later in a career (b=0.12). Moreover, those who taught greater numbers of courses in their earlier career were more likely to teach a greater number of classes later (b=0.10), and those who taught actively continued to do so, albeit this effect was relatively weak (b=0.08). Overall, these results showed the impact of early decisions on later career practices.

The third part of the model described and adjusted for differences between Project NExT alumni and the matched comparison group (Selection Model, Fig. 1 lower left predicting Program variable). In general, PN alumni were more likely to be less experienced teachers (b= -0.11) and women (b=0.07) and took part more in professional development at their institutions

(b=0.12). They were also more likely to work in departments with less expectation to lecture (b=-0.10). This component of the model is useful at showing differences between groups that persist after matching and holds these differences constant when assessing the Program variable.

Finally, the program effect for Project NExT was statistically significant, indicating that PN alumni were more likely to teach in an active manner. The raw effect size comparing PN to the comparison group was δ=0.63, considered a large effect (Bakker et al., 2019). Using Program in a regression model holds all other variables constant and determines if any remaining variance in active teaching can be explained by group membership. Here, PN group membership persisted in its association with active teaching with a standardized regression coefficient of b=0.13.

Discussion and Conclusions

We highlight a few key findings. First, the results demonstrate that faculty adoption of student-centered active approaches to teaching is not the product of any single influence or decision (Austin, 2011; Henderson & Dancy, 2007). Future faculty make choices that launch them toward a career in an institution that centers teaching (Austin et al., 2009). New faculty may be encouraged to use active teaching methods by social norms and departmental evaluation criteria that value teaching (Ebert-May et al., 2025). They may pursue professional development on teaching, early in their career and later, in the discipline and at their institution. When these activities are rewarding and supported by peer behaviors and norms, people feel fulfilled and more likely to continue to pursue them. Using a momentum lens, these factors are mutually reinforcing and all build faculty momentum toward use of active teaching methods.

Our finding about the association of the TDO construct with active teaching is new to the literature: faculty who take up active teaching are also more likely to take up other scholarly and professional activities that are likewise centered on student experiences and inclusion, such as outreach, diversity and equity work, and engagement with teaching change initiatives. We suggest that active teaching and TDO activities may reinforce one another and may gain further momentum in institutional settings where they are valued and rewarded.

Also salient is the role of early-career experiences and choices in influencing outcomes up to 25 years later in our sample. Pursuing and securing academic work that emphasizes teaching exemplifies the role of choice, while graduate school experiences of teaching may involve a mix of choice, constraint, and chance: all of these are positively associated with active teaching in the model. Moreover, past participation in Project NExT has a measurable and positive association with use of active learning that persists even when other factors are held constant. In text comments that we separately analyzed, PN alumni indicated that both their initial learning experiences and ongoing interaction with a disciplinary peer cohort were influential in shaping their current teaching. Ongoing analyses will elucidate all these associations and link them to additional data sets from this study and to prior research, seeking to build a full picture of how momentum toward use of active teaching practices waxes or wanes among mathematics faculty.

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Transposition Instability Due to Novel Mathematical Organizations for Life Sciences

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This study examines how institutions navigate praxeological uncertainty when implementing Modeling Life, a mathematical modeling curriculum for life sciences students. Using the Anthropological Theory of the Didactic, we analyzed implementation across five institutions where didactic transposition processes remain under active negotiation. We identify four interconnected patterns: positioning uncertainty, epistemological conflicts, human resource demands, and systematic adaptations. These findings illustrate how novel mathematical organizations whose requirements exceed existing institutional conditions both generate and are shaped by institutional adaptation processes during ongoing curricular development.

Keywords: mathematical modeling, ATD, biology majors, institutional perspective

The growing importance of mathematical modeling to the research methodologies and conceptual frameworks of the life sciences has prompted numerous calls for reforming the mathematical experiences of life science majors (Brewer & Smith, 2011; Steen, 2005). However, research has documented persistent barriers to implementing modeling-focused approaches in undergraduate mathematics education (Barquero et al., 2015; Bennoun, 2022). Studies identify institutional constraints including faculty's epistemological commitments to *applicationism* and *monumentalism* (Barquero et al., 2015), rigid corpora of mathematical knowledge at university level (Bosch, 2014), and institutional conditions that favor traditional calculus approaches over modeling-centered curricula (National Research Council, 2013). With highly standardized topics like calculus or linear algebra, "the scholarly mathematical *praxeologies* that are selected as knowledge to be taught and the *praxeologies* that are actually taught in the classroom constitute a very stable corpus of knowledge" (Bosch, 2014, p. 115). This stability is not present with bodies of knowledge that are new to mathematical programs at institutions, creating opportunity to study transposition processes during active development. The UCLA *Modeling Life* curriculum represents a modeling-centered approach that has been adopted by multiple institutions. Previous work documenting challenges to implementing novel curricula indicate that the Anthropological Theory of the Didactic (ATD) would allow for examination of the institutional dynamics that underly curricular transformation processes. Adopting this lens, we analyze *Modeling Life* implementations across five institutions to examine these dynamics.

Theoretical Perspective

ATD explains how mathematical knowledge transforms as it shifts between institutions, such as scholarly communities that use knowledge and scholastic communities that teach it. ATD concerns itself with how institutional conditions affect mathematical teaching and learning by coordinating analysis of didactic transpositions, praxeologies, levels of codetermination, and operating epistemologies (Bosch, 2014; Chevallard 2002). These tools aid in understanding why curriculum innovation often encounters systemic, institutional resistance and how

implementation challenges arise from mismatches between innovative praxeological requirements and existing institutional conditions.

Knowledge also undergoes systematic transformation as it moves between institutions; this *didactic transposition* reconstitutes scholarly knowledge as knowledge-to-be-taught. The transformation is not so simple as articulating a list of topics and techniques. It emerges from activity within the noosphere, the institutional sphere where curriculum designers, policymakers, textbook authors, and educational experts select, organize, and adapt mathematical content for institutional contexts. External didactic transposition refers to processes within noospheric activity while internal didactic transposition subsequently transforms this prescribed knowledge into "knowledge actually taught" and ultimately "knowledge learned" through classroom interactions between teachers, students, and mathematical content (Bosch & Winsløw, 2020). Both the process and the mathematical knowledge are shaped by the institutional conditions.

ATD's primary object of analysis is *praxeology*. These are organized blocks of knowledge and practice that describe how mathematical work is accomplished within institutions. A praxeology comprises two interconnected components: (i) a praxis block consisting of types of tasks (T) that individuals encounter and techniques (τ) used to solve them and (ii) a logos block containing the technology (θ) that describes, explains, and justifies the techniques, supported by theory (Θ) that provides broader theoretical foundation (BarbÉ et al., 2005). Mathematical organizations (MOs) are structured collections of praxeologies that make up the mathematical knowledge, and they operate within specific institutional contexts. An important undertaking for ATD is to elaborate the ecological conditions that enable praxeological development, maintenance, and evolution, as well as which institutional factors support or hinder the construction and survival of MOs (Bosch & Gascon, 2006). Chevallard's (2002) levels of didactic codetermination show how each level influences lower levels, explaining why curriculum implementation challenges stem from inter-level conflicts that require coordinated institutional responses rather than isolated pedagogical changes.

ATD has also examined the epistemologies underlying mathematical instruction that influence how MOs develop and function within institutions. For example, the "visiting monuments" paradigm treats mathematics as established works to be consumed and appreciated. This creates praxeologies where tasks are phrased in terms of procedures and techniques focused on algorithmic implementation of those procedures. In contrast, a "questioning the world" paradigm treats mathematics as tools for addressing genuine problems, creating praxeologies where tasks emerge from authentic inquiry and techniques developed in response to problem-solving needs (Bosch & Winsløw, 2020). Similarly, "applicationism" views mathematical modeling as mere application of pre-established mathematical knowledge to external domains (Barquero & Bosch, 2015). This leads to modeling appearing as end-of-chapter exercises following content presentation, creating a weak logos block where the technology and theory present modeling as supplementary to mathematical development rather than as mathematically generative. In contrast, an epistemology that positions modeling as integral to mathematics itself enables praxeologies with strong logos blocks where the technology and theory establish modeling processes as fundamental to mathematical understanding and concept formation.

Whereas in previous studies, external and internal transposition occur simultaneously (Bergsten et al., 2010; Bosch et al., 2021; Bosch & Winsløw, 2020; Strømskag & Chevallard2024; Winsløw et al., 2014), we use didactic transposition and praxeology to describe situations where external transposition (noospheric selection and prescription) has not settled into a reference MO by the time instructors begin internal transposition. We refer to this situation

as *transposition instability*. Under this condition, institutions attempt to position, implement, and legitimate courses in the absence of reference MOs, even though instructional practice is already underway. As a result, stakeholders face *praxeological uncertainty*, understood here as indeterminacy regarding which praxeologies should be taught, how they relate to existing curricular pathways, and on what grounds they can be justified within the institution. In this study, we examine how praxeological uncertainty emerges when institutions attempt to adopt novel MOs under conditions of transposition instability.

Mathematical Organizations of Modeling Life

The MOs of *Modeling Life* are constructed around dynamical systems and biological modeling. The praxis block centers on task types that require students to formulate, simulate, interpret, and revise mathematical models for phenomena like disease transmission, population dynamics, and gene regulation, in order to predict system behavior. The tasks demand techniques that combine analytic methods (linear stability analysis and equilibrium classification), numerical approaches (Euler's method, simulations), and computational tools (Python, SageMath). Freshman life science students construct meaning for differential equations by identifying feedback relationships in biological systems, entailing interdisciplinary reasoning about biological processes and their mathematical representations simultaneously. The logos block comprises technology that emphasizes concepts directly relevant to life sciences: state space representations, vector fields, bifurcation theory, and covariational reasoning. This technological discourse positions derivatives as linear approximations of vector fields rather than rules or slopes, and integration as accumulated change rather than antiderivatives. The underlying theory proposes an epistemological shift because the tasks emerge from biological questions (e.g., what is a genetic switch?) and the justifications for the mathematical techniques follow. Two characteristics distinguish these MOs from calculus praxeologies. First, the primacy of modeling over mastery of procedures entails the creation of a logos block that explicitly bridges disciplinary boundaries. The technological discourse brings to the fore both mathematical concepts and biological processes, and the theoretical framework positions mathematical validity as dependent upon biological plausibility. Second, the computational praxis block assumes simulation, visualization, and manipulation of parameters as standard mathematical practices, which enable tasks about sensitivity analysis, for example. The presence of these praxeologies create specific institutional demands: faculty must possess or develop interdisciplinary expertise, computational infrastructure becomes important, and assessment practices must evaluate conceptual synthesis rather than procedural execution. The MOs themselves thus create the conditions for transposition instability, as their requirements often exceed existing institutional capacities while their epistemological foundations challenge established mathematical teaching cultures.

Methods

We visited all five sites implementing *Modeling Life* materials, which represent variation in departmental context, institution size, and implementation maturity. We focused exclusively on adopting institutions since non-adopting cases provide limited insight into implementation processes (Ragin, 2006). At each site, we conducted semi-structured interviews with stakeholders including *Modeling Life* instructors, faculty teaching related courses, administrators, teaching assistants, and support staff (site characteristics in Table 1). Interviews elicited

stakeholders' salient experiences pertaining to developing, implementing, and sustaining the course (Leech, 2002). We treat absence of a reference MO as an observable condition when stakeholders report contested or changing learning goals or prerequisite expectations. We interpreted stakeholders' reports of adaptations and adjustments to curriculum, contested positioning of the course, or shifting academic oversight as evidence of transposition instability.

Table 1. Site characteristics

Institution	Years Operating	Department	Approx. Enrollment	# Interviews
UCLA	10+	Life Sciences Core	2400	12
UC Santa Cruz	1	Mathematics	160	7
U. Arizona	6	Mathematics	120	7
Utah State	1	Mathematics	60	10
Harvard	1	Mathematics	30	10

Results

We present short vignettes of each site, organized to show evidence of the state of external transposition (e.g., noospheric activity) and institutional responses that instantiate praxeological uncertainty (e.g., positioning moves, epistemological tensions, resources adaptation). We obscure interviewees' identities and attribute conditions they mention to their institution.

At Utah State, praxeological uncertainty emerged through transposition instability operating under conditions of perceived external disruption. At the time of our site visit, a state legislative mandate for general education, including Quantitative Literacy (QL), was interpreted by stakeholders as consistent with the university's land-grant mission to prepare citizen scholars for workforce needs. Participants anticipated a potential reconfiguration of curricular priorities towards critical thinking, data science, and meaningful quantitative work. Although the concrete implications of the mandate for the mathematics department were not yet fully specified, the transfer of curricular authority from home disciplines to a general education center was understood by some as an opportunity for reform, while others perceived it as a destabilizing force for established content and pathways. As one administrator noted, the legislation would break established curricular structures, disrupting what what Diaz-Eaton (under review) identified as "forces that hold the 'canon' sacred." These legislative changes, including significant budget reallocations, functioned as concrete, enacted constraints and new powers, shaping anticipatory positioning around curricular innovations. In this context, the *Modeling Life* approach was viewed by stakeholders as aligning with the direction signaled by the mandate. However, departmental leadership was hesitant to commit sustained resources amid budget pressure and unresolved questions about course evaluation and students' revised pathways. Within this environmental uncertainty, instructors endeavored to predict the eventual audience for the course and align the MOs with their anticipated needs, including retaining, excluding, modifying, or emphasizing MOs from the traditional and modeling focused versions of the course. Overall, *Modeling Life* occupied an ambiguous institutional position: it was sufficiently aligned with emerging priorities to proceed, but its long-term stabilization and the securing of additional, dedicated resources for its full development remained a concern.

At Harvard, a different kind of positioning uncertainty emerges through consideration of students' pathways through mathematical programs. There, *Modeling Life* was conceived as an

alternative to the precalculus/calculus requirement, but not as calculus preparation, since the praxis blocks contrast starkly. This reflects institutional uncertainty about how MOs with fundamentally different praxeological structures should relate within mathematical programs. However, the introduction of *Modeling Life* creates system-level effects throughout the mathematical program that can impact other students, as it draws life science students away from traditional courses and subsequently relieves pressure on the precalculus courses regarding what praxeologies are now suitable. Additionally, students' hesitancy to take a course that hadn't been offered before indicates their perception of instability due to the institution's inability to provide clear positioning information about MOs still undergoing noospheric negotiation. These uncertainties both emerge from, and contribute to, broader system-level effects throughout mathematical programs.

Stakeholders across multiple sites evidenced epistemological conflicts between the praxis and logos blocks of *Modeling Life* and institutional commitments to traditional mathematical praxeologies. At UCLA, the initial response from the mathematics department was fierce and vigorous opposition to the new course viewing it as "not real math" or "not rigorous enough." Faculty resistance surged from both the logos blocks of *Modeling Life* positioning mathematics as emergent from biological inquiry, and the praxis blocks using geometric and numerical methods. Chevallard (2015) noted that teaching systems establish an "authorized" version of knowledge and, Diaz-Eaton (under review) argued the language of rigor is often wielded to justify deviations from that authorized knowledge. It is beyond the scope of this paper to argue the mathematical rigor of *Modeling Life*, however we note that its technology integrating feedback mechanisms, state space representations, dynamical systems, and covariational reasoning are all rigorously articulated areas of mathematical practice. Indeed, multiple stakeholders explained that taking a pure mathematics approach as being "not in the right frame of mind" since teaching *Modeling Life* requires "letting go" of the usual mathematical practices. Thus, it seems that productive integrations of this course into the ecology of mathematics departments might further entail an epistemological reorientation towards the course's theoretical underpinnings, namely, positioning mathematics as emergent from biological inquiry rather than as monuments to visit (Chevallard, 2012). Instructors willing to engage in this shift may yet feel "nervous" or "surprised" by the textbook's approach, as did an instructor at Utah State.

These epistemological conditions create institutional pressure on the survival of *Modeling Life*'s praxeologies by demanding faculty orientations that mathematics departments may find difficult to supply, forcing institutions to either modify their implementation approach or develop new human resource strategies to support MOs whose requirements exceed traditional faculty preparation. At our sites, we observed plural strategies to address faculty knowledge and epistemological gaps, with a commitment to maintaining praxeological integrity. At UCLA dedicated teaching lines were authorized to support course development and maintenance. U. Arizona dedicated resources to a co-teaching model in order to induct new instructors to the implementation cycles. This positions co-teaching as a mechanism for developing faculty capacity to implement MOs whose praxeological requirements exceed traditional preparation. Although it is expensive, administrative stakeholders there recognize that courses like *Modeling Life* cannot be sustained with a "rotating cast of characters," and that co-teaching, particularly when paired with a dedicated lead instructor, provides for continuity, recruitment, and is essential for the long-term viability of the course. Additionally, administrative and instructional stakeholders asserted that co-teaching facilitates a collaborative environment which becomes engaging for students. UCLA and Harvard rely heavily on undergraduate learning assistants, but

how those resources are used differ. At UCLA, learning assistants have successfully completed the course and can help students who have GTA's that are not as familiar with the interdisciplinary MOs. At Harvard, the learning assistants have not completed the course, but they are drawn from a pool of students with advanced applied mathematics knowledge. Investment of resources in undergraduate learning assistants created a low student-to-teacher ratio, creating opportunities for the instructor to focus on generating pedagogical notes that would make the course "teachable by others."

The positioning of *Modeling Life* within degree plans and mathematical programs also places demands that create incoherence, where praxis blocks diverge from the philosophy enacted through the logos blocks. For example, at Harvard, concern for students' success in later physics courses necessitated inclusion of some tasks from traditional calculus praxeologies, such as trigonometric functions, with companion differentiation and integration techniques. The instructor reflected that these changes were "not very closely tied to anything else we're doing," suggesting that the praxis block was not aligned with *Modeling Life*. Similar adjustments were made to *Modeling Life* at Utah State to satisfy constraints attributable to downstream statistics courses. An instructor at U. Arizona observed that these kinds of modifications "weakened the course a little bit." Interpreting their comments in terms of the ATD, the ecologies in these institutions created a divergence where the logos block supporting the traditional calculus tasks was largely absent (see BarbÉ et al., 2005) and, when obliged, in conflict with the praxeologies of *Modeling Life*. These adaptations preserve prerequisite functions while introducing praxeological tensions between modeling-centered and traditional procedural approaches.

Resource constraints systematically modified the computational praxeologies of the MOs. At Utah State, stakeholders explained that time limitations inhibited integrating coding into the course work because there were only three (credit) hours a week allocated to the course. The full realization of the computational praxis block would require a 4 or 5 credit-hour course, however an existing mathematics course was repurposed to realize *Modeling Life* there. Thus, to suit the institutional conditions, the computational praxeology was removed entirely. Yet, the instructor excluded treatment of limits, despite protestations from colleagues, to avoid "just teaching typical calculus." They opposed specific praxis blocks to preserve logos blocks they perceived as fundamental to the course's philosophy. Student population characteristics also drove systematic adaptations to praxis blocks as institutional conditions forced modifications to task types and techniques. Institutions expanded praxis blocks to include foundational quantitative reasoning tasks (proportions, percents) or adjusted the pace of technique development to accommodate local student preparation, demonstrating how demographic constraints shape praxeological boundaries during external transposition. At UCLA, programming components were "softened" with labs becoming "much easier on the coding" because they could be "brutal" for students with no prior coding experience. Here, the expectation shifted from students writing Python code to modifying provided code to run simulations. This transformation altered both the task types (T) from code generation to code interpretation and modification, and the associated techniques (τ) from syntax mastery to parameter manipulation. The logos block adapted correspondingly, with the technology (θ) repositioning computational work from teaching coding to using simulations to emphasize important ideas from the course. Harvard adopted Chat GPT for coding to bypass syntax learning constraints. These modifications preserve the logos block's emphasis on the value of simulation while adapting the techniques to institutional realities.

UCLA differed from the other sites in that sustained noospheric activity located *Modeling Life* within the Life Sciences (LS) Core, a separate academic unit. There, faculty possessed

epistemological commitments to the praxeologies underlying *Modeling Life*, such as praxis blocks centered on biological questions giving rise to logos blocks positioning mathematics as a tool. Within the unit, resources were allocated towards course releases for material development, funding for ongoing evaluation, and strategic hiring of quantitative biologists dedicated to course maintenance. This permitted both the iterative refinement of MOs as specific praxeologies were adopted and discarded, and the production of concrete instructional materials to unify multiple course sections. These artifacts then served as a reference MO (BarbÉ et al., 2005), at least provisionally as new instructors were inducted to the course and the faculty of the LS Core sought to secure ongoing resource allocation. As a result, many of the tensions observed at other sites were attenuated. The pre-med and life science majors' willingness to satisfy degree requirements with *Modeling Life* further legitimized it. A secondary consequence was that downstream courses could build upon the praxeological foundation of *Modeling Life*, reinforcing the institutional integration of its MOs.

Implications & Future Directions

Our analysis identifies four interconnected patterns of praxeological uncertainty when institutions adopted *Modeling Life*'s MOs. Institutions experience positioning uncertainty as they struggle to locate *Modeling Life* within existing academic structures. Epistemological conflicts emerge between faculty commitments to traditional mathematical praxeologies and modeling-centered approaches. Novel MOs create human resource demands that exceed faculty's preparation. Finally, institutional constraints drive systematic adaptations to MOs as they strive for ecological fit within the local conditions. These findings demonstrate transposition instability creating praxeological uncertainty about *what* is being adopted, *how* it should function institutionally, and *for whom* it was offered. The literature identifies broad challenges to to innovative mathematical education such as "monument" paradigms (Barquero et al., 2015), institutional inertia (Bosch et al., 2020), and faculty preparation gaps (Díaz Eaton, 2017), but provides limited analysis of underlying mechanisms. Our findings reveal specific processes through which these barriers operate. *Modeling Life*'s interdisciplinary MOs create institutional tensions operating across multiple codetermination levels. Unlike content-focused innovations that accommodate existing structures, integrating dynamical systems with biological reasoning requires coordinated changes spanning disciplinary, pedagogical, and institutional levels. UCLA's LS Core created dedicated institutional space where hybrid MOs could develop without competing against traditional departmental expectations, addressing what Winsløw et al. (2014) identified as constraints from mathematical domain isolation. Yet, at each of the sites, pressures to adapt *Modeling Life* arose from the noosphere. ATD permitted insights into how and why challenges to innovative MOs operate through specific praxeological mechanisms, rather than general resistance patterns (e.g., Shadle et al., 2017). We acknowledge that some praxeological uncertainty emerged from the interdisciplinary characteristics of *Modeling Life*'s logos blocks rather than its shift in mathematical content. This presents challenges to potential instructors trained in analytic and symbolic techniques, at the pedagogical level of codeterminacy, that could be addressed through collaboration with life science colleagues (Bennoun, 2022). It is yet unclear what features of collaboration could support a bridge between disciplinary domains.

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Students' Perspectives of Proof Community Values and Goals

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Undergraduate students' views on proof provide insight into how they engage with mathematical practices, yet little is known about the values and goals they associate with proof. We surveyed 32 students at two Hispanic-Serving Institutions and identified themes related to values and goals of the proof community across three domains: the proof itself, the utility of proof, and the prover. Students' perspectives often echoed mathematicians' values such as rigor and explanatory power, but also emphasized less-documented dimensions, including creativity, problem solving, and educational aims. By highlighting both alignment and extension of disciplinary norms, these findings suggest ways to better support students' participation in proof-based mathematics.

Keywords: Proof, values and goals, norms, proof community

Undergraduate students' experiences with mathematical proof offer important insights for the mathematics education community, yet this area remains relatively underexplored. Prior research has examined what students retain from proof-based instruction (Krupnick et al., 2018; Lew et al., 2016) and their beliefs about various dimensions of proof, such as reading, writing, and comprehension (Guajardo et al., 2025; Lew & Mejia-Ramos, 2019; Weber & Mejia-Ramos, 2014). These studies illuminate students' perceptions and learning outcomes, but further work is needed to understand how students engage with the norms and practices of proof-based mathematics throughout their coursework.

Several aspects of mathematical proof present challenges for students, including abstraction (Rowland, 2001; Weber, 2004) and unfamiliar writing norms and conventions (Dawkins & Weber, 2017; Lew & Mejia-Ramos, 2019; Melhuish et al., 2025). For example, prior research has documented that students may struggle with making implicit arguments explicit (Samkoff & Weber, 2015; Selden & Selden, 2003). These difficulties can impact students' self-efficacy and their sense of identity within the mathematical community (Wrightsman, 2025). Additionally, a common narrative in mathematics is that ability is innate and success comes easily for those with high aptitude (Melhuish et al., 2024). Together, these challenges and perceptions can contribute to an exclusionary and elitist culture in mathematics.

Community values and goals serve as foundational elements for the norms that shape how individuals see themselves, and are seen by others, within the mathematical community. The values and goals students perceive as important to the proof community influence which norms become salient in their experiences in proof-based courses. While existing literature has identified values that professional mathematicians associate with proof (e.g., Dawkins & Weber, 2017), we know less about how students perceive these values and goals. This is especially relevant as instruction in proof-based courses increasingly emphasizes student-student interaction, creating more opportunities for societal narratives about mathematical ability and participation to influence classroom dynamics (Battey & McMichael, 2021). To address this, we investigated the following research question: What values and goals do students perceive as important to the proof community?

Theoretical Perspective

For this study, we leverage Lauden's (1984) work on scientific communities' norms, values, and goals. A scientific community has various accepted results that it produces, these are referred to as theories of the community. A value can then be defined as the "good" attributes of these theories (Laudan, 1984). For instance, for the mathematics community, proofs are considered part of their theory and as such, a *value* would be for a proof to be rigorous and follow deductive logic (e.g., Dawkins & Weber, 2017). Additionally, a community has certain goals that they aim to achieve which then guide the rules and methods of the community (i.e., norms, Laudan, 1984). These norms are upheld by the shared values and goals that a community aims to achieve (Dawkins & Weber, 2017). Goals and values are inextricably linked, in that, goals are specific aims that a community or individual hopes to achieve and values are the underlying motivations and rationale for wanting to reach those goals (Laudan, 1984). Within the math education community, specific mathematicians' values and norms have been identified, however we are interested in understanding students' perspectives of the proof community's goals and values.

In proof classrooms, students are expected to embrace mathematicians' proof norms, yet they may not be aware of the mathematicians' value and goals underlying those norms (Dawkins & Weber, 2017). For example, the norm that "justification in proof should be deductive and not admit rebuttals" (p. 129) reflects the shared value that mathematical knowledge is a priori. Although students are often expected to write proofs in this way (Dawkins & Weber, 2017), they may perceive this practice as forced upon them without understanding the rationale (i.e., value) behind it. Without a shared understanding of the values and goals tied to these norms, students may find them arbitrary or meaningless (Dawkins & Weber, 2017). Further, Dawkins and Weber, citing Solomon (2006), noted that "asking students to adapt norms without corresponding values can leave students feeling passive and marginalized with respect to their learning of proof" (p. 133). Understanding the values and goals students associate with the proof community can help make these norms feel more meaningful and less arbitrary or imposed.

When goals and values of proof and proving are not explicitly discussed, students are left to infer them on their own. For example, while *a priori truth* is a valued aspect of mathematical proof (Dawkins & Weber, 2017), conciseness is also valued—unnecessary justifications are often omitted in published proofs (Selden & Selden, 2013). Students may not recognize this norm, as Weber and Mejia-Ramos (2014) found that students often believe every step should be included in a proof and that readers should not be expected to fill in gaps. Students may, in fact, prioritize clarity over conciseness. Another commonly held value is the exclusion of references to the author of the proof (Dawkins & Weber, 2017; Melhuish et al., 2025), as proofs are intended to be "autonomous objects," independent of their authors. Additionally, de Villiers' (1990) roles of proof—verification, explanation, systematization, discovery, and communication—highlight goals (and values) focused on the proof itself, such as its structure and presentation. However, these goals provide limited insight into values and goals related to other aspects of proof, such as the role of the prover or the proving process. Exploring the values and goals students identify can further efforts to make these aspects more explicit.

Methods

Data collection

We designed a survey that asked students to identify values and goals that they associate with the proof community. The survey included a short introduction to our study and described what we meant by a community's perceived values and goals. We then asked them: 1) *In your*

experience, what are some things that are valued about proof within the mathematics community? and 2) *In your experience, what are some goals related to proof that people work toward within the mathematics community?* Following each question, we asked students to describe a situation from their experience as math students that reflected their identified values and goals. Responses to this question gave us insight and context into the students' personal experience with these values and goals. The survey was distributed to students across two institutions in the southwestern United States, both of which are Hispanic-Serving Institutions with one school having over 90% Hispanic student population. Students were asked to complete the survey if they had taken at least one proof-based course. We received a total of 32 responses from students, see Table 1 for description on the students' proof experience (students selected 1-2 or 3+ courses), gender, and race/ethnicity (both self-reported). All 32 respondents answered the question on values and 28 of the 32 respondents answered the question on goals.

Table 1. Student demographic information (number of students indicated in parentheses)

Proof Course Experience	Gender	Race/Ethnicity*
1-2 courses (14)	*Female (10)*	*African American (1)*
3+ courses (16)	*Woman (2)*	*American Indian (1)*
	Male (16)	*Asian American (1)*
	Non-binary (2)	*German (1)*
	N/A (2)	*Hispanic (19)*
		Latinx (3)
		Mexican/Mexican American (2)
		White (8)
		White Hispanic (3)

**Some students self-identified as multiple race/ethnicities*

Analysis

To analyze our data, we used an open coding approach to identify themes related to proof community values and goals. We started by collaboratively coding the values: the authorship team reviewed each survey response and assigned tags summarizing the student-identified value(s). Students' personal stories of their experiences were used to help contextualize their identified values and resolve disagreements among the team. Since students often reported multiple values, we applied as many tags as needed to capture their full response. Disagreements were addressed in real time during the group coding process. After completing the value coding, each author individually open coded the goals. We then met to compare tags, discuss discrepancies, and reach consensus on the final coding.

After tagging all responses for values and/or goals, the first author synthesized the codes into overarching themes. During the initial coding, the team noticed that students' responses often referenced different aspects of proof—such as the proof itself versus the person constructing it. With this in mind, the first author categorized each theme according to the aspect of proof it described. For example, when students noted that the proof community valued *the structure and strength of a proof,* this was classified as a theme related to the proof itself. Once the themes were developed and grouped, the full authorship reviewed them, resolving any disagreements through discussion until agreement was reached among the full team. These finalized themes are presented in the next section.

Results

We found that the students identified values and goals of the proof community referenced three aspects of proof: (a) the proof itself, (b) the utility of a proof, and (c) the prover behind the proof. Within each category, we identified themes pertaining to both values and goals. The results are organized accordingly, with themes related to values presented first, followed by those related to goals. Table 2 summarizes these themes.

Table 2. Values and Goals Identified by Students

Aspect of proof	Value Themes	Goal Themes
The Proof Itself	*Truth and Objectivity of a proof* *Structure and Strength of a proof* *Clear Communication & Language of a proof*	*For proofs to be accessible and approachable* *For proofs to be clear and rigorous* *For proofs to be logically sound*
The Utility of the Proof	*Proofs' explanatory power* *Proofs' power to justify the validity of a claim* *Proofs' General Applicability*	*Use proofs to prove general statements* *Use proofs to understand why mathematical concepts work*
The Prover	*Problem Solving Skills* *Creativity and Ingenuity* *Knowledge and Expertise*	*To have a mathematics research career* *To improve proof skills* *To educate others*

The Proof Itself

Values and goals that fell into this category reference attributes about the proof itself. In other words, these are values and goals that students identified that related to the aesthetics of a proof or features that a proof should have.

Values. We found three themes related to attributes that are valued in a proof. The first value was related to the *truth and objectivity of a proof*. Students highlighted the axiomatic basis of proofs and the objectivity of deductive logic. One noted that the proof community valued "that the steps to get to the proof are logical [...] proof is a logical based construction that follows order and does not require opinions," while another emphasized that a proof "establishes certainty through logical rigor, ensuring that conclusions follow inevitably from accepted assumptions." Students also identified values related to the *structure and strength of a proof*. Students saw value in rigor, resilience, and adherence to precise structural rules. Some pointed to the value on "how organized [proofs] can be," with one recalling a professor describing an entire course on proof organization. Others emphasized rigor, such as a student reflecting on epsilon–delta proofs: "Being almost right wasn't enough. Every inequality and bound had to be airtight." Lastly, students identified values related to *clear communication and language of a proof*. Students saw value in clarity, conciseness, and precise mathematical language. One recalled an instructor's reminder "that while proofs do require logically sound steps, they are still meant to be read," while another noted losing points on "proofs that are correct in every way, except they are 'too long/complicated'". Precision with notation also mattered; as one student explained, "I

mistakenly switched up delta and epsilon [...] it was not 'wrong,' but my instructor noted it's best to stick to traditional convention."

Goals. Similar to the values, we found three themes related to goals that students identified that the proof community has for a proof. The first was a goal for proofs *to be accessible and approachable*. Students saw this as the goal of making proofs intuitive and understandable, with one student explaining that they "try to make my proofs easy enough to understand so that everyone can be able to be involved within a proof whether they're a contributor or simply learning from it," while another emphasized the importance of this goal as they felt "discouraged by terms like 'obviously' or 'clearly'" when intermediate steps were omitted. Students also noted there is a goal for proofs *to be clear and rigorous*. That is, students saw it as a goal to construct rigorous proofs that cleanly use mathematical symbols and notions. One student noted that proofs should "establish rigorous truths," another noted that "it's only through rigorous definitions and components that proofs are constructed," while another described relying on notation "to make things better and clearer for the reader." Lastly, students identified a goal for proofs *to be logically sound*. This goal primarily referred to constructing proofs by building on axioms and deductive reasoning to establish truths, as one explained, this involves "a showcasing how certain truths can be used to further demonstrate assumed truths (or contradictions) logically," a goal they connected to their practice of structuring proofs through lemmas and theorems.

The Utility of the Proof

Values and goals that fell into this category reference attributes about the utility of a proof. In other words, these are values and goals that students identified that related to the purposes and uses of proofs.

Values. We found three themes related to attributes that are valued in the utility of a proof. First, students valued *proofs' explanatory power,* noting how proofs can deepen understanding of mathematical concepts. For instance, one student noted "proofs give us a window into WHY math works the way it does, allowing in-depth understanding of mathematical concepts." They explained how they have seen this in action as they have "been able to apply concepts I learned from proofs presented to me in a Complex Variables course to solve proofs in other classes." Second, students saw value in *proofs' power to justify the validity of a claim,* highlighting their function in establishing the truth (or falsity) of a claim. As one student noted "a proof ensures that a statement that can be taken as fact is actually factual" and then notes the power this has in their own schoolwork: "theorems and lemmas are what power my work. It ensures that what I'm 'stating' is actually meaningful and correct." Lastly, students valued *proofs' general applicability*. This refers to a proofs' power to establish general results that extend across contexts. For instance, one student stated that proofs are valued "for their ability to provide general statements to solve a certain type of problem," and that their ability to "provide a statement that solves a multitude of problems is essential for math to function."

Goals. Students identified two goals related to the utility of a proof. First, they noted that a goal for the proof community is that proof should be used to *prove general statements,* highlighting how proofs can be particularly useful the more general the statement is that they prove. For instance, one student mentioned that by proving general statements they can "actually put proofs to use, using them to solve problems and such." Second, many students identified that a primary goal of proof is to *use proofs to understand why mathematical concepts work.* This goal refers to how proofs can provide insight to the prover about how the proofs work and the mathematical concepts they can be applied to. One student noted how proofs can provide insight

into different algorithms they use in their math courses, stating that these algorithms are "based on a strong understanding of mathematical structures, which are founded by theorems that are justified by proofs." Another student recalled how one of their math professors researched algebraic partitions and how proofs helped them "better understand how they worked inherently." Lastly, one student reflected on how "proofs have helped me [understand mathematics] such that I feel it adds meaning to mathematics."

The Prover

Values and goals that fell into this category referenced attributes about the prover. In other words, these are values and goals that students identified that related to the person who creates the proofs.

Values. We found three themes related to attributes that are valued about the prover. First, students valued provers' *problem-solving skills*. This value addressed skills like critical thinking, deductive reasoning, and the ability to approach a proof in different ways. One student commented how "being critical in questioning proofs is of high value" explaining how their professor encouraged them to question "our inherent trust in the proofs and challenged us to think deeply about why it worked." A different student highlighted the value of approaching a proof in multiple and unique ways recalling a class where they "would all provide our own version of the proof. This made us think about how to go about problems and different ways we could look at these things." Second, students identified *creativity and ingenuity* as valued traits in a prover, referring to how provers who have original ideas and can generate novel proofs are valued. For instance, one student stated that "original thoughts and ideas are greatly valued within proofs" while another reflected that in advanced mathematics, value lies in "being able to create not only an accurate proof, but also an elegant one, which, if possible, is a novel way of solving the problem." Lastly, students captured the notion that *knowledge and expertise* are valued in a prover. In other words, deep content knowledge and mastery of the proving process is of great value. As one student mentioned, "[w]hat matters is the steps that are taken to get there, the knowledge that the mathematician has to make the proof clean and beautiful," while another recalled a professor's remark that "the great graduate students and researchers always take the extra step and review proofs and extra problems in the textbook to receive those skills" underscoring the effort required to develop expertise.

Goals. We found three themes related to goals students identified for a prover. The first goal for a prover is *to have a mathematics research career*. This goal encompasses many aspects of being a research mathematician like advancing the field, making new discoveries, and proving new results. One student reflected that their professors "are also excited to discuss proofs of new or recent results, emphasizing that this is the ultimate goal" while another noted how their professor often shared their "post-doctoral experience and opportunities to receive jobs and how to achieve these goals." The second goal focused on *improving proof skill*. Here, students often reflected on their own educational journey and noted that the goal was to continuously improve. For example, one student described a personal milestone: "Be able to read and understand a proof independently," recalling a time when they were reading a proof on infinitely countable sets, saying "I read along this proof understanding the argument, and when I asked the professor if my interpretation was correct he said yes. I was so joyous and proud of myself and being able to understand the proof." Lastly, students recognized that there was a goal *to educate others*. This goal referred to both the classroom and peer support. As one student put it, the goal was to "train others to think critically" recalling class discussions where students were asked to critique sample proofs and propose improvements. Another, reflecting on tutoring and helping out their

peers, commented that a goal would be to clearly explain "details of a proof for student understanding."

Discussion

In this study, we investigated the values and goals that students perceived as important to the proof community. Their perspectives were both varied and nuanced, reflecting attributes of the proof itself, the utility of proofs, and attributes of the prover (see Table 2). The themes we identified connect with and extend findings from existing literature. For instance, the value themes *Truth and Objectivity of a proof* and *Structure and Strength of a proof*—highlighting the value of axiomatic foundations, deductive logic, and rigor in proofs— align with the mathematician value that "mathematical truth is a priori" (Dawkins & Weber, 2017, p. 128). Similarly, the value theme *Proofs' Explanatory Power* directly corresponds with the value that "proofs should increase mathematical understanding" (p. 130). Finally, the theme *Clear Communication and Language in a proof*, including consistent use of notation, resonates with the value "mathematicians desire a consistent set of norms and practices" (p. 131). These parallels suggest that many student perspectives are well already aligned with the broader mathematical community.

At the same time, our findings introduce dimensions not previously documented in the literature. While much prior work has focused on values related to the proof itself (e.g., Dawkins & Weber, 2017; Melhuish et al., 2025; Selden & Selden, 2013), our participants also identified values connected to the utility of the proof and attributes of the prover. Notably, their attention to the prover reflects an awareness of the proving process, recognition of the human element in proof production, and appreciation of skills such as problem solving, creativity, and expertise. These responses challenge the tendency to treat proofs as "autonomous objects" divorced from their authors (Dawkins & Weber, 2017, p. 130). Although not often explicitly captured in prior literature, we believe these values are ones many mathematicians would likely endorse. Additionally, our study extends the field by documenting student perspectives on the goals of proof—an area often left implicit. Closely tied to values, goals illuminate the purposes behind proof values and provide insight into how students make sense of the proof community.

An important observation from our findings is that the themes related to the values and goals students identified often aligned with their respective groupings—namely, aspects of the proof, its utility, and the prover. For instance, the value themes *Proofs' explanatory power* and *Proofs' power to justify the validity of a claim* correspond with the goal theme to *Use proofs to understand why mathematical concepts work.* Similarly, the value theme *Problem-Solving Skills, Creativity and Ingenuity, and Knowledge and Expertise* aligned with the goal theme *To have a mathematics research career*, as these attributes are central to advancing the field. While these connections are unsurprising given the intrinsic relationship between values and goals (Laudan, 1984), it is notable that this alignment emerged clearly in the students' responses. This suggests that although students may sometimes view mathematical norms as arbitrary or meaningless (Dawkins & Weber, 2017), they are attuned to many of the values and goals upheld by the proof community. Moving forward, the challenge is to help students make these connections more explicit as they deepen their engagement with the community.

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MGTA Professional Growth: Aligning Structural Supports With Symbolic Value

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Mathematics graduate teaching assistants (MGTAs) are central to undergraduate instruction, yet their professional development (PD) is often limited. The Engaged Learning & Intentional Teaching Experiences PD program is a multi-institutional effort to strengthen MGTA PD around active learning, equity, and inclusion. To examine how such initiatives can be enacted and sustained, we studied departmental culture at two participating universities (Beta and Gamma) using Bolman and Deal's (2017) Four Frames model, focusing on the structural and symbolic dimensions. Interviews with nine departmental leaders revealed five cross-cutting categories shaping MGTA involvement. Findings highlight how sustainable change requires alignment between structures (e.g., coordination, PD requirements) and symbols (e.g., values for equity and active learning). At Beta University, mathematics coordination structures reinforced symbolic commitments, embedding MGTAs in active learning reforms. At Gamma University, symbolic commitments appeared stronger but lacked structural reinforcement.

Keywords: graduate teaching assistants, institutional change, active learning, department culture

Mathematics graduate teaching assistants (MGTAs) are often a primary point of contact for undergraduate students in introductory mathematics courses. Serving in roles such as instructor of record or recitation instructor, these graduate students are responsible for delivering course content to thousands of undergraduate students each year (Selinski & Milbourne, 2015). MGTAs impact the learning experiences of a large population of undergraduate students making their professional development (PD) for teaching critical. How MGTA PD is enacted within a department varies across institutions, but literature suggests the need for sustained PD beyond a pre-term orientation (Beisiegel et al., 2019; Defranco & McGivney-Burelle, 2001; Miller, 2018). However, establishing such a program requires a supportive department culture and structures in place to sustain these efforts.

The authors of this paper are part of a team of researchers that designed and implemented a PD program for MGTAs. The Engaged Learning & Intentional Teaching Experiences (ELITE) PD program is a large-scale, multi-institution project supported by a collaborative National Science Foundation grant. Key components of the work involve examining the graduate PD culture at participating institutions and introducing the ELITE PD program. Each ELITE PD MGTA participant, regardless of institution, attends a minimum of one year of PD. This work aims to prepare MGTAs to teach using active learning and enhance equitable and inclusive teaching practices. Understanding the impact of these changes requires an initial baseline understanding of the culture at the different institutions.

This paper uses Bolman and Deal's (2017) Four Frames of Leadership: Structures, People, Power, and Symbols, which has been used to understand change in STEM education (e.g., Couch et al., 2024; Reinholz et al., 2019). For this manuscript, we focus on the structures and symbols that are evidenced through interviews with department leaders and analysis of the institutions. In particular, we aim to investigate the following:

1. What structural and symbolic features of departmental contexts shape the development and involvement of MGTAs?
2. How are symbolic commitments and structural realities aligned (or not)?

Literature Review

MGTA Development in Mathematics Departments

MGTAs play an essential role in many mathematics departments. Formal responsibilities often include grading student work, staffing tutoring centers, teaching recitation sections that accompany courses led by an instructor of record, and, in some institutions, serving as instructors of record themselves (McLeod et al., 2025). Informal responsibilities can involve gathering insights on student mathematical thinking, conceptions, and misconceptions to share with instructors of record, as well as serving as role models who help bridge the gap between faculty and undergraduate students.

The support, training, and challenges MGTAs experience can vary widely between institutions; in some, the training might be as little as a short few days before the term starts, while in other departments, MGTAs may receive multi-year professional development with weekly sessions (Cho et al., 2011; Ghalichi et al., 2023). Additional support can include having a course coordinator meet with them regularly, separate from formal PD. However, despite the support MGTAs receive, they also face a number of challenges that can impact their ability to, in turn, support their students. These challenges can include navigating power dynamics with faculty and students, managing financial stress since most graduate programs provide modest funding that may be insufficient to meet living expenses, and meeting unclear or even competing expectations in how they work with students and perform their role, especially when departmental or institutional culture and values are not aligned between faculty or administration (Cramer, 2019). Additionally, MGTAs have responsibilities as students themselves with their coursework, theses, or dissertations.

The ELITE PD program is designed to support MGTAs through such challenges as they take on teaching roles, providing at least one year of sustained PD. However, implementing pedagogical practices – even those supported by research, such as active learning and equitable and inclusive teaching – is well established to be difficult, with "[e]nacting and sustaining educational change requir[ing] a systemic approach, focused on culture" (Reinholz & Apkarian, 2018, p. 2). To this end, we focus on understanding departmental culture, particularly regarding the role of MGTAs and their work within departments, to understand how the ELITE PD program may be enacted and sustained within local contexts.

The Four Frames Model

For this study, we utilized the Four Frames Model as a theoretical framework. The Four Frames, developed by Bolman and Deal (2017), is a theoretical framework that has been used in STEM education to understand departmental culture, particularly to support departmental change initiatives (Couch et al., 2024; Reinholz et al., 2019), such as the ELITE PD program.

The four frames provide lenses for understanding an organization's culture: structural, people, power, and symbolic. The structural frame focuses on formal roles, expectations, and policies. The people frame emphasizes individual needs, relationships, and employee development in support of organizational goals. The power frame highlights dynamics such as competition for resources, authority, and influence. The symbolic frame captures culture, shared meaning and values, and identity, making it especially relevant to how initiatives are carried out.

Structural changes are most effective when paired with symbolic shifts that reinforce their meaning (Reinholz & Apkarian, 2018), highlighting both the relationship between these frames and the importance of symbols. For example, if different sections of Calculus 1 provide varied student experiences and content, a structural adaptation could be creating a course coordinator role or requiring a common syllabus. However, without shared value for horizontal integration among faculty and MGTAs, structural changes are less likely to be impactful.

Symbols Shared beliefs, values, and priorities guiding MGTA work	**Structures** Formal division of labor, role definitions, and responsibilities for MGTA work	**People** Promotion, tenure, training, mentoring, work plans	**Power** Expertise, authority, agenda-setting

Figure 1. Framework: Four frames (MGTA Context). Adapted from Reinholz & Apkarian (2018)

Methodology

As part of the broader ELITE PD program focused on MGTA professional development in active learning, equity, and inclusion, we conducted interviews with departmental leaders at two participating institutions: Beta University and Gamma University. Both of these institutions are large public universities that offer graduate programs (i.e., master's and PhD) in mathematics or closely related fields and employ graduate students in teaching roles within the mathematics department, with roles ranging from recitation leaders to instructors of record. Additionally, at the time of data collection, Beta was designated as a Minority-Serving Institution.

Between Beta and Gamma, a total of 9 departmental leaders were interviewed. These individuals were selected based on their roles in shaping MGTA teaching and professional development. For example, department chairs and associate chairs often oversaw TA contracts and assignments, while course coordinators and PD providers regularly worked with MGTAs, often serving as their first point of contact for teaching support. At Gamma, this included the department chair, associate chair, graduate program chair, and the PD provider; while at Beta this included the department chair, PhD director, two course coordinators, and the PD provider, who was also involved in course coordination.

The nine interviews were semi-structured, 60-minute interviews and conducted virtually via Zoom during the 2021-2022 academic year. The interview protocol included questions about MGTAs' roles, teaching practices, and support structures. Participants were also asked to reflect on their beliefs about active learning and equitable teaching, departmental values related to these approaches, and any specific indicators or metrics of their institution's or department's commitment to diversity, equity, and inclusion.

These interview transcripts were analyzed using Bolman and Deal's (2017) four-frame model as an interpretive lens. We initially coded each transcript using Bolman and Deal's (2017) four frames as discrete categories but found this approach limiting because the frames often overlapped (e.g., PD could simultaneously appear as a structural requirement and as a human resource aimed at supporting people). We therefore shifted to an inductive analysis (Miles & Huberman, 1994) that identified five categories that reflect cross-cutting areas that shape departmental culture: (1) agenda setting and authority, (2) autonomy and constraints, (3)

expertise and resources, (4) collaboration and alliances, and (5) institutional and departmental initiatives.

We used the five categories from the broader study's analysis to organize and interpret patterns in how departments understood and engaged with MGTAs, applying Bolman and Deal's (2017) four frames as an analytical lens. To ensure consistency, we developed a structured codebook that was co-constructed and negotiated between the two coders to define how each frame could be expressed within each category. Table 1 shows examples from the data illustrating the *autonomy and constraints* category. For example, when examined through the *symbolic* frame, an interviewee expressing desire to allow MGTAs to choose if they wanted to implement active learning in their classroom was interpreted as giving *autonomy* to MGTAs. In contrast, MGTAs roles as *structured* in a department may be *constrained* in ways that impact their pedagogical choices.

Table 1. Interpreting the Autonomy and Constraints Category in Terms of Symbols and Structures

Frame	Codebook Definition	Example
Structural	Formal policies and role definitions that set limits on individual decision-making authority.	"MTGAs usually aren't involved much above the 100 level… almost everything they're doing is with our 100 level....In those courses… there's usually a coordinator, it's pretty structured. We're usually writing every exam…sometimes even common worksheets." [constraint]
Symbolic	Cultural values and expectations that shape perceptions of autonomy and conformity.	"So, is it active learning? Is it lecturing? That is really up to the instructor or the TA. So if a TA is teaching their own section of a course, they can lecture if they want to. They can do active learning if they want to." [autonomy]

For this manuscript, we focus on the structural and symbolic frames, which Reinholz and Apkarian (2018) highlight as central to sustainable instructional change. This focus allowed us to examine how formal systems and values shaped MGTA roles, expectations, and support across Beta and Gamma.

Results

Our results highlight how Beta and Gamma's departmental contexts shape the development and involvement of MGTAs, revealing both alignments and misalignments between structures and symbols that can respectively support or constrain change associated with MGTA PD.

Agenda-Setting and Authority

This category captures how departmental agendas are established, prioritized, and enacted. At Beta, agenda-setting is most visible in the structure of College Algebra and Precalculus

courses. Both of these courses, which are frequently taught by MGTAs, were coordinated. These courses are coordinated by designated faculty pairs who oversee grading, office hours, and course implementation, reflecting a structural–symbolic alignment aimed at improving DFW rates in introductory mathematics. The Ph.D. Director described the coordinators as administrators, with the coordinated courses being "very coordinated" with exam, course content, and schedules determined by the coordinators. Notably, the College Algebra course at Beta seemed prioritized by the department with College Algebra described as having active learning directly embedded in the course structure using worksheets, unlike Precalculus.

At Gamma, fewer such structures exist. Course structure and MGTA responsibilities – such as facilitating recitations – are largely determined by individual instructors of record. Some coordination does occur, with College Algebra and Calculus described as the "most tightly coordinated." As one department leader explained: "the instructors… together with the undergrad committee… sit down and say, 'Okay, this is the text …and these are the topics we're gonna cover.'" This informal coordination was also enforced via course assignments: "the active learning occurs in classes that are coordinated, so if the person is not going to be participating in that coordination, then you'd say, 'Well, you're not going to teach this class.'"

Across both institutions, MGTAs are excluded from formal agenda setting. As one departmental leader put it: "TAs… tend to be… roped into… very late in the game." However, Beta provides more structured oversight and embeds its value for active learning into the formal design of College Algebra, a course MGTAs often teach as instructors of record.

Autonomy and Constraints

This category addresses how individual autonomy is shaped and limited by departmental policies and expectations. At Beta, MGTA autonomy varies by course and supervising instructor. Active learning is encouraged and embedded into introductory courses such as College Algebra, though no specific teaching methods are formally mandated. MGTAs teaching College Algebra are required to attend regular meetings to discuss their instruction. Symbolically, instructor-of-record autonomy is valued, as reflected in the lack of formal active learning requirements, yet coordination structures exist to facilitate active learning. As the course coordinator stated: "So there's been a push at the 100 and the 200 level…to do active learning...College Algebra is the one that does the most [activities]."

At Gamma, MGTA autonomy depends on experience and the preferences of the instructor of record. First-year MGTAs are required to attend an orientation and teaching seminar. Some lead their own sections, while others follow prescribed recitation plans. Symbolically, autonomy is framed as a professional right, which can translate into freedom without clear guidance. Active learning is valued in principle but interpreted broadly, resulting in no shared vision for teaching. As one leader stated: "what active learning should do is it should activate and engage a learner… And I include a great mesmerizing lecture."; whereas other leaders described active learning as activities that engaged learners like exit tickets and in-class problem-solving. Overall, Beta offers clearer expectations and more structural support for active learning, while Gamma provides fewer formal guidelines and lacks a cohesive symbolic vision for active learning.

Expertise and Resources

Expertise and Resources highlight how individual expertise is recognized, developed, and leveraged within a department. At Beta, MGTAs sometimes receive feedback or are observed based on content and student engagement, though course structures vary by course and instructor. For instance, coordinated courses offer more support, with MGTAs teaching College

Algebra attending weekly coordination meetings to discuss upcoming content and address teaching challenges. Coordinators also run workshops for faculty and MGTAs who teach College Algebra. However, recent changes to coordination roles have created uncertainty about who is responsible for MGTA support. Symbolically, course structures emphasize introductory courses and reflect valuing MGTA growth through workshops, meetings, and observations, though equity is not a focus: "equity, diversity and inclusion doesn't show up on the observation protocol anymore. It's mostly focused on the content, the teacher, there's a little bit about student engagement."

In contrast, Gamma offers structured teaching support through a weekly seminar and teaching workshops emphasizing inclusivity, equity, and professional skills, though these are required only for first-year MGTAs. Symbolically, MGTAs are positioned as professionals-in-training, with development tied to both classroom competence and long-term career preparation: "I'd like them, when they graduate, to feel that they have… been given the opportunity… to grow professionally in ways… meaningful… in terms of their career." While Gamma's teaching support is more formalized and integrates instructional development, its influence beyond the teaching seminar is unclear due to inconsistent symbolic support for active learning and the absence of stronger coordination structures. Beta's support, by contrast, is more localized and tied to coordination structures.

Collaboration and Alliances

Collaboration and Alliances identifies formal and informal partnerships that support shared goals, shaped by collaborative structures and community-building practices. At Beta, MGTAs–particularly those teaching College Algebra–participate in weekly coordination meetings for formal support and share office spaces which allows for additional opportunities for informal exchange. Leaders described the weekly meetings as vital for discussing teaching and fostering interaction between MGTAs and faculty: "they're allowed and encouraged to interact with the teaching faculty… that interaction is encouraged."

At Gamma, collaboration occurs through PD activities, shared spaces, and faculty-led groups. Participation is largely voluntary and not structurally supported yet was reported as functioning well: "I would say my estimation would be that TAs talk to each other about 85 to 90% of the time, and that they reach out to people outside of their TA friendships." Overall, Beta offers more consistent collaboration opportunities within coordinated courses but limited structural support outside them, while Gamma's informal community appears strong despite fewer structural supports, especially beyond the first year.

Institutional and Departmental Initiatives

This category encompasses reforms that reflect and advance broader priorities through structural change and symbolic commitments, particularly around diversity, equity, and inclusion (DEI) and active learning in mathematics. At Beta, DEI and teaching-related initiatives (e.g., workshops, surveys, infrastructure investments) are largely faculty-focused, with MGTAs inconsistently included. Symbolic gestures–such as MGTA teaching awards–exist, but DEI in particular has been absent from MGTA teaching support and classroom observations (e.g., equity is missing from feedback forms), reflecting symbolic tensions in the department: "our chair actually pushed back on [our DEI committee]... his belief is that it is everybody's responsibility to attend to those things… not just a single committee." Departmental discussions are underway regarding DEI-focused PD for MGTAs and more inquiry-based approaches to teaching precalculus.

At Gamma, many ongoing DEI initiatives were described, including workshops, DEI committees, and related communications. Some leaders reported encouraging MGTAs to participate: "But I see the leaders, you know, above me. I see [them encourage MGTAs], definitely." Hence, Gamma integrates graduate students into some DEI initiatives both symbolically and structurally more than Beta. However, the extent to which these initiatives influence everyday MGTA instructional practice was unclear.

Discussion

Across the five categories we identified, Beta demonstrated stronger structural-symbolic alignment in its introductory courses, with MGTAs playing a central role in advancing active learning and equity goals aimed at reducing DFW rates. Structures such as coordination meetings, embedded activities in College Algebra, and PD through observations, workshops, and teaching awards position the department to build on emerging initiatives like inquiry-based precalculus and the ELITE PD program. The teaching support that Gamma provides reflects stronger symbolic commitments to DEI and active learning, yet these values are not consistently reinforced through departmental structures. This is evidenced by Gamma's coordination structures, which were often described as informal or unofficial, and a lack of a shared vision for active learning within the department. Without that alignment, Gamma's ability to sustain MGTA-focused initiatives may be less stable than Beta's. However, importantly, this study provides only a snapshot of these two institutions, which continue to evolve through interventions, faculty turnover, funding, and policy shifts. Future work will revisit these sites after ELITE PD's implementation and extend analysis to include the other two frames (power, people) frames.

Even within this snapshot, however, these cases suggest that departments like Beta may be more likely to sustain initiatives when their formal structures (coordination, PD, incentives) reinforce their symbolic commitments (e.g., equity, active learning). This resonates with the findings from Reinholz et al. (2019) where "insufficient changes to culture" (p. 1140), such as the lack of structures, can impact sustainability. Departments with concrete structures may be better positioned to institutionalize new practices like the ELITE PD program. Symbolic commitments alone (e.g., valuing DEI, active learning) will likely not be enough for long-term persistence of initiatives without accompanying structural supports. Where structures are absent or fragmented, even strong symbolic endorsements risk remaining rhetorical, leaving initiatives perhaps less likely to be sustained or fully realized. Finally, MGTAs can play a critical role advancing departmental goals around active learning, equity, and inclusivity, but their influence may depend on whether both structures and symbols support their PD and the enactment of those values. Department leaders are well-positioned to make a positive impact on teaching culture and support MGTAs by making decisions and setting policies that bring structures and values into alignment (Couch et al., 2024).

Lastly, this study also contributes methodologically by highlighting the value of examining departmental culture through each of the four frames rather than using the frames as discrete codes, which facilitated our understanding of how individual aspects of the departments contributed to the culture around MGTA development at Beta and Gamma University through these different lenses.

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"You Can't Unring the Bell:" Examining Relationships Between Objects, Tools, and Outcomes From a NIC Working Toward Critical Transformation in Undergraduate Mathematics

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Developing reflective practitioners can lead to and sustain new practices in undergraduate mathematics. In this study, we used Cultural Historical Activity Theory (CHAT) to examine relationships between objects, tools, and outcomes from a Networked Improvement Community (NIC) seeking to critically transform introductory mathematics. By examining interviews and observer field notes, we found three pathways that connected NIC members' individual and collective goals (objects), NIC activities and resources (tools), and NIC members' perspectives on teaching and students (outcomes). We found that sometimes objects, mediated by tools, led to outcomes, but not always. Although the transformation that NIC members envisioned was not realized, the experience shaped their thinking about teaching and students in impactful ways such that "You can't unring the bell."

Keywords: Institutional change, reflective practitioners, Cultural Historical Activity Theory, introductory mathematics

Despite evidence that traditional teaching practices do not serve a diversity of students, practices in undergraduate mathematics remain largely unchanged (Seymour, et al., 2019, Stains et al., 2018) and disproportionately impact the academic success and persistence of students belonging to historically underrepresented groups (e.g., racial/ethnic minorities, women, first generation students) (e.g., Olson & Riordan, 2012; Theobald et al., 2020, Totonchi, et al., 2023). Research shows that one way to effectively cultivate change is through engaging mathematics faculty in experiences that develop them as reflective practitioners. This grass-roots change strategy encourages and supports faculty in trying new instructional practices and sustains their commitment to change (Borrego & Henderson, 2014). Communities of faculty, administrators, and students can be particularly effective in prompting such changes. One structure for forming such teams is a Networked Improvement Community (NICs; Penuel et al., 2020). NICs engage in multiple cycles of inquiry and improvement (e.g., observe-plan-act-reflect cycles) to collect and analyze data and make data-informed decisions (Martin et al., 2020).

The project described herein involved members of a NIC at one institution working together toward critically transforming introductory mathematics. The purpose of this qualitative study was to understand how the original objectives (objects) of the NIC compared to its outcomes as mediated by activities and resources (tools). This NIC was part of the ACT UP Math grant, a research practice partnership that engaged institution-based NICs comprised of mathematics personnel (e.g., faculty, instructors, administrators, and students) to explore data on student performance and experiences in introductory-level mathematics courses in order to make critical transformations within their mathematics departments. The goal of ACT UP Math was to help mathematics departments achieve critical transformations in their undergraduate programs, specifically by increasing members' awareness of how historically marginalized students experience the dominant culture of mathematics.

Theoretical Framework

Drawing primarily from participant reflections and researcher field notes, we use Cultural Historical Activity Theory (CHAT; Engeström, 1987; Vygotsky, 1978) to examine relationships between the objects, tools, and outcomes of the NIC. CHAT is an anthropological framework used to explain the object-oriented activity of an individual or group of individuals (Engeström, 1987; Vygotsky, 1978). The CHAT framework has multiple iterations, for the purpose of this study we focus on the second generation CHAT framework (Engeström, 1987) as it highlights the importance of collective activity, including the social, cultural, and historical components that shape said activity. In this iteration of the CHAT framework, an activity system is conceptualized as involving seven components: subject, tools, object, outcomes, rules, community, and division of labor. A subject (i.e., an individual) uses tools to make progress toward an object (i.e., a goal). Through the successful use of mediating tools, objects are turned into outcomes of the activity system (Engeström & Sannino, 2017), but an individual's actions are shaped by the broader community in which they're situated, and the rules (or norms), and division of labor within that community which collectively govern how that community operates.

The CHAT framework has been applied to study how activity systems operate as they work toward transformative action. Reinholz et al., (2017) examined the efforts of departmental action teams (DAT) within university STEM departments and compared their objects, tools, and outcomes to a faculty learning community via an activity theory lens. Hancock and Miller (2018) focused on an activity system of K-12 teacher candidates within the Department of Special Education. The object of their activity system was learning to teach with inclusive practices, and alongside the governing rules, surrounding community, and division of labor, the tools collectively mediated the outcome of preparing teacher candidates for inclusive education.

In this study, the community of the activity system consists of the Tau NIC, the subjects are the NIC members, the objects are the members' collective and individual goals for their work, the tools are the resources, activities, and opportunities throughout the experience, the rules are the explicit and implicit norms determining how NIC members interact with one another, and the outcomes are the results of the NIC's efforts. We address the research question: *What is the relationship between the objects, tools, and outcomes of a two-year networked improvement community that sought to critically transform undergraduate mathematics?*

Methodology

Tau University is a public, research-oriented university in the Southeast United States. The NIC included 11 mathematics department members, nine instructors and three (not mutually exclusive) administrators. In the following paragraphs, we provide more context about the Tau NIC and describe how Subjects (Tau NIC members) made progress toward an Object (their collective and individual goals for the NIC) through the use of tools (resources, activities, and opportunities within the NIC).

Our study methodology is critical ethnography (Palmer & Caldas, 2015). Within this research-practice partnership (Coburn et al., 2021), we were embedded within the NIC community as ethnographic researchers and as members who regularly attended NIC meetings (e.g., Hagman et al., 2024). The primary source of data for this study is individual interviews conducted with NIC members during Jan-Feb 2025, which occurred at the conclusion of their (formal) work as a NIC within the ACT UP Math project. We triangulated quotes from NIC members with a secondary data source, observation field notes that two researchers completed during each monthly NIC meeting. Since the NIC operated as an activity system, the observation

notes allowed us to look up specific tools and events that NIC members mentioned in interviews and create a cohesive story of the NIC's journey. All names are pseudonyms; most NIC members opted for the research team to select a pseudonym for them.

We operationalized the second generation of CHAT as an analytical framework (Engeström & Sannino, 2017) to describe the work of the Tau NIC as an activity system. Our data analysis consisted of several iterations of analyses for each data source (e.g., interviews, observation field notes), similar to narrative coding (Saldaña, 2016). In previous work, we described the community of the Tau NIC and focused on their early efforts during the first year, as described by the NIC's three co-leaders (Bennett et al., 2025). Our prior analysis focused on the implicit rules that existed in the NIC (e.g., collegial, honest discussions during meetings), which led to the leaders establishing an explicit rule that deficit-oriented discourse about students, such as student lack of preparedness, was "off the table" for NIC meetings. The co-leaders shared that dwelling on topics about which they did not have agency hindered their progress as a NIC, thus, creating a rule that limited unproductive conversations helped the NIC focus on achieving their goals. The analysis of this study continues through the end of the project and captures NIC members' reflections on their overall experience.

Engeström & Sannino (2010) elaborated on how to describe the object of an activity system, emphasizing that "one needs to distinguish between the generalized object of the historically evolving activity system and the specific object as it appears to a particular subject, at a given moment, in a given action" (p. 6). Similarly, in our analysis of individual member interviews within the collective NIC experience: we found that individual-level objects did not always match the collective NIC goals, similar to the idea that "the generalized object is connected to societal meaning, [while] the specific object is connected to personal sense" (p. 6). Thus, we analyzed parallel planes within the NIC activity system (Grimalt-Álvaro, & Ametller, 2021).

Grimalt-Álvaro and Ametller (2021) describe how each element of the second generation of CHAT can be viewed through different planes, or grains of focus. For instance, the Tau NIC is an activity system situated within a mathematics department; thus, the subject could be defined as an individual member or the entire NIC (or perhaps, the entire department, if the department was working toward similar goals). Selecting a particular plane of focus determines other components of the system, that is, selecting the individual plane dictated that objects correspond to individual NIC members' goals. However, as Grimalt-Álvaro and Ametller (2021) note, the planes cannot be held fully separate, as they are still part of the entire activity system. Thus, we analyzed the NIC activity system along the individual-level plane, but recognized that NIC members were situated within the NIC community and had some shared objects, tools, and outcomes. The relationship between these components are described in our findings.

Findings

In this section, we share characteristics of the Tau NIC as an activity system. We focused on the individual plane of analysis to identify Objects (goals), Tools, and Outcomes of NIC members and then organized them into salient themes they expressed in common. Then, we explore the relationship between these components by providing rich examples of the connections between them.

Objects

When the Tau NIC formed, their stated, collective goal (object) was to achieve critical transformations in their undergraduate mathematics program. In reality, "critical transformations" were interpreted and operationalized in various ways; namely, each NIC

member held a different idea about the goal of the NIC. NIC members operationalized "critical transformations" as a "massive, structural change" that was "truly innovative" and gained "buy-in from the whole department." Beyond this notion of grandiosity and overall improvement, there was not a tangible object. In this way, the specific object varied as it related to a specific subject (each NIC member) at a given time. Here, we share individual-level objects that at least three NIC members held in common.

1. Change course content and policies and develop shared resources for teaching,
2. Help students learn and develop metacognitive skills,
3. Challenge perspectives about students and "change hearts and minds"
4. Learn how to collect and analyze student data, and
5. Develop relationships and collaborations with colleagues.

In a later section, we describe in more detail the challenges, barriers, and frustrations that NIC members faced as they worked toward critical transformations.

Tools

Faculty described several tools that influenced them or their experiences in the NIC. We organized these tools into four primary themes:

1. opportunities to reconsider traditional approaches to undergraduate mathematics teaching and learning and the impact on students overall and on historically marginalized populations in particular,
2. reflection exercises to surface attention to cultural biases and their own teaching practices,
3. qualitative and quantitative data collected from their students, and
4. resources to support student learning.

The NIC read and discussed articles on traditional approaches to mathematics and the impact on students' sense of belonging, motivation, and academic outcomes. They took cultural bias assessments and viewed recordings of themselves teaching class with attention to their interactions with students and how different students were engaging in the learning experience. Qualitative data about students' experiences in NIC faculty's courses and suggested improvements were collected from students through interviews, focus groups, small-group discussions, and end-of-course surveys. NIC faculty also collected qualitative and quantitative data by surveying their students at midterm and toward the end of the semester about their experiences during class (e.g., opportunities to ask questions) and outside of class (e.g., working, care-giving), and their academic engagement, self-efficacy, and sense of belonging. These data were compared to students' grade predictions and performance on assessments at mid-term and at the end of the course. The NIC discussed patterns and trends in the data and reflected on what they expected and what the data revealed. Then, NIC faculty discussed ways to address challenges that students were having in mathematics courses and shared ideas related to cultivating a growth mindset and teaching students how to learn. Through the use and discussion of these tools, faculty were regularly reflecting on their own beliefs and practices and students' learning experiences, identifying instructional changes they could make to better support their students, and then supporting each other as they planned, implemented, and revised their practices.

Outcomes

Individual NIC members reflected upon their journey within the NIC, and in so doing shared multiple outcomes from their experience. The majority of outcomes related to an individual or

personal outcome rather than a collective outcome; for instance some members felt that their own awareness of student perspectives increased. However, many of these individual outcomes were shared in common by (at least three) members of the NIC. Here, we share the outcomes which were most salient:

1. an increased knowledge and appreciation of student perspectives based on data,
2. a shifting perspective on one's relationship with students,
3. increased critical consciousness,
4. closer relationships with colleagues, and
5. an increased willingness to experiment and adapt instructional practices in their own classroom.

For example, some NIC members described having a greater understanding of the factors impacting their personal relationships with their students. One NIC member, Ethan shared how being involved in the NIC increased his awareness that "different people have different needs," and also emphasized that this knowledge made interactions with students more complicated, or a "mixed bag".

Relationship between Objects, Tools, and Outcomes

Recall in the CHAT framework, tools are intended to mediate from the activity system's object to the outcome. In our analysis of NIC members' objects, tools, and outcomes along the individual plane, we found the common themes listed above for each CHAT component. To explore the connections and relationship between the Tau NIC's objects, tools, and outcomes as an activity system, we now look across components and focus on specific examples from NIC members. We observed that the three CHAT components were evident in different combinations. Sometimes, tools clearly mediated the path from an object to an outcome. However, there were also times when specific tools led to an outcome, but an initial object was not explicitly stated, or other times, objects were realized as different outcomes due to a lack of mediating tools. We organized the examples of the relationship below to illustrate these relationships as reflected in the data. Figure 1 illustrates three different pathways we observed that connect CHAT components within the Tau NIC activity system. Each pathway is indicated by a different color.

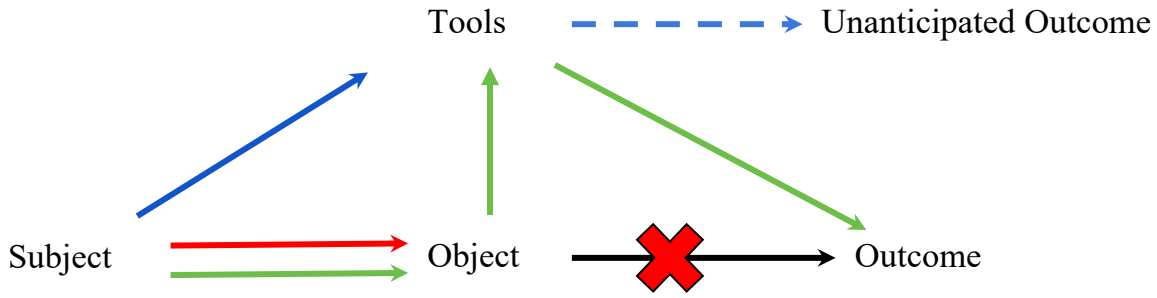

Figure 1. Pathways of connections between CHAT components within the Tau NIC activity system

Objects, mediated by tools, led to outcomes. For some members, all CHAT components aligned in the ideal relationship that the framework emphasizes. Kayla's experience illustrates this connection: her goal of supporting students' metacognitive development (Object #2) shaped how she approached student data (Tool #3). She explained that she would need to have a goal in mind before looking at the data, so that she could better focus her analysis. By organizing and interpreting student data through the lens of fostering students' metacognitive skills, she identified concrete strategies for course development (Outcomes #1 & #5), later noting, "I have

opportunities to do instructor meetings with my college algebra instructors [and metacognitive skills] will be the focus of the meetings, because the data said, this is what our students need. So to me, that's the biggest takeaway." Kayla identified a goal and reached her desired outcome through the mediation of proper tools provided by the NIC. This relationship between subject, object, tools, and outcome is indicated by the green pathway in Figure 1.

Specific tools led to a [new] outcome. In other cases, tools were the most salient link between the NIC experience and outcomes, but there was no explicitly stated object at the outset. Jordan, for instance, described how metacognitive discussion prompts (Tool #4) and student surveys (Tool #3) provided through the NIC informed his classroom practices (Outcome #5): "This allowed me to think about ways in which I can foster better cognition in my classroom." Similarly, Adele emphasized how the collection of student data (Tool #3) prompted a shift in her instructional priorities (Outcomes #1 & 5) to focus on teaching students how to learn: "After seeing the numbers… it made me shift my attention in other directions instead to work more on how to help students with their study skills and their learning skills and all that." This relationship seemingly excluded the object component of CHAT, which led to an unanticipated or secondary outcome. We indicated this relationship between subject, tools, and outcome (2) by the blue pathway in Figure 1.

Lack of [the right] tools led to a disconnect between object and outcome. Ethan joined the Tau NIC with a goal of collecting and analyzing student data (Object #4) to better understand the relationship between student identity and learning, specifically, how a student's identity might impact their learning of mathematics. However, the NIC struggled to find the relevant tools to address this goal; they collected survey data about student identity (e.g., demographics, learning disabilities; related to Tool #3) but were unable to come to any conclusions about their analysis that were clearly connected to Object #4. Ultimately, Ethan reflected that the outcome of his experience in the NIC was a personal change in himself (Outcome #3):

> You ring a bell, and you can't unring the bell. And so I think [the NIC] has changed the way that I think about things. […] I think that the way that an individual's identity impacts them and everything, including their learning in a math class is– it's very complicated, and it's delicate. [...] Different people have different needs.

Ethan noticed that he became "overly aware" of his students in a more holistic way and that his increased critical consciousness would continue to influence how he interacted with students in his classes; he "can't unring the bell."

Other NIC members experienced a disconnect between their goals and outcomes. Abel shared, "when I joined the project, I thought we were going to, you know, create something that we could use in the course. And I saw that that wasn't happening right away, and there wasn't any sign of a concrete deliverable that we could get." Abel first states his goal of creating materials to use in his course (Object #1), then expresses his realization that the object would not come to fruition as an outcome. Similarly, Garrett hoped for the NIC to accomplish a "massive structural change," like new departmental policies on "using calculators and notecards" or "content changes to courses to make them more accessible to students (Object #1). He drew on initial tools, such as reading articles about other institutions enacting similar departmental changes (Tool #1) and then attempted (unsuccessfully) to get buy-in from Tau administrators. Ultimately, his object was not realized as an outcome but did lead to unexpected outcomes: "I didn't really necessarily pull off a big, huge structural change. But I think I made a significant impact to some students in my classes" (Outcome #2). The relationship between CHAT components described by Ethan's, Abel's, and Garrett's experiences is indicated by the red

pathway in Figure 1. We note that there was an attempt to use tools in these cases, but they were not used successfully to mediate a desired outcome.

Discussion

We addressed the research question: *What is the relationship between the objects, tools, and outcomes of a two-year networked improvement community that sought to critically transform undergraduate mathematics?* We investigated this relationship by first conducting analysis on the individual plane (Grimalt-Álvaro, & Ametller, 2021) to identify objects, tools, and outcomes that were meaningful to each NIC member, as well as relationships between these three components. This analysis aided us in describing how each of these three components were connected (or not) within the work of the NIC. Our analysis of pathways between components (see Figure 1) revealed three distinct relationship types: (a) a clear connection between object, tools, and outcome, (b) tools leading to an outcome (without an explicit object), and (c) a disconnect between objects and outcomes (due to a lack of the right tool mediating this relationship). To describe the third relationship, we shared examples of how objects fail to be realized as outcomes when the appropriate tools are not present. For a community aimed at achieving critical, institutional transformation, it may be imperative that the activities, resources, and opportunities available to individual members are aligned with explicitly stated objects.

We note that several of the tools used were chosen by the Tau NIC leaders or implemented due to recommendations from the larger ACT UP Math team. Additionally, Objects #3 and #4 were emphasized by the ACT UP Math project. These objects were realized as outcomes for a subset of individual NIC members, but as externally driven goals, they were not taken up by the NIC as a whole. Research on departmental action teams (DAT) to transform undergraduate STEM programs emphasize that departments are key locations for initiating sustainable change within the complex university system, but it is crucial that the "DAT's focal issue is not externally defined" (Reinholz et al., 2019, p. 66). Reinholz and colleagues argued that DATs should focus on the collaborative process of their work, suggesting that "the DAT begins with a series of activities to develop a shared vision, consensus on the focal issue, and a set of goals" (p. 66). In our past work (Hagman et al., 2024; Tremaine, 2024), we acknowledged the challenge of aligning individual goals to the collective efforts of the NIC and highlighted tensions that arose when there was a disconnect between the vision of the NIC and that of the project leadership team. In this study, we again observed this tension between competing goals. As others have recognized (e.g., Reinholz et al., 2017), there are advantageous and disadvantageous ways of including external influence on a community working toward change (e.g., via facilitators), which could be explored further in future studies.

In spite of challenges, the experience of participating in the Tau NIC helped to develop reflective practitioners. We found that through the NIC, faculty reflected on their own beliefs and practices and on students' learning experiences and outcomes, which in combination with their objects and tools, influenced outcomes impacting their perspectives on teaching and approaches to supporting students (e.g., Borrego & Henderson, 2014). For instance, Ethan's critical consciousness development, while not his initial intention, significantly impacts how he interacts with students and views them as whole, complex individuals. By first understanding the personal and intrinsically motivated goals of faculty, we can purposefully select the tools and objects of collective transformation efforts. This intentionality and alignment may result in sustainable changes and have long-lasting impacts on faculty members' teaching and views of students. We encourage those who work with mathematics department faculty toward institutional change to consider the alignment of goals, tools, and objects as they develop reflective practitioners.

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Teacher Professional Development as a Mechanism for Cultivating Competence and Professional Identity in Graduate Student Instructors' Transition to Faculty Roles

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In the past decade, teacher professional development has gained popularity as a tool for preparing graduate student instructors. Research demonstrates the immediate positive impacts of this training for both the undergraduate students the graduate students teach and the graduate students' own professional teacher identity development. However, much less is known about its long-term impact. For this study, we interviewed eleven former graduate student instructors who participated in such a training program and now have a position in higher education with teaching responsibilities. We leverage Lankveld et al.'s (2016) framework of psychological processes and contextual factors for teacher identity development to explore their sense of competence as mathematics educators as they transition into their new higher education contexts. Findings suggest that a sense of connectedness to new peers and past training experiences supports productive comparison and reflective practice that promotes identity growth in new academic environments.

Keywords: Teacher identity, graduate student instructor, teacher professional development

Introduction

Teacher professional development for graduate student instructors (GSIs) has been an important research priority for the Mathematical Association of America (MAA)'s Special Interest Group Research in Undergraduate Mathematics Education; these efforts are codified in evidence-backed resources published by the MAA (e.g. the 2018 Instructional Practices Guide or the 2020 College Mathematics Instructor Development Source) (Yee et al., 2023). The duration of these efforts provides us with a unique and vital opportunity to conduct longitudinal research on the impact teacher professional development has on GSI's as they transition into faculty roles.

We are particularly interested in this transition phase because professional development for novice instructors positions them as future change agents (Hauk & Speer, 2023) within their new institutions. Additionally, GSIs, in particular, are receptive to learning and implementing student-centered strategies (Ellis et al., 2016). As GSIs transition to new institutions, they must acclimate to a new departmental culture and navigate changes in their teacher identity.

Theoretical Perspective

Developing a professional identity is an essential process for novice mathematics educators as they become members of the professoriate. Professional identity – which shapes an individual's beliefs, values, and motives as they navigate membership of their community – is constituted and reconstituted by their experiences in the profession (Beisiegel & Simmt, 2012). This process is accelerated with the transition from graduate school to faculty roles (Colbeck, 2008). Lankveld et al.'s (2016) systematic review of teacher identity literature identified four contextual factors (interaction with students, staff development activities, direct work environment, and the wider context of higher education) and five psychological processes (a

sense of appreciation for teaching, a sense of connectedness to other educators, a sense of competence as an educator, a sense of commitment to educate the next generation, and the ability to imagine their future teaching career trajectory) that impact the development and maintenance of teacher identity in higher education. For this report, we will focus on a sense of competence and how it is related to the other strengthening and constraining factors.

In Lankveld et al.'s framework, competence represents an educator's feeling of being well prepared/equipped to teach courses assigned, as well as their feeling of being a valued contributor, collaborator, or resource to the instructional mission of their department. Competence also includes feelings of preparedness for instructional leadership roles, such as directing a tutoring center or volunteering for a textbook adoption taskforce. Competence captures a teacher's overall sense of confidence in what they "bring to the table" as an educator.

Within our study, staff development activities are drawn from the teacher professional development program, *Promoting Success in Undergraduate Mathematics through Graduate Teaching Assistant Training (PSUM-GTT)*, which was implemented at three large, public universities in the United States between 2015 and 2022. Each location adapted the five program components to fit the constraints of their local context. The development program spanned three to four years and incorporated a pedagogical seminar, a Critical Issues in Undergraduate STEM Education workshop series hosted by visiting researchers, active learning support from a teaching assistant coach, regular one-on-one peer mentoring (Manzanares et al., 2023; Lanius et al. 2025), and regular opportunities for written and oral reflection (Lanius et al., 2024).

Research Question

Our overarching research question is: In what ways does this multifaceted teacher professional development experience shape the teacher identity of GSIs transitioning into faculty roles? However, in this report, we narrowly focus on the question: In what ways are new faculty's sense of competence within their new institution impacted by past participation in a teacher professional development experience?

Methods

Data Collection

A graduate student on the research team conducted interviews with 11 former program participants utilizing an interview protocol designed by the training program's research team. The protocol aimed at soliciting participants' historical perspectives on the components of the professional development program (i.e., mentorship, teaching, program didactics), their insight into their experiences of their current position, and the ways that previously engaging with the program impacted their recent experiences. The interview protocol underwent one round of revision after the first interview to edit questions for clarity and interview length. Each interview took approximately 1.5 hours. Interviews were first transcribed with Otter.ai, with the graduate student and two research team members editing the transcripts for accuracy and anonymization.

Participants

The eleven interview participants had transitioned into a career with a teaching component after participating in the teacher training program while earning their graduate degree from one of the three universities. Four participants had undergraduate teaching experience when the teacher training program began, and therefore served only as mentors. Six participants began the training program as mentees and later became mentors. One participant was a mentee, a mentor

and a teaching assistant coach. One participant was employed where they earned their graduate degree, one was employed at the institution where they earned their undergraduate degree, and the remaining were employed at four-year institutions they had not attended. Two participants had been in a faculty position with teaching responsibilities for over two years, three were in their second year as faculty, five had completed one semester as faculty at their new institution, and one was a post-doctoral researcher with a one-course per semester teaching load.

Analysis

As the interviews were quite extensive, for the purposes of this report, the research team narrowed the corpus to portions of the interview transcripts that were specifically addressing the faculty members experiences at their new institution. The research team used a priori coding (Saldaña, 2013) with a codebook we developed from Lankveld et al.'s framework. In addition to marking codes where they applied in the text, each coder took analytic memos to capture how these constructs uniquely manifested and how the codes related to one another. For example, a passage of text might be marked "staff development activities → competence". As the purpose of this project is exploratory and does not aim to make any frequency claims, we did not conduct interrater reliability statistical analyses. Instead, we had open discussions for consensus building where coders could share their independent findings and receive feedback.

Results

Because the constructs within the teacher identity framework interact in unique ways for each junior faculty member, we discuss results at the individual level before generalizing.

Alex

Alex reflected that as a novice GSI they only lectured, which mirrored the lecture-heavy learning experiences they had as an undergraduate. With support of the training program, they developed comfort with utilizing engagement techniques like group work and think-pair-share, and came to appreciate the value of active learning strategies. In their new faculty role, they conveyed a continued sense of competence in their teaching practices and expressed a curiosity to learn more teaching strategies (sparked in particular by an exposure to alternative grading in their new department). Although they are confident in their instructional practices and enjoy discussing teaching with others, they are careful not to present their approach as "*my way is the best*". A sense of connectedness, to both the GSI training program and to their new colleagues, is key to their growing sense of competence because it allows them to reflect on their teaching and gain insight by observing their peers. In the language of the teacher identity framework, Alex's professional identity and competence is heavily influenced by the skills they developed in the training program and a sense of connectedness derived from participating in the program's community of practice and engaging in teaching discourse with new department members.

Bojing

For Bojing, the sense of connectedness they developed while a GSI in the training program supported their sense of competence, particularly through the process of observing a peer teach and also being observed by a peer. They reflect that their teaching experiences as a GSI have helped them to thrive in their new faculty position and to be less stressed in comparison to others at their institution. Bojing is in an interesting position, now serving as a faculty member in the department where they earned their undergraduate degree. They report struggling to be seen as a

colleague with teaching knowledge by some people in their department and instead are seen as a former student. This tension is reflected in how they engage in teaching-related discourse with new department members: "*We talk about teaching usually. We're pretty open about it. I'm at more of a teaching university. I don't know if I'd say I'm a resource per se, but sometimes I give out ideas or suggestions.*" In the language of the teacher identity framework, Bojing's professional identity and competence is positively influenced by a sense of connectedness to others, which allows productive comparison, and is negatively mediated by the direct work environment which introduces a struggle to be perceived in a new way by department members.

Carmen

For Carmen, the self-reflection practices they developed while a GSI in the training program have directly influenced their sense of competence as a teacher. While a GSI, this reflection was facilitated by observing another GSI teach. As a faculty member, their connection to students in a large lecture setting has inspired further reflection on how to improve their teaching through an equity lens. Carmen shares that they have a strong teacher identity, explaining "*I think I'm a better teacher than researcher.*" Their sense of connectedness to other educators in their department has changed since they were a GSI. As a novice GSI, when they felt like they didn't know something, they were reluctant to find support. The training program was integral in developing their competence as an effective educator and they recognize they will always be learning. As a faculty member, they don't mind being the new person, and "*feeling dumb*". They are comfortable connecting with people in their new context and addressing problems early. They reflect, "*…you're supposed to have all the answers, quote, unquote, because you're now the doctor, faculty member.…But if you've been in the situation, you know, you're never above it. And there's always going to be something within every situation, you could do better.*"

Domingo

For Domingo, participating in the training program gave them the confidence to be an instructor. They speak of coming into their new context as a faculty member with a "*collaborative mindset*", which reflects a readiness to connect with other educators. Comparing their training to other new faculty further enhanced their sense of competence, particularly among colleagues who had no prior experience as an instructor. In their new context, they shared experiences of refining existing department resources to share and incorporating inclusive teaching practices and active learning strategies and reporting that it worked well.

Edric

For Edric, being given teaching responsibilities as a graduate student was intimidating because they felt a duty to try their best to do well. Their current feelings of competence are derived from those early teaching experiences, especially as they compare their training and experiences to those of graduate students at their current institution. As a faculty member, they do not see themself as a resource within their department, but do frequently talk to others about teaching. In terms of their teacher identity, they describe themself as a "*new teacher*", as they are struggling with classroom time management and teaching large lectures. They report a drive to try new approaches for their new context, like incorporating iClickers for large lectures.

Fadi

As a GSI, Fadi quickly employed the things that they learned in the training program and talked to others about them. This participant's sense of competence directly feeds into their sense of connectedness and drive to engage with others around teaching. The converse was also true, with them sharing an experience where they felt less connected and consequently felt less competent as an educator. Fadi felt like a "*spokesperson*" for the goals of the training program and has brought this forward into the future, still trying to advocate for the use of active learning strategies in the classroom. Fadi has a strong sense of self as an instructor, explaining "*I was confident that I could be a good teacher, I had a lot of supporting evidence, right?*" Despite being the newest hire in their department, they further share "*I definitely feel confident that I have something to contribute to these spaces, even though… I'm the newest hire in the department. I still feel like I have a space and some skill set that I can share with others*".

Gerard

Before participating in the GSI training program, Gerard had a couple of years of teaching experience and recognized that they weren't growing as an educator. Through the connections they made with the community that grew around the training program, they started trying active learning techniques: "*I mean I was told by many [program] mentors how important active learning is, and I really started implementing them. And it turns out that they are. They're super important and super engaging for students.*" At their new institution, they report that teaching is very different than at a big university and that they're doing much more active learning in the classroom and are exploring developing new skills, like implementing inquiry-based learning into the classroom. Fostering a sense of connectedness remains important for Gerard. For example, when teaching statistics for the first time, they found a colleague who was willing to share their materials and talk to them frequently about teaching the course. Additionally, in comparing themself to other new faculty, they feel more prepared and thus competent as an instructor, but also explain, "*I'm growing, and I'm learning always.*"

Harley

As a faculty member, Harley feels a strong sense of competence in effectively implementing group work and other activities that they learned while participating in the GSI training program. A sense of connectedness from participating in the professional development program also contributed to their growing sense of competence in the classroom. Their reflection on the training highlights the role of connectedness in developing competence:

> *"When there were seminars, that was good… I appreciate those things. And also the time that we got to share with each other because when you're teaching, it's a local thing. It's changed to university, university, university. So it's really important to share tools, strategies, and get to know about classes. So having that kind of conversation is really good. So I enjoyed that. And I got to know some ideas."*

In their new departmental context, they view themself as a resource for others, for example substitute teaching for other faculty.

Ivon

Participating in the GSI training program helped Ivon to feel knowledgeable and like they were a better teacher. The sense of connectedness they developed within the program community directly contributed to this confidence, with them reflecting "*I participated in many meetings,*

where we were sharing each other's opinion about active learning…It gave me various ideas…it comes with experience, and it comes with meeting your mentor, or attending those critical issues seminar meetings. So, I was feeling pretty confident as a teacher". In their new context they have mixed feelings about their competence as a teacher. They shared two issues that have preoccupied them: (1) communicating with students because some students have left feedback in course evaluations about their accent, and (2) making class more engaging. Engagement is of particular concern for them in the large lecture format, where they are teaching a large course for the first time and feel less prepared than their colleagues. On the other hand, they do feel like a useful resource to the teaching assistants helping with the course. Ivon continues to seek connection with others, reflecting "*there is no unique way of teaching. So it really is interesting to see how my friends are teaching. I sometimes wish to go to another person's class to see how. Although I'm teaching pretty well, sometimes students are not super excited because it's math.*"

Jasper

Jasper candidly shared that they did not want to participate in the GSI training program. However, now as a faculty member, they see how much they have learned. This appreciation is further deepened by comparing their experiences as a GSI to the teaching experience of others on the job market: "*I was way more aware of a lot of the education research and how to deliver materials and issues like that, compared to a lot of the people who I was working with…I was like, 'holy cow, like I got a great foundation from this seminar', because other people were just like, 'Oh, I've never heard of that before.'*" Although Jasper has a strong sense of competence, their new context gave them new challenges to navigate, such as multiple course preps, time management, and balancing teaching and research. In terms of connecting with new colleagues and engaging in discourse around teaching, they shared "*I feel like right now, I'm just like taking a lot of stuff in, but I feel like I will be able to participate more in the future.*"

Kyla

For Kyla, participating in the training program supported their sense of competence as an educator and gave them the language to express what they were doing in the classroom with others. As a GSI, they felt competent to generate new tactile activities and to try new teaching or grading approaches. In comparing themself to other faculty at their new college, they feel "*worlds ahead*", with particular confidence in their ability to effectively take on a large teaching load. They are currently working with others on writing an open access book. In summarizing their perspective on being a teacher, they shared "*I tend to be very modern in my thinking. I tend to be very, 'oh, let's just try this. That didn't work. Okay, let's try something else.'*"

Discussion

As GSIs, program participants were members of a community of practice that fostered a sense of connectedness. Participants frequently cited the relationships they cultivated through the program, specifically mentoring interactions and peer discussions in seminars, as central to their growth. These training experiences provided a foundation for reflective practice and normalized teaching-related dialogue within their academic communities. Many participants report continuing these practices within their new context. For a majority of our interviewees, productive comparison with new colleagues – on prior training and teaching experiences – supported self-reflection, which bolstered their sense of competence as a new faculty member and inspired them to grow their teaching practices. This process of social comparison enabled

them to locate themselves within the broader institutional teaching culture and engage more meaningfully in pedagogical discourse. For many, it was through navigating a new professional environment, making these comparisons, and engaging in reflective dialogue that participants further developed their confidence and competence in their teacher professional identity.

Reflecting on their experiences as GSIs, many participants noted the value of constructive feedback through peer teaching observations. Participants noted this feedback directly contributed to their instructional improvement, and when it was absent in the new environment, they expressed disappointment. Jasper shared, *"And so I've gotten a few helpful insights from faculty mentoring but I felt like I got a lot more from the peer mentoring."* This may indicate that new faculty mentors and department chairs should implement an observation protocol designed to provide constructive feedback, such as the red-yellow-green feedback system (Yee et al., 2022) or teaching TRIOs (Bleiler-Baxter et al. 2020).

For two participants, mathematics research was their only career goal when they entered graduate school, and participation in the teacher training inspired a shift in their career goals. All participants acknowledged that the GSI training program substantially supported their job search process because it enabled them to effectively articulate their teaching philosophies and to effectively discuss pedagogy with faculty search committees.

Conclusion

The PSUM-GTT training program functioned as a formative site for teacher identity construction, where GSIs developed not only a sense of competence but also a sense of belonging to a professional teaching community. For many participants, their strong teacher identity inspired them to pursue teaching-focused careers.

As participants transitioned into faculty roles, their prior participation in the PSUM-GTT training program continued to have a lasting and transformative impact on how participants perceived themselves as educators. The program instilled instructional confidence, fostered pedagogical curiosity, and provided language and frameworks for articulating and refining teaching practices. Further, we saw that a sense of competence as a teacher can be deepened by the experience of transitioning itself, when GSIs-turned-faculty are given a new context to situate themselves within and to compare themselves to. Many participants described entering their new departments with a stronger foundation of teaching skills and a clearer understanding of how to reflect on and improve their teaching practices compared to peers. On the other hand, comparing themselves to faculty with a much broader range of teaching experience also provided new opportunities for reflecting on where they currently are and for identifying opportunities for growth.

In conclusion, we found that a sense of competence is not a static psychological process, but rather a dynamic, socially constructed aspect of teacher identity, deeply influenced by prior training experiences, interpersonal connections, and institutional context.

Statement on Generative AI Use

Our first draft of this manuscript was over the page limit. ChatGPT-5 and Claude.ai were asked to suggest ways to condense. All suggestions were thoroughly discussed as we have an ethical responsibility to accurately represent the participants appearing in our report.

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The Role of Syllabi in Measuring Teaching: Comparing Planned and Enacted Behavior

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Syllabi contain information about an instructor's teaching and have been used as a measure of change in response to professional development. Syllabus scores have been linked to more widely used and resource intensive measures of change, including observations and surveys. We used the theory of planned behavior to examine the relationship between syllabus scores and self reported teaching practices and tested for changes in these measures in response to professional development. We found syllabi scores can approximate instructor teaching intentions, which seemed to manifest in their use of teaching practices. Instructors' teaching intentions changed in response to PD, but their teaching practices did not. These findings are consistent with the theory of planned behavior, indicating that intentions change before behaviors, which supports the use of syllabi as a measure of change that can be economically deployed and detect nascent changes in teaching practices.

Keywords: syllabus, teaching, professional development, theory of planned behavior

Non-didactic, student-centered teaching practices are well documented as effective for enhancing student learning and achievement in undergraduate STEM education (Freeman et al., 2014). Evidence also indicates that student-centered teaching practices contribute to reducing achievement gaps among students historically underrepresented in STEM (Ernst & Colthorpe, 2007; Haak et al., 2011; Eddy & Hogan, 2014; Freeman et al., 2014). Despite these benefits, widespread adoption remains limited; estimates suggest that only about 20% of STEM instructors consistently implement student-centered teaching practices in their teaching (Stains et al., 2018; Eagan, 2016). Consequently, many students have little exposure to student-centered teaching practices and miss out on their potential advantages.

Professional development (PD) has been identified as an influential strategy in increasing instructors' use of student-centered teaching practices in undergraduate STEM education (Khatri et al., 2013; Laursen et al., 2019). However, relatively few studies have examined the relationship between teaching-focused PD and the instructional practices of STEM faculty across disciplines such as math, chemistry, physics, and biology (Archie et al., 2022; Benabentos et al., 2020; Borda et al., 2020; Chasteen & Chattergoon, 2020; Manduca et al., 2017; Bathgate et al., 2019). Research in this area may be limited, at least in part, by the research methods commonly utilized in this research, including classroom observations, surveys, and interviews. These research methods are time-consuming and labor-intensive and thus make this type of research expensive to conduct. More economical research methods have the potential to encourage new research, especially given the current funding constraints in STEM education research.

Syllabi are a rich data source that can be collected easily and analyzed quickly using syllabus scoring tools such as the Measuring the Promise (MtP), a syllabus rubric that been shown to detect changes in teaching practices (Mumpower et al., 2024), including changes in response to professional development (CTE, 2017; Favre et al. 2021; Schmid et al., 2021). While the MtP authors emphasize that the "syllabus is only a proxy to actual classroom practices and student

learning and is in itself neither a measure of teaching effectiveness nor necessarily an accurate reflection of an instructor's values," it can be argued that the syllabus can reflect an instructor's intentions for teaching: it is created before an instructor performs any teaching and describes the intent of the course and what student will learn (Palmer et al, 2014a). Research has shown that syllabi and classroom observations are positively associated (Laursen & Archie, 2018), suggesting that instructor intent indeed translates to their teaching.

The temporal relationship between syllabi and teaching practices is well aligned with the Theory of Planned Behavior (TPB) in which behavioral intentions are the primary determinants of actual behavior (Ajzen, 1991). Intentions, in turn, are shaped by three key constructs: attitude, subjective norms, and perceived behavioral control. Attitude refers to an individual's overall positive or negative evaluation of a given behavior. Subjective norms represent an individual's perception of social pressure to carry out or avoid the behavior. Perceived behavioral control reflects the extent to which an individual believes they can perform the behavior and exercise control over it. Importantly, for intentions to translate into action, individuals must possess the requisite ability to perform the behavior. Consequently, perceived behavioral control is hypothesized to exert a direct influence on actual behavior, in addition to its effect on intentions.

The TPB has been employed to examine shifts in teaching and learning within STEM education (Reinholz & Andrews, 2020). Specifically, it has been used to explain mathematics and physics instructors' adoption of research-based instructional strategies after participating in professional development programs (Archie et al., 2022; Chasteen & Chattergoon, 2020). In this study, the TPB is used to explore the relationship between the syllabus and instructors' teaching practice, using the MtP syllabus rubric as an indicator of change in practice. In this study, we answered two research questions:

1. What is the relationship between instructors' behavioral intent (syllabi scores) and their behavior (use of student-centered teaching scores)?
2. Do instructors' behavioral intentions and behavior change after participating in professional development?

Methods

The study design was approved by the university's IRB and all instructors gave informed consent.

Study Population and Survey Sample

Study participants were early-career undergraduate mathematics instructors who participated in MAA Project NExT), New Experiences in Teaching, professional development program. Briefly, this full-year program provides strategies and advice for teaching and navigating the academic environment, with an emphasis on a variety of inclusive and student-centered approaches to teaching (MAA, n.d.; Higgins, 2012). Because this analysis focuses on syllabi as a measure of change, rather than on outcomes of the PD, we do not describe the program in detail.

Participants completed a pre-survey at the end of the academic term preceding participation in the PD program, a mid point survey 1 year into the PD program, and a final survey at the end of the PD program. The survey asks respondents to report on a specific course they taught and to upload the syllabus for their selected course. A total of 171 instructors completed the pre survey and provided a syllabus, 110 completed the midpoint survey and provided a syllabus, and 72 completed a final survey and provided a syllabus. To maximize statistical power, we computed a "post" syllabi score from the final time point (n = 72) or the mid time point for survey

participants who provided a syllabus at the mid time point, but not at the final time point (n = 70), for a total of 142 respondents.

Of the 171 unique respondents, 168 provided some demographic information. 48% identified as men, 46% identified as women, 5% identified as other than man or woman. 72% identified as white, 15% identified as Asian, 8% identified as Hispanic, and 3% identified as Black. 61% were assistant professors and 34% were non-tenure-track instructors or visiting professors. 41% taught at a PhD granting institution, 44% at a 4-year institution, and 15% at a Master's granting institution. 16% taught at a minority-serving institution, and the remaining 84% did not or did not know if they taught at a minority-serving institution.

Survey measures. In this study, the TPB construct of "behavior" was operationalized using self-reported teaching practices. Respondents were asked to report the percentage of time they used 11 teaching practices in a specific course over the course of the term, totaling to 100%. We then computed "behavior": as the proportion of time using three student-centered teaching practices (student group work + student presentation + class discussion). Syllabi scores were used as a measure of the TPB construct "behavioral intent", and the scoring procedure is described in the following section. Departmental expectations to use non-lecture based teaching strategies represented the "subjective norm" construct. The item was worded as "How much do colleagues in your department have formal or informal expectations for instructors' teaching? - Expectation to use a variety of techniques" and was coded on a five-point scale (1=not at all, 5=a great deal).

The TPB construct, "perceived behavioral control", was operationalized by one item, "How much effort have you made to implement active learning strategies in your teaching?" and was coded on a five-point scale (1=no effort, 5= a great deal of effort).

Instructors were asked three growth mindset questions (Dweck, 2006) that have been shown to be related to the use of student-centered teaching practices (Aragón et al., 2018). These items were reverse coded on a five-point scale (1 = strongly agree, 5 = strongly disagree).

Syllabus scoring. Prior to coding, syllabi were deidentified with respect to instructor name(s) and contact info, higher education institution, and other wording that might identify the course. Calendar year dates were removed to mask whether the syllabus was part of the pre-program or follow-up sample.

Two of the authors (SL, SW) followed an iterative process to test and modify the Making the Promise (MtP) rubric. They first applied the MtP rubric to four annotated reference syllabi from the rubric authors (CTE, 2017) and discussed findings. They then applied the MtP rubric to 15 college-level course syllabi collected in prior mathematics education studies. The authors scored several syllabi individually, then discussed their scoring rationales until they came to consensus. They revised the rubric to specify the levels of evidence for each item, to clarify rubric language, and to distinguish items so they were non-overlapping, and repeated this process several times across the set of 15 syllabi. This yielded a detailed modified rubric outlining three scoring levels (low, medium, and high) for each of 15 items. An overview of the modified MtP rubric items and scoring is provided in Figure 1.

Like the original MtP rubric, our modified rubric contained five subsections: learning goals, assessment, schedule, classroom environment, and class activities. Added language specifies the kinds of evidence that support scoring and allows the rater to reliably assess whether the strength of evidence for each item is low, medium, or high. These descriptions also support the rater in making decisions around evidence that can be inferred from the syllabus text. Such evidence can be used to support scoring an item as medium, but cannot be used to support a high score.

Following this iterative process, the two authors individually rated a new set of ten mathematics syllabi, five collected for this study. Author TW calculated interrater reliability statistics on the results, finding an interrater agreement of 65%, an intraclass correlation (ICC) of 0.73, and a Kappa of 0.67. These statistics are in acceptable ranges for interrater reliability; a Kappa of 0.618-0.8 is considered "substantial" (Cohen et al., 2000). One rater subsequently applied the rubric to the entire dataset of mathematics syllabi (n = 380). Eight syllabi were excluded due to substantial missing information (e.g., grading scheme) that made the syllabus impossible to score, or a lack of alignment with inclusion criteria (e.g., not an undergraduate course). A total of 353 syllabi were scored on each of 15 individual items. Low, medium, and high item scores were converted to 0, 1, or 2 points, respectively, then multiplied by the item weight (Figure 1) to arrive at a numerical score. We adjusted the weighting of items #9 and #14, as described in Figure 1. The total maximum score for each syllabus was 54.

Any rubric is potentially influenced by bias, and bias can be magnified by differential weighting of items. We reduced item weighting where our syllabus set tended to lack information important to evaluating the item, but did not alter the weighting of the original authors' judgments of the relative importance of different items. Overall, the original MtP rubric puts significant weight on learning goals, learning objectives, and the alignment of those goals with assessments and classroom activities, and less heavily weights the quality of classroom activities and the classroom environment. A differently weighted rubric could generate score patterns and analyses with slightly different implications.

Item	Short definition	Modifications made	Item weight	Max item score
1	Big picture learning goals and measurable smaller learning objectives are articulated or implied. Together, goals and objectives address all of Fink's six dimensions of learning (Fink, 2013).	Focused on breadth of learning goals, operationalized as number of Fink's dimensions included in the set of learning goals and objectives (rather than mere presence of learning goals).	3	6
2	Learning goals and learning objectives are separately defined and clearly articulated. Learning goals be may written broadly, but learning objectives are measurable and use specific action verbs.	Reduced overlap with item #1 with a focus on the degree to which learning goals are clearly and measurably articulated (rather than the presence of learning objectives).	3	6
3	Learning objectives are appropriately pitched to the course level, class size, position of the course within the curriculum, and characteristics of students taking the class.	Not assessed for this study, due to a lack of needed contextual information.	0	0
4	Assessments are aligned with learning goals and objectives.	Focused on alignment only; removed considerations of grade weighting.	3	6
5	Major assessments are well defined, rationalized, and involve "authentic tasks".	No changes.	2	4
6	There is frequent formative assessment and scaffolding of learning that encourages students to attend to past learning and iterate their efforts.	Added wording about scaffolding and iteration of effort.	1	2
7	Assessments are adequately paced, spreading out the student workload and giving students information in a timely manner to make decisions about enrollment.	Focuses on how well pacing information aids students in making enrollment decisions. Removed wording about scaffolding, reducing overlap with #6.	1	2
8	The grading scheme is weighted appropriately for the assessment design.	Focus is placed on grade weighting. Removed wording about goals alignment. Both changes reduce overlap with #4. Removed wording about where the grading scheme is placed in the syllabus.	1	2
9	Schedule is fully articulated and provides evidence for a logical progression of learning.	Reduced weighting from 3 to 2 because schedule information was often provided in a course LMS, not accessible to this study	2	4
10	Invites the learner and indicates a positive and collaborative class environment.	Removed wording about "big questions" addressed by course content.	2	4

11	Supports connected and lifelong learning. Answers the question: Why should you learn this for your life?	Removed considerations about grading schemes to reduce overlap with #8.	2	4
12	Signposts that the work will be challenging, and students will be supported to enable their success.	Focused on methods for supporting students through challenging work. Removed wording about rigor and expectations to reduce overlap with #1, 2, 4, and 8.	1	2
13	An organized, learning-supportive, interactive document.	No changes.	1	2
14	Classroom activities are aligned with assessments.	Removed wording about alignment with learning objectives, to reduce overlap with #4. Reduced score weighting from 3 to 2 because information needed to score this item was rarely provided.	2	4
15	Learning activities are derived from evidence-based practices.	No changes.	2	4
16	Class activities engage students in a variety of ways.	No changes.	1	2

Figure 1. Rubric items, definitions, and modifications to the original Making the Promise rubric and scoring.

Data Analysis

A structural equation model was used to answer RQ1. (AMOS 31, SPSS Inc., Chicago, IL, USA). This analysis was limited to 99 survey responses with complete data and only included one response from any individual respondent across time points. The model was calculated using maximum likelihood estimation and model fit was evaluated using two indices: the Comparative Fit Index (CFI) and the Root Mean Square Error of Approximation (RMSEA). CFI values range from 0 to 1, with values ≥ 0.95 indicating good fit (Hu & Bentler, 1999). RMSEA values ≤ 0.05 are interpreted as evidence of close fit (Browne, 1993).

RQ2 was answered using independent samples t-tests to test for differences in pre and post syllabus scores and average use of student-centered teaching practices.

Results

A path model was used to answer RQ1. The model fit indices were within acceptable ranges (CMIN = (df) 3.830 (2) $p = 0.147$, CFI = 0.99, RMSEA =0.035). As shown in Figure 2, we found positive, statistically significant standardized direct effects in all but one of the specified paths in the structural model. Among the four variables hypothesized to influence behavioral intention (syllabi scores), both perceived behavioral control (effort in implementing student-centered practices) and attitude (growth mindset items) showed comparable moderate effects, while subjective norm (departmental expectations for teaching) showed a minimal and non-significant direct effect. Behavioral intention and perceived behavioral control were both expected to directly influence behavior (use of student-centered teaching practices), with each showing a moderate positive association with behavior. The model explained moderate amounts of variability in both behavioral intention ($R^2 = 0.23$) and behavior ($R^2 = 0.31$).

T-tests used to answer RQ2 revealed statistically significant positive change (Table 1) in overall syllabi scores and in two of the five syllabus categories: learning goals and assessments. All other categories showed positive minimal effects that were not statistically significant. Instructor use of student-centered teaching practices shifted from 32% to 35% from pre to post, a minimal and not statistically significant difference.

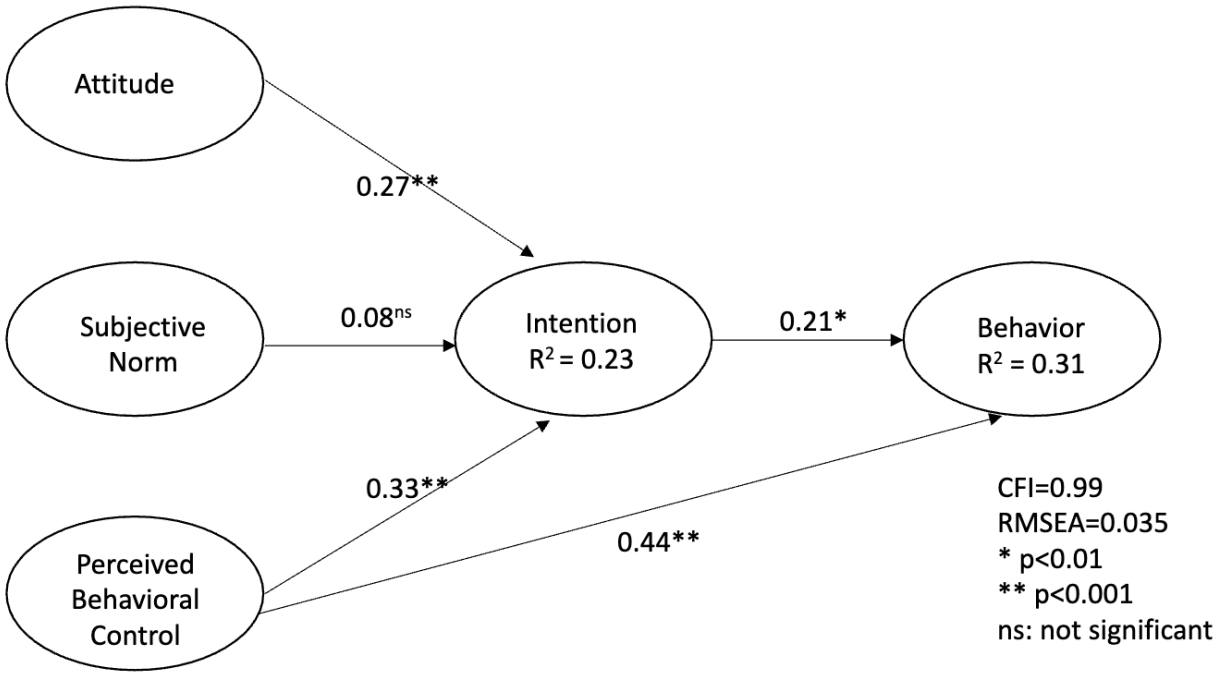

Figure 2. Structural equation model of the theory of planned behavior with standardized direct effects.

Table 1. Pre/post comparison of syllabus scores, subscores, and average percent course time using student-centered teaching practices: T-test results

Category	Pre	Post	*t*	*df*	*p*	*d*
Overall score	24.3	26.7	2.724	324	0.007	0.31
Learning goals	5.98	7.12	3.300	324	0.001	0.37
Assessments	8.73	9.58	3.100	324	0.002	0.35
Schedule	1.70	1.65	0.455	324	0.650	0.05
Environment	4.37	4.51	0.511	324	0.610	0.06
Learning activities	3.52	3.83	0.961	324	0.337	0.11
Teaching practices	32.35	34.55	0.792	324	0.429	0.09

Discussion

RQ1 results showed that syllabi scores are positively associated with instructors' use of student-centered teaching practices and are consistent with the TPB and prior research. These findings indicate that instructors' teaching is aligned with their teaching intentions as communicated in their syllabi and is consistent with prior TPB research, which has found behavioral intent to be a strong predictor of behavior (Armitage & Connor, 2001). Also consistent with prior research, we observed that behavioral intent and behavior were moderately influenced by perceived behavioral control. We found that the greater the effort of instructors to use active learning teaching practices, the more likely their syllabus and teaching were student-centered. Attitudes about students also showed a positive relationship with syllabus scores, indicating that an instructor's attitude influenced their intentions for the course and, subsequently, their teaching practices. Lastly, we found no significant association between subjective norm and behavioral intention, indicating that perceived departmental expectations for using a variety of teaching practices did not relate to instructors' syllabus scores. While this finding is unexpected and inconsistent with some prior research (McConnell et al., 2020,

Bathgate et al., 2019), it is consistent with prior applications of the TPB that have also found no meaningful association between subjective norms and behavioral intent (Armitage & Conner, 2001; Archie et al, 2022).

For RQ2, the results showed statistically significant positive change in overall syllabi scores and in two of the five syllabus categories: learning goals and assessments. All other categories showed positive but minimal effects that were not statistically significant. These results suggest that the overall shift in the syllabi scores was driven primarily by these two categories. It is plausible that some PD content was directly or indirectly related to articulating their learning goals in their course materials. The lack of meaningful change from pre to post average use of student-centered teaching practices was unexpected. However, because much of the data used in this study was collected during the pandemic, these findings are not likely to be fully reflective of their teaching constrained by the challenge of adapting student-centered teaching practices to remote teaching formats.

The findings from this study are encouraging as it provides evidence that syllabi can be a useful measure of an instructor's teaching practice. While the syllabus may not reflect enacted teaching practices, our findings indicate that a syllabus scored with the MtP rubric can approximate the type of teaching practices used by a given instructor. A syllabus can be a more accessible and cost-effective means of characterizing teaching and measuring differences across time points in response to teaching-focused interventions such as professional development.

While the findings from RQ1 indicate the intention that instructors infuse into their syllabi manifests in their teaching behaviors (and use of student-centered teaching practices), the findings from RQ2 showed a positive shift in intention, but no significant change in teaching practices. These findings suggest that while instructors intend to change their teaching practices, they may need more time to gain skills and knowledge to use the teaching practices they aspire to use. Thus, the syllabi and corresponding MtP score may be a more useful and realistic measure of short-term PD outcomes than more resource-intensive methods such as observations or surveys.

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An Analysis of Connections Between Epistemological Obstacles Experienced Within the Quantification Unit of a Transition-to-Proof Course

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Students' conceptions play a large role in how they learn math. When students experience challenges, or epistemological obstacles, they may struggle to adapt their ideas that previously served them well. Little research has been done regarding interactions between co-occurring epistemological obstacles. This report utilizes a proposed adjacency matrix methodology for analyzing the co-occurrence of epistemological obstacles. Our results show that students' difficulty noticing hidden quantification had strong thematic connections to two other challenges: quantifying the negation and transforming logical implications into related forms.

Keywords: Epistemological Obstacles, Quantification, Proof and Proving, Logical Implication

Prior research has documented students' persistent challenges when reasoning about logical statements in transition-to-proof courses (TTP; e.g. Dawkins & Roh, 2020; Epp, 2003; Norton et al., 2025). Many of these challenges persist even in response to research-based instruction, and researchers characterize such challenges as necessary in students' development of logical structures, such as logical implication (LI; Norton et al, 2025; Steffe, 1992).

Antonides et al. (2025) observed that existing mathematics education research usually studies students' learning challenges independently, even though students might experience multiple challenges in tandem. Norton et al. (2025) found that students' ways of reasoning impact the profile of challenges that they experience. To support researchers, Antonides et al. (2025) proposed a methodology for mapping connections between co-occurring learning challenges.

This paper aims to add to the literature by investigating challenges experienced concurrently by students during research-based instruction on quantification. Quantifiers, such as "for all," play an essential role in mathematical study as they, when taken together with predicates, allow for reasoning about mathematical objects, their properties, and relationships between objects. Moreover, there is a wealth of literature documenting students' persistent challenges with quantification in logical statements (e.g. Dawkins & Roh, 2020; Epp, 2003; Shipman, 2016).

We asked the following research questions: (a) During research-based instructional interactions on quantification, what challenges do students experience concurrently? and, (b) What ways of reasoning do students demonstrate when challenges are experienced concurrently within the same interaction? In response, we analyzed video of the first six classes of the quantification unit in an undergraduate TTP course. This paper reports on the study's results and shares themes in student reasoning within the two pairs of EOs that co-occurred most frequently.

Theoretical Framework

Following Brousseau (2002), we characterize learning challenges that persist in response to research-based instruction as *epistemological obstacles (EOs)*. They are "the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (p. 82). Students may experience an EO when they become aware of a conflict or perturbation in their thinking, but teachers may notice this tension first.

We situate our investigation of concurrent EOs within Antonides et al.'s (2025) proposed methodology for mapping co-occurrences of EOs experienced within instructional interactions.

This six-step process begins with a literature review of EOs and it culminates in the generation of an adjacency matrix whose rows and columns are these EOs and whose entries indicate the number of times pairs of EOs co-occur in a single instructional interaction. The resulting adjacency matrix can then be visualized as a weighted graph, allowing "researchers to visualize connections between EOs and the strength of each connection" (p. 1321). We elaborate on this six-step process within our Methodology section below.

Methodology

To collect data about EOs, we used the 6-step methodology outlined by Antonides (2025).

Step 1: Literature Review

An extensive literature review was conducted within a larger study, "The Proofs Project" (e.g., Norton et al., 2025). We summarize here the EOs pertinent to our study.

Students consistently struggle with implicit, or "hidden," quantification (coded Qh; Shipman, 2016; Durand-Guerrier, 2003; Epp, 2003). For example, "If $x > 2$, then $x^2 > 4$" implicitly refers to *all x*. When quantification is unhidden, students struggle to quantify a statement's negation (coded Qneg; Dawkins & Roh, 2020). Multiple quantifiers introduce additional struggles (coded Qmult; e.g., Adiredja, 2020; Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000).

Students also experience difficulties when proving quantified statements. For example, they may question why the *Principle of Universal Generalization* (see Copi, 1954) proves a "for all" statement (coded PUG; Dawkins et al., 2022; Norton et al., 2023). Students may also conflate the value of a variable (e.g. $\varepsilon = 1$) with its role (e.g. the distance between two points; coded Qrv; Dawkins & Roh, 2022).

Challenges related to LIs of the form $p \rightarrow q$ have also been documented. Students might treat LI as biconditional (coded LIff; Epp, 1999, 2003; Rumain et al., 1983; Wagner-Egger, 2007) or struggle to transform an LI into, for example, its converse or negation (coded LIt; e.g., Antonini, 2004; Hawthorne & Rasmussen, 2015; Hub & Dawkins, 2018). Furthermore, students may conflate the negation of $p \rightarrow q$ with its opposite statement, $p \rightarrow \sim q$ (coded NO; Epp, 2003; Arnold et al., 2024), and they may experience difficulties drawing distinctions between the negation and a disproof of a statement (coded NF; Norton et. al., 2025).

Step 2-4: Data Collection and Analysis

Data consist of a subset of six video-recorded 50-minute classes within the quantification unit of an inquiry-oriented undergraduate TTP course taught by the second author. Table 1 outlines the topics discussed during the classes analyzed within this study (Class 15 was a review day and was not analyzed). Students sat at tables in groups of approximately five students, engaging in group discussion captured on iPads via audio- and screen-recording. We refer to the five groups as the Hexagons, Rectangles, Circles, Squares, and Triangles. We analyzed the recorded videos,

Table 1. The classes analyzed for this report and the respective subjects covered.

Class	Discussion
13	Introduction to the existential quantifier
14	Writing existential proofs, writing nonconstructive existence proofs
16	Comparing constructive vs. nonconstructive existence proofs
17	Introduction to the universal quantifier
18-19	Writing "for all" proofs, constructive vs. non-constructive for all proofs

Table 2. A table with descriptions of EOs that occurred.

EO Code	Description
Qh	Overlooking hidden quantifiers
Qneg	Demonstrating uncertainty about the quantification of the negation
NO	Treating the negation $p \rightarrow q$ as its opposite statement $p \rightarrow \sim q$
PUG	Questioning or neglecting the *principle of universal generalization*
LIt	Difficulty transforming LIs into related forms (converse, negation, etc.)
Qrv	Conflating the role of a variable with its value
NF	Struggling to articulate the difference between negation and disproof
LIff	Interpreting LIs as biconditional (e.g. assuming $p \rightarrow q$ means $q \rightarrow p$)
Qmult	Struggling to make meaning of statements with two or more quantifiers

noting timestamps where EOs were elicited. We marked EOs based on a codebook (see Table 2) developed from our literature review. When students were perturbed, we coded the interaction with an EO that most closely captured their experience. After coding, we demarcated our data into chunks by instructional task. Video from the task launch, group discussions about the task, and subsequent whole-class discussion comprised a single chunk.

Step 5-6: Generating and Interpreting Results

We counted co-occurrences within each chunk as follows. If no EOs were coded, our matrix A remained unchanged. Otherwise, for each distinct pair of EOs (i,j) within a chunk, we updated our matrix A, increasing by 1 the entries in row i column j (A_{ij}) and row j column i (A_{ji}). For example, when Qh, Qneg, and PUG co-occurred within a chunk, regardless of the number of each code's occurrence, a 1 was added to $A_{Qh,Qneg}$, $A_{Qneg,Qh}$, $A_{Qh,PUG}$, $A_{PUG,Qh}$, $A_{Qneg,PUG}$, and $A_{PUG,Qneg}$. All distinct codes experienced by any group were counted as co-occurrences. Using this adjacency matrix, we generated a graph in which each vertex was an EO and weighted edges between EOs represented the pair's number of co-occurrences. Then, we analyzed the strongest connections visible in the graph for themes across chunks in which those connections appeared.

Results

The adjacency matrix and weighted graph, shown in Figure 1, identify which EOs (see Table 2) students experienced concurrently (RQa). We omitted from our graph those EOs not elicited or with no connections. 8 chunks had concurrent EOs, 5 chunks had a single EO, and 2 chunks had zero EOs. Several EOs co-occurred at least once, with Qh and Qneg exhibiting the most connections with other EOs (17 and 11, respectively). The graph highlights two prominent

	Qh	Qneg	NO	PUG	LIt	Qrv	NF	LIff	Qmult
Qh	–	3	2	2	3	2	2	1	1
Qneg	3	–	2	1	1	2	2	0	0
NO	2	2	–	1	1	1	1	0	0
PUG	2	1	1	–	1	2	0	0	0
LIt	3	1	1	1	–	0	1	1	0
Qrv	2	2	1	2	0	–	0	0	0
NF	2	2	1	0	1	0	–	0	0
LIff	1	0	0	0	1	0	0	–	0
Qmult	1	0	0	0	0	0	0	0	–

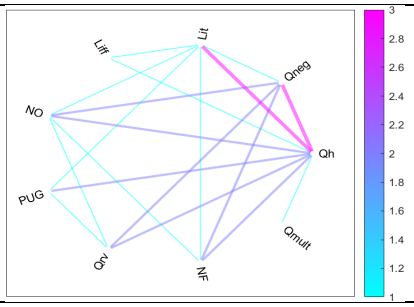


Figure 1. Our study's resulting adjacency matrix (left) and weighted graph of EO connections (right).

Table 3. All chunks where Qh, Qneg, or LIt co-occur. Gray rows indicate tasks reported on in this paper.

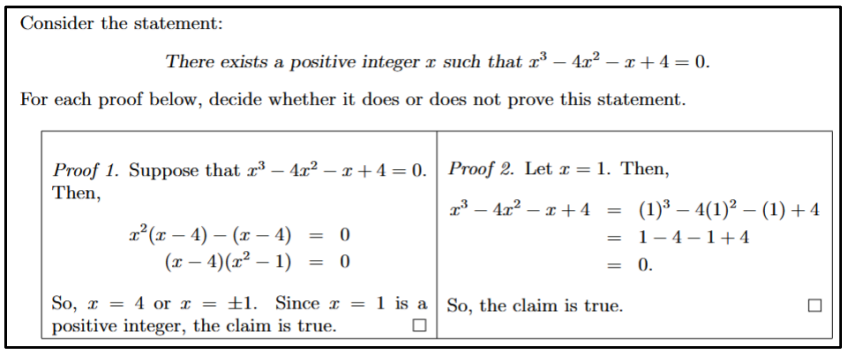
Class, Chunk	Task Name	Co-Occurring EOs
13, 3	Proof Comparison Task (PCT)	Qh, LIt, Liff
14, 4	Vector Multiplicity Task (VMT)	Qh, Qneg, NO, PUG, Qrv
16, 1	Types of Existence Proofs	Qh, Qneg, NF
16, 2	Proof Evaluation Task (PET)	Qh, Qneg, LIt, NF, NO
17, 1	Proving vs. Assuming Existence Task	Qh, LIt, PUG

connections that appeared within three chunks each: (a) hidden quantification and quantifying the negation (Qh-Qneg), and (b) hidden quantification and transforming an LI (Qh-LIt). The chunks and associated tasks in which Qh, Qneg, or LIt co-occurred are outlined in Table 3. We elaborate on results from the Proof Comparison Task (PCT), Vector Multiplicity Task (VMT), and the Proof Evaluation Task (PET) because they exemplify each prominent connection twice.

Proof Comparison Task (PCT)

In PCT (Figure 2), students considered whether two proofs did or did not prove the given existence statement. *Proof 2* is valid, but *Proof 1* assumes the existence of x: "Suppose that $x^3 - 4x^2 - x + 4 = 0$" implicitly means "Suppose *there exists an x such that* $x^3 - 4x^2 - x + 4 = 0$."

Every group's discussion about PCT was coded as Qh because students experienced difficulty unhiding the quantification of x in *Proof 1*. In the Triangle group, a student said, "If this was in the way of if-then, then *Proof 1* might not be correct because it started with the conclusion. Since it's a 'there exists,' it's fine. You just need to prove the existence." Because *Proof 1* began with the equation, she compared this to beginning a proof of an LI by assuming the conclusion. Whereas she understood this was a flawed approach to proving an LI, she believed it was acceptable when proving an existence statement. Another student said, "Yeah, I think they both work. Just, one of them gives you a single answer and the other gives you three answers." She distinguished the two proofs only in terms of how many "answers" were provided. Following group work, the instructor explained why the first proof assumed the existence of x. This prompted one student from the Rectangle group to ask, "Would it be possible to, like, reword it into, like, 'there exists an x such that if the equation equals 0… x =1 or something? [...] If you were to transform [PCT existence statement] into an implication, then, like how would *Proof 1*…would it still hold?" When the instructor observed that *Proof 1* would be a proof of the converse of the proposed LI, the student was surprised–prompting us to code the interaction as LIff. We also coded LIt because the student experienced difficulty attempting to transform the given statement into an equivalent LI.


Consider the statement:

There exists a positive integer x such that $x^3 - 4x^2 - x + 4 = 0$.

For each proof below, decide whether it does or does not prove this statement.

Proof 1. Suppose that $x^3 - 4x^2 - x + 4 = 0$. Then,

$$x^2(x-4) - (x-4) = 0$$
$$(x-4)(x^2-1) = 0$$

So, $x = 4$ or $x = \pm 1$. Since $x = 1$ is a positive integer, the claim is true. □

Proof 2. Let $x = 1$. Then,

$$x^3 - 4x^2 - x + 4 = (1)^3 - 4(1)^2 - (1) + 4$$
$$= 1 - 4 - 1 + 4$$
$$= 0.$$

So, the claim is true. □

Figure 2. The PCT from Class 13, chunk 3.

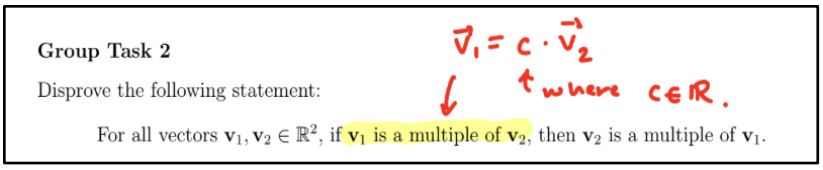


Figure 3. The Vector Multiplication Task (VMT), given during Class 13, Chunk 4.

Vector Multiplicity Task (VMT)

VMT is shown in Figure 3. Before students began work on this task, the instructor provided the definition of "v_1 is a multiple of v_2." Note that both the hypothesis and conclusion hide the quantification of a scalar c within their definitions. Moreover, c need not be the same value in each statement. Four groups arrived at the equation $v_2 = 1/c\ v_1$ and then spent significant time attending to this scalar; the Triangle group focused on recalling linear algebra concepts and never derived the equation. Of the four groups, two groups did not discuss c's quantification (coded Qh) or attempt to write a proof and two groups experienced difficulty attending to quantification of c in the conclusion (coded Qneg) while writing a proof.

The Circle group determined that a disproof entailed providing a counterexample.

Student 1: So since it says for all vectors v_1 and v_2, could we just find a counterexample? [...]

Student 2: Or you can do it specifically. v_1 equals like 〈1,1〉 and you multiply it by 2 …

Student 1: Yeah, or I guess you could say, like, so $v_1 = cv_2$, then that means $v_2 = 1/c\ v_1$ and since 1/c is not a real number… I mean it's a real number… but it's not like an integer.

Student 2: I think it's supposed to be that c is an integer, otherwise it would be true.

While discussing, the group's scribe wrote the statement and its negation symbolically (Figure 4). After solving $v_1 = cv_2$ for $v_2 = 1/c\ v_1$, they questioned whether $1/c$ was a valid multiple. There was no mention of c's quantification in either definition. Their inclusion of "where $c \in \mathbb{Z}$" for both the original statement and their negation (Figure 4) further corroborates that: (a) they overlooked the shift in quantification of c in "v_1 is a multiple of v_2," and (b) c's quantification remained unchanged in the negation. We also hypothesize that their writing, "where $c \in \mathbb{Z}$," seemed only to mimic that of the instructor's (Figure 3), and we conclude that the existential quantification of c remained implicit in the word "where" (coded Qh).

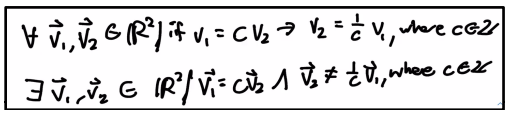


Figure 4. The Circle group's understanding of the VMT's original statement and its negation.

In contrast, the Rectangle group discussed the quantification of the scalar, which they called k. After identifying a counterexample wherein v_1 should be the zero vector, they noted $v_1 = kv_2$ when $k = 0$. However, they struggled with how to show precisely that v_2 was not a multiple of v_1.

Student 1: ...prove that v_2 times $1/k$ does not equal v_1 because… 1 over zero can't exist.

Student 2: Honestly, it doesn't have to be k.

Student 1: Well, it can be any letter.

Student 2: Well, the k has to be different. […]

Student 1: It'd just be $1/k$. Like you'd write, so you'd, like, let $kv_2 = v_1$ and then, like, you show that is $1/k\ v_2$ and let $k = 0$ and you prove that that doesn't exist. If k is 0, $1/k$ can't exist. You don't have to use two separate…

Student 2: But, I'm saying like it doesn't have to be $1/k$. It could be anything. […]

Student 1: Yeah, yeah, but I'm saying, like, the general case.

The group seemed to agree that they must show no scalar existed, but their understandings of "there does not exist" seemed to influence their reasoning differently; Student 2 understood it as "for all," whereas Student 1 seemingly did not. Instead, he viewed $1/k$ as addressing "the general case," perhaps because k was a variable. However, his focus on algebraic manipulation overshadowed his ability to operationalize the meaning of "there does not exist." We characterize this as Qneg, as Student 1's quantification of k in "v_2 is not a multiple of v_1" prevented his noticing that $1/k$ introduced a dependency that removed arbitrariness of the scalar.

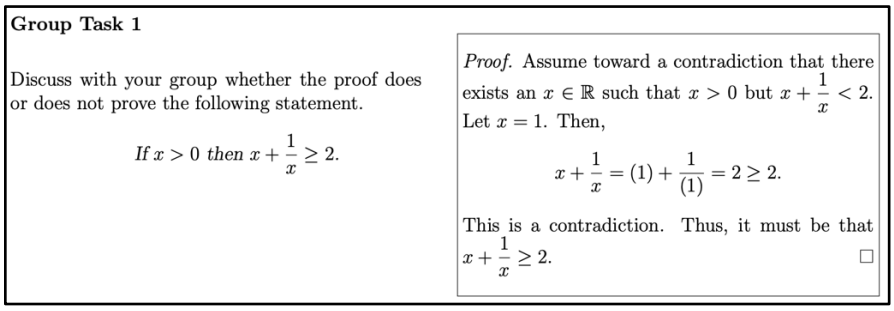


Figure 5. The Proof Evaluation Task, given during Day 16, chunk 2.

Proof Evaluation Task (PET)

PET (Figure 5), adapted from Shipman (2016), invites students to analyze a flawed proof by contradiction. The task is designed to perturb students' thinking regarding whether they can or cannot choose the value of a variable x when it is assumed to exist. When one assumes an x exists, its value is fixed, yet unknown. This stands in contrast to the logic of proving existence, wherein we may choose a specific x-value. Students experienced a variety of EOs when working on PET, but most notably our strongest connections, Qh-Qneg and Qh-LIt, co-occurred within this task. We note that only one group's discussion was coded as Qh with all others coded Qneg.

A student in the Square group began discussing saying, "'If P, then Q'… negation is, what, P and not Q?" The group agreed it was, and one student explained, "Yeah, so it's assuming the negation, then showing the negation is false, so the original statement must be true." In attending to the predicate of the negation (P and not Q), the group overlooked the quantification of x (coded Qh). With quantification hidden, they concluded the proof was valid.

In the Rectangle group, one student observed, "I feel like if you're assuming toward a contradiction that there exists an x-value then you can't give a constructive example…because there's nothing guaranteeing that the example you give is the assumed existing value." Another student questioned, "Are you allowed to use just one example or do you have to prove for all x-values?" expressing uncertainty about quantification (coded Qneg). Discussion then shifted to what statement the proof *did* prove. One student asserted it was a valid proof of the converse which prompted one student to disagree, and another to say, "That's the negation." They experienced difficulty transforming the LI to fit the logic of the proof (coded LIt).

Discussion

Our results revealed, via number of connections, the centrality of Qh and Qneg in the challenges students experience within instructional interactions on quantification. While we might expect Qh and Qneg to have a strong connection (as negating quantification requires unhiding it first), the equally strong connection between Qh and LIt was surprising. In the discussion that follows, we characterize students' ways of reasoning when the connections Qh-Qneg and Qh-LIt occurred (RQb).

For students to experience the challenge of quantifying the negation, they must first unhide the quantification. Therefore, because challenges are EOs only when they *persist*, students cannot experience both Qh and Qneg simultaneously. So, when Qh and Qneg appeared together, it indicated that, for some groups, quantification remained hidden, and for others, attending to quantification was challenging.

PCT was the only task discussed above in which Qh appeared without Qneg. Because all five groups experienced Qh, we conjecture that students' existing ways of operating on algebraic equations (solving for x), prevented them from noticing implicit quantification. In VMT, only those groups who attempted to write a proof experienced Qneg. Without reaching this stage, quantification of the scalar remained hidden. In PET, only the Squares experienced Qh, as they were focused instead on the logical structure of an LI and its negation. This finding echoes Norton et al.'s (2025) claims that one must conceptualize LI as a single object before they can attend to its quantification. All other groups were perturbed (coded Qneg) by the proof's use of a single example x=1.

Our Qh-Qneg analysis validates existing literature (e.g. Dawkins & Roh, 2020) that students still experience difficulties with quantification after unhiding it. Students' existing ways of reasoning also seem to impact students' abilities to unhide quantification, but the act of proof writing appears to assist students in attending to quantification.

The two tasks, PCT and PET, in which this connection appeared engaged students in determining whether a given (flawed) proof was valid. Each proof contained implicit quantification that perturbed students and, when hidden, resulted in their conclusion that the proof was valid. In the case of PET, those groups that successfully unhid quantification determined the proof was logically flawed. LIt was coded, during whole-class discussion in PCT and group discussion in PET, when students tried to determine a statement that the proof *did* prove.

Our analysis of the Qh-LIt connection suggests tasks that engage students in proof evaluation are effective in eliciting experience of Qh. And, when a proof is invalid, students have opportunities to analyze the logical structure of the proof to determine what, if anything, the proof does prove. This echoes a study by Hub & Dawkins (2018) who used proof evaluation tasks to elicit student reasoning in clinical interviews.

Conclusion

Our implementation of Antonides et al.'s (2025) methodology for mapping connections between EOs supported our identification of key themes in student reasoning. Hidden quantification was highly connected to all challenges experienced by students, and most strongly connected to Qneg and LIt. Analysis of these two connections revealed three key findings. First, students' existing ways of reasoning (e.g. algebraic reasoning and treatment of LI) influenced their ability to unhide quantification. Second, engaging students in proof writing assisted students in noticing quantification. Lastly, proof evaluation tasks in which quantification is implicit are effective in eliciting students' experience of Qh and in offering students opportunities to transform statements to determine the logical structure of a proof.

Studying co-occurring EOs highlighted nuances in students' reasoning and offers perspective on the challenges related to quantification documented in prior literature. Our study showed students' experience of hidden quantification was essential for them to have opportunities to experience other necessary challenges (e.g., Qneg and LIt). This finding suggests the importance of instructors' abilities to notice, elicit, and address head-on EOs during instruction.

Acknowledgments

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Is Equivalence Just "Same Number"? College Students' Numerical vs Non-Numerical Conceptions of Equivalence

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As coursework extends beyond arithmetic, students encounter forms of equivalence beyond numerical equality. This study examines how college students characterize equivalence in mathematics generally, and specifically for equations and expressions. Responses from 198 students were analyzed and coded as numerical or non-numerical. Across questions, most responses appealed to numerical criteria, treating equivalence as equality to a single number, with higher prevalence of non-numerical criteria correlating with higher-level courses. We observed multiple variants of both numerical and non-numerical criteria, including converting algebraic expressions to numbers to justify equivalence. Findings highlight the need for educators to support students in distinguishing types of equivalence and their associated criteria.

Keywords: equivalence, equality, algebra, expressions, equations, solution set

Equivalence has been identified as a "big idea" in mathematics and is formalized via the notion of an equivalence relation (Charles, 2005; Cook et al., 2022; Siemon, 2022; Tay, 2019; Toh & Yeo, 2019). Student understanding of equivalence is crucial for their *inferential knowledge*–their understanding of the validity of mathematical procedures (Mirin et al., 2025). As students progress from arithmetic to algebra, and then to more advanced mathematics, they move beyond numerical equality and regularly encounter equivalence in many contexts, involving different types of objects and equivalence criteria. For example, equivalence criteria for algebraic expressions and equations go beyond determining whether two numbers are equal.

Here we consider students' characterizations of equivalence, both generally and in the context of expressions and equations—attending to the extent their equivalence criteria link to numerical equality or non-numerical equivalence. This provides insight into how students' criteria for the familiar notion of numerical equality intersect with their equivalence criteria in other contexts. Better understanding how college students conceptualize equivalence is critical to improving instruction that supports understanding of equivalence across mathematical contexts.

Equality and Equivalence in Mathematics

The terms "equality" and "equivalence" have been used in many ways in research and instruction (Asghari, 2019). To clarify the focus of this research, we define our use of this terminology (while recognizing that others might use terms differently). Here, we define *numerical equality* as a relationship where two objects are equivalent if (and only if) they are each equal to the same numerical quantity. Numerical equality is one type of equivalence relation where every equivalence class could be represented by a canonical element that is a single numerical value. Equivalence of numerical expressions (e.g., $6(2)$ and $9 + 3$) is one type of numerical equality. For example, one way to determine equivalence is to conclude that both $6(2)$ and $9 + 3$ can be represented by the canonical element, 12, of the equivalence class.

When equivalence involves something other than numerical equality—i.e., when it compares

objects that cannot be represented by single numerical values—we call this *non-numerical equivalence*. In algebra, students encounter non-numerical equivalence in the context of equivalent algebraic expressions and equations. Two *algebraic expressions are equivalent* if they are equal to the same numerical output for every domain input (i.e., for every possible combination of substitution of numerical values in for the variables; *insertion equivalence* [Zwetzschler & Prediger, 2013])[1]. For example, $3x - 2x$ and x are equivalent because they are equal for all x-values; however, neither $3x - 2x$ nor x are always equal to a single numerical value. Relatedly, two *algebraic equations are equivalent* if they have the same solution set; i.e., they have the same truth values for every possible domain input. For example, $|3x - 2x| - 2 = 12$ and $|x| = 14$ are equivalent because they are true for the same values of x (only $x = \pm 14$).

Before algebra, numerical equality is often the most prevalent type of equivalence.[2] As a result, many K-12 studies have investigated whether students conceptualize the equals sign from an *operational* (as announcing the answer to a calculation) vs. a *relational* (as describing an equivalence relationship) view (e.g., Kieran, 1981; Knuth et al., 2008; McNeil et al., 2006; Stephens et al., 2013; Stylianou et al., 2024); studies with college students show they may also hold operational views of the equal sign (Fyfe et al., 2020; Weinberg, 2010). However, much less research has explored how students conceptualize non-numerical equivalence criteria.

Some studies (e.g., Cook et al., 2022; Jones et al., 2012; Mirin, 2019; Tondorf & Prediger, 2022; Wladis et al., 2022; Zwetzschler & Prediger, 2013) have considered how students relate *transformational* activity (i.e., equivalent symbolic representations can be transformed into one another) or *substitutive* activity (i.e., equivalent symbolic representations can be substituted for one another) to other equivalence criteria that justify this activity (e.g., *descriptive* criteria: representations describe the same situation; *insertion* criteria for algebraic expressions). The literature on students' work with algebraic expressions and equations suggests that they tend to focus on transformation without attending to underlying descriptive, insertional, or other criteria that are a *reason* for the validity of transformation (e.g., Cook et al., 2022; Sfard & Linchevski, 1994). It is thus unsurprising that for many students view an equation's "solution" as the result of transformation rather than a property of the equation (e.g., Frost, 2015; Tsamir & Tirosh, 2023).

While there is substantial research on students' thinking on numerical equality, there is much less on non-numerical equivalence. What literature does exist has thus far tended to focus on describing types of equivalence that students attend to or invoke in specific contexts, such as while solving equations or transforming expressions (e.g., Cook et al., 2022; Tondorf & Prediger, 2022). Our work complements this by describing what students think "equivalence" means, how they characterize "equivalent expressions" and "equivalent equations", and by explicitly considering how this relates to numerical vs. non-numerical equivalence criteria.

Method

Open-ended items on equivalence were administered to 198 students at a U.S. community college in 18 different courses, ranging from developmental elementary to linear algebra. Courses also included statistics, discrete mathematics, mathematics for elementary teachers, intermediate algebra, precalculus, and Calculus I-III. We operationalized course level based on algebra prerequisites: "low" = no algebra prerequisite: "mid" = elementary algebra prerequisite; "high" = intermediate algebra prerequisite. Items were administered during class. Some students

[1] We note that, with *n* variables, the "inputs" involves *n*-tuples.

2 We note that non-numerical equivalence relations do exist in earlier grades (e.g., shape similarity or congruence).

were given all three questions (40), some only 1 and 2 (77), and some only 1 and 3 (81).

Q1. What do you think equivalence means in mathematics?
Please give an example or two to show what you mean.
Can mathematical equivalence mean anything else?
Circle one: *Yes* *No* *I'm not sure*
Please explain how you know.
Please give an example or two to show what you mean.

Q2. How could you check whether two mathematical **expressions** are equivalent?
(An expression is a mathematical phrase that does **not** contain an equals or inequality sign.)
Please give an example or two to show what you mean:
Are there any other approaches you could take to check whether two mathematical expressions are equivalent?
Circle one: *Yes* *No* *I'm not sure*
Please give an example or two to show what you mean:

Q3. How could you check whether two mathematical **equations** are equivalent?
(An equation is a mathematical phrase that **contains an equals sign**.)
Please give an example or two to show what you mean:
Are there any other approaches you could take to check whether two mathematical equations are equivalent?
Circle one: *Yes* *No* *I'm not sure*
Please give an example or two to show what you mean:

Six researchers collaboratively employed *reflexive thematic analysis* (Braun & Clarke, 2006). This led to the emergence of several themes, one of which was "equivalence as numerical equality vs. non-numerical equivalence relationship" (details of other themes are discussed in other work, e.g., Wladis et al., 2022c, 2022b). After themes were developed, they were used as the basis for generating a codebook for formal coding, including codes indicating whether students cited numerical equality (i.e., examples or explanations describing numerical equality criteria) and/or non-numerical equivalence (i.e., examples or explanations appeared to illustrate non-numerical equivalence criteria). Not all student work merited one code or the other (e.g., because of ambiguity). Some students showed evidence of both; thus, codes were not mutually exclusive.

Results

First, we consider frequencies indicating whether students, when asked to describe and give examples of what equivalence means in mathematics (Q1), provided examples of numerical equality and/or of non-numerical equivalence relationships (Table 1).

Table 1. General Equivalence Item (Q1) Results (subset of students given Q1 and Q3).

	course level				*p*-value		
Numerical equality vs. non-numerical equivalence	low	mid	high	total	mid vs. low	high vs. mid	high vs. low/mid
cited numerical equality	91%	87%	91%	89%	0.2491	0.6862	
cited non-numerical equivalence	0%	0%	14%	2%		0.0015**	0.0001***
n	37	62	22	121	$^{**}p \leq 0.01$; $^{***}p \leq 0.001$		

Roughly 9 of 10 students cited numerical equality in Q1, while only 2% referred to non-numerical equivalence (those that did were all enrolled in higher-level courses). Just because students did not volunteer non-numerical equivalence examples does not mean they can't provide them; however, these patterns do suggest that college students' primary equivalence associations may be numerical.

Conceptualizing equivalence as numerical equality was also common in response to Q2 (equivalent expressions) and Q3 (equivalent equations); coding frequencies are given in Table 2. We then proceed to characterize students' examples in more detail. Due to space constraints, we

illustrate just a few subcategories with a few brief examples of student work.

Table 2. Type of Equivalence Criteria Applied to Q2/Q3 (Some students provided more than one type.)

Numerical equality vs non-numerical equivalence criteria	Q2: expression item				Q3: equation item			
	low	mid	high	total	low	mid	high	total
numerical equality, arithmetic calculation	56%	41%	33%	44%	65%	34%	21%	40%
numerical equality, transform to numeric	44%	19%	56%	32%	20%	40%	50%	37%
other type of numerical equality	0%	22%	0%	12%	0%	9%	7%	6%
any kind of numerical equality	**94%**	**78%**	**89%**	**84%**	**80%**	**81%**	**79%**	**80%**
same solution set					0%	0%	7%	1%
transformation equivalence	6%	16%	22%	14%	5%	11%	0%	7%
other possible non-numerical equivalence	0%	3%	11%	4%	0%	4%	7%	4%
total possible non-numerical equivalence	**6%**	**19%**	**33%**	**18%**	**5%**	**15%**	**14%**	**12%**
"same solution" but ambiguous whether numerical/non-numerical					15%	2%	7%	6%
n (provided examples to compare)	16	32	9	57	20	47	14	81

Numerical Equality Criteria

Students attended to numerical equality criteria in different ways. In some cases students provided entirely numerical examples. Iota, a Calculus I student, stated for Q1 that equivalence means "when a answer is equal to each other", for Q2 wrote "we can substitute or solve the math expression and see if it is similar" and for Q3 wrote "solve the problem on both sides and if *they equal to each other* [emphasis added] they are equivalent". Figure 1 shows examples they gave.

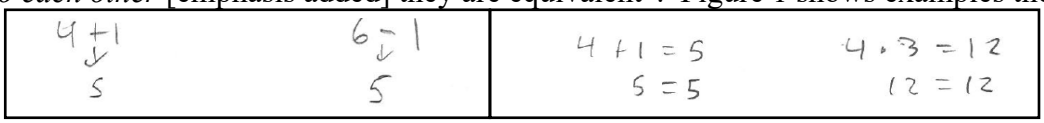

Figure 1. Examples given by Iota for Q2 (left) and Q3 (right)

In Q2, Iota gave numerical expressions with arrows to show numerical equality (Figure 1, left); whereas in Q3 they wrote $4 + 1 = 5$ and $4 \cdot 3 = 12$ and on the next line showed that the left and right side of each equation are equal. They write their work differently in Q3 vs. Q2, suggesting that they might view expressions and equations as different. Although it's somewhat ambiguous which objects Iota is comparing in Q3, their criteria for equivalence involves *comparing individual numbers*, suggesting a focus on numerical equality.

In some cases, students provided algebraic rather than numerical expressions or equations but then *converted* them to numerical values. Lambda, a student in elementary algebra, answered Q2 by writing "plug in numbers in the expressions" and then gave the examples shown in Figure 2.

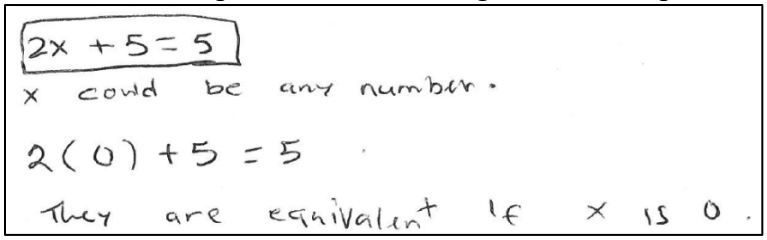

Figure 2. Examples given by Lambda for Q2

Lambda appears to have substituted a single value for x into an algebraic expression $2x + 5$ and, performed numerical calculation to demonstrate that $2x + 5$ is equal to 5 if x is 0. Although Lambda provided an algebraic expression, their demonstrated *criterion* for equivalence is

consistent with numerical equality because it involves comparing single numerical values. This example, though numerical, incorporates some aspects of insertional equivalence; they assess equivalence by checking that expressions are equal for a *particular* value of x. The extent to which Lambda might conceptualize the more generalized notion of insertion equivalence of algebraic expressions can't be determined from this answer alone. However, their focus on a single numerical value (vs. considering all values in the domain) does suggest that they may be more comfortable using numerical equality criteria over non-numerical equivalence criteria.

Some students seemed to impose numerical equality on algebraic expressions differently—by setting them equal to a fixed value (e.g., Mu, linear algebra student, Figure 3). In Q1, Mu stated "Two things are equal to each other. Whatever is on the left side is equal to the right side."

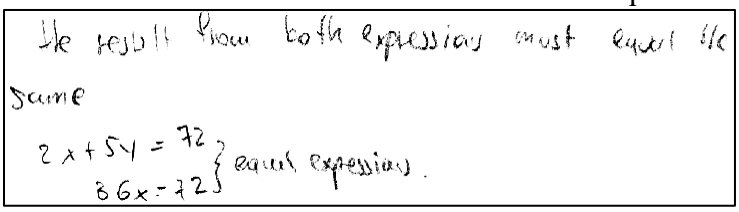

Figure 3. Example given by Mu for Q2

In Q2, Mu describes that the "result from both expressions must equal the same", and then they set two algebraic expressions equal to 72, calling them "equal". Like Lambda, Mu has taken two algebraic expressions and made them numerically equal, rather than considering how algebraic expressions represent larger sets of values and therefore are related by non-numerical equivalence criteria. As with prior students, Mu may be able to produce examples of non-numerical equivalence criteria in other contexts; however, their focus on setting algebraic expressions equal to a single number to illustrate equivalence suggests that they may be more comfortable with numerical equality criteria than non-numerical equivalence criteria.

When students answered Q3, they often talked about "solving" equations to determine equivalence, but it was also often ambiguous exactly which objects they were comparing. For example, one student in a combination intermediate algebra/precalculus course wrote for Q3, "You can check by solving the equations and see if they are equal to each other"; in this case, it is unclear what is "equal" (equations? x-values? solution sets?) and thus this response was coded as ambiguous. In other cases, there was evidence that students appeared to conceptualize equations as representing a single numerical value, suggesting an orientation around numerical equality criteria. Xi, a Calculus I student, wrote for Q3 "If they both add up to the same number" and then wrote the work in Figure 4.

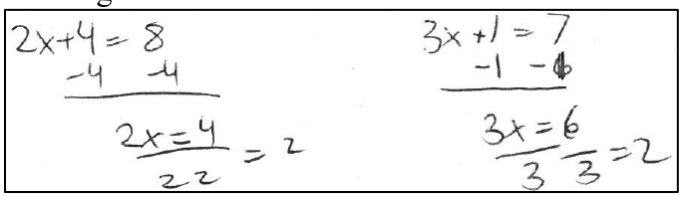

Figure 4. Example given by Xi for Q3

Xi solves two algebraic equations correctly, but their framing of equations "add[ing] up to the *same number* [emphasis added]", and their use of equals signs suggest a possible operational view of the equals sign, or even perhaps that the equations themselves are conceptualized as equal to the number 2 (it is ambiguous from their writing whether this is the case—taken together, this suggests a likely orientation towards numerical equality rather than non-numerical equivalence when considering the equivalence relationships between these algebraic equations.

Non-numerical Equivalence Criteria

While it was not always completely clear when students were using non-numerical equivalence criteria, we did observe various responses that suggested students may be attending to these criteria. For example, one student (Rho, in Calculus III) explicitly referred to equivalent equations having the "same solution set" in Q3 and provided the work given in Figure 5.

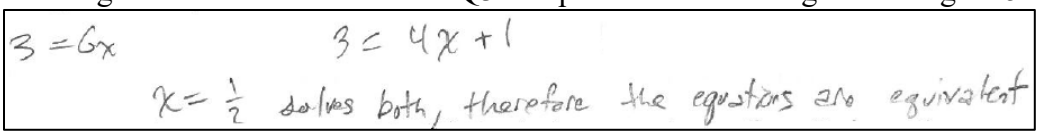

Figure 5. Example given by Rho for Q3

Rho uses the term "solution set" and provides the "solution" in equation form rather than representing it as only a single numerical value (e.g., like Xi in Figure 4). This suggests that Rho is using non-numerical equivalence criteria when considering the equivalence of equations, even though they, like other students, focus on a solving process that generates a single solution.

As expected, some students described *transformation* of algebraic expressions/equations as a type of non-numerical equivalence criteria. For example, Sigma, a student in a combined introductory algebra/statistics course, wrote in response to Q2 "find the value of each expression and see if they are the same value", giving examples in Figure 6.

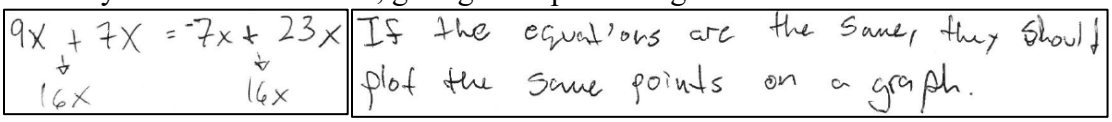

Figure 6. Sigma's Examples in Q2

Unlike students who converted algebraic expressions to numerical examples (e.g. Lambda, Mu), Sigma did not convert expressions to single numerical values; instead, for them, getting the "same value" appears to mean transforming expressions algebraically into the same algebraic expression representing a generic value (e.g., achieving the same "simplest form", $16x$, through transformation). While Sigma does not explicitly describe how these expressions have the same outputs for all the same inputs (i.e., non-numerical insertion equivalence for algebraic equations), they also provide a second example (Figure 6, right), describing "equations" as "plot[ting] the same points on a graph", which may be referring to insertion equivalence in the sense that $y = 9x + 7x$ and $y = -7x + 23x$ represent the same line (i.e., have the same solution set; i.e., $9x + 7x$ and $-7x + 23x$ have the same outputs for all the same inputs).

Some students illustrated another potential type of non-numerical equivalence criteria by describing equations as equivalent when they represent the same relationship, perhaps appealing to a type of descriptive equivalence (Figure 7).

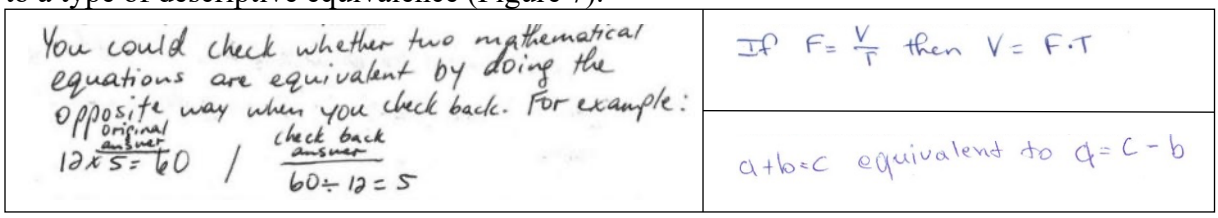

Figure 7. Examples of "Same Relationship". Upsilon (Q3, left), Phi (Q3, upper right), Chi (Q1, lower right)

Upsilon, an elementary algebra/statistics student, Phi, an elementary algebra/math for elementary education student, and Chi, an intermediate algebra and precalculus student, all provided examples of equations that appeared to represent the same "relationship" between the numbers/letters. While Upsilon provides arithmetic equations, we note that if they are comparing

the two equations ($12 \times 5 = 60$ and $60 \div 12 = 5$), their equivalence criteria (whether the numbers in the equation represent the same multiplicative relationship) are not about comparing individual numerical values. (As with other students, it is not completely clear which objects Upsilon is comparing—they may be comparing the numerical equality of the expressions 12×5 and 60 by performing division; however, their use of the term "equation" and similar patterns in examples given by Phi and Chi suggest that some students may be using this type of "same relationship" non-numerical equivalence criteria to determine the equivalence of equations, even if it is not completely clear whether Upsilon is personally making this comparison in this excerpt). Both Phi and Chi appear to be comparing equivalent algebraic equations where the equations represent the same multiplicative or additive "relationship" between the same variables. This type of criteria is not typically called "equivalence" in the curriculum, but it does represent an important non-numerical equivalence relation that could be productively leveraged during equation transformations. The work of these students may illustrate an opportunity for the mathematics community to consider alternate ways of characterizing certain types of equivalent equations using non-numerical equivalence criteria that some students may already be noticing. This could be a potentially productive route to explore for improving instruction.

Discussion

Across our sample, students predominantly described numerical equality criteria. The frequency of non-numerical equivalence criteria increased with course level; however, even in higher-level courses, numerical equality criteria still dominate. This occurred even in contexts where students were explicitly asked if equivalence could have other meanings, when they were asked specifically about equations and expressions, and when they themselves provided examples of algebraic expressions/equations. This is somewhat surprising, given the centrality of non-numerical equivalence (e.g., of expressions and equations) in algebra and beyond. On the other hand, the broad use of a single term, "equivalence", to describe different types of objects and criteria across the curriculum could lead to ambiguity in these distinctions for students. In particular, if existing instruction does not address the ways that the term "equivalence" is used to describe *different* equivalence criteria in different contexts, this may predispose students to try to make sense of all equivalence criteria under the lens of numerical equality (the one type of criteria that may be the most overtly addressed in existing curriculum and instruction).

One limitation of this study is that it asked students about their meanings for the term "equivalence". Although students encounter many examples of equivalence relations in the K–16 curriculum, the term "equivalence" may not always be used, and therefore, even when students recognize non-numerical equivalence relationships, they may not associate them with the term "equivalence". A helpful area for future research could be to investigate how students enact equivalence criteria in comparison to what they themselves call "equivalence".

This research demonstrates a critical need to explore more carefully how instruction shapes the development of student conceptions of non-numerical equivalence criteria, especially those for equivalent algebraic expressions and equations. In particular, it is important to better understand which kinds of instructional approaches support students in understanding the context-dependent nature of equivalence criteria, including distinctions among numerical expressions, algebraic expressions, and algebraic equations.

Acknowledgments

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Verbalized Transformational Reasoning in Small Group Discourse

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Transformational reasoning is a powerful form of reasoning that can be leveraged during collaborative mathematical activity. However, little research has been done to investigate how transformational reasoning supports collaboration, if at all. Using Brown's (2025) framework for identifying verbalized transformational reasoning and Staats' (2021) poetic structure methodology, the present study sought to investigate how transformational reasoning supported students to collaborate. The results demonstrate that students spent the bulk of their time collaborating on a collective solution by warranting the operation(s) they envisioned transforming the objects.

Keywords: Transformational reasoning, poetic structures, discourse analysis

Inquiry classrooms provide many opportunities for students to engage in collaboration, which has been shown to improve student success. Verbalizing one's reasoning and building on others' reasoning is what drives collaboration. Researchers studying collaborative mathematical activity are interested in understanding how the individual's reasoning, once verbalized, contributes to the collective solution of the mathematical activity. Hence, analytical tools that allow the researcher to track the developing collaborative activity while remaining focused on the students' shared reasoning are needed. Staats (2021) makes the case that common discourse analysis methods "discourse away" the mathematical reasoning of the students (p. 1). To address this concern, she introduced a method of discourse analysis that focuses on student reasoning, called *poetic structures* analysis.

The term poetic structure comes from the work of the linguistic anthropologist Roman Jakobson (1960). Staats comments that in Jakobson's conception, "any time a repetition causes listeners to attend to the form of the statement, or to use the form of a statement to construct meanings, the poetic function of language is in play" (2008, p. 29). A poetic structure is a verbal repetition that involves "similar syntax and at least one word in common" between two phrases (Staats, 2021, p. 5). This definition provides other scholars with a means for identifying meaningful repetition that share mathematical ideas.

Staats (2008; 2021) argued that students communicating their reasoning is the foundation of understanding. Furthermore, "…if communication is a central process for mathematical learning, we must account for the linguistic processes that allow students to construct mathematical ideas" (Staats, 2008, p. 26). Repetition with similar syntax "allows a speaker to recognize an existing idea and modify it in a way that is concise and easy to understand" (Staats, 2008, p. 29). Hence, the use of poetic structures supports the researcher in tracking the ways in which various warrants evolve through conversation between individuals as a collective product is developed (Staats, 2021).

For this study I adopt a socio-cultural perspective of learning. A socio-cultural perspective of learning takes into account that classrooms are socially and culturally situated environments that are not removed from one's prior experience and not separate from one's peers. Coordinating the notions that communication is the foundation of understanding (Staats, 2021), and learning is actively participating in a community of practice (see Lave & Wenger [1991]), communication supports individuals to engage in authentic activity of the community in which they are seeking

membership. Furthermore, mathematics classrooms are micro-communities that approximate the mathematics community at large; students learn by becoming active *doers* of mathematics. This suggests that there are two intertwined processes that take place in the individual, a cognitive and a social component of their learning.

I conducted a multiple-case study to understand how transformational reasoning (Simon, 1996) supported students to develop a collective product. I coordinated Brown's (2025) framework for identifying and describing verbalized transformational reasoning with Staats' (2021) poetic structure analysis to understand how verbalized transformational reasoning was used by the students as they developed a collective solution. In this report, I will emphasize the cognitive component of transformational reasoning and demonstrate that the students' verbalized reasoning supported collaboration. As was evident in the ways in which various phrases were repeated and evolved into the collective solution that was produced.

Theoretical Perspective and Relevant Literature

Transformational Reasoning

Simon (1996) coined the term transformational reasoning to describe a type of mathematical reasoning that he noticed students were spontaneously engaging in, that resulted in the students coming away with a sense of how a model worked. Simon defines transformational reasoning in the following way:

> Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. (1996, p. 201)

This definition includes (a) an object (or objects) that is (are) being transformed, (b) the operation(s) that affect the object (or objects), and (c) results of the operation(s) on the object(s). The process of transformational reasoning involves identifying (a) and (b), which allows one to *envision* the transformation taking place, and once (c) is attained one can examine them together to draw conclusions about the entire process to understand why it works. Envisioning the operation(s) on the object(s) is what constitutes transformational reasoning, but the components serve as signals of it.

Scholars have studied transformational reasoning in a variety of math contexts. This scholarship demonstrates that transformational reasoning can support students to: (1) learn about rates of change (Carlson et al., 2002; Johnson, 2012), (2) identify the invariant algebraic properties in related rates problems (Engelke Infante, 2021), (3) ascertain evidence in both proving and proof-related activity (Brown, 2025; Harel, 2001; Harel & Sowder, 1998), (4) continually refine and generalize their arguments (Ellis, 2007), and (5) identify key ideas (Raman, 2003; 2004) for a proof (Zandieh et al., 2008). All of the research here supports the claim that transformational reasoning can foster the development of a deep conceptual understanding.

I studied five pairs of students engaged in different implementations of an inquiry-oriented mathematics task and sought to understand how transformational reasoning contributed to the collective math activity. Specifically, I investigated the following research question: *how does the use of verbalized transformational reasoning facilitate the evolution of the collective product of the students' mathematical activity?*

Methods

Context and Background

The data analyzed in this report come from the NSF-funded project, *ASPIRE in Math* (DUE# 1916490). The *ASPIRE in Math* project sought to support students in their transition to advanced math while also supporting instructors to shift to inquiry-oriented instruction. Among the project deliverables is a curriculum for the guided reinvention of fundamental concepts in real analysis. The curriculum development efforts were guided by the design theory of realistic mathematics education (Gravemeijer, 1994). The curriculum tasks take students step by step through developing a conjecture equivalent to the Intermediate Value Theorem (IVT), developing an approximation method to find roots of continuous functions, and then leveraging the approximation method to introduce concepts like sequences and convergence, among others (Larsen et al., 2022).

All of the data comes from two different public institutions in the Pacific Northwest; these included a research-focused public university and a community college. All of the participants were math majors or minors, and the names in this report are all pseudonyms. The data analyzed consists of videos of each of the pairs of students as they worked on a task, transcripts of the video, and Google Docs of the students' work.

For this report, I analyzed student work on the *fifth-degree polynomial task*, which states: true or false, every fifth-degree polynomial has at least one real root. Students were paired up and asked to justify their answer. The goal of the task is to generate a conjecture that is logically equivalent to the IVT. Often students will observe that the claim is true and justify it using algebra or calculus knowledge. First, the students may recall that odd degree polynomials have end behavior that tends to infinity in one direction and negative infinity in the other, so it must pass through the x-axis. Alternatively, students may use an algebraic approach that leverages factoring to identify potential complex roots in pairs. While both solutions are valid, the class proceeds with the end behavior argument since the goal is to reinvent the fundamentals of analysis.

Once the arguments have been generated, the teacher then works with the students to scrutinize the main components of the argument to tease out the IVT conjecture assumptions. After these activities, the teacher asks the students to write a claim in the form: If [blank], then f has at least one real root. One example of this claim is "if a function f is continuous with both a positive and negative output, then f has at least one real root".

The Cases

The overarching methodology used in this study is a multiple-case study (Yin, 2018). A case consists of a pair of students working on the fifth-degree polynomial task. The case begins when the students share their initial reasoning and ends when the focus shifts to writing a conjecture. See table 1 below for an overview of the cases.

Table 1. Description of the cases

Case	Student Names	Context	Date
1	Chloe & Gabe	Teaching Experiment	Summer 2018
2	Will & Kevin	Whole Class Teaching Experiment	Spring 2020
3	Shay & Justin	Introduction to Proof Class	Winter 2021
4	Andrew & Jovan	Bridge to University Mathematics	Winter 2022
5	Harriett & Josue	Introduction to Proof Class	Fall 2022

Poetic Structures Analysis

Staats (2021) details her methodology for identifying and describing poetic structures in transcript data. A poetic structure is a phrase that "shares syntax and at least one word in common" (Staats, 2021, p. 5). Repeated syntax may take the form of Subject-Verb-Object, or something similar.

In alignment with Staats' (2021) approach, I went through the transcripts line-by-line. In a spreadsheet I recorded (a) the line number, (b) current speaker, (c) phrase that I determined was a repetition, (d) line number of original speaker, (e) original speaker, (f) source of the repeated phrase, and (g) whether or not the repeat was internal or across talk turns. A poetic structure was only recorded if it was on the same topic, so repeats like "3 minus 2" and "4 times 8" would not be considered poetic structures (Staats, 2021). Each poetic structure was the longest stretch of talk that captured the idea the student expressed; this improved the validity in the analysis since it allowed me to capture more of the students' verbalized reasoning (Staats, 2021, p. 6). Staats notes that the method cannot be relied upon to identify all of the poetic structures present in the data, rather the goal of the method is to "value consistency over completeness" (Staats, 2021, p. 8). Lastly, I identified each poetic structure according to Staats' (2021) framework, with the addition of Brown's (forthcoming dissertation) poetic structure of *elaboration* (see table 2).

Table 2. Types of Poetic Structures

Type	Definition
List	Saying a pattern
Echo	Revoicing a short phrase to confirm listening or validate an idea
Comparison	Saying two things are associated
Contrast	Posing two things as alternatives
Interposed List	Two lists are collated
Consolidation	Two or more previous poetic structures are combined
Expansion	A previous phrase has a novel clarifying phrase inserted
Reversal	Two phrases switch places syntactically
General	Repetition mobilizes a prior comment without a distinctive structure
Elaboration	A previous phrase that gets repeated with some words replaced in order to clarify an idea

Transformational Reasoning Analysis

Once the poetic structures analysis was complete, each transcript was coded according to components of Simon's (1996) definition of transformational reasoning. My previous work (Brown, 2025) demonstrated the utility of operationalizing Simon's definition of transformational reasoning. I identified objects, operations, and outcomes. Objects were mainly nouns or pronouns that named or referred to mathematical objects. Operations were adjectives or verb phrases that indicated that the student had modified a mathematical object. Outcomes were most commonly in the form of a claim that included a reference to the object. Each instance of transformational reasoning was recorded in a memo along with a description of my model of the students' envisioning activity. The combination of objects, operations, outcomes, and the description of what a student may be envisioning constitutes (at best) a second order model of student thinking.

Final Steps

Once the poetic structures and transformational reasoning analyses were completed, I overlaid them to analyze how verbalized transformational reasoning was communicated via a poetic structure, if at all. Finally, I conducted a cross-case analysis to extract meaningful patterns in the ways in which the transformational reasoning was used to advance the collective product of students' mathematical activity.

Results

In this section, I will discuss the ways in which students used poetic structures when verbalizing or refining verbalized transformational reasoning as they developed a collective product. I begin by showing how the various components of transformational reasoning were expressed using poetic structures.

Objects and Outcomes

Of the three components of transformational reasoning, objects and outcomes are the most obvious for the observer to witness. Students often make statements about objects and in the claims that they make, one can find the outcome of their transformational reasoning. Objects were not a major component that needed to be negotiated in the fifth-degree polynomial task since the students were largely working with the object as a fifth-degree polynomial as a graph or an equation.

Outcomes were often unchallenged by the students. However, within a students' own explanation the outcomes were commonly rephrased utilizing a poetic structure. For example, consider Shay's (they/them) initial justification for the fifth-degree polynomial claim.

> It'll always have… it will always have at least one real root, because the two… because it's an odd degree. One end will always approach negative infinity and the other will approach positive infinity. So, it'll have to approach… it will have to cross zero at one point on the negative side.

Underlined above is the poetic structure that Shay used to *elaborate* on the meaning of having one real root. The single underline represents a "direct" repeat, and then the double underline demonstrates what was changed from the first sentence to the last. Shay's transformational reasoning likely involved mentally comparing several generic odd-degree polynomial graphs, where they focused on the tails of each graph and observed that one tends to positive infinity and the other to negative infinity. Then it is likely that Shay envisioned tracing a point along the curve. Shay's new outcome "it will have to cross zero…" indicates a shift in their thinking from "always have at least one real root" which likely resulted from envisioning the curve passing through the x-axis.

Similarly, in Will and Kevin's group, Will provided his intuition for the IVT by using a poetic structure to verbalize his transformational reasoning. Will says, "[The IVT] just basically says, if it goes two different places then it has to go to the place in between those two places". In this quote, the double underlined portion is an *expansion* poetic structure. Will expressed his intuition for the IVT as an IF-THEN statement. The transformational reasoning that he verbalized indicates that Will was imagining two points separated on a coordinate-plane. Then the operation "to go" indicates he likely envisioned connecting the two points as one would by drawing a graph on paper. The expansion poetic structure served the purpose of connecting not only his hypothesis to his conclusion, but the objects and operation to the outcome he envisioned.

As was discussed in the two examples here, the operation that one envisions transforming the objects plays a major role in explaining one's reasoning. In what follows, I will discuss how the operation was warranted through poetic structures.

Warranting Operations

Many of the conversations during the fifth-degree polynomial task involved articulating operations that transform objects. Here I will discuss two examples of pairs of students warranting a claim by making the operation they envisioned transforming objects explicit.

Chloe and Gabe were tasked with determining whether or not $p(x) = x^5 - x - 5$ has a real root on $[0,2]$. The pair began by seeking to factor the polynomial. Once that attempt proved fruitless, they shifted gears to graphing the polynomial. The pair proceeded as expected by evaluating the polynomial at integer values zero and one, followed by $x = ½$ and one of the teacher-researchers observed that the pair appeared to be seeking a sign change.

Teacher-researcher: So, you're just trying to see if the output is positive or negative?

Gabe: Right, to see if it crosses the, uh, x-axis.

Teacher-researcher: So, how would that tell you that it crosses the x-axis?

Gabe: For this one it gives insight into seeing what areas, like, if we see that, if we saw that this was positive [pointing with left hand to the y-intercept $(0, -5)$ and with the right hand to a location above the x-axis where $x = ½$] we'd be like, somewhere between here and here [moves hand from the location of $x = 0$ to $x = ½$ on the x-axis], oh it definitely crossed for sure.

Teacher-researcher: Mm, why's that?

Gabe: Cuz it's a continuous function and it's not just going to skip a chunk of the y-axis. So, it would have to cross the x-axis at some point.

There are a number of poetic structures in this short transcript excerpt. The most salient poetic structure to pay attention to is Gabe's elaboration in line 4 above (double underlines). Gabe elaborated on his explanation in line 2, after the teacher-researcher sought to ascertain the goal of Chloe and Gabe's activity. In line 4, Gabe expanded on the operation he envisioned as transforming the objects, but still he glossed over the hidden assumption. In line 5, the teacher-researcher probed Gabe's thinking further. This prompted Gabe to make the operation explicit: he was using continuity. Gabe's statement about continuity is an *elaboration* (dotted underline) given that he replaced the warrant from line 4 but kept the same conclusion. Importantly, the explication of this warrant proved to be a critical point in the students' activity, and it later showed up as a major warrant in their written argument.

Andrew and Jovan. The bulk of Andrew and Jovan's activity involved trying out different operations on fifth-degree polynomials and determining whether or not that operation preserved the fifth-degree polynomial. Consider the following exchange that occurs between Andrew and Jovan.

Andrew: Um, ok so like, if it's a negative 5th degree polynomial, that still has one real root, it's just on the opposite side of the y-axis.

Jovan: Yeah, it would just be going. It would start going down.

Andrew: Mhmm. It would begin going down, do your bumps [as he is speaking, he moves his right hand from left to right, and wiggles his fingers to indicate the 'bumps'], and then continue going in the direction it started, right?

Jovan: Well since it's odd, it would just be, instead of going up from left to right, it would be going… right? [While Jovan is speaking, Andrew draws a curve in the air with his stylus]

Andrew: Right, like the whole function would be coming from the top, squiggly squiggly, and then continue downward through the x-axis [as Andrew is speaking, he sketches the function he is imagining, doing the squiggles as he says squiggles, and showing the overall trend of the curve]. Or if it already was, it would just be downward, but at some point, it will cross…[[Andrew goes on to compare odd and even degree polynomials]]

Jovan: Yeah.

Andrew: So, a negative wouldn't change that…

Throughout this entire exchange, Andrew and Jovan are negotiating Andrew's operation of "negative". Andrew's transformational reasoning likely involved a series of transformations on different objects. For starters, it is likely Andrew appended a negative to the leading coefficient of some polynomial equation he was working with in his mind. Then, given his routine use of DESMOS in class, he likely plotted the equation he had in mind. Once he and Jovan began discussing the claim, it is likely Andrew then shifted to thinking of the graph as emergent, given the way he sketched the graph in the air and described the shape of the graph.

Jovan began by trying to warrant Andrew's operation by describing what he envisioned happened to the graph after operating on it via "negative". Andrew *expands* on Jovan's verbalization in line 3 by describing what he envisioned in terms of the outcome of the transformation. Jovan is still seeking clarification, and he now compares the shapes of fifth-degree polynomials to negative fifth-degree polynomials. Andrew's final *expansion* in line 5 of the excerpt satisfied Jovan's uncertainty, as evidenced by Jovan's "yeah" in line 6. The negotiation of this operation revealed the key point for Andrew and Jovan: that a fifth-degree polynomial has a real root whether its graph emerges up from the left to the right, or down from the left to the right.

Discussion and Conclusion

The findings presented above suggest that the teacher can anticipate that conversations about transformations of objects are likely to be a key part of mathematically productive collaborations. In terms of components of transformational reasoning, these transformations are what I referred to above as the operations. As the students collaborated on the collective solution to the fifth-degree polynomial task, they sought to warrant their belief by verbalizing the operation with which they transformed the objects. Gabe warranted that the polynomial was continuous. This was a pivotal connection with which Gabe and Chloe argued that a root existed. For Andrew and Jovan, the operation that Andrew expanded on allowed the pair to warrant that the fifth-degree polynomial claim was plausible at all.

Additionally, the findings suggest that a signal of successful verbalized transformational reasoning is actively connecting the claim to the observation of the outcome of the transformation that was envisioned. For example, Shay began by stating confidently that a root always exists, but through use of the operation "to approach", they were able to connect the claim to the definition of a root by stating that "it must cross zero". Despite the encouraging findings from this study, more research is needed to ascertain productive ways to support students to engage in transformational reasoning actively.

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Examining the Self-Regulation Strategies of Students in a Developmental Mathematics Course

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Self-regulated learning (SRL) skills have been linked to academic success, particularly in developmental mathematics courses. This study investigates student SRL strategies in a developmental College Algebra course, taught using Inquiry-Based Learning (IBL) and supplemented with ALEKS. While SRL strategies have been investigated in a variety of contexts, research is lacking in IBL courses with ALEKS support. Data were collected from seventy-four students using open-ended surveys, which were used to answer the research question: What SRL strategies do students use in an IBL with ALEKS course? This was answered in three learning contexts: in-class, ALEKS time, and outside studying. Findings indicate that seeking social assistance and seeking information, especially from ALEKS were the most reported strategies, followed by note-taking and practicing problems. However, very few students reported using SRL strategies related to goal-setting and planning. This suggests that students may need more support in developing SRL strategies, such as through interventions.

Keywords: self-regulated learning, inquiry-based learning, ALEKS, developmental mathematics

Over the last few decades, there have been concerns about the college-readiness of students entering colleges and universities, including discussions on how to prepare students for university level mathematics. One of the principle ways to approach this issue is by including developmental mathematics courses in their catalog, which often have large enrollment rates (Chen, 2016). However, there is often a high withdrawal rate for the students in developmental mathematics courses (Adelman, 2004).

One vital component of student success in mathematics classes is students' ability to self-regulate their learning. Self-regulated learning (SRL) is the ability of a student to take an active part in their learning process, by setting goals and motivating themselves, using different strategies to complete their learning task, and reflecting on their learning and strategy usage (Zimmerman, 2002). Students with more SRL strategies have often improved in their mathematics scores (Kramarski & Gutman, 2006; Mevarech & Fridkin, 2006; Throndsen, 2011), while students who do not possess these SRL strategies often struggle more in mathematics courses (Bol & Garner, 2011, Isaacson & Fujita, 2006; Ley & Young, 1998; Zimmerman, 2002).

SRL interventions have yielded positive results, helping both improve students' SRL skills, as well as their mathematics scores (Bol et al., 2016). Another possible approach to improving students' SRL skills is by creating more active learning-based classes (McDonough & Sullivan 2008; Zimmerman 2002). For example, classes using a problem-based learning model showed increases in SRL skills (Blumberg 2000; Sungur & Tekkaya, 2006). This also connects with the university efforts to create more active learning-based mathematics courses, as the benefits of active learning have been well documented (e.g. Freeman et al., 2014).

One specific type of active learning model is known as Inquiry-based learning (IBL), where students are given investigative tasks that allow them to be engaged in the creation of important mathematics (Laursen & Rasmussen, 2019). Generally, in IBL classes students are practicing mathematics as a mathematician would. For example, creating and testing conjectures, making generalizations, providing justifications, and reflecting on their conclusions (Mesa et al., 2020). IBL generally involves student collaboration, where students work together to build their

knowledge, while the instructor facilitates the learning rather than being the sole source of knowledge. IBL has generally seen positive effects on academic achievement in mathematics classrooms, as well as reducing learning gaps for underrepresented students (e.g. Kogan & Laursen, 2014; Laursen et al., 2014).

Another common change universities are making is by implementing technology like intelligent tutoring software. A specific type of software that does this is Assessment and Learning in Knowledge Spaces (ALEKS), which has been used more in the last decade in mathematics courses (Cosyn et al., 2021). ALEKS provides examples and tutorials to students while assessing their understanding. Research on ALEKS has been mixed, with some research finding ALEKS was at least as good as a traditional lecture course (Sun et al., 2021) but some research finding ALEKS did not affect the passing rates (Childers & Lu, 2017). However, in mathematics classes, ALEKS seemed to have a more positive effect when used as a supplement to an active-learning style classroom (Boyce & O'Halloran 2020; Sun et al., 2021).

Theoretical Perspective

One prominent model of SRL is Zimmerman's (2002) model, which has three phases: the forethought phase, the performance phase, and the self-reflection phase (Zimmerman, 2002). The forethought phase includes analyzing the task at hand and motivating oneself to perform the task. Analyzing the task includes setting goals and determining specific strategies to accomplish these goals. Motivation includes self-efficacy, which is a student's belief in how well they can complete a task. Self-efficacy is often affected by the student's perceived skill level, outside factors like feedback from instructors or peers, and observing peers performing the task (Bandura, 1977). Motivation also includes intrinsic motivation, or how much the student is interested in the task and what they hope to achieve by completing the task.

The performance phase occurs during the completion of the task. This phase involves how a student completes a task, such as how they focus, the time they spend on the task, and the different strategies they use. The performance phase includes self-observation, where the student monitors their progress, tries different strategies, and changes or adopts them depending on their perceived effectiveness. Examples of strategies may be working with peers or taking breaks.

Once the task is complete, the self-reflection phase begins. In this phase, the student evaluates themselves, for example by comparing their task completion to the task standards. This also includes a self-reaction, where the student may feel satisfied with the task and adapt the new learning strategies or become defensive and not change their strategies. This then leads into and affects the forethought phase of the next task, completing the cycle.

This study sought to explore the SRL strategies of students in developmental mathematics courses. My research questions are: What self-regulated learning strategies do students use in an IBL with ALEKS course?. Specifically, (1): What self-regulated learning strategies do students use during the IBL class-time? (2) What self-regulated learning strategies do students use during the ALEKS homework time? (3) What self-regulated learning strategies do students use during outside-class time, not including homework?

Methods

The course studied is a developmental but credit-bearing mathematics course, College Algebra with Review, taught at a mid-sized, PhD granting university in the Midwest. This course has the same content as the regular College Algebra course but has more credit hours and therefore more in-class time. This extra in-class time allows students to work through the same content at a slower pace and alleviates some concerns about active learning requiring more class

time. The content of this course is basic algebra, coordinate geometry, and functions. The course description on the syllabus includes "[the] course is taught using Inquiry-Based Learning with a focus on collaboration". Instructors of this course are given advice by the course coordinator and materials to use during the semester, but the day-to-day activities and teaching methods are up to the instructor, allowing variability in the IBL with ALEKS implementation by class section.

Five instructors and their students participated; one instructor taught two sections and the others each taught one section. Students from all sections were chosen using self-selection sampling, where the participants could participate in a Qualtrics survey. This survey was administered after the first exam. The number of students participating in each section were: Section 1 (both sections from one instructor): 23 (28%); Section 2: 16 (41%); Section 3: 9 (23%); Section 4: 12 (30%); and Section 5: 14 (35%), for a total of 74 participants.

Data was collected using student questionnaires containing open-ended questions to provide in-depth understanding of the students' SRL strategies. The questionnaire was designed to avoid providing examples of strategies to the students. The questionnaire was designed around Zimmerman & Pons' (1986) structured interview, which was developed to assess students' use of SRL strategies, and included questions about the following categories of learning strategies: self-evaluation, organizing, goal-setting and planning, seeking information, keeping records and monitoring, environmental structuring, self-consequences, rehearsing and memorizing, seeking social assistance, and reviewing records. For this project, I am interested in the dichotomy of the class setup, where students learn with IBL in the class but have ALEKS software guide them for homework. Therefore, I modified the categories to include both in-class and ALEKS contexts to illustrate and shed light on these contextual differences.

I initially coded the responses from questions 1-3 using the modified Zimmerman & Pons' (1986) codebook. These questions were phrased by giving a context and asking students to write out their methods. For example, question 1 was "[The math course] has in-class sessions that may involve listening to the instructor, working on problems individually, collaborating in groups, or having whole-class discussions. What particular methods, if any, do you use to help you learn and understand the mathematics content during class?" Each participant response could receive multiple codes, including the same code multiple times, in the case participants described two different strategies that fit with the same code. As the responses were coded, the categories were split into a few different subcategories or codes, based on the student responses, and some categories were added. After completing the codebook for questions 1-3, questions 4 and 5 were then coded with the same codebook. Questions 4 and 5 asked more broadly about their study habits. For example, "Think about your general process of preparing for and studying for [this math course]. How do you prepare for and study for this class?". Therefore, the responses were generally longer and more complex but still ranged between one and five sentences. When coding questions 4 and 5, two extra categories were added.

Results

What SRL Strategies do Students use During the IBL Class Time?

Table 1 lists the frequencies of codes in relation to each question. Seeking social assistance occurred most frequently with 58 participants using at least one of the strategies in this category. The prevalence of the strategies focused on social assistance is consistent with the class structure, which emphasized collaborative learning. 40 participants described learning with their peers, which were coded as Peer-Learning or Peer-Seeking. Peer-Learning was used when participants described working with groups generally and often had statements concerning

collaborative learning. For example, one participant described this as "finding an answer and comparing it with another person, if we get different answers, then I solve using their method and see if it would apply better." The code Peer-Seeking was used on statements describing asking group members for help if they were struggling, such as "ask my group if I don't understand". With this mixture of codes, the participants appeared to be using the group work sessions both to get help from their peers, as well as for collaborating with others, suggesting students may use group work differently. In addition, 18 participants were coded with Instructor-Seeking and Tutoring-Seeking, with 16 and 2 participants respectively. This is consistent with the class style, as the instructor walked around during group work, providing help for groups.

Table 1. Codes and Frequencies

Category	Code	Q1	Q2	Q3	Q4-5	Total
Seeking social assistance	Peer-Learning	29	1	9	6	36
	Peer-Seeking	11	6	2	15	23
	Instructor-Seeking	16	0	3	20	29
	Tutoring-Seeking	2	3	2	7	11
Seeking information	ALEKS-Seeking	8	2	5	16	29
	Explanations-Seeking	0	39	0	10	41
	Notes-Seeking	6	11	0	27	32
	Video-Seeking	2	6	5	13	22
	Internet-Seeking	2	9	4	8	19
Keeping records and monitoring in class	Taking-Notes	19	0	0	5	22
	ALEKS-Notes	0	16	0	5	17
	Problem-Notes	0	5	0	1	6
Organizing	Summarizing	2	0	0	1	2
Self-Evaluation	Self-Evaluation	3	8	5	3	15
Rehearsing and memorizing	Rehearsing-Flashcards	1	0	1	0	2
	Rehearsing-Formulas	1	1	1	5	6
	Rehearsing-Problems	13	3	7	0	22
	Rehearsing-Steps	0	1	0	0	1
	Rehearsing-Review	2	0	8	10	14
Environmental structuring	Music-Structuring	0	1	0	0	1
	Library-Structuring	0	2	1	2	5
Reviewing records	Reviewing-Quizzes	0	0	4	6	9
	Reviewing-Notes	0	0	10	0	10
Goal-Setting/Planning	Goal-Setting	0	0	0	7	7
Effort Regulation	Effort-Regulation	0	2	0	8	10
	Paying-Attention	0	0	0	2	2

The next most frequent categories were Keeping records and monitoring in class (19), Seeking information (17), and Rehearsing and memorizing (17). The only code that was utilized in Question 1 was Taking-Notes. The category Seeking information had 17 instances, spread across four codes: ALEKS-Seeking (8), Notes-Seeking (5), Video-Seeking (2), and Internet-

Seeking (2), while Explanations-Seeking had no instances. However, Question 1 asked about in-class strategies, so it is possible that the students interpreted the question as asking about the course generally, rather than only their in-class activity. Finally, six participants were coded with Note-Seeking, where they described using previously made notes to help with their understanding, such as "[Using] notes from previous work, either in class notes or ALEKS".

For the Rehearsing and memorizing (17) category, four different codes were used: Rehearsing-Problems (13), Rehearsing-Review (2), Rehearsing-Flashcards (1), and Rehearsing-Formulas (1), with no participant being coded with Rehearsing-Steps. 17 participants described using practice problems to help with their understanding, such as "writing the same question repetitively and troubleshooting until I find a method that applies to each question.". This was the most common code, with only a few other participants using the other strategies.

The categories Summarizing and Self-Evaluation were coded to two and three participants respectively. Summarizing most likely referred to outside of class activities, such as "write examples/notes to revise later if I need to remember how to do something". Self-evaluation was only described twice, such as "Finding an answer and comparing it with another person, if we get different answers, then I solve using their method and see if it would apply better.". The four categories Environmental structuring, Reviewing records, Goal-Setting and Planning, and Effort Regulation were not coded to any participants.

What SRL Strategies do Students use During the ALEKS Work Time?

The category Seeking information was the most frequently used category of codes, with a 67 participants. The most common code was Explanation-Seeking, with 39 participants. This referred to the ALEKS showing explanations. Notes-Seeking (11), Internet-Seeking (9), and Video-Seeking (6) showed that participants were also referring back to their classroom notes or worksheets to help study and using the internet and video websites as well. While ALEKS helps explain topics and has videos, some participants were still using outside resources. Finally, ALEKS-Seeking was used twice, where participants used ALEKS more often than required.

The next most frequent category is Keeping records and monitoring in class, with 21 participants. With 16 participants making statements about taking notes while working in ALEKS, note-taking appears to have a similar importance to students as note taking in class, which has 19 such codes. This strategy also shows more involvement with the software, as those students are not only trying to solve the problems and read the explanations but trying to record important information. The second code is Problem-Notes, as six participants considered writing down the problem in their notebook to solve, a strategy to help them learn.

The category Seeking social assistance was coded for 10 participants. Seven participants were coded with Peer-Seeking or Peer-Learning, implying that some students work together on their ALEKS work. Three participants were also coded with Tutoring-Seeking. Eight participants also used Self-Evaluation strategies during their ALEKS time. For example, one participant described evaluating their work as: "[I] try solving an example problem on my own before looking at the answer". This strategy shows more involvement with self-guided learning, where the student is working through the problem themselves, and evaluating the work afterwards.

Rehearsing and memorizing had five participants coded, with three participants describing using more practice problems in ALEKS to study. One participant described writing out the steps to help, while another participant described writing down equations. This may imply students are focusing on working through the problems in ALEKS as strategies.

Finally, three and two participants respectively were coded in the category Environmental structuring and Effort Regulation. These focused on working in the library. The two participants

coded in Effort-Regulation described taking regular breaks from ALEKS. No participant was coded with the categories Summarizing, Reviewing Records, or Goal-setting and planning. While Summarizing and Reviewing Records may not be as relevant to ALEKS work, it was surprising that there were no mentions of Goal-seeking and planning. With the course requirements of three hours of ALEKS a week and a certain number of modules completed, setting goals seems like it would be a common strategy to be utilized.

What SRL Strategies do Students use During Outside-class, Non-ALEKS Time?

The category Rehearsing and memorizing contained seventeen participants, with the majority of codes being Rehearsing-Problems and Rehearsing-Review. The students described practicing problems to help them review. For exam 1, which was administered prior to this survey, the students were all given study guides covering review problems. Eight participants specifically described using these to help review, such as "I do the optional review work and check my answers with the key". One participant also described reviewing using flash cards, and another by writing down all the formulas multiple times.

Sixteen participants were coded with Seeking social assistance, with 11 students seeking peer support specifically. These included statements about study groups, such as "helping and explaining to a friend". The category Seeking information had fourteen participants, with relatively equal amounts of codes for using ALEKS, the internet, and online videos to review.

Fourteen students were categorized in Reviewing records. Ten students were coded with Reviewing-Notes, using their past lecture or ALEKS notes for better review. For example, one participant described using their notes by: "looking through the notes taken during class and retrying problems that I got wrong and seeing if I could get them correct", where their in-class notes were helpful, using the important problems they recorded. In addition, four students were coded with Reviewing-Quizzes, where they used their past quizzes and feedback to review.

Finally, five students were classified using the code Self-Evaluation. For example, this participant describes working through problems they struggled with, then assessing their work: "[I] write a question on a worksheet I struggled with and try to answer it without looking back at the worksheet". One student was also coded Environmental structuring, having studied in the library. Summarizing, Goal-setting, and Effort-Regulation were not described in question 3.

What SRL Strategies do Students use in an IBL With ALEKS Course?

The purpose of questions 4 and 5 were to help students provide strategies regarding goal-setting and planning, as well as giving them a chance to list more learning strategies they may have forgotten. Seven participants described goal-setting across questions 4 and 5, such as: "I work with my group outside of class on the assignment given to us for the next class at the library". Here, they had a goal of preparing for class beforehand, and planned study sessions with their group members to help be prepared when class starts. The category Effort-Regulation was described by 10 participants. For example: "I get up frequently and walk away from my computer, either to go outside or grab food to give myself a break". Two participants were also coded with the code Paying-Attention. For example, one participant wrote "I prepare [for class] by getting as much sleep as I can", which may help the student stay focused during class.

The category Seeking information saw higher rates of ALEKS-Seeking, Notes-Seeking, and Video-Seeking in questions 4 and 5, compared to questions 1-3. This implies that there are still a variety of sources students look to when the material is difficult, with their notes being a more common source, but also ALEKS and videos online. The most frequent category of SRL strategies is Seeking information. Using ALEKS to help with studying contained the most

substantial codes. Video-Seeking and Internet-Seeking had frequencies of 22 and 19 respectively. A similar number of participants (33) described using the notes they took, either in-class or during ALEKS, to help them study. Seeking social assistance was the next most common category, focusing on their peers, with 23 Peer-Seeking codes and 36 Peer-Learning codes. 29 and 11 participants were coded with Instructor-Seeking and Tutoring-Seeking respectively; thus, some students did seek out their instructors if they needed additional support. However, most of the codes involving asking the instructor for help were in Question 1, so it is possible the Instructor-Seeking occurred mostly during class.

Keeping records and monitoring in class, as well as Rehearsing and memorizing were the next most frequent categories, which was also described in previous questions. Rehearsing and memorizing focused mostly on using and practicing with practice problems (22), which is consistent with mathematics courses. 14 participants also described using instructor created exam review sheets. Although other strategies for memorization were listed (such as using notecards), they were only described by a few participants. Self-Evaluation was mentioned by only 15 participants. Most of these instances revolved around using ALEKS. Reviewing quizzes (9), Effort-regulation (12), Goal-setting (7), Environmental structuring (6), and Summarizing (2) had less frequencies compared to the other codes and categories. It was expected that Goal-setting, Environmental structuring, and Effort-regulation would have more instances, as the course had relatively strict requirements with ALEKS. However, it is possible that the participants did not consider these strategies or thought they did not apply based on the questions in the survey.

Discussion

Overall, the SRL strategies used by the participants matched the categories originally given by Zimmerman and Pons (1986). The most common SRL strategies used were Seeking social assistance and Seeking information, focusing on ALEKS and online sources. The strategies of taking and reviewing notes from lecture and ALEKS were also commonly used strategies, as well as practicing problems to help with memorization. However, other memorization strategies, as well as the SRL strategies of Self-Evaluation, Organizing, Environmental structuring, Goal-Setting, and Effort-Regulation were not utilized by many participants. Instead, participants seemed to be focusing on strategies to learn the in-class material and using ALEKS for practice. This also connects to the issue that students did not describe many goal-setting strategies, even when prompted, which is an integral part of the SRL cycle. This may suggest that the course structure is beneficial for the Performance Phase of Zimmerman's (2002) cycle, but students are struggling with the Forethought Phase. However, this might be addressed by incorporating literature on goal setting, such as weekly goal-setting journals (Bol et al., 2016).

One limitation is that the survey was administered close to the start of the semester. This is the time students are still getting used to the course, have just completed their first exam. Therefore, after getting their exam scores, students may reflect on their study strategies and adapt as the semester goes on. As the in-term exams are the largest assessments students receive (outside of the final exam), students may not have been focusing much on the Self-Reflection phase. In this case, all of the students' feedback came from ALEKS and smaller weekly quizzes, so they may not have been evaluating their general study habits, and instead focused on smaller issues they were having with specific content. This could also explain some of the issues with goal setting described above; that is, it is possible students may have started adding more goals after evaluating themselves after the exam. It would be beneficial to do more research specifically on goal-setting in this style class, or to see if any of these SRL strategies change throughout the semester.

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Understanding the Implementation and Impacts of an Innovative Corequisite Math Course

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This paper explores findings from a quasi-experimental study, focusing on the implementation and impacts of an innovative corequisite mathematics intervention. Conducted in a large U.S. community college, the study is evaluating the impacts of the Statway Corequisite course on student outcomes compared to the business-as-usual statistics corequisite course. Preliminary analyses show positive student outcomes in courses using the curriculum. The paper investigates the extent to which key implementation fidelity components are associated with those outcomes. These components include the use of a contextualized curriculum designed to be relevant to students, incorporating student collaboration in online and in-person classes, and addressing the social-emotional and academic needs of students. Understanding the impact of these components on student outcomes has implications for the design of corequisite mathematics coursework.

Keywords: Corequisite, fidelity of implementation, mathematics curricula, social-emotional learning, pedagogy

This paper focuses on understanding the implementation and impact of an innovative corequisite mathematics intervention in a large-scale quasi-experimental study. Conducted in a large U.S. community college, the goal of the study is to evaluate the impacts of the Statway Corequisite course on student outcomes and to understand the implementation components that relate to those outcomes.

Background

Over the last two decades, evidence has shown that structural changes such as math pathway implementation and corequisite model adoption have helped more students advance to, and succeed in, college-level math than traditional models of developmental remediation (e.g., Jenkins et al, 2021). That evidence has led to many states, including Tennessee, Texas, Louisiana, and California, championing corequisites in their developmental education strategies and to many colleges incorporating multiple math pathways for students to earn college math credit (Education Commission of the States, n.d., Hodges et al., 2020).

Statway Corequisite is a statistical reasoning curricular offering developed by Carnegie Math Pathways (CMP) at WestEd. It is designed as a one-term college-level course with a corresponding corequisite support course that aims to accelerate students through a college-level statistics course in a single semester. The college component of Statway Corequisite covers the learning outcomes of standard introductory college statistics courses, supporting both in-person and online formats. The foundational concepts and skills needed for success in the college-level lessons are included in the corequisite component (e.g., percentages, simplifying expressions, probability, etc.). Statway Corequisite combines a contextualized curriculum grounded in real-

world application with an active, collaborative learning approach and student supports that help build learner mindsets and a sense of belonging.

Statway Corequisite instructional design is based on research on teaching for understanding and the development of mathematical practices (Bransford, Brown, & Cocking, 2000). The Statway Corequisite pedagogy emphasizes productive struggle with challenging problems (Schmidt & Bjork, 1992); making conceptual connections explicit (Hiebert & Grouws, 2007); deliberate as opposed to routine practice (Ericsson, 2008; Pashler, Rohrer, Cepeda, & Carpenter, 2007); opportunities for rich mathematical discourse (Moschkovich, 2007); and collaborative learning in promoting mathematical sense-making (Esmonde & Langer-Osuna, 2013; Webb, 2009).

Statway Corequisite Design Principles

The design principles for Statway Corequisite approach to curriculum, pedagogy, and student supports include the following:

Students feel that the course has value. Many students who have struggled with math often question the value of mathematics in their daily lives and long-term goals. Statway Corequisite is designed to help students see this value through a curriculum that includes relevant contexts that students see as personally engaging (Hulleman & Harackiewicz, 2009). Given the curriculum's emphasis on relevance and engagement, instructors are discouraged from introducing other, more traditional materials and approaches. In implementing the course, the focus is on depth into core ideas and skills rather than coverage of the full curriculum.

Instructors engage students in collaborative group work. Modeled after typical mathematics lessons in Japan (Shimizu, 1999), the instructional approach relies on small group, collaborative work within the four phases of a Problem Cycle: The Problem Launch to prepare students for productive struggle, Working the Problem in small groups, Discussing the Problem to make public students' thinking, and Conclusion to highlight the key concepts drawn from student thinking. The Problem Cycle provides a structure for instructors to facilitate group work by preparing students for collaboration and bringing ideas together that surface during the collaborations. Students collaborating in groups is key to this approach and is often an indicator of its use. The online version of the course requires students to meet synchronously to collaborate on the lessons' problem situations. Student interactions throughout the online lessons and debriefs are recorded and shared with the instructor, who then responds accordingly.

Students demonstrate Productive Persistence, or the combination of tenacity and good strategies. Research has shown that psychological aspects such as mindsets, "grit" or passion, and perseverance for long-term goals, self-control, and engagement are important predictors of academic achievement (Dweck, Walton, & Cohen, 2014). Statway Corequisite's approach supports these non-cognitive factors through a collection of Productive Persistence activities and routines, including a growth mindset writing activity, a contract activity to develop a sense of belonging, supports for forming groups to strengthen student ties, and instructor email routines that foster a sense of community.

Framing Fidelity of Implementation

Program fidelity is typically defined as "how well an intervention is implemented in comparison with the original program design" (O'Donnell, 2008, p. 22). The level at which a program is implemented with fidelity is often related to the results (Hill & Erickson, 2019). Various frameworks for fidelity exist in literature (e.g., O'Donnell, 2008; Century et al., 2010; Bos et. al, 2023), with most rooted in the five factors proposed by Dane and Schneider (1998):

Adherence to the intended program, exposure to the program, quality of delivery, program differentiation between conditions, and student responsiveness to the program.

For this project, we focused our definition of fidelity on the elements of the course related to its design principles, roughly aligning them with Dane and Schneider's framework elements of adherence, student responsiveness, and program differentiation, as summarized in Table 1. Assessing the quality of delivery was outside of the scope of the project. Elements of exposure, such as students engaging in Productive Persistence activities, overlapped with adherence and were not analyzed separately. In addition, since the focus of the curriculum is depth, not breadth, traditional indicators of exposure, such as the number of lessons completed, do not directly apply. For **adherence**, we focus on the teachers' implementation of the course design principles: (1) the primary curricular resource used is Statway Corequisite, designed to be relevant and valuable to the student population; (2) whether or not teachers use group work, and (3) whether or not teachers teach the Productive Persistence lessons that designers designated as essential. **Student responsiveness** focuses on whether students report the math class, as implemented, being relevant, and whether they show improvement in non-cognitive outcomes related to productive persistence. As well as being a fidelity indicator, improvement in non-cognitive outcomes is also a secondary outcome measure of the full study. Student responsiveness would also include whether students engaged in collaborative work, which is being measured in adherence. **Program Differentiation** looks at what is the same or different between the treatment and control conditions for the key elements of fidelity. We will look at similarities and differences in the curricula taught, rates at which teachers' use group work, differences in students' report of math class relevance, and differences in students' non-cognitive outcomes.

Table 1. Fidelity of implementation of the Statway Course.

Fidelity factor	Key components		
	Students feel that the course has value	Instructors engage students in collaborative work	Students demonstrate productive persistence
Adherence	Teacher uses Statway	Teacher uses group work	Teacher teaches required productive persistence activities
Student Responsiveness	Students report math class is relevant	[See Adherence]	Students' non-cognitive outcomes improve
Program Differentiation	Difference of curricular materials Difference in students' report of math class relevance	Difference in rates teachers implement group work	Difference in students' non-cognitive outcomes by condition

While student responsiveness and program differentiation are important, this paper focuses on describing the fidelity in terms of adherence, with the presentation also including treatment impact analyses using adherence fidelity as a moderator.

Methods

Study Context

This fidelity of implementation analysis is set within the context of a large-scale quasi-experimental study of the impact the Statway Corequisite course has on student outcomes, such as grades and persistence. The research team partnered with a large community college system in the Southwestern United States and spanned eight academic terms. The study team collected baseline data on students' prior math achievement and demographics, and course-taking outcomes (grades, credits). In addition, the team collected data outlined in the "Methods" section to assess fidelity of implementation, non-cognitive student outcomes, and to understand other contextual factors.

Results of the quasi-experimental design (QED) study, with a sample of over 1,800 students, indicate that students in the treatment course, in comparison to business-as-usual students, earned higher course grades (ES = 0.51, $p < 0.001$); were almost 50% less likely to earn a grade of D, F, incomplete or withdraw from the course (DFWI) compared to business-as-usual ($p < 0.001$); and, though non-significant, were about 20% more likely to complete the course. (Walters et al., in press). The current paper explores fidelity components that may have led to these promising results, with the presentation also examining whether the treatment impact varies by level of fidelity.

Study Design

The preliminary quasi-experimental analysis was conducted with 1,861 students (771 in Statway and 1,090 in business as usual) across multiple campuses of a large community college system. This paper focuses on adherence fidelity components directly under teacher control, which applies only to the treatment teachers in the main sample. The research questions are:

- To what extent is Statway Corequisite implemented with adherence fidelity?
- Does the treatment impact vary by the level of adherence fidelity?

Analysis related to the second question will be included in the presentation.

Teachers were chosen as a unit for analysis because we did not see meaningful variation from term to term with respect to the fidelity components. The sample for fidelity analysis included 17 treatment instructors, where 10 taught in person and seven taught both in person and online. Each instructor taught up to three sections each of the eight study terms. Class sections varied between three and 18 students, with a total of 49 in-person sections and 14 online sections.

Over the course of each study term, all teachers were asked to complete logs and then complete a post-term survey at the end of the term. Most, but not all, teachers of in-person sections were observed one time each term and interviewed soon after the observation. Teachers of online sections were interviewed in the second half of the term. At the end of the study, all treatment teachers were asked to complete a fidelity survey. In addition, treatment students' online platform usage records were collected for all but two terms.

Data Sources

The data sources are detailed below. Unless otherwise noted, treatment and control teachers completed both using the same protocol.

Instructor Fidelity Survey. This survey was administered to treatment teachers after the completion of other study activities. Teachers were asked the percentage of course materials used

that were Statway Corequisite, if group work was used, and which Productive Persistence activities (e.g., email routines and forming groups) they implemented in their classroom.

Post-Term Instructor Survey. The post-survey was administered at the end of each semester and included questions about teaching practices, such as the use of small groups.

Instructor Logs. Logs were administered six times in the fall and spring semesters and four times in the summer terms, approximately every two weeks. Log questions included how often students worked in small groups. Responses were averaged for each teacher and each term that they taught a study course.

Observations. Trained researchers observed classes during all semesters, excluding the short summer terms. They observed each teacher's in-person class at least once over the course of the study and often once within each term they taught. The team coded the observations using the Toolkit for Assessing Mathematics Instruction – Observation Protocol (TAMI-OP), a matrix covering a variety of teacher and student behaviors in the classroom divided into two-minute intervals (Hayward, Laursen, & Weston, 2021). The two main indicators of fidelity were the percentage of class time in which the variables "WG" (Working in groups on structured group work) and "OG" (Other group activity, such as a Think/Pair/Share) were coded.

Interviews. Most teachers participated in one interview a term. Interview topics included describing a typical lesson, differences between a typical lesson and the observed lesson, any modifications made to the curriculum, and any social-emotional lessons taught.

Course Usage Data. Usage data from the online homework platform included records of which students attempted college and corequisite exercises. The number of exercises attempted per student for the college and corequisite components was then averaged for each teacher and term.

Fidelity Measures

For each adherence fidelity component, teachers' implementation was rated as "Meeting Standards" or "Not Meeting Standards," as specified below. In rare instances, interview data were used to adjust ratings if there was strong evidence of meeting or not meeting the metrics. The metrics for meeting standards are:

Uses Statway. Reported on the Fidelity Survey, that 90% or more of the course materials were Statway course materials, and, on average, students completed 20 or more Statway exercises per term. If usage data are unavailable for all terms a teacher taught, the rating is based on the Fidelity Survey.

Uses Group Work. Meets three or more of the metrics in Table 2 below. Using multiple measures helps to account for missing or inconsistent data.

Table 2. "Uses Group Work" Fidelity Metrics

Fidelity Metric	
Online	In Person
Fidelity Survey: On a 3-point scale, teachers reported students working in groups "Occasionally" or "Regularly."	**Observations:** Group work codes WG and OG comprised 25% or more of the observation class time. **Logs:** On average across responses, teachers reported students spending 25% or more of class time working in groups.
Logs: On average across responses, teachers agreed or strongly agreed, on a 4-point scale, that "The majority of my students are consistently engaged [while] working in small groups."	
Post-Survey: On a 4-point scale, teachers reported students working on math problems together in small groups, "Most of my classes," or "Almost all of my classes."	

Teaches required Productive Persistence activities. Reported on the Fidelity Survey, teaching three of the four required Productive Persistence lessons.

Findings

All treatment teachers' implementation met metrics for at least one adherence fidelity component, with all but two teachers (92%) meeting metrics for two or more adherence fidelity components. However, roughly half of the treatment teachers' implementation met metrics for all three adherence fidelity components. Metrics indicate a higher percentage of in-person teachers implemented each fidelity component than online teachers, though the percentage difference is minimal (2% – 7%). There were also high rates of meeting fidelity metrics across all adherence fidelity components, with the total percent of teachers meeting metrics ranging from 75% to 88%. These results are summarized in Table 3.

As mentioned above, the main study's findings indicate students have stronger positive outcomes in the treatment course as compared to the control course, likely due to the implementation of these three adherence fidelity components: using the Statway curriculum, using group work in lessons, and teaching the required Productive Persistence lessons. Additional analyses incorporating these components into the analytic model will be included in the RUME presentation.

Table 3. Teachers meeting fidelity metrics across the sample

		Fidelity component			Number of fidelity components		
		Uses Statway	Uses Group Work	Teaches Required PP Lessons	One or more	Two or more	Three or More
Online	N Meeting	5	5	6	7	6	3
	% Meeting	71%	71%	86%	100%	86%	43%
In Person	N Meeting	15	13	15	17	16	10
	% Meeting	88%	76%	88%	100%	94%	59%
Total	N Meeting	20	18	21	24	22	13
	% Meeting	83%	75%	88%	100%	92%	54%

Implications

The positive QED findings, coupled with the high rates of fidelity of implementation, indicate the key components of the treatment course are likely related to positive student outcomes. Beyond the adoption of this particular course, this has implications for improving all corequisite mathematics courses. Use of curriculum with problem contexts designed to be relevant to students increases interest and engagement, while also helping students connect mathematics use to their lives, to their studies, and future goals. Incorporating collaborative learning strategies that foster mathematical discourse, give students' voice and agency, and provide opportunities to build habits of mathematical reasoning. Practices and routines that promote safe and supportive group learning nurture students' sense of belonging in the classroom, helping them to productively struggle and persist through the course. Addressing all are important for improving student outcomes (Huang, M., 2018).

More broadly, this study demonstrates the importance of measuring fidelity in large impact studies. Without such data, it is impossible to understand the factors that may contribute to an intervention's success and can be used to scale the intervention in other contexts.

Conclusion and Next Steps

This study demonstrated positive student outcomes associated with the Statway Corequisite course, likely due to the course design and use, incorporating student collaboration in online and in-person classes, and addressing the social-emotional and academic needs of students. Additional work will investigate fidelity components related to student responsiveness to confirm and uphold the results from adherence. Program differentiation will be analyzed to further confirm and illustrate any meaningful differences between the treatment and control implementation.

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Artificial Intelligence and Cognitive Demand During a Domain and Range Mathematics Task

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This study examines students' cognition during high demand mathematics tasks when an AI tool is available. Using video recordings, we analyze 20 undergraduates' cognitive demand on a task involving a nonroutine roller coaster application of domain and range. Our findings show wide variability in how students approach the task, how and how often they elected to use ChatGPT (half chose to use it, half did not), and the levels of cognitive demand that students exhibit in their solutions and thinking. Students that solve the task with high demand tend to grapple with its nonroutine aspects and challenge AI output, while students that solve with low demand give decontextualized prompts, struggle to use AI output, and/or trust AI despite hallucinations. We discuss implications that undergraduate mathematics researchers and educators might consider to preserve opportunities for students to experience high demand problem solving.

Keywords: Domain and range, cognitive demand, AI in education

Artificial intelligence (AI) tools are becoming commonplace in higher education. Although the notion of using computers to support, transform, or even replace mathematical thinking is not new (Kaput, 1992; Pea, 1985), AI tools present a new frontier for teachers and students as they can mimic human reasoning at most stages of problem solving. Prior and recent works that have studied AI in mathematics education have reached varying conclusions. For instance, AI can serve as a helpful scaffold when used deliberately (Yoon et al., 2024) and in ways that preserve human demand for thinking. On the other hand, AI tools can stunt learning opportunities when students use them in ways that disable their ability to master mathematics on their own (Bastani et al., 2024). Regardless, findings from these works suggest that ignoring the impact of AI tools on students' thinking and learning opportunities will pose serious detriments to researchers and mathematics practitioners. It is our goal to draw attention to this phenomenon by studying the nature of students' mathematical thinking when AI tools are made available to them.

Examining how students think with AI tools matters because AI prompting is a cognitive activity. When querying an AI chatbot, a student must *envision* what it is they need to know and write a query that will likely generate the information that they need to reach resolution (Subramonyam et al., 2024). At present, we lack research that details the extent to which students (a) engage with mathematical envisioning as they create AI prompts, and (b) how the presence of AI impacts their own cognition as they reason mathematically on demanding tasks. This study addresses both of these gaps. Using video analysis and attending carefully to students' thinking during AI interactions (e.g., Stein & Delaney, 2025), we observed 20 undergraduate students solve high demand tasks when given access to ChatGPT. We ask, *what is the nature of the cognitive demand when students solve a nonroutine mathematics task with AI?*

Theoretical Framework

We examine the changes in cognitive demand during a mathematics task and the interactions between students and AI throughout its enactment. Stein and colleagues refer to a *mathematics*

task as "activity, the purpose of which is to focus students' attention on a particular mathematical idea" (*note:* we omit the "classroom" aspect of their definition as our study was conducted with individual students) (1996, pg. 460). Historically, mathematics tasks have been considered in the context of what students are expected to produce with a given set of resources (Doyle, 1983). AI tools may potentially change this conception, since they function as general-purpose cognitive resources that remove the need for focused student attention, depending on how they are used.

High demand tasks have functioned as a primary source from which students can learn (Shavelson et al., 1986). These tasks demand cognitive processes, such as identifying key information and activating prior knowledge that is useful for constructing solutions (Doyle, 1983). *High cognitive demand tasks* require students to make conceptual connections, generate nonalgorithmic solutions, and consider multiple solution strategies (Boston, 2012; Stein & Smith, 1998). Again historically, technologies have influenced mathematical activity and cognition, including the cognitive demand of tasks (Hollebrands, 2003; Sherman, 2014). However, technologies' potential to mediate student learning has been contingent on students' knowledge about how to use them productively toward the task's goals (Delaney & Kinsey, 2021). As students can use AI tools without sophisticated knowledge of how they work, it is not yet known if, for whom, and to what extent AI tools will influence students' cognitive demand as they solve rigorous math tasks.

Methods

Four researchers with expertise in undergraduate lower-division mathematics education and AI tool integration designed and collected data for this project in spring 2025. Our goal was to investigate the nature of students' mathematical thinking and reasoning when ChatGPT-4o was made available to solve a mixture of closed-form and non-routine questions. The questions were given to students in three separate 20-minute tasks. In this paper, we analyze 20 students' responses to one component of the algebra task (see Figure 1), and the ways in which students' uses of ChatGPT impacted the cognitive demand of the task as they solved it.

We recruited students via physical campus advertisements. To participate, students needed to have taken a mathematics course in the past 12 months to reduce the likelihood that they would be completely unfamiliar with the tasks' mathematics concepts. We also collected information about students' academic year, major, and undergraduate math courses. We recruited from a variety of majors, and 9 of the 20 participants were not explicitly STEM students, although some listed 4-5 math intensive courses on their screener survey. They studied biology, geography, liberal studies (2), marketing, business management, economics, food/nutrition, and psychology.

During a 10-minute pre-task interview, one researcher asked participants additional questions about their typical AI use on school assignments, beliefs about AI, and the extent to which their instructors demonstrated or encouraged AI use in their courses. A detailed account of students' AI beliefs and practices can be found in (Delaney et al., 2025). Following the pre-interview, each participant was instructed to "think aloud" as they solved the math tasks on an iPad screen that recorded their responses temporally. A lead interviewer reminded students to think aloud, asked open-ended questions (e.g., "what are you noticing right now?") during problem solving, and assured participants that we were primarily interested in studying their thinking and reasoning. A second researcher managed video and audio recordings, kept time during the sessions, and took field notes that captured instances where the student appeared to be grappling with the mathematics. We concluded with a post-task 10-minute debrief.

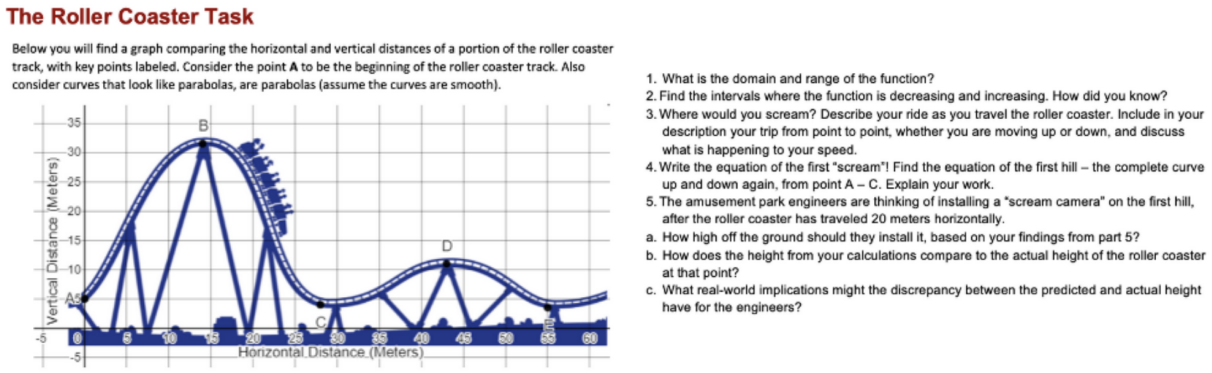

Figure 1. The roller coaster task, adapted from the Virginia Department of Education.

Data Analysis

We focus on the domain and range subcomponent of the Roller Coaster task because it is a routine question (*"What is the domain and range of the function?"*) with a nonroutine solution. We conjectured that if students maintained the cognitive demand of the task, they would attend to the ambiguous right-side of the graph (a high demand feature), which continues despite a fifth point labeled "E" at around (4, 55). We further hypothesized that students' *in situ* experiences with roller coasters would enable them to reason about its domain and range holistically, rather than anchoring on canonical domain/range representations and processes from math courses.

The first and second authors analyzed the Roller Coaster task video recordings (Derry et al., 2010) of each student using the Mathematics Task Framework (Boston, 2012; Stein & Smith, 1998). We contextualized the Mathematics Task Rubric to the Roller Coaster task, first solving it ourselves and mapping students' solutions to definitions from the original framework (see Table 1). We individually rated students' cognitive demand as they gave their solutions, noting that some students experienced higher levels of cognitive demand as they were thinking about domain and range compared to their demand as they wrote down their domain/range solution. After giving our initial ratings, we collectively compared ratings, discussed our interpretations with respect to the framework, and resolved disagreements. For example, we determined criteria for "complex, nonalgorithmic thinking" and reviewed the ratings we gave iteratively to ensure that we were interpreting them consistently. Once initial agreement was reached and ratings stabilized, the first author reviewed all ratings one final time to check for internal consistency.

Table 1. Cognitive demand rating of the roller coaster task, adapted from the Mathematics Task Rubric

Definition	Domain Definition and Example	Range Definition and Example
0 - Not applicable *Guessing, non-math thinking*	Guessing without procedures or mentioning previously learned domain facts, rules, definitions.	Guessing without procedures or mentioning previously learned range facts, rules, definitions.
1 - Memorization *Reproducing previously learned facts, rules, or definitions.*	Reproducing previously learned values for the domain. ***Example:*** "I would say negative to positive infinity…it was 100% on the AP Calc test."	Reproducing previously learned values for the range. ***Example:*** "I can remember from high school…range just keeps going, unless it ends at zero."

2 - Procedures without Connections *Focused on correct answers without connecting to underlying meaning.*	Finds values for the domain without connecting to underlying meaning or ambiguity of where the track ends. ***Example:*** "So the domain is like the x, so it goes from 0 to 60 meters."	Finds values for the range without connecting to underlying meaning or grappling with multiple local nonzero minima. ***Example:*** "The range is the y, so it goes from zero to…34? 32.5? This is the maximum, it's not quite 35."
3 - Procedures with Connections *Procedures for the purpose of developing deeper levels of understanding of mathematical concepts.*	Reasons about domain by connecting to ambiguity about where the track ends. ***Example:*** "...Assume it starts at zero, there's a closed endpoint …and it goes up until 60? 55? I'm going to say infinity, because it goes on with nothing else."	Reasons about both range values (starting and ending) in the context of the graph's min. and max. ***Example:*** "It looks like the lowest point is at 5 meters, and the highest is at the peak, point B, which looks like 32 meters. So yeah, 5 to 32."
4 - Doing Mathematics *Complex, nonalgorithmic thinking, use relevant knowledge and experiences.*	Reasons about the domain in the situated context of a roller coaster. ***Example:*** "It says consider point A to be the start. But the track has to connect with this, eventually, come around to point A."	Reasons about the range in the situated context of a roller coaster. ***Example:*** "B is the highest peak on the graph. There could be a higher peak beyond there, but at least on this graph, the peak is 33."

Findings

Students' strategies, ChatGPT uses, and cognitive demand varied drastically. Most students spent 0:30-4:00 minutes solving for domain and range, although some grappled with its meaning for 4-10 minutes and one aerospace engineering student spent all 20 minutes on it. Figure 2 shows students' cognitive demand, the time they spent, and whether or not they used AI. Table 2 shows students' written responses to the task, if they used AI, and how many prompts they used. Completely coincidentally, 10 students used ChatGPT and 10 did not.

*Table 2. Students' written solutions to the domain and range portions of the roller coaster task. Students who chose to use ChatGPT are listed in the left two columns with number of prompts given in parentheses. *Indicates STEM.*

Student (no AI)	Written Solution	Student (used AI)	Written Solution
Student_1	D: $(-\infty, \infty)$, R: [5, 30]	Student_4 (1)	D: n/a, R: $3 \leq y \leq 32$
Student_2	D: (0, 55), R: (5, 32)	*Student_5 (1)	D: [0, 55], R: [4, 32]
Student_3	D: $(0, \infty)$, R: (5, 32)	*Student_6 (5)	D: (0, 55), R: (5, 30)
Student_9	D: (0, 60), R: (0, 35)	Student_7 (3)	D: [5, 10], R: $[5, \infty)$
*Student_10	D: (0, 63), R: (5, 32)	Student_8 (3)	D: $(0, \infty)$, R: (5, 32)
*Student_11	D: $[0, \infty)$, R: [4, 32]	Student_13 (2)	D: $(-\infty, \infty)$, R: (4, 32)

*Student_12	D: {x | x ≥ 0), R:{y | y ≥ 0)	*Student_14 (1)	D: [0, ∞), R: [3, 33]
*Student_15	D: [0, ∞), R: [3, 32]	*Student_16 (1)	D: (0, ∞), R: (5, 32)
*Student_18	D: [0, 60], R: [0, 32]	Student_17 (8)	D: [5, 55], R: [-5, 35]
*Student_20	D: [0, 62.5], R: [4, 32]	*Student_19 (1)	D: [0, ∞), R: [0, 32]

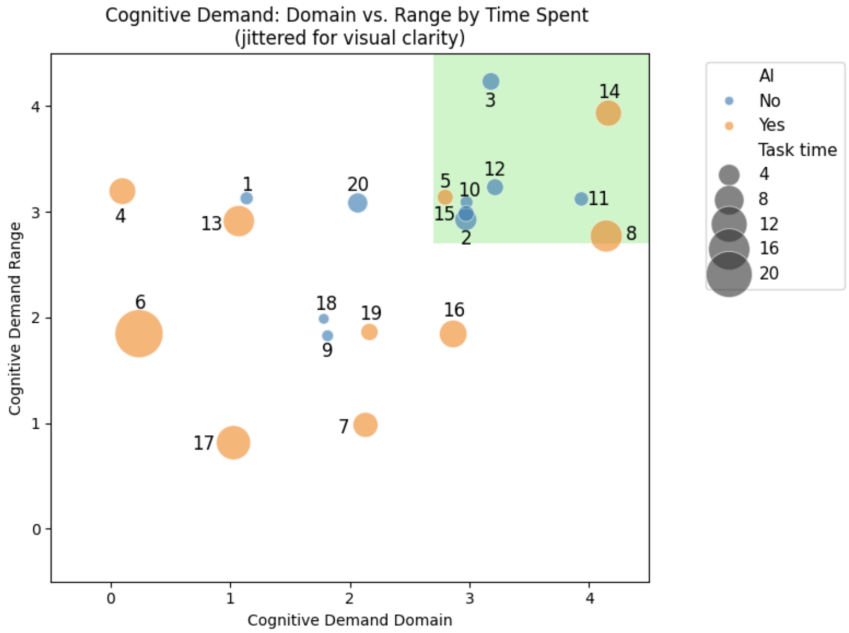

Figure 2. Cognitive demand (domain and range), time spent solving, and AI use of 20 student participants. The numbers in the figure indicate the student ID number (as listed in Table 2).

High Cognitive Demand

9 of 20 students maintained high cognitive demand for both domain and range (a 3 or 4 on the Mathematics Task Rubric, Stein & Smith, 1998), indicated by the green shaded region in Figure 2. A majority of these students (6 of 9) did not use ChatGPT to assist them with domain and range. Rather, they tended to remember what domain and range were and had ideas about how to apply them in the novel roller coaster setting. If they did use ChatGPT, it was typically a single prompt to remind themselves which axis domain or range corresponded to (e.g., Student_14 prompted, *"Does range measure the y axis?"*). They noticed at least one discrepancy between the plotted points (A-E) on the roller coaster track, the track visually continuing beyond the figure boundaries, and they generated questions pertaining to the domain's ambiguity about where the track ended (and *if* it ended). To illustrate, Student_8, a business management major who first used ChatGPT to recall domain and range concepts, said:

> "But you can see it continues this way. But [*in the problem statement*] it says consider point A to be the beginning of the roller coaster track. But this (*gestures to the track*) has to connect to the beginning eventually…come around to point A. So this makes me think that the domain is all real numbers. But at the same time, a roller coaster is not infinite."

Here, Student_8 explained her struggle to find a reasonable endpoint for the domain of the roller coaster. She knew it could not be positive infinity, because the roller coaster "has to connect eventually", yet she was uncertain what number would make sense, apart from numbers

visualized on the graph itself, which could not be the domain's endpoint because the roller coaster continued past them. The nature of her struggle showed high demand thinking (level 4, doing mathematics) because she grapples with non-algorithmic solution strategies, such as attempting to visualize the rest of the non-infinite roller coaster.

Student_8 and others struggled to generate a reasonable endpoint for the roller coaster's domain. As shown in Table 2, students created one of three responses: 55 (the x-coordinate of point E), 60-63 ("some point beyond 55"), or positive infinity, even though many students who wrote ∞ stated aloud that the roller coaster could not go to infinity. Student_16, a mechanical engineer, explained this discrepancy in terms of how she would approach it in her major:

> "I know that the roller coaster does not go on forever, but I have no information to suggest where it will end. So I mean, in the context of engineering, if this goes on for…65, 60 meters of it, and it's a 1000-meter roller coaster, you theoretically would consider that infinity, because it's so much higher. But I just don't have that information."

Low Cognitive Demand

11 of 20 students solved the task with low cognitive demand for domain, range, or both (typically it was domain - see Figure 2). Of these 11, 5 were STEM majors and 6 were not, suggesting that students with more math exposure through coursework did not appear to solve the task while maintaining higher cognitive demand than their non-STEM peers.

Students who solved with low cognitive demand tended to not grapple with the situated ambiguity presented by the roller coaster. For instance, Student_18 (an electrical engineering major) spent only 39 seconds on the task, reading the graph and immediately declaring, "So the domain is x, and it goes from 0-60 meters. The range is y, and it goes from 34? 32.5 meters. Because this is the maximum." He did not explain where 60 meters came from, but rather scanned the graph and selected domain values (level 2, procedures without connections). Student_13 (a biology major), on the other hand, spent almost nine minutes puzzling about the domain, before concluding, "Am I allowed to guess on this? My guess would be from negative infinity to infinity…it was 100% on the AP Calc test…if you were given a parabola and had to determine all these things about it." (level 1, memorization).

7 out of 11 students used ChatGPT to solve the task, but only three used it in ways that led them to generate a high demand answer for either domain or range (Student_4 and 13 for range, and Student_16 for domain). The others, and in particular Student_6, Student_17, and Student_7, struggled to obtain helpful information about the task via time intensive back-and-forth interactions with ChatGPT. Their struggle to use ChatGPT productively originated from one of three potential causes. First, they did not know how to make use of the information ChatGPT gave them, because it gave generic facts and examples about parabolas' domains and ranges in response to students' generic questions (Student_13's prompt: "determine domain and range on a parabola."). Student_13 struggled to relate ChatGPT's hints about utilizing a function in vertex form, to the roller coaster, which did not give a function equation:

> "It gave me the equation of a parabola, then it put it into vertex form…and then h and k is the vertex…And then it tells me the domain, based on, plugging in x and getting any real number. I mean, it's kind of clarifying, but I don't have the, like, function for it."

Other students, such as Student_7 (a food and nutrition major), interpreted ChatGPT's hints incorrectly. Student_7, reading from ChatGPT, inferred that domain is, "where it starts and where it finishes," but applied this definition in such a way that she obtained two answers, "and I

want to say that the domain would be x=5 and x=10." Her response reflects an understanding that a domain is discrete values, which she explained came from ChatGPT's illustrative example.

Finally, although hallucinations did not appear to occur frequently, Student_17 struggled when ChatGPT gave her negative range values after she prompted it with a screenshot of the roller coaster from the task. Student_17, a liberal studies major, realized that the graph visualization did not contain negative values, but did not know how to prompt ChatGPT in such a way that it would produce positive range values:

> (*reading AI output*) "y is the line that goes up and down, from the lowest point to the highest, drops as low as -5 meters and climbs to 35 meters. (*stops reading*). No it doesn't. Because, right here (*points to B, vertex*), looks like it's maybe 32. So, wrong, incorrect."

However, Student_17 was unsure how to find the domain and range absent a visually logical response from ChatGPT. After about 9 minutes and eight prompts, she resigned, "Okay, I'll just say what it tells me" (see Table 2), even though she suspected that ChatGPT's response was incorrect ("I don't know what -5 has to do with it") and had her own emerging ideas about how to find the range without AI (" *positive*) five is on this line, at the same place as zero").

Discussion

Consistent with prior work that has studied technology as a mediator of cognition (e.g., Hollebrands, 2003; Sherman, 2014), our findings suggest that AI tools do not, in many cases, deterministically lower or raise cognitive demand. Rather, how students solve the task depends on how they interpret it, construct prompts that bridge their understanding with what they need to learn, and evaluate AI output in response to the task's goals. Importantly, we observed that evaluating AI involved students' willingness and ability to challenge ChatGPT as a technological form of authority. Student_8 did so successfully, using it to remember domain and range then moving toward the demanding, ambiguous aspects of the task herself, while Student_17 did not, writing down ChatGPT's incorrect solution as her own even though she did not agree with it.

What students wrote down as their domain/range solution did not unilaterally represent their thinking during the task. Many students, for instance, did not know how to turn "this roller coaster cannot go on forever" into a domain statement even though they exhibited high demand thinking. Furthermore, ChatGPT did not tend to provide clarification because it gave canonical parabola domain examples that did not apply to the roller coaster. Rather than follow up with prompting that alluded to a need to *envision* a roller coaster as a function (Subramonyam et al., 2024), students with low demand thinking translated ChatGPT's generic output into ideas they believed were usable, often subverting qualities that made the task high demand in the first place.

These findings hold implications for the undergraduate mathematics education community. First, we hope that our findings illustrate that some students need to learn how to use AI tools productively to solve problems by becoming more attuned to their own sense-making and discernment. Second, educators must help students learn how to navigate open-ended, ambiguous problems, particularly as AI tools become more accurate in solving routine math problems (Pardos & Bhandari, 2024). Finally, as our study was conducted with individuals outside of any course they were enrolled in, future work should explore the nature of students' cognition and AI use within courses, where their AI use habits may have a stronger impact on their grades and eventual career trajectories. Future work might also consider how to design student-AI interactions, particularly with frontier AI models that students use on a regular basis, so that student thinking is amplified rather than automated.

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Examining Novice and Experienced Programmers' Use of Python in a Combinatorial Context

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We explore the use of computational tools and techniques to enrich student's combinatorial reasoning by contrasting novice and experienced programmers in their exploration of combinatorial tasks in the context of Python. Drawing upon Lockwood's model of combinatorial thinking, we present data from interviews with students who were engaging with Python modules designed to help students solve counting problems. We contrast pairs of students who had different levels of prior experience with coding, and we highlight the experienced coders' solution that reflects how their experience with coding arose in their combinatorial solution.

Keywords: combinatorics, discrete mathematics, computing, programming.

Introduction

The use of computational tools and techniques has long supported mathematical activity (e.g., Papert, 1980), and there have been recent calls for studies that explore the role of computing in enriching students' mathematical reasoning (e.g., Geraniou et al., 2024; Lockwood & Morken, 2020). There is a growing body of work that highlights the potential of computational settings and activities to support student's mathematical thinking in a variety of contexts (e.g., Buteau et al., 2019; Sinclair & Patterson, 2018). In this report, we seek to add to this literature by contrasting novice and experienced programmers in their exploration of combinatorial tasks in the context of Python. The work in this report contributes by illuminating some aspects of programming that more experienced programmers may draw upon in such contexts.

Literature and Theoretical Perspectives

Counting Processes, Sets of Outcomes, and Formulas/Expressions

To characterize what we mean by combinatorial thinking, we draw on Lockwood's (2013) model of students' combinatorial thinking, which includes three main components of counting processes (the processes one engages with in solving a counting problem), formulas/expressions (the numbers, variables, and operations that often give the answer to a counting problem), and sets of outcomes (the set of objects being counted by a counting problem; the cardinality is the solution to the counting problem). In this model and in subsequent work, Lockwood and others (e.g., Soto et al., 2022) have made a case for the value of giving students experiences to connect counting processes they employ with the sets of outcomes those processes generate. It is desirable to us for students to make connections among these components, and the use of the computer is a motivation to foster connections between counting processes and outcomes.

Notably, in some prior work we have explored students' programming in the context of counting problems (e.g., De Chenne & Lockwood, 2022; Lockwood & De Chenne, 2020), but such work has focused mostly on novice programmers with no prior programming experience. Some work has examined an experienced coder with no combinatorial experience (De Chenne & Lockwood, 2023), but we have not yet reported on such tasks with students who had both combinatorial and coding experience. There is a need in this literature for more insights into the work of what more experienced coders bring to these combinatorial tasks.

Programming and Machine-based Computing

In this report, we focus on *machine-based computing*, as defined by Lockwood and Morken (2021): "the practice of developing and precisely articulating algorithms that may be run on a machine" (p. 405). We are interested in approaches that foreground the development of an algorithm as opposed to only its performance or implementation. In our study, this machine-based computing was programming, which entailed both creating code and editing existing code (in alignment with Weintrop et al., 2016). In the context of coding, we conceive of the programs as representing counting processes (in the sense of Lockwood 2013), as they are representative of the process of generating outcomes. The output that the computer then generates from that code represents the sets of outcomes. This is in line with other work (e.g., De Chenne & Lockwood, 2022; Lockwood & De Chenne, 2021) that has connected the computer to components of combinatorial thinking and activity in this way. We seek to answer the following research question: *What kinds of computational activity did experienced coders engage in as they solved counting problems in a computational context?*

Methods

Data Collection

The work described in this paper was part of data collected during a whole-class implementation of materials involving Python in the context of combinatorics. We developed a set of modules in which students engaged with Python and solved counting problems. These were implemented in a classroom setting of an upper-level mathematics course designed for pre-service teachers (the second author was also the instructor for the course). The modules covered six 80-minute class sessions, and they began with an introduction to Python and progressed to increasingly advanced combinatorics problems. The entire class worked through the modules, and we video- and audio-recorded two pairs of students for those six sessions. In those recordings, we both recorded the students as they worked, and we recorded their screens; we spliced together those inputs for analysis. As they worked in pairs, the instructor and another researcher would occasionally check in on how they were doing and would ask questions about their thinking and their work.

For this report, we focus particularly on one of the pairs of students we recorded, Zach and Levi (all names are pseudonyms). As a contrasting case, we briefly describe the solution of another pair Taylor and Kirk. Taylor, Kirk, and Levi were math majors enrolled in the secondary education program, Zach was a math major not enrolled in the secondary program. They were all senior-level mathematics majors with strong mathematics backgrounds. Taylor and Kirk had taken discrete mathematics but no advanced combinatorics; Zach and Levi students were taking an upper-division combinatorics class at the time of the interviews.

Taylor had no prior experience; Kirk had seen VPython, Matlab, and RStudio but said, "I don't remember how to do anything." Levi did not articulate prior experience with programming, and his work suggested he was a relative novice but felt comfortable trying to program. Zach had a strong background in programming, having coded previously in Python. At the time we did not ask what specific courses he had taken, but he talked about and demonstrated prior experience with coding in Python. This is borne out in their work, and we consider a relative comparison of *novice* for Taylor, Kirk, and Levi, and *experienced* for Zach; we are not making a claim about Zach's relative expertise to expert coders, but Zach was certainly more experienced than the other students and students we have reported on previously (e.g., Lockwood & De Chenne, 2021). Practically, this meant that he drew upon techniques (such as using functions and structures) that suggested familiarity with Python programming.

Data Analysis

For data analysis for this report, we began by examining Zach and Levi's work to investigate ways in which a pair of students that had more experience (in Zach) engaged with the problems. Specifically, the authors first watched the spliced videos of the interviews with Zach and Levi and regularly met to discuss broad themes that emerged from them. Then, the first author reviewed all of the interviews again and identified and collected episodes where the students specifically engaged with the code in a way that reflected their counting process. From these instances, the authors collaboratively wrote the episodes and together chose a representative example that highlighted an interesting solution to a counting problem that reflected the student's advanced knowledge of coding. The first author then reviewed the example and analyzed the students' thought processes. Finally, the first author analyzed Taylor and Kirk's solution to the same counting problem to see how they worked through the problem.

Results

We present data on the Language Books problem, which states: *There are 3 different French books, 4 different Norwegian books, and 5 Spanish different books. Create a computer program to list all of the possible book combinations of two books, both of which are not in the same language.* The students were asked to write code to solve the problem. We begin by only briefly (due to space) presenting Taylor and Kirk's solution, which is the solution we anticipated when we wrote the problem. We show this mainly to highlight a contrast with Zach and Levi's work, which we present in more detail. Then, we present Zach and Levi's solution, which demonstrates what the additional coding experience facilitated for their work. The point of the comparison is not to criticize the novice coders, but rather to give some sense of the kinds of tools and resources that might arise if such tasks are presented to more experienced students.

Novice Coders' Solution to the Language Books Problem

The first solution that Taylor and Kirk tried was a single nested for loop. This solution loops through all the French, Norwegian, and Spanish books, but it will repeat pairs causing the code to overcount. It appears in Figure 1a below, and the partial output shows some repeated pairs (such as F1 N1 appearing multiple times). The students were not sure how to fix their code in order to keep a single for loop and still avoid the overcounting that arose. In order to combat this overcounting, the novice coder group decided to use three separate for loops (Figure 1b). This code uses three separate, consecutive for loops that separately handle pairs of French and Norwegian, French and Spanish, and Spanish and Norwegian books. This case breakdown provides a cleaner way to organize the problem that avoids the overcounting that previously arose, without having to have the code check to see if there are repeats. As a result, Taylor and

```
French = ['F1','F2','F3']
Norwegian = ['N1','N2','N3','N4']
Spanish = ['S1','S2','S3','S4','S5']

books = 0
for i in French:
    for j in Norwegian:
        for k in Spanish:
            print(i,j)
            print(i,k)
            print(j,k)
            books = books+1
print(books)
```

```
F1 N1
F1 S1
N1 S1
F1 N1
F1 S2
N1 S2
F1 N1
F1 S3
N1 S3
F1 N1
F1 S4
N1 S4
F1 N1
F1 S5
N1 S5
```

```
French = ['F1','F2','F3']
Norwegian = ['N1','N2','N3','N4']
Spanish = ['S1','S2','S3','S4','S5']

books = 0
for i in French:
    for j in Norwegian:
        print(i,j)
        books = books+1
for i in French:
    for k in Spanish:
        print(i,k)
        books=books+1
for j in Norwegian:
    for k in Spanish:
        print(j,k)
        books=books+1
print(books)
```

Figure 1a and 1b. Figure 1a shows Kirk and Taylor's initial attempt at a single for loop. Figure 1b shows their solution for code that uses three consecutive for loops.

Kirk are utilizing sets of outcomes from Lockwood's model (2013). By thinking about how to do this process, they relied on knowing that the final set of outcomes would have pairs of French

and Norwegian, French and Spanish, and Spanish and Norwegian books. This is a correct answer to the problem, and indeed it was the solution we expected when we designed the task.

The output of the code in Figure 1b is shown in Figure 2. Note that each nested for loop creates one line of these lists (the first nested for loop created pairs of French and Norwegian books, the second created pairs of French and Spanish books, and the third created pairs of Spanish and Norwegian books). To reflect formulas/expressions of the model, we can consider an expression to reflect this process. For each nested for loop, we multiply the number of items in the outer array and the number of items in the inner array, which is also how Taylor and Kirk thought of it. Since we are counting up by one for each time the loop goes through, we end up adding up all the times each of the nested-for loops completes a loop, and so we sum up the products. This code thus reflects the sum of products, 3*4 + 3*5 + 5*4, which is a result that Taylor and Kirk found after they had written their code using the three consecutive loops.

```
F1 N1  F1 N2  F1 N3  F1 N4  F2 N1  F2 N2  F2 N3  F2 N4  F3 N1  F3 N2  F3 N3  F3 N4          12

F1 S1  F1 S2  F1 S3  F1 S4  F1 S5  F2 S1  F2 S2  F2 S3  F2 S4  F2 S5  F3 S1  F3 S2  F3 S3  F3 S4  F3 S5          15

S1 N1  S1 N2  S1 N3  S1 N4  S2 N1  S2 N2  S2 N3  S2 N4  S3 N1  S3 N2  S3 N3  S3 N4  S4 N1  S4 N2  S4 N3  S4 N4  S5 N1  S5 N2  S5 N3  S5 N4          20
```

Figure 2. The output of the code from Figure 1b. This was printed horizontally to save space.

Experienced Coder's Solution to the Language Books Problem

We now share results from Zach and Levi as they solved this problem. As noted, Zach especially was an experienced programmer. The students attempted a different solution than we had anticipated. Zach and Levi also attempted to create a single nested for loop, and we highlight what aspects of coding they needed to draw upon in order to do this successfully.

Zach and Levi create a single nested for loop. The first thing when creating their for loop was making sure that the code "looked forward" in the list; this was a term Zach used. By this, we infer that he meant that if X is the 5th item in the list, Y would only be the 5th or higher item in the list. The excerpt below shows Zach's reasoning about this, writing the code in Figure 3.

Zach: So, I was gonna do for X in range, uh, 1, 13.

Levi: Oh, yeah.

Zach: And then for, um... 'Cause I don't wanna have F1, R1, and then, uh, at... back at R1 go back to F1. So, I can actually do for Y in range X13. Right, so now if I'm... if I... so now if I start with like three- I'm not gonna look backwards. [...] It can only look forwards.

```
books = ['F1','F2','F3','R1','R2','R3','R4','S1','S2','S3','S4','S5']

for x in range(1,13):
    for y in range(x,13):
```

Figure 3. Zach's use of the range function in the code.

To do this, Zach used the range function so that the code would look at the first element of the list ("for x") and then the second element of the list ("for y") by putting an "x" into the range so that it would look at the number that the first for loop is on and only after that number. We suggest that this approach is indicative of someone who is more advanced in coding. While more novice coders may know that the range function can be used to create a list of numbers, we argue that Zach is using more advanced knowledge of coding here. In particular, he uses a nested for loop with range functions in order to force the second for loop to only look at the element that x is on and after that element. The use of range(x,13) suggests that he understood conceptually how the range function works, as well as understanding that this will use the x created in the outer for loop as a variable within the range function.

However, Levi and Zack recognized that this code would not ensure that they would not repeat the same language. As seen in the excerpt below, Levi recognizes that "F1, F2" is still a possibility. They discuss this below, and Zach tries to use nested indexing (which we would argue is a more advanced coding practice) to check that he can isolate the first elements of the string. In Figure 4 we see his initial work on comparing the first letters of each pair.

Levi: But then you can still get F1, F2.

Zach: Right. And then I can say, uh,

Levi: If-

Zach: If books at X at one is equal to... is... I'm sorry, is not equal to books at Y at one. All right, this should be a zero, shouldn't it? This is a zero.

Levi: Oh yeah, yeah, yeah.

Zach: So, this is like, the X inden- Uh, indices [...] Actually, this should be zero as well, shouldn't it? [...] So then it's gonna probably pull the first things of that string. So, when... if I pull F1 and F2-

```
books = ['F1','F2','F3','R1','R2','R3','R4','S1','S2','S3','S4','S5']

for x in range(0,12):
    for y in range(x,12):
        if books[x][0] != books[y][0]
```

Figure 4. Zach using nested indexing to look at the first letter of the x-th element of the array and making sure it is not the same as the first letter of the y-th element of the array.

Zach then checked this work to test whether the code is in fact isolating the first letter. He successfully "pulls" just the first element of the string, as seen in the column of Fs in Figure 5.

```
books = ['F1','F2','F3','R1','R2','R3','R4','S1','S2','S3','S4','S5']

for x in range(0,12):
    for y in range(x,12):
        print(books[x][0])

F
F
F
F
F
F
F
F
```

Figure 5. Zach testing the nested indexing function to see if it does what he thinks it does.

Zach: [...] It's gonna read it as, um, FF. So that... So, books... well, here. You could do like, print books at X at zero. Let me see what that does. So that- that's printing F. So that's seeing it as looking at the string- [...] First then look- so books X looks at the string and then zero pulls the first element of the string, which is just F.

Levi: Oh, okay.

Zach and Levi conclude their work on the problem in Figure 6, and in the following excerpt Zach explains his reasoning. Having successfully isolated the first letter, he can include a conditional if statement that only counts pairs where the first letter (the books[x][0] and books[y][0] are not equal). Here, they first make sure that the language of the books would not be the same by having the code look at the first letter and only keep going if the first letter of the first book selected and the second book selected were not the same, as Zach explains.

Zach: So, if we have book... so, as long as the letters aren't the same, books X-[...] at zero, as long as it's not equal to books, our yth component at zero... We gotta be careful with that one. I'll do plus one there I think […] Yeah. 'Cause like, as long as I have this, this is gonna say F1 and then for Y in range zero, it's gonna look at F1 again [...] It's gonna compare it to itself, but obviously it- it can't be the same as itself.

Then we do count equals, uh, zero. And then if they're not the same, the letters aren't the same […]

```
books = ['F1','F2','F3','R1','R2','R3','R4','S1','S2','S3','S4','S5']
count = 0
for x in range(0,12):
    for y in range(x+1,12):
        if books[x][0] != books[y][0]:
            count += 1
            print(books[x],books[y])
print(count)
```

Figure 6. Zach and Levi's final code.

In arriving at this final code, Zach also did a few fixes to make sure that the code was operating correctly. He first had to change the range to go from 0 to 12 instead of 1 to 13 since the indexing of the array starts at 0 and not 1. He then made the range for the y go from x+1 instead of starting at x so that it won't have a repeated book and it will just start at the next one. These practices highlight Zach's prior experience with coding. There are also examples of them utilizing the set of outcomes to obtain their final code. For example, they pay close attention to repeats like F1F2 and Zach utilizes a function of the code for reading the first letter to make sure that their set of outcomes does what they want it to do.

Zach and Levi's explanation of how the code works. In the excerpt below, Zach and Levi explain their work, and the explanations demonstrate how well that Zach understands how the code works. Notably, Levi also seems to agree and is able to articulate aspects of what the code is doing as well.

Levi: For X in range between 0 and 12 for Y and range X plus 1 to 12, and then, uh, that's just if that... if the F... if the letters don't match up.

Zach: Yeah.

Levi: Then you can continue [...].

Zach: If the letters aren't the same then they're the same... if they are the same then, they're like, the same language so I don't wanna count it.

Levi: Mm-hmm (affirmative).

Zach: And then the for range X plus 1 I'm not... I'm not gonna count, look behind me anymore, which you... it removes duplicates. So, I'm not gonna get R1 F1 and F1 R1 […]

Interviewer: Okay, good. And then what is making it so you don't get F1 R1 and also R1 F1?

Levi: The range on Y because once you... 'cause the range on Y is, it only looks farther down the line-[...] on the books, so...

Interviewer: I see, because you're just... you're in the same set. You have one set here. [...] So by looking at Y is only gonna be further down […]

Zach: Yep, so X picks like the fourth one and then my Y is only gonna be from the fifth on.

Relating the counting process to an expression. In terms of the model, we see the students connecting the code to expressions for the numerical total. In Figure 7, we see Zach wrote out the total of 27+20 = 47. To represent how the counting process was reflected in their code, they started with F being the first letter, and they noted that each of the "Fs" has 9. With 3 "Fs", there would be a total of 27. Then, looking at the "Rs", since we have already accounted for the pairs with French and Norwegian, each "R" is going to have 5 pairs. With 4 "Rs", there would be a total of 20. There are 47 combinations. In this way, they connected the process in their code to the outcomes being counted, and to mathematical expressions that reflected that code.

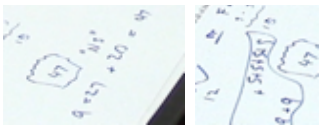

Figure 7. Zach's numerical solutions of how he counted the number of combinations of language books.

Discussion and Conclusion

By leveraging a couple of more advanced programming ideas (specifically, a sophisticated use of the range function and nested indexing), Zach arrived at a creative and efficient program that solved the Language Books problem in a single for loop. By using these techniques, Zach was able to account for making sure that they would not repeat the same language and ensured a complete list of desirable outcomes. We also highlight computational practices that Zach employed that suggest he drew upon his coding experience. Zach used the code to troubleshoot while making a prediction of what it would do. He wanted the code to specifically look at the first letter so that he could later compare the letters. To do this, he tested the code so that it would print out what he hoped was the first element of the code (this is reflected in Figure 5). We interpret that not only is the use of function itself using knowledge that an experienced coder would draw upon, but his testing of it to make sure that it does what he wants it to also highlights a practice of a more experienced programmer.

In comparing Taylor and Kirk's code and Zach and Levi's code, the overall structure of their sets of outcomes and final formula were different. Because Zach and Levi combined the code into one for loop, it forced French to be paired up with Spanish and Norwegian all at the same time, creating a group of where Spanish and Norwegian are almost like its own array. Meanwhile, Taylor's and Kirk's method forced French and Spanish and French and Norwegian to be completely separated groups. This is also reflected in their formulas, since Taylor and Kirk's group had three separate additions (3*4 + 3*5 + 5*4) while Zach and Levi's code had two separate additions (3*9+4*5). In all, they created the same result, but their Zach's experience as a programmer, they were able to see a different set of outcomes and formula/expressions.

In terms of components of the model of students' combinatorial thinking, both pairs of students were able to relate their counting processes with a structure of the set of outcomes and a corresponding mathematical expression. We note that their explanations suggest Zach and Levi understood the counting processes, and they ultimately expressed mathematical expressions that reflected the process of their creative, unconventional code. This highlights the fact that alternative approaches may meaningfully be connected to expressions that suggest a process.

In the undergraduate mathematics education literature, there is relatively little work on ways in which experienced programmers engage in mathematical tasks in computational settings. The episodes highlight ways that experienced coders may leverage certain advanced coding practices in their work. Both pairs successfully solved the problem, but the advanced techniques allowed for Zach to program the problem in one for loop, something Taylor and Kirk attempted but did not do. Zach's advanced coding practices emphasize why it is not surprising that Taylor and Kirk were not able to use a single for loop. Future research could explore whether there are mathematical implications for more advanced programmers, and subsequent studies could design tasks that could utilize certain computational tools that might not be available to novice coders.

Acknowledgments

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Readers' Movement of Attention Through a Calculus Text

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Mathematics textbooks demand specialized forms of literacy, yet little research has examined how readers actually move through them. Drawing on the concepts of agency and didactical disciplinary literacy, this paper explores how readers allocate attention across textbook components and the reasons they give for doing so. We also propose an "attention continuum" in which shifting patterns of engagement represent agentive choices. These findings highlight how textbook reading intertwines literacy practices, disciplinary positioning, and agency, offering new insight into mathematics learning.

Keywords: Mathematics Textbooks, Agency, Didactical Disciplinary Literacy

Textbooks have long been a staple of college-level mathematics classes. Mathematics textbooks—dense combinations of technical prose, symbolic notation, graphs and figures, and examples—require students to engage with them in particular ways. Textbook authors offer instructions to students about how to read their textbooks, with exhortations to read closely and to attend to the various components of the textbook in particular ways and in a certain order (e.g., Stewart, 2016). Similarly, university learning centers advise students on reading strategies such as skimming and close reading for academic texts in general and scientific textbooks in particular (e.g., University of North Carolina, 2025). Yet, the mathematics education research community has paid relatively little attention to how students read mathematics textbooks.

Following Wiesner et al. (2020), we argue that literacy is central to disciplinary practice, and that textbooks are different from other disciplinary texts and deserve particular attention. Learning from reading a textbook depends both on the reader's background knowledge and the literacy practices they employ, including how they allocate their attention across the various components of the text. Moreover, these reading activities are situated within the disciplinary community and in relation to the reader's positioning within it.

To investigate these ideas, we present results from a study of undergraduate mathematics students and non-mathematics professors as they read an excerpt from a calculus textbook. Using a stimulated recall approach, assisted by eye-tracking software, we investigate how and why readers move their attention through the text. Specifically, we investigate the questions

- How do readers of a calculus textbook excerpt move their attention through the reading?
- What are the stated reasons and motivations for the choices that readers make about their movement through the text?

Literature Review

The concept of *disciplinary literacy* recognizes that disciplinary experts engage with texts in their field in ways that draw not only on their content knowledge, but also on their knowledge of the disciplinary community norms and concepts of knowledge and inquiry, and the reader's positioning within the community (Moje, 2007; Shanahan et al., 2011). *Didactical disciplinary literacy* extends this idea to didactical texts (texts whose principal purpose is didactical, such as textbooks); it encompasses "the habits of mind of the individual immersed in didactical practice and on the literacies of the discipline in which the text and reader are embedded" (Wiesner et al., 2020).

Agency emerges from an "individual's perception and enaction of control and choice" (Weinberg et al., 2024). The concept is a fundamental piece of disciplinary literacy (Moje, 2008) and has been highlighted as essential in both math education and literacy communities (Fulmer et al., 2022; Vaughn, 2014). We adopt the perspective put forward in Fulmer et al. (2022), who reconcile these perspectives by viewing agency as the "interplay of autonomy, identity, power, and knowledge" (p. 632). Agency in mathematics is naturally in tension with the structure and permanency of the discipline (e.g. Pickering, 1995); textbooks—"inherently hegemonic" (Fulmer et al., 2022, p. 633)—offer a particular challenge to the personal agency of the reader (Vaughn, 2014). In this context, we argue that the way a reader directs their attention through the reading offers an important opportunity to act agentively. In particular, as Weinberg et al. (2024), note, "actions that run counter to the status quo are evidence of agentive moments" (p. 669).

Theoretical Framework

The concept of *disciplinary literacy* recognizes that disciplinary experts engage with texts in their field in ways that draw not only on their content knowledge, but also on their knowledge of the disciplinary community norms and concepts of knowledge and inquiry, and the reader's positioning within the community (Moje, 2007; Shanahan et al., 2011). *Didactical disciplinary literacy* extends this idea to didactical texts (texts whose principal purpose is didactical, such as textbooks); it encompasses "the habits of mind of the individual immersed in didactical practice and on the literacies of the discipline in which the text and reader are embedded" (Wiesner et al., 2020).

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Methods

Participants

We selected participants to capture a wide range of disciplinary and didactical disciplinary literacy practices. To recruit participants, we sent invitations to all math majors and minors who had not yet completed numerical analysis (where Newton's method is a standard topic); all students who had completed first-semester calculus with a grade of C- or better in the previous semester; and all (non-math) STEM faculty at our institution. All interested individuals who were available during interview times were included in the study. A total of 15 individuals particicpated in the study, five from each of the invited groups. We subsequently refer to participants by their group (MM for math major, NMM for non-math major, and NMF for non-math faculty) and a number.

Materials

Participants in the study read a four-page excerpt from a commonly used calculus textbook (Stewart, 2016) on Newton's method. The excerpt included introductory prose that attempted to motivate the need for Newton's method; a derivation of a general formula and description of the iterative process and associated limit; and three examples intermixed with text that elaborated on the examples. There were also seven figures that illustrated the ideas presented in the text. This organization of the excerpt reflects a common template in mathematics didactical texts; it informed the interviewer's questioning during the stimulated recall phase and served as a map for tracing readers' navigation through the text during data analysis.

The excerpt was chosen because it relied on knowledge that is generally taught in a first-semester calculus course, but Newton's method itself is often not taught in introductory semester calculus. Thus, we predicted that the topic itself would be novel to most participants but that they would possess the necessary background knowledge to engage mathematically with the reading.

Data Collection

Participants were each interviewed for a 75-minute session, conducted by the third author, a professor in the mathematics department. Following Schindler and Lilienthal (2019), the interview incorporated a stimulated recall of participants' reading. Participants were first asked to read the textbook excerpt for the purpose of learning Newton's method, reading an electronic version of the excerpt with an eye-tracker recording their gaze. After reading, the participant and interviewer jointly viewed a playback of their fixations. The participant was asked to narrate their fixation patterns and describe what they were thinking as they looked. In particular, if the participant's gaze was fixed on a particular component of the text for an extended period of time, if the participant skipped or skimmed over a component, or if the participant moved back to an earlier portion of the excerpt from a later portion, the interviewer prompted them to pause and describe their thinking and motivation.

Analytical Methods

We used a process of open and axial coding (Strauss & Corbin, 1994) to generate codes. Each author reviewed the interview transcripts and identified turns where the participant addressed how their attention moved throughout the text. Using both the turn itself and adjacent turns, the authors individually characterized each turn into an initial code, then worked collectively to refine the codes. This resulted in a collection of behavior codes and reason codes for each behavior code. The authors then individually coded the flagged turns (for both behavior and reason) and collaboratively reconciled differences.

Results

We identified five categories of behaviors–reading straight through, skimming, skipping, close reading, and reading order– and a classification of reasons the participants provided for each type of behavior. Between six and 17 turns were identified for each participant. Additionally, each participant was coded for at least two of the main behaviors we analyzed–skimming, increased attention, and reading order–and eight of the 15 participants were coded for all three.

Reading Straight Through

A turn was coded as *reading straight through* when the reader indicated that they had read the text in the order presented with no indication of more or less attention than expected. For

example, NMM1 stated, “I was locking in, reading straight through.” These moments were not elicited or probed by the interview protocol and were infrequent (five out of 146 turns were assigned this code.) In each case, the reader did not provide any justification for their behavior of reading straight through.

Skimming and Skipping

Turns were coded as *skimming* if the turn indicated rapid reading or reduced attention. For example NMF5 stated, “I just glossed over those formulas.” We classified reasons offered by participants for engaging in skimming as follows: efficiency, familiar, don’t understand, disciplinary reasons, overview, and no reason.

A skimming turn was coded as *efficiency* when the participant explicitly referred to moving through the text at increased speed. In all but one instance, turns that were coded as efficiency also received a secondary reason code. For example, NMM2 stated, “I think I probably read through this one quicker because I kind of understand more how the numbers plugged in.” This was coded as efficiency and familiar; in general, we identified skimming as *familiar* when the reader expressed that they already understood the material sufficiently. Other readers skimmed a portion of the text when they struggled to make sense of it, coded as *don’t understand*. For example, NMM3 explained their choice to move quickly through a segment of text: “I couldn’t figure it out.”

We used the code *disciplinary reasons* to identify turns where the reader justified skimming in terms of what they valued mathematically or based on disciplinary norms. For example, MM5 skimmed the first part of the reading–motivating Newton’s method–so that she could move on to the derivation of the method, explaining that she found the motivation section less important than the derivation section.

Skimming was classified as *overview* when participants indicated they were trying to get an overview or preview of the reading. For example, NMF5 stated, “Well, I was just trying to figure out, like, normally you don't know—in a textbook—you don't know what the best place to start is. So you're looking for a heading, or like, now we're going to learn about this and I guess that's at the top, I'm just seeing right now, so Newton's method.”

Skimming (as well as other behaviors) was coded as *no reason* when a participant did not offer a reason themselves and the interviewer did not prompt them for a reason, or authors could not identify a reason from the participant’s explanation of their behavior.

We coded separately as *skipping* instances where a participant gave an explicit indication that they did not look at a particular portion of text at all. For example, after being asked if they had looked at a figure, NMM2 stated that they had not noticed the figure. In each instance, the participant either did not supply a reason or indicated that they missed the piece or text or were not sure why they had skipped it; these were all coded as *no reason*.

Close Reading

The code *close reading* was applied to turns where the reader’s movement through the text appeared to slow or stop on a particular component of the text. We classified close reading using the reasoning codes understanding, familiar, disciplinary reasons, and no reason (defined as above).

The code *understanding* was applied when participants indicated they were working to make sense of part of the text that they found confusing or that they deemed needed additional time to

understand. For example, NMF1 said: “When you had that notation that's confusing, you look at it more and then—I don't want to go too fast.”

Similar to the skimming behavior, a close reading turn was coded as *familiar* when the participant indicated they were engaging in the behavior because a piece of text was familiar to them or a type of text that they felt comfortable with. This gave them greater access to or interest in the text, motivating their increased attention. For example, MM2 focused on Example 2, explaining the set-up of the problem (finding the sixth root of 2 using Newton’s method) reminded them of work he had done in a previous class.

Complementing the skimming-disciplinary reasons code, the close reading-*disciplinary reasons* code referred to turns where the reader indicated that their increased attention was motivated by what they valued mathematically or based on disciplinary norms. This happened in MM1’s report of close reading of the blue-boxed equation, stating that it was significant in understanding how and why Newton’s Method works.

Reading Order

The code *reading order* was used to categorize turns where the reader did something other than read in the intended order (as perceived by the paper authors). This included reading a piece of the text multiple times, jumping forward in the text (over a portion of the reading), jumping backward in the text, or moving back and forth between two different pieces of the text. We grouped these behaviors under one code as they often involved overlapping behaviors that were difficult to tease apart. For example, when a participant jumped backward in the text, they were likely rereading the earlier section but they may not have explicitly indicated this. Justifications for this behavior were classified as understanding, connections, disciplinary reasons, overview, zoning out, and no reason.

The code *understanding* was applied when participants indicated they were working to make sense of part of the text that they found unclear, to check their understanding, or to build understanding iteratively. For example, MM2 explained their choice to move back to an example earlier in the reading: "I wanted to make sure that like the x_1, that it was more of a choice of what to choose for x_1. You can choose one or something else, yeah I just wanted to make sure."

We coded reading order as *connections* when the reader was attempting to make connections between different pieces of text (such as graphs and text) without indicating they were attempting to resolve a confusion. For example, after starting to read Example 1, MM1 returned to the general equation earlier on the page to coordinate the calculation in the example with the general equation.

Participants also indicated that reading order was motivated by disciplinary norms and values, coded as *disciplinary reasons*. For example, NMF1 explicitly referenced her discipline when explaining her choice to look at the figure captions before looking at any of the text: “It’s how you read a [biology research] paper…. And when I read biology papers—and part of it is the method I was taught, and I like it—is you read the abstract and the figure legends, and then you go back and read the paper.”

Readers also engaged in reading order to get an overview or preview of the reading (coded *overview*). For example, NMF4 explained jumping around the text:“I wanted to see what came next.”

Finally, in three instances of reading order, a participant indicated that they reread a piece of the text because they had “*zoned out*”; this was coded as *zoning out*.

Discussion

This study highlights participants' movement of attention as a realized opportunity for agency. All readers employed a variety of reading behaviors. These movements demonstrated intentional choices about how to move through the excerpt, both through their own explanations and the authors' perceptions of the text ordering. Participant activity was constrained by the text itself and the goal of learning about Newton's Method. However, the participants' explanations for their choices reflected their own goals, understanding, and sense of themselves in relation to the text and interview context. This reflects the notion of agency as the "interplay of autonomy, identity, power, and knowledge" described in Fulmer et al. (2020, p. 632) .

In detailing participants' movement of attention, we have come to conceptualize the behaviors as having fuzzy and overlapping boundaries; and many of the behaviors can be viewed as existing along a continuum, as displayed in Figure 1.

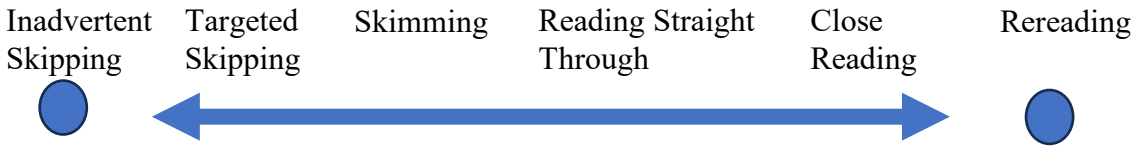

Figure 1. The attention continuum

Here the transition between skimming and close reading represents a dialing up and down of attention, reflecting the speed at which a person moves their attention and how much and what grain size of the text they give attention to. Moreover, skimming itself operates on a spectrum, where high-speed, large-grain size skimming might typically be described as skipping. In particular, we interpret instances when a reader identifies the intellectual/conceptual content of text component and make an active choice to not read it as skimming; this is labeled as "targeted skipping" in Figure 1 to highlight this extreme end of the continuum. We distinguish this from unintentional skipping, where a reader is not aware of the text they skipped. The turns in this study coded as skipping appear to fall into the category of unintentional skipping. The reading order-zoning out may also be a consequence of a lack of attention similar to unintentional skipping. At the other end of the model, rereading–while a different behavior from close reading–arises from similar motivations as close reading and represents a similar dialing up of attention. Not represented in this figure are other kinds of reading order behavior, namely jumping forward and backward. This type of behavior appears to arise, in part, from combining of skimming, reading straight through, and close reading.

The attention continuum contrasts with other research (for example, Phillips and Phillips, 2007) that presents skimming and close reading as in opposition to each other. We argue that viewing these behaviors as a continuum is useful for understanding a reader's movement through the text as a web of choices they make and coordinate throughout the reading. Moreover, this perspective positions all behavior other than reading straight through as inherently agentive, because it, "run[s] counter to the status quo" of reading straight through (Weinberg et al., 2024, p. 669).

The attention continuum can, potentially, be thought of in terms of various forms of attention. Posner and Petersen (1990) point to awareness of stimuli, planning, and goal-directed behaviors. For example, sustained attention, which enables an individual to concentrate on a particular stimulus for an extended time period (e.g., Chen & Wu, 2015; Roda, 2011), could be connected to close reading. Conversely, alternating attention, which supports rapid shifts among

a set of stimuli (Lezak et al., 2015; Parasuraman, 1998) could support behaviors like skimming and targeted skipping.

Participants' stated reasons for attentive behaviors reveal a variety of ways that readers used attentive movement to engage in a learning experience. Moreover, we believe that didactical disciplinary literacy provides a useful lens for understanding these reasons. While some of the readers' moves through the excerpt could be characterized in terms of general literacy strategies (such as skimming to get an overview), their explanations were often tied to the specific didactical structure of mathematics textbooks (for example, coordinating an example and general equation, as part of developing understanding). In some cases, we explicitly connected participants' reasons to disciplinary norms and values (via the disciplinary reasons code). Although this study focused on common themes among the participants, we also found considerable variation between participants in the particular ways that they engaged in behaviors and how these behaviors connected to their reasons. Although the small size and complexity of the dataset makes it difficult to draw clear distinctions between the three groups of participants, we believe the variation between participants suggests that the individual's positioning in the disciplinary community and as a learner may offer a useful perspective on how they move through a text.

The results of this study display a varied and nuanced set of behaviors that readers engaged in while reading a mathematics textbook. Readers' explanations for how they navigated the excerpt appear to be connected to their prior knowledge, their perceptions of the excerpt as a didactical text, their positioning in the discipline, and their understanding of disciplinary norms and values. While this study provides an initial description of reader's attentive movement and associated motivations, it also highlights this as an area for further investigation. In particular, additional research could reveal a richer description of how individuals organize their movement through a text, how this movement is guided by the text itself (as both a disciplinary object and a didactical text), the reader's background knowledge, and the reader's disciplinary positioning and identity.

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Moving Beyond Total Scores With Skill-Proficiency Profiles on Concept Inventories

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Research-based assessments (RBAs) are widely used to evaluate student learning in STEM, yet most provide only total scores. We illustrate the value of moving beyond overall performance by applying a cognitive diagnostic (CD) model to student responses on the Precalculus Concept Assessment (PCA) and the Calculus Concept Inventory (CCI). Using 1,366 pre/post responses collected through the LASSO platform, we modeled five key calculus skills: prerequisites, limits, derivatives, applications of derivatives, and introductory integration. Traditional analyses revealed small but statistically significant score increases (d = 0.09 for PCA; d = 0.26 for CCI). CD modeling also showed limited shifts in skill proficiency profiles: students who lacked prerequisite skills did not gain those skills. These results underscore how CDs provide finer-grained evidence than total scores and motivate the development of a Calculus Cognitive Diagnostic to support timely instructional interventions.

Keywords: Calculus, Assessment, Learning, Cognitive Diagnostic

Research-Based Assessments (RBAs) have become foundational in DBER because they are developed through iterative qualitative and quantitative studies that build a cumulative validity argument aligned to field standards. This systematic approach positions RBAs to measure learning and to inform instructional decision-making at scale. In physics, RBAs enabled early demonstrations that interactive engagement markedly outperforms lecture (e.g., Hake, 1998), guided the design and evaluation of Modeling Instruction (Brewe, 2008), and later underpinned cross-disciplinary meta-analytic evidence that active learning improves achievement and lowers failure rates (Freeman et al, 2014). More recent work shows that although students *feel* they learn less during active engagement, they in fact learn more (Deslauriers et al., 2019), and that lecturing still predominates in undergraduate STEM (Stain et al., 2018). These results underscore the need for sustained, RBA-informed reform, and RBAs designed to meet the needs of educators and researchers pursuing those reform efforts.

Mathematics education researchers have developed at least six RBAs covering precalculus and calculus content (Epstein, 2006: Carlson et al., 2010; Carlson et al., 2015; Thompson et al., 2018; Lee et al., 2025). Researchers tend to use these fixed-length instruments to measure learning across a semester though they have played a smaller role in the RUME community than similar instruments have in physics (e.g. Hake, 1998). This limited use may result from subsequent studies that questioned the psychometric validity arguments of some of these instruments (Gleason et al., 2019; Jones, 2021; Lee et al., 2025).

We are developing a calculus cognitive diagnostic (CCD). Whereas current fixed length RBAs primarily assess overall ability, a cognitive diagnostic (CD) assessment measures student proficiency on individual skills and overall ability (Leighton & Gierl, 2007). The CD classifies students into skill profiles to provide instructors, researchers and students with finer grained information about student learning (de la Torre & Minchen, 2014). Unlike many fixed-length tests, CDs can assess a subset of skills and overall ability whenever needed, even multiple times

a semester, without compromising the instrument. By drawing from a larger item bank, CDs also prevent over exposure of items. Our CCD development uses a computerized adaptive testing platform (LASSO, 2025) that optimizes the information each question provides based on students' responses to earlier questions (Yasuda et al., 2021). Combining a CD with computerized adaptive testing ensures accurate and efficient measurement of each student's proficiency level (Morphew et al., 2018) with the fewest questions necessary.

In this paper, we apply a CD model to student responses on the Calculus Concept Inventory (CCI, Epstein, 2006) and the Precalculus Concept Assessment (PCA, Carlson et al., 2010). Prior work establishing acceptable fit for the CD student model used in this study (Roberge et al., 2025). This application illustrates the additional resolution provided by skill-level proficiency estimates. We contrast pre–post changes in total scores with changes in skill proficiency profiles, including transition patterns across skills, to show what these two outcomes reveal and obscure.

Research Questions

To that end, we asked the following research questions for the PCA and CCI:

1. How much do students' scores change (learning) from pre to post on each of the instruments using mean shifts?
2. Using the CD student model, what are students' pre- and post-instruction skill-proficiency profiles, and to what extent does instruction increase proficiency probabilities and transitions into proficiency for calculus skills?

Theoretical Framework

Building on prior work (Vy et al, 2025), we drew on Evidence Centered Design (Mislevy et al., 2003) to develop the CCD. The conceptual assessment framework within Evidence-Centered Design has five components: student model, evidence model, task model, assembly model, and delivery system. In this article, we focus on the student and evidence models, as these components shape the assessment output. The student model includes the skills that the instrument aims to assess. The evidence model establishes the criteria for evaluating the assessment and defines how to score the student's answers to the questions. For the evidence models, we applied a CD model to interpret performance in terms of skill proficiency. The task model defines the format of the items (multiple-choice questions in our case). The assembly model integrates the student models with task models to assemble the assessment and assess student's performance. The delivery system (LASSO, Van Dusen et al, 2021) administers the assessment online.

Literature Review

Calculus can function as a gatekeeper in STEM education (Seymour et al., 2019). In fact, Seymour et al. (2019) found that calculus was one of two of "…the first and second year courses with the highest numbers of students receiving a D, F or incomplete/withdrawal grades…" at a rate of 20% (p. 199). Other studies have found DFWI rates as high as 34.3% (Koch & Drake, 2018). Calculus failure rates may disproportionately affect women (Ellis et al., 2016) and students of color (Treisman, 1992) contributing to a lack of representation in STEM fields (Sanabria & Penner, 2017). Evidence based instruction can greatly improve learning outcomes (Kramer et al., 2023) and such methods require evidence to assess. To this end, researchers have developed at least six RBAs to assess student readiness for calculus or learning during first and second semester calculus and provide data for researchers to evaluate.

Beginning in 2006, Epstein created the Calculus Concept Inventory (CCI) to measure learning in a first semester calculus course. In 2018 Thompson and Ashbrook began developing the Calculus 1 Concept Inventory (C1CI) and the Calculus 2 Concept Inventory (C2CI). More recently Lee et al. (2025) have released the Revised Calculus Concept Inventory (RCCI) which combined the CCI and C1CI. Carlson et al (2010) developed the Pre-calculus Concept Assessment (PCA) to measure student learning in precalculus. She and collaborators would later extend this work to the Calculus Concept Readiness (CCR) in (2015) to measure student proficiency on prerequisite concepts important for calculus.

Because researchers use a wide range of instruments and inconsistent reporting practices, we found it difficult to characterize learning gains in calculus. Researchers sometimes use RBAs such as the CCI (Bagley, 2014; Villalobos-Camargo, 2021; Dagley et al, 2018; Epstein, 2007 Thomas & Lozano 2013) but often they create their own instruments for pre and post assessments (Adams and Dove, 2018; Cronhjort et al., 2013; Kramer et al., 2023; Young et al, 2011). Reporting of descriptive statistics varies across these studies. Normalized gains in lecture based courses tend to be small and range from 0.1 (Villalobos-Camargo, 2021) to 0.3 (Young et al., 2011; Adams and Dove, 2018). Normalized gains (Hake et al., 1998) are not without issues (Nissen et al., 2018).

We (Roberge et al., 2025) identified a list of calculus skills by examining the learning outcomes from the OpenStax first semester calculus textbook (Strang & Herman, 2016) and from the Advanced Placement AB calculus test (*AP Calculus AB and BC: Course and Exam Description*, 2020). We initially developed a list of 27 skills that we used to code the items from both RBAs. Each instrument had too few items to cover 27 skills. After several iterations, we decided to focus on five broader categories: Prerequisite skills, Limits, Derivatives, Applications of Derivatives and Introduction to Integration. Tallman et al. (2016), after analyzing 254 different final exams for post-secondary, first-semester calculus courses, came to a similar conclusion about common calculus content. Our results (Roberge et al., 2025) supported the student models of the five skills fitting the data well. Both fit characteristics (RMSEA and SRMSR) were well below cutoff values and classification accuracies were either acceptable or good for each learning outcome.

Methods and Materials

The data was supplied by the LASSO platform (LASSO, 2023). The data consisted of 1,366 college student responses to both a pre and a posttest from the PCA (25 items – 897 students) and CCI (22 items – 469 students). Data for the PCA came from six institutions across thirty-three unique math courses that ran between the fall of 2016 and the summer of 2018. Data for the CCI came from five institutions, eight instructors, and eighty-six courses taught between the Fall of 2021 and the Winter session of 2024. Following previous data cleaning processes on LASSO data (Morley, Nissen & Van Dusen, 2023; Van Dusen & Nissen, 2020) we excluded students who took less than five minutes to complete the assessment or attempted fewer than 80% of the questions.

Descriptive statistics on pre/post scores for the PCA and CCI suggest only small changes. These scores are on a 100 point percentage scale. The PCA pre and post score means were 44.4/46.0 with standard deviations 17.1/17.3 respectively. The CCI pre and post score means were 34.2/38.7 with standard deviations 15.8/18.2 respectively. The PCA effect size was $d = 0.09$ and the CCI effect size was $d = 0.26$.

To determine if the differences in mean scores were significant, research question 1, we used a hierarchical linear model. Pre and post scores are nested within students and students are nested

within courses. Neglecting nesting in data can violate the assumption of independence of many statistical methods such as multiple regression and ANOVA (Van Dusen & Nissen, 2019). For our models the intraclass correlation coefficients (ICCs) were, for PCA/CCI, students 0.54/0.56 and for courses was 0.18/0.069 respectively. Greater between group variance indicated we needed to treat groups (in this case: students and courses) separately and the hierarchical linear model accounts for both within-student variance and within-course variance.

To answer research question 2 we used CD models to produce finer grained information than traditional concept inventories. The model used test items, student responses, and a binary Q-matrix that codified item-skill dependencies. The Q-matrix is an $n \times m$ matrix for n items and m skills (Chen et al., 2015; Cui, Gierl & Chang, 2012). A one in the *ij-th* location means that skill j is required to correctly answer item i. An item may require more than one skill so its row in the Q-matrix will be a string of zeroes and ones. The resulting CD fitted to data categorized students into skill profiles and provided a variety of classification accuracies (Ravand & Robitzsch, 2015).

Skill profiles are binary vectors of length m (for m skills) where each '1' represents a proficiency and each '0' represents a missing proficiency. For example, a skill profile of 10100 for the CCI represents proficiency on prerequisite material and derivatives but missing proficiency on limits, applications of derivatives and introductory integration. We used alluvial plots to show how students transitioned from one skill profile to another from the pretest to the posttest. The quantity of students following any particular path from left to right is depicted by the size of the band connecting the left node to the right node. Additionally, when presenting alluvial plots of student proficiency groups we filtered to include only groups of ten or larger. For the PCA, this resulted in 24 students removed and six skill profiles removed. For the CCI this resulted in 39 students removed and fourteen skill profiles removed.

Table 1. The number of items and the classification accuracy (CA) for each skill for each RBA.

		Prerequisites	Limits	Derivatives	Apply Deriv.	Integration
PCA	items	20	3	0	5	1
	CA	0.92	0.68	-	0.87	0.91
CCI	items	6	3	6	6	1
	CA	0.94	0.91	0.94	0.95	0.98

Skill classification accuracy is a value between 0 and 1. Values between 0.8 and 0.9 are considered acceptable and values above 0.90 are considered high. For the five skill student model applied to the PCA and CCI data, Roberge et al. (2025) reported the results in Table 1. The reader may notice that the skill classification accuracy for Introductory Integration is considered high even though each RBA has only a single item coded for this skill. When a skill is represented by only one or two items, we may be skeptical of a high classification accuracy. On the PCA only 17% answered item 8 correctly on the pre-test (coded for integration) and only 18% on the post-test. On the CCI only 20% of students correctly answered item 18 (coded for integration) on the pre-test and only 24% on the post-test.

Findings

The results, shown in Table 2, indicate that post-test scores for both RBAs were larger than pre-test scores. The average post-test score on the PCA, accounting for student level and course level nesting of data, is 1.57 percentage points higher than the average pre-test score and this

difference is statistically significant at the level of $p < 0.001$. The average post-test score on the CCI, similarly accounting for nested data, is 4.54 percentage points higher than the average post-test score and this difference is statistically significant at a level of $p < 0.001$.

Table 2. Model summary for each of the hierarchical linear models

		coefficient	st. error	df	t	p
PCA	intercept	39.31	1.74	26.84	22.56	$p < 0.001$
	post-test	1.57	0.45	897.00	3.52	$p < 0.001$
CCI	intercept	34.52	0.95	107.03	36.40	$p < 0.001$
	post-test	4.54	0.66	474.11	6.92	$p < 0.001$

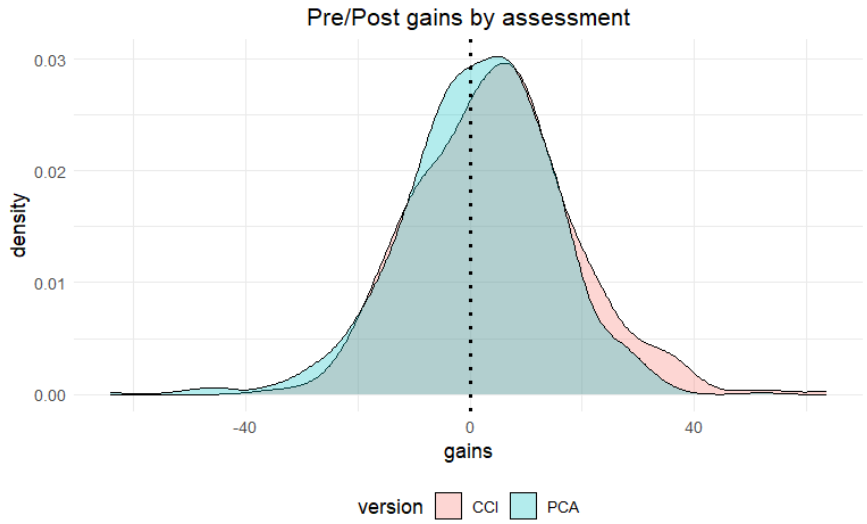

Figure 1. Density plots by gain (post score - pre score) for the CCI and PCA.

Addressing RQ2, Figure 2 depicts the changes in student skill profiles from pretest to posttest. The results for the PCA are located on the left and the results for the CCI are located on the right. Pretest skill profiles are depicted on the left-hand side of each alluvial plot and posttest skill profiles are depicted on the right-hand side of each plot. For example, in each alluvial plot the profiles 0000 or 00000 (we did not code any items from the PCA as assessing "Derivatives") would represent a lack of proficiency for all of the skills. Profiles 1111 or 11111 would represent proficiency in all the skills. The size of each rectangle represents the quantity of students in each group, for example in both the PCA and CCI the largest skill profile group on the pretest is 0000 or 00000.

The two figures portray overall similar trends of small changes in skills that were consistent with the results from research question 1. The PCA also differs because the size of the 0000 group decreased by more than half (51%) whereas for the CCI the 00000 group only decreased by 12%. Large chunks of the 0000, 1100, 1110, and 1111 groups on the pretest stayed at that same skill proficiency on the posttest. For the 0000, and 1100 groups, a notable number of students gained additional skill proficiency on the PCA posttest. This was much smaller for the transition from 1110 to 1111. Almost all of the students that ended with proficiency of all 4 skills, 1111, started there or with proficiency of the first three skills.

The shifts on the CCI were smaller than on the PCA and there were fewer skill proficiency profiles, which was likely due to the smaller sample size. We treat skill proficiency profiles of

00000 and 00001 as equivalent because that last skill is integration and the CCI only has one item that assesses integration. Only mastering integration is unlikely. When we equate skill profiles 00000 and 00001, out of the 378 students who start with no proficiency, 336 finish the course in the same situation. That's 89% of students acquiring no increased proficiency. Though enough students transitioned from 00000 to either 11110 or 11111 to make both of those groups increase in size. The total number of students who finished with proficiency in four or five of the skills was 17% (37) of the sample.

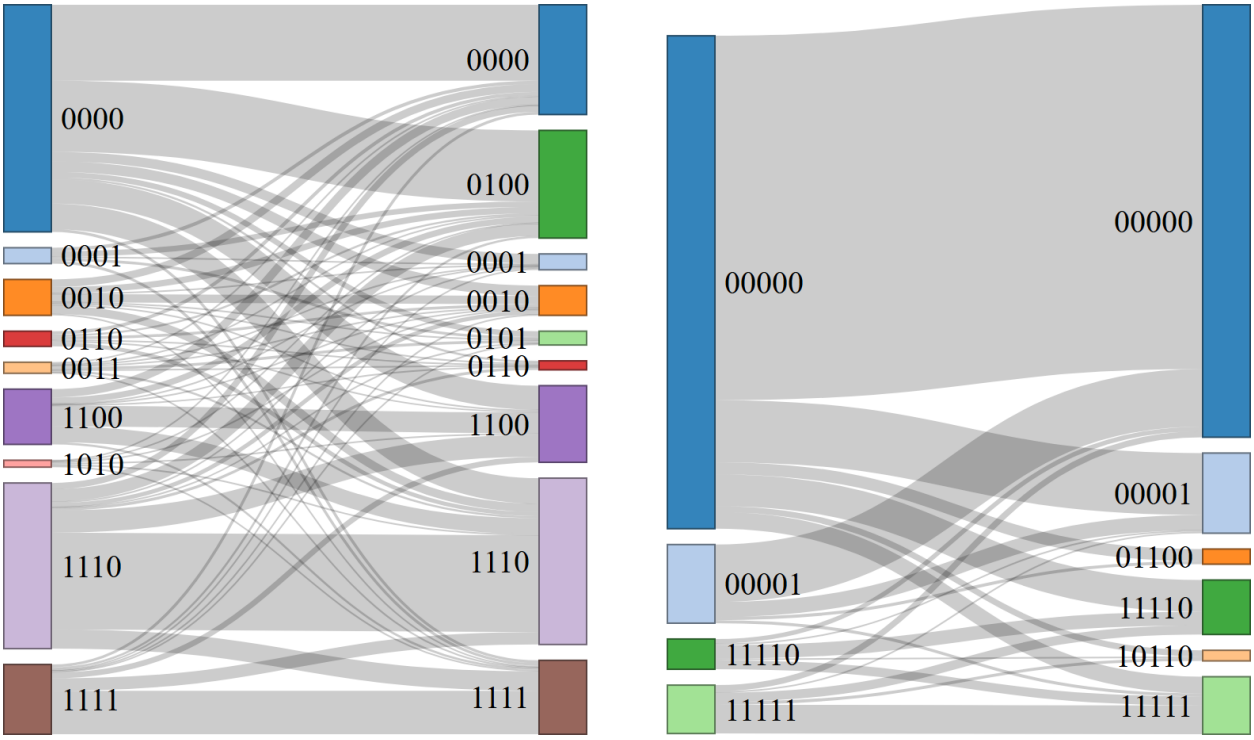

Figure 2. Sankey diagrams of pre-test and post-test data for the PCA (left) and CCI (right). Pre-test skill profiles are on the left of each plot and post-test skill profiles are on the right. For the PCA the binary 4-vector represents student proficiency on prerequisites, limits, applications of derivatives and introductory integration. For the CCI the binary 5-vector represents student proficiency on prerequisites, limits, derivatives, applications of derivatives and introductory integration. For example, on the CCI plot the skill profile 01100 would represent lacking proficiency in prerequisite concepts, proficient in limits and derivatives while lacking proficiency in applications of derivatives and introductory integration. Note: the data was filtered to include only skill profiles that contained at least ten students. Care should be taken when interpreting these plots as they do not represent individual courses but a population of students from many courses.

Limitations

These results represent the outcomes of students in the dataset and do not generalize to other courses or data. The contrast of skill proficiency profiles and overall scores, however, does generalize beyond this data. While binary skill proficiency profiles may seem to oversimplify student knowledge, they are a tradeoff of the finer grain measures that CDMs provide, and our development of the CCD seeks to better leverage this trait of CDMs. The timeframe of pre and post scores misses within-semester dynamics that CDAs can assess. Finally, neither RBA was designed as a cognitive diagnostic and the PCA was not designed with calculus content in mind though the retrofitting of a non-diagnostic assessment in this way is common practice (Ravand & Robitzsch, 2015) and serves to illustrate the information CDAs can provide.

Discussion

While both post-test scores were larger than the pre-test scores, the difference shows limited learning. The effect size on the PCA was d=0.09 and on the CCA was d=0.26. Kraft (2020) points out that a typical gain in secondary school is d=0.5 over an academic year and in introductory college physics the typical gain is closer to d=1.0 (Nissen et al., 2022). The density plots in Figure 1 rule out both floor and ceiling effects, which also indicates that the instruments did not grossly misalign with instruction. The results of the PCA and CCI data are consistent with existing literature on learning gains in calculus classrooms even though those gains are hard to compare given the variety of instruments. This points to a need to develop a standardized, yet flexible, instrument that can meet a broad range of teaching and research needs.

The CDM provides a finer grained perspective than the overall scores. Some students did make small gains in their skill proficiency. This is notable in the decrease of the no skill proficiency group from pre to posttest on the PCA and the increase in the 4 and 5 skill proficiency groups on the CCI. These small gains, however, sit within a larger picture of stagnation where instruction maintains proficiency for those who start with it and leaves those without proficiency in that state. It is particularly concerning that there does not seem to be a greater increase in the number of students gaining proficiency on prerequisite skills.

Consistent with the results from our first research question, either the instruction in these courses did not support the development of student proficiency or the RBA was not a good fit for the content of the course. Our five skills are consistent with commonly covered content areas in a first semester calculus course (Tallman et al. 2016) and most students who start with no proficiency (00000) end the course with no change to that skill profile. Most students who began with full proficiency retained that throughout the course, though some students transitioned from partial/no proficiency to full proficiency. This is consistent with Bressound's (2021) description of calculus classrooms in higher education having both students who are encountering these ideas for the first time and for whom this may be their last math course and those taking calculus again, after high school preparation who are building towards a math intensive STEM discipline.

Conclusion

The data for the PCA and CCI pre and post test scores are consistent with the literature that students experience relatively little learning in introductory calculus courses. Students who begin the course with low proficiency have minimal opportunity to improve while those who already possess high proficiency maintain that level.

The post-test score is too little too late to support teaching and learning. The CCD will support instruction by providing timely assessments that are fine grained enough to be acted upon. For example, instructors can use CCD results to inform what content they cover in their lectures or on their homework and to inform timely interventions to better ensure that students had the skills they needed before moving on. Student groups for in-class work could also be structured using CCD results to foster peer collaboration. The CCD, delivered with computerized adaptive technologies on the LASSO platform, has the capacity to support this vision.

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"I think most of them would say they don't understand yet": Mathematicians' Views on the Benefits of In-class Time

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Through analysis of 7 mathematicians' perspectives teaching proof-based linear algebra, we found that participants wanted to offer benefits to students during class which were analogous to the benefits documented in previous research that mathematicians have when attending and giving mathematics lectures at professional events (e.g., conferences, symposia, or workshops). Findings suggest that mathematicians may see attending mathematics lectures as an authentic mathematical practice in ways that education researchers have not previously attended to.

Keywords: teaching practices, proof-based courses, lecture, active learning

Efforts by the mathematics education research community to improve teaching have led to the development and testing of reform initiatives throughout the undergraduate mathematics curriculum. However, as documented by Fukawa-Connelly et al. (2016) and summarized by Artigue (2016), there has been "very limited influence of our research on university teaching practices. Reading recent publications… I find nearly the same description of standard university practices at undergraduate level as decades ago" (p. 12). One explanation for this is that "standard university practices" may be fulfilling the instructional goals of mathematicians in ways that reform initiatives championed by mathematics education researchers are not. By investigating what mathematicians see as valuable uses of class time, we gain insight that may allow us to propose pedagogical changes in ways that are better aligned with mathematicians' goals and have broader impacts at the practitioner level. To this end, we report on a collaboration between mathematics education research and mathematicians on the implementation of specific evidence-based active learning strategies (EBALS) in an advanced linear algebra course.

Background and Framing

Surveys on reports of how mathematicians structure their in-class time have consistently shown that lecture is the dominate form of instruction. For instance, Eagan (2016) found that about two-thirds of mathematics teachers use "extensive" lecturing in all or most of their courses, Johnson (2019) stated that "77% of pre-calculus courses, 82.3% of Calculus I courses, and 86.3% of Calculus II courses are taught as either 'lecturing and answering student questions' or 'lecture with some active learning techniques', and Fukawa-Connelly et al. (2016) found in their study that 85 percent of respondents reported using "lecture" to teach abstract algebra, as opposed to reporting that they use a "non-lecture pedagogy". Research on instructor's beliefs about teaching and learning, however, suggest that the predominance of lecture may not simply follow directly from their beliefs. For instance, in comparison to the 85% of respondents who reported that they lecture in their survey of abstract algebra instructors, Johnson et al.'s study (2019) only about 59% of the respondents agreed with the statement "*Lecture is the best way to teach*", whereas the overwhelm majority of them, 87%, agreed with the statement *"I think students learn better when they do mathematical work (in addition to taking notes and attending to the lecture) in class*". This raises a natural question: If instructors believe that students learn best when they are doing mathematical work in class, then what do mathematicians see as the benefit of lectures?

A similar question was the focus of a study conducted by Weber and Fukawa-Connelly (2022). Instead of looking at what mathematicians perceived as benefits of attending a lecture for their students, they instead investigated what benefits they perceived for themselves as lecture attendees in a workshop on inner-model theory. As mathematical learners in this setting, the mathematicians did not necessarily expect to learn everything they needed to know from watching a lecture. Rather, they saw key features of the lecture as beneficial to their own independent study of the content. Indeed, the benefits Weber and Fukawa-Connelly (2022) found belonged not only to those attending the workshop lectures, but also to those who provided them. It is natural to wonder whether the same benefits mathematicians see in attending lectures for themselves as learners are also considered by mathematicians as they think about how to spend class-time as teachers.

Analytic Framing, Methods, and Data

A key premise of the NSF funded *Collaborating with Mathematician's to Enhance Teaching (COMET)* project, of which this work is subsidiary, was that the reasons mathematicians have for rejecting or accepting EBALS provide insight into what they see as beneficial or valuable. Similar to Herbst and Chazan (2011), we aim to understand stable systems of practices by understanding how those systems react to perturbations. By proposing changes to mathematicians' face-to-face instructional practice, we created discussions focused on their considerations when planning class, revealing the reasoning behind and justification of their use of class-time. Here we analyze these reasonings and justifications to answer the research question*: When mathematicians are considering what is a valuable use of class time, do they attend the same benefits that they attribute to the mathematics lectures that they present and attend (e.g., at conference presentations, symposia, and workshops)?*

The data was collected over the course of two semesters at a large American university. Over the course of six semi-weekly meetings throughout each semester, three mathematics education researchers met with a group of mathematicians teaching a proof-based linear algebra course. The first group had four instructors and the other three. In this paper, the year one instructors are assigned pseudonyms A1, A2, and so on, and the year two instructors are assigned B5, B6, and so on. The goal of such meetings was to strategize about incorporating EBALS in the linear algebra course. The course presented an axiomatic treatment of abstract vector spaces and linear transformation on those spaces and covered topics such as eigenvectors and eigenvalues, the Jordan Canonical Form (JCF), its associated theorems, and inner product spaces. Prior to the start of regular meetings, the mathematicians were also interviewed about their current teaching practice. Each of the interviews and collaborative planning meetings were initially transcribed using Otter.ai. First, we conducted a thematic analysis (Braun & Clarke, 2006) to understand the mathematicians' considerations about how to spend their class time. Then, we analyzed how the themes we found relate to conventions of lecture in a professional context.

After editing the AI generated transcripts for accuracy, the first author made an initial pass through the data, identifying any excerpts where participants discussed the purpose of in-class time, either by describing what they wanted to do during class or by explaining why they would not want to do a particular activity. These excerpts were logged in a spreadsheet where the excerpt was described and initial comments about the content of the excerpt were made. Then, these initial descriptions became the focus as we sought commonalities between comments and then arranged them into larger themes. These themes became a codebook that was used to code each transcript during the second pass through the data. After collecting all instances of each theme into a single document, the first and second author met to resolve any disagreements.

Finally, we compared these inductive themes to the benefits of lecture identified in Weber and Fukawa-Connelly (2022). Specifically, we sought analogues to the benefits *exposing participants to interesting results, providing a big picture of the domain, highlighting results and techniques that are important,* and *illustrating how the lecturer thinks about a topic.* That is, we hypothesized that these mathematicians were attending to benefits analogous to those identified by Weber and Fukawa-Connelly when they considered how to spend class time, and then looked for confirming and disconfirming evidence in our thematically coded data.

Findings

When discussing their plans for class, these mathematicians did not expect that the students would develop deep understanding of the material based solely on what happened during class time. Instead, they often indicated that they believed that students needed to study the material outside of class to understand it thoroughly[1]. Six of the seven mathematicians made comments relating to this idea. When asked how students best learn in the course, A1 said,

> "By doing mathematics. . . **Watching me doing it a million times, I don't think has much of an effect and seems like a different skill**. You know, **they could watch me prove a theorem and be really amazed at it**, how clever that is, and so on. **And they will never know how to do that themselves**."

We note the distinction made by A1 between the skill of studying independently and the skill of watching a lecture. Students could be amazed by watching a lecture, but that is not enough for students to learn the mathematics themselves. A2 and A3 echoed this sentiment, with A2 noting that "30 seconds or one minute is not enough" to absorb complex ideas during class, and A3 saying that, if you asked students what they understood at the end of class for an *exit-ticket*, "I think most of them would say they don't understand yet". It makes sense that if you believe that significant time and effort are needed for students to develop a deep understanding of the content, then developing such an understanding would not be a reasonable goal for the few hours a week spent in class.

Instead, when we asked what was most important to do during class time, responses fell broadly into four categories. For instance, B7 stated:

> "The first thing is that **I give a very organized lecture so they know what is important**. The most important formulas, or theorems, and hopefully some of the arguments. And second, I think **hopefully it'll be a bit inspiring**, so that **they will find it interesting**. For example, in today's class, I did one example about the derivative for polynomials. So they can sort of relate differentiation and integration as linear operators to the matrix representation. So **this is some connection between calculus and analysis to algebra and matrix algebra, as well as how it's related to physics**. So I hope I try to be a little bit inspiring at times."

These mathematicians wanted to spend class time interesting, inspiring, and motivating their students; they wanted to provide an overview of the big ideas in linear algebra, including how they relate to other domains; to highlight important results and techniques; and to illustrate appropriate thinking about the content.

[1]While they expected students to follow along with calculations and proofs of lemmas or simpler theorems, these discussions were often referencing abstract concepts, or theorems which required more involved methods of proof.

Class Time Should Inspire Students

Inspiring students by discussing interesting results during lecture came up frequently during meetings. Furthermore, participants' commentary often focused on motivating the mathematical material presented in the course. Five of seven mathematicians made relevant comments.

A1 referenced wanting to "sell" the material to the students, as well as ways he generated student "buy-in" to the course, to "get [students] hooked" by explaining how a vector is an essential object with applications "not just to linear systems, but to differential equations, to probability and statistics, to game theory, [and] to particle physics . . ." Later, he argues, "the reason you can have all these different applications is that [a vector is] fundamentally an abstract mathematical thing. So that's how I actually put the hook in there. And I think students are buying it so far." Several mathematicians in our study prioritized showing students interesting results from linear algebra that apply to other sciences. We interpret A1's reference to "buy-in" as expressing a desire for students to be willing to engage in the more abstract reasoning required in proof-based linear algebra.

In addition to motivating students, these mathematicians also wanted to motivate the mathematics presented in class. Specifically, they prioritized discussions that developed intellectual need for the material they covered. All of the year 1 collaborators considered motivation when they discussed the lectures on the Jordan Canonical form, complaining that the textbook's treatment was inappropriate for presenting in class. A1 commented, "It's written in a very "mathematician" style: definition, theorem, proof . . . for twenty theorems." A2 echoed this sentiment and explained,

> "It's unfortunate that this book starts with proofs . . . so I also had to adjust this. But **for motivation**, what I told the students is that we always want to find the simplest possible matrix. So that's actually a good way to **explain why we need Jordan canonical form**."

B6, A1, and A2 all noted that motivation was an essential product of lecture that was often missing from the textbook. B6 even contended that this kind of motivation supported students' learning outside of class, saying, "Because students have a textbook that they can read for theorems, I think an important thing is to just motivate things because it's a lot easier for them to learn if they know why you would ever define these things." In short, the mathematicians prioritized motivation in two senses: inspiring students to be curious about the content, and discussing the origins and context of the material to create an intellectual need for learning it.

Class Time Should Provide an Overview of the Content

We found that the mathematicians in our collaborative planning meetings wanted to provide students with an overview of the high-level ideas of linear algebra; they did not intend to spend class time fleshing out every detail exhaustively, because they saw the details as time-consuming and students as unable to digest them in real time. All seven participants made comments in this theme. For example, A3 remarked, "I don't think that's an efficient use of class time to listen to someone prove very slowly this theorem, because I think **you just want to get the main idea**." He did see presenting material in class as communicating to students which content to account for on future assessments:

> "You have to spend time on the proof, because they're responsible for knowing that. You can't just say read the book. **That doesn't mean you have to go through every detail**, you know. I think you have to do the hard direction [of the biconditional], or **you have to introduce the main idea**, right?"

In addition to demonstrating which concepts students are "responsible for knowing", participants frequently commented that class time should be spent showing students how important concepts

are related to each other, as well as situating the course content within broader scientific contexts. A1 remarked, “Sometimes it's not fully connected in their minds, these different concepts. So maybe that's something that I could stress more in my lectures, try to make these connections clearer.” We see this attitude as analogous to the mathematicians interviewed in Weber and Fukawa-Connelly (2022) in that, for our participants, emphasizing the high-level ideas and their connections to each other was a benefit of lecture.

Class Time Should Highlight Important Concepts and Techniques

Participants frequently accepted or rejected a suggested EBAL based on whether it concerned content that was important enough to spend class time on, with five of seven making comments in this theme. What warranted importance often varied; content was “important” if it was worth understanding more deeply, introduced a new skill, or if it was seen as either more sophisticated than the prerequisite linear algebra course’s content or as necessary for the preparation of math majors. After accepting a task concerning the JCF, mathematician A1 commented,

> “I think in this case, you made the right choice. The theorem that you chose questions for here, that's actually the main important thing to get the block diagonal structure. That is a really important theorem. And it takes a long time [to prove], because it has several parts, and none of the parts are trivial to do. And then everything else kind of follows from that.”

B5, A2, and B6 often referenced wanting to spend class time on demonstrating new skills, with A2 commenting that he would not “give the full details” because “the main thing is to teach them how to do the calculations” which were new content for the course. We interpret the reference to “the main thing” to be synonymous with “the important thing.” He sees understanding how to find a linear transformation’s JCF as important enough to spend class time on because, in contrast to other calculations, this one is specific to the advanced linear algebra content.

That is, an activity was often considered important if it was appropriately sophisticated for the level of the course. A3 agreed, saying that the use of more abstract vector spaces of functions was “really important if they want to go into a differential equations class.” As well as attending to whether a task was appropriately sophisticated for the 350-level course, participants also weighed importance based on relevance to other courses in the math major, like differential equations. B5 added, “I think it just ties in with so many different courses in the math major that it's worth doing.” Finally, A3 commented, “I think in terms of the applications [math majors] are interested in, the inner product stuff is much more important.”

These participants wanted to show students “important results” that were conceptually rich, worth understanding deeply to use in practice later. The pedagogical context offered more nuance; a concept was also considered important if it was appropriately sophisticated for the level of the course, or if it was relevant to future courses in the math major.

Furthermore, this decision-making apparatus took the form of a biconditional statement. Not only did these mathematicians want to spend class time discussing important concepts, they warned that the amount of time spent on a concept during class *indicated to students* which content was important. When discussing whether he valued having students create their own conjectures during class, A3 commented that many theorems in linear algebra would not be appropriate for such an activity, noting the rank-nullity theorem as an exception. Then, he said, “But I don’t know, I don’t really feel like the rank-nullity theorem is that important. I don’t want to spend that much time on something that I don’t really care about that much.”

Time spent during class on a particular topic corresponded to its importance relative to the course content. Although A3 saw conjecturing as a valuable practice for students, he did not

wish to emphasize a theorem that was not important enough to spend that amount of class time on. A2 made similar comments:

> "Especially for some really, really important theorems, for example, the proof that every linear transformation has a Jordan canonical basis. The goal for the students to get these arguments, it's very good. But on the other hand, we cannot use it for all the theorems, because we do not have enough time for this course. We can only do it for the important theorems."

It is notable that these mathematicians sometimes saw active learning as having an undesirable potential to emphasize results or techniques which were not important enough to warrant the time expenditure.

Class Time Should Illustrate How to Think About the Content

The mathematicians in our study wanted to model how to do linear algebra according to the norms and values of the broader mathematical community, beyond showing how the lecturer as an individual approached the content. This included demonstrating appropriate rigor and precision, showing students which ways of thinking may be productive, and giving students the opportunity to have their misconceptions corrected by an expert. All seven mathematicians made comments in this theme.

A2 observed that "the nice thing" about class was the chance he got to show students how to express their correct ideas abstractly and rigorously, saying "I point out that okay, yes. Roughly, you're right. But see, this is not precise. The precise way is [this]." It was important for participants that time spent in class not only showed students appropriate rigor, but also focused on material that was sufficiently sophisticated for the course. For example, it was important that examples used vector spaces other than $\mathbb{R}^n$. A3 commented,

> "In things like signal processing, we really use vector spaces of functions that are usually not polynomials, polynomials very rarely come up in in applications. It's usually either periodic functions, or exponentials, or logarithms, things like that. I think this course is really an opportunity to talk a little bit more about those kinds of functions."

B5 and B6 both described tension regarding the use of notation aligned with the textbook that was unconventional in the mathematical community, with B6 saying "I'm really torn between whether to use the book's notation or to use the notation that everyone else uses." A1 was willing to spend time in class on tasks that developed productive ways of reasoning about the content, saying, "there should be a way of making that part of their habit of thinking."

Our participants wanted to spend class time demonstrating what it meant to do math at this more advanced level. We see this benefit as analogous to the benefit Weber & Fukawa-Connelly (2022) found that the mathematicians in their study valued observing the habits of mind of a professional, experienced practitioner of a niche subfield. Because the audience members our participants referred to were students, it is sensible that they felt a responsibility to include *informal content* (see Fukawa-Connelly et al. 2017) like attending to precision and rigor, or correcting misconceptions, where the lecturers in the former study may not have.

Conclusion and Implications

There are many motivations for mathematics education researchers to advocate for the use of EBALS, and many benefits of their use recognized by mathematicians. These motivations and benefits vary based on the specific strategy, and the implementation of these strategies may be more or less compatible with lecture. For instance, Johnson et al. (2025) distinguished between "two types of active learning pedagogy in advanced mathematics: pedagogy to *replace* lecturing

and pedagogy to *augment* lecturing" (pg. 147, emphasis in original). They go on to argue that, while there has been comparatively limited research in undergraduate proof-based mathematics courses on the use of EBALS that augment lecture, there has been substantial research in undergraduate mathematics education on initiatives designed to replace lecturing.

In this push to replace lecture, a common sentiment is that class time should be used to engage students in authentic mathematical activity. For instance, in providing a joint characterization of Inquiry Based Mathematics and Inquiry-Oriented Instruction, Laursen and Rasmussen (2019) state that a

> "distinguishing characteristic of inquiry is the nature of students' mathematical work. In inquiry classrooms students reinvent or create mathematics that is new to them. They do so by engaging in mathematical practices similar to those of practicing mathematicians: conjecturing and proving, defining, creating and using algorithms, and modeling" (p. 4).

However, this description of the practices of mathematicians leaves out some of the ways that mathematicians learn about new areas of mathematics, by attending lectures at workshops and studying the foundational texts of that area (Weber & Fukawa-Connelly, 2022). And indeed, we found that our mathematician collaborators attended to the similar benefits mathematicians attending – and giving – professional lectures when evaluating how to use class time.

We see two ways in which this finding can inform the work of mathematics education researchers looking to have a broader impact on undergraduate instructional practice. First, many mathematicians have goals like covering course content and improving students' abilities to write conventional proofs (e.g., Johnson et al., 2013; Johnson et al., 2025; Wu, 1999). Our findings add nuance to these goals. When we attending to the goal of providing students with a "big picture", we can make sense of content coverage as a priority for mathematicians regardless of department pressure (see Johnson et al., 2018). Content coverage may be seen as a prerequisite for situating the high-level ideas and techniques in relation to each other. Notably, active learning pedagogy designed to replace lecturing tends to prioritize depth over breadth in terms of content (Roth McDuffie & Graber, 2003). Additionally, demonstrating proof conventions like abstraction and rigor during class may be important to mathematicians not only because it prepares students to write formal proofs, but because it gives the audience access to the habits of mind of an expert. Modeling a formal proof by providing a lecture may be a more effective way of meeting that goal than active learning strategies that are designed with the intention that "the teacher must help transition the developing mathematics of the classroom, from that of informal student reasoning to that of formal mathematics" (Kuster et al., 2018). While we see the desire for more student-centered instruction as legitimate, it seems that there are places in which that desire may be at odds with some of the goals that mathematicians have for class time.

Second, in problematizing lecture-based instruction, we implicitly disavow a disciplinary practice that they themselves turn to when they want to learn (i.e., listening to lectures). This stance disavows a practice that is seen as beneficial by many in the community we are trying to reach. Further, we may be asking them to replace that practice with something that fails to provide them with those same benefits. This information not only allows mathematics education researchers to better understand why so many of our instructional innovations have failed to be adopted at scale, it also provides additional considerations for designers and disseminators of active-learning innovations as they think about how their products do (or can be adapted to) provide benefits more aligned with the goals of practitioners while preserving their own values.

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Memengful Calculus

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This study explores student-generated memes submitted as a part of a Calculus II homework assignment. Drawing on a twofold theoretical perspective that combines Bini et al.'s (2023) heuristic model for the creation of a mathematical meme and Bolman and Deal's (2008) Four Frames framework of an organizational culture, we analyzed 32 memes and corresponding reflections. We found that via memes, students reflect on and communicate their struggles with Calculus and draw connections between Calculus concepts. Memes also became a vehicle for reflecting and shaping the classroom culture, offering students an opportunity to "laugh back" at the course authority. Together, these findings illustrate how memes can serve as reflective artifacts that capture both cognitive and socio-cultural dimensions of learning. We argue that meme-based assignments hold promise for fostering higher-order thinking, humanizing instruction, and promoting engagement in a challenging Calculus course.

Keywords: memes, Calculus, assignments, classroom community and culture

The term *meme* was first introduced by Richard Dawkins in his book *The Selfish Gene* (1976) to describe a cultural unit transmitted between people in a manner analogous to genes. With the rise of the internet, memes initially appeared as simple captioned images circulating in online communities for humorous purposes. Today, they encompass a wide range of formats, including photographs, videos, and hashtags, and play an increasingly prominent role in modern digital culture. Beyond entertainment, memes now function as tools for communication, creativity, and self-expression, offering ways to comment on current events, challenge mainstream narratives, and participate in shared community life.

While research on memes in education is still emerging, scholars have increasingly recognized their significant educational potential. Studies demonstrate that memes can be a valuable and effective addition to higher education classrooms, fostering student engagement, critical thinking, and creativity across a range of disciplines, such as psychology (Kath et al., 2024), pharmacy (Brown, 2020), nursing (Bradley et al., 2022), and others.

Mathematics educators have likewise begun exploring the role of memes in teaching and learning. One line of work focuses on *mathematical memes* (memes containing a mathematical statement that can be validated) circulating in global online communities (e.g., Benoit & Salopek, 2023; Bini et al., 2022). For example, Benoit et al. (2025) found that engagement with mathematical memes can influence students' mathematical identity and their perceptions of who can become a mathematician. Another line of research examines *pedagogical memes* – memes intentionally integrated into mathematical instruction. Studies suggest that such memes support effective learning practices, enhance students' motivation, and reduce math anxiety (Akin et al., 2025; Sidekerskienė & Damaševičius, 2025). In particular, Sidekerskienė and Damaševičius (2025) presented a case study of a university Calculus instructor who incorporated a set of memes into her lessons as visual aids, thereby making complex Calculus concepts more accessible and engaging, which in turn improved student understanding and motivation.

The purpose of the present study is to extend the exploration of memes in undergraduate mathematics education by focusing on student-generated memes. Specifically, we analyze memes created by undergraduate students as a part of a homework assignment in one section of Calculus II, taught by the first author during the Spring 2025 semester. This study is guided by the following research questions:

1. How can undergraduate students use memes to reflect on and communicate their experiences with Calculus? In particular, what Calculus concepts and learning experiences do they reflect on and communicate through memes?
2. What pedagogical implications emerge from incorporating meme creation into a homework assignment in a Calculus course?

Theoretical Perspective

In answering these research questions, we adopt a twofold theoretical approach. First, we draw on Bini et al.'s (2023) heuristic model for the creation of a mathematical meme. This model views meme-making as an iterative process of linking between a specific meme template and a mathematical concept. According to Bini and colleagues, mathematical memes integrate three partial meanings: structural (the visual and textual conventions of meme formats), social (the cultural resonance of the chosen template), and specialized (the mathematical content to be expressed). The creative act lies in identifying a *linking idea* that bridges these domains. Although this model was originally developed for mathematical memes, we extend its application to encompass memes that capture both Calculus concepts and students' learning experiences.

As it was pointed out by Benoit and Salopek (2023), "… memes do not exist in a vacuum; they rely on a shared cultural and semiotic understanding for collective interpretations. They are entangled in their sociocultural environments and cannot exist outside of the events and practices in which they appear" (p. 45). In our study, memes were created by the members of a specific sociocultural environment with its own rules, practices, and shared meanings. Therefore, we adopt another framework that complements Bini et al.'s model by providing in-depth insights into classroom culture – the *Four Frames* perspective. Originally, this perspective was developed in organizational science (Bolman & Deal, 2008) and later adapted for undergraduate STEM department contexts (Reinholz & Apkarian, 2018). In this study, we extend its application to an even more local setting – a single classroom community.

In the Four Frames perspective, culture is defined as "a historical and evolving set of *structures* and *symbols*, and the resulting *power* relationship between *peopl*e" (Reinholz & Apkarian, 2018, p. 3). This definition highlights four interrelated dimensions of organizational culture. Table 1 summarizes the definitions of these dimensions and illustrates how they manifest in a mathematical classroom setting. Specifically, we view classroom *structures* as the syllabi, assignments, routines, and assessments; *symbols* as the cultural and humorous artifacts, informal rituals, and shared meanings; *people* as the permanent and temporal members of the classroom community (i.e., students, teacher, and the substitute teacher); and *power* as the authority and explicit (and implicit) hierarchy in a mathematical classroom.

Table 1. Definitions and aspects of the elements in the Four Frames framework, and their applications to a mathematical classroom setting.

Frame	Definition (adapted from Reinholz & Apkarian, 2018)	Application to a Mathematical Classroom
Structures	Formal arrangements, policies, and practices that organize how people work.	Syllabi, assignments, routines, and assessments.
Symbols	Cultural artifacts, language, rituals, stories, and meanings that community members use to guide their reasoning.	Cultural and humorous artifacts, informal rituals, and shared meanings.
People	Individuals and groups within the community.	Students, course instructor, and substitute teacher.
Power	Status, control, position, and political coalitions which mediate interactions between people.	Authority and explicit (and implicit) hierarchy in a mathematical classroom.

Methods

The data for this study stem from a second-semester Calculus course (Calculus II) taught by the first author in Spring 2025. The course enrolled 33 students, the majority of whom were STEM majors. Throughout the semester, the instructor regularly incorporated memes into his instruction, using them in lecture notes, handouts, online announcements, and a specific homework assignment called *Memengful Calculus*. This assignment, which accounted for 3% of the overall course grade, required students to submit a meme accompanied by a brief written reflection. In their reflections, students were asked to explain the choice of meme context, describe the Calculus concept or learning experience they were referring to, and elaborate on the connection between the two. By the time the assignment was due (around the middle of the semester), the following topics had been covered: Riemann Sums, volumes of solids of revolution, various integration techniques, and improper integrals.

The data corpus consisted of 32 submissions (32 memes and 32 corresponding reflections). All submissions were completely anonymized. The analysis proceeded in multiple iterations. In the first round, we conducted open coding, focusing on three dimensions of each submission: the context of the meme, the Calculus concept or learning experience it referenced, and the links between them. The submissions were divided into four equal sets, with each researcher coding the assigned set independently. During weekly research meetings, we reviewed and discussed all codes as a team, reaching consensus through the process of constant comparisons (Glaser & Strauss, 1967). Related codes were then grouped into categories, which served as the foundation for developing higher-level themes through thematic analysis (Braun & Clarke, 2006).

The final step of the analysis involved interpreting the resulting themes through the lens of the Four Frames framework. This allowed us to examine how students' submissions reflected their positioning within the social and cultural dynamics of the course.

Results

Our thematic analysis revealed four themes: two *cognitive themes* that captured students' engagement with mathematics and two themes related to different aspects of the *classroom community and culture*. In what follows, we describe each theme and illustrate it with the most salient examples from the students' memes and corresponding reflections.

Cognitive Themes

Through memes, our participants expressed their struggles with Calculus, particularly in relation to quizzes and exams. At the same time, the Memengful Calculus assignment provided a space for students to reflect on their mathematical knowledge by comparing various Calculus concepts and even making connections across different branches of Calculus.

Struggles with Calculus. A persistent myth surrounds mathematics–undergraduate mathematics, and Calculus in particular–that portrays it as "nightmarish, inherently difficult, and something to be avoided" (Darragh, 2018, p. 197). Many of our participants' memes echoed this characterization. Figure 1, for instance, illustrates students' struggles with two integration techniques – integration of rational functions by partial fractions and trigonometric substitution. As one student explained in their reflection, "Trig sub has been my biggest downfall in this class, so I would rather do any type of integration over that."

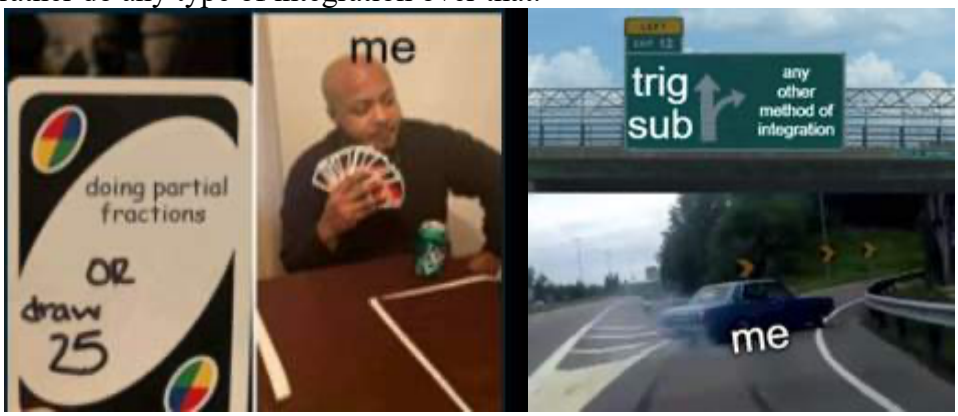

Figure 1. Students' memes illustrating struggles with two integration techniques.

In addition to communicating struggles with particular Calculus concepts, our participants' memes often conveyed the stress and frustration they and their peers experienced in relation to quizzes and exams. For example, one student described the discouraging experience of forgetting to adjust the bounds when applying the substitution rule in a definite integral on a midterm (Figure 2, left). Another student created a meme capturing their classmates' struggles with an in-class quiz on partial fraction decomposition (Figure 2, right). As they explained in their reflection, "I made this meme in the wake of the ***devastating*** news of quiz 5. While I personally did not struggle with the question, I know others who did and thought this would be a lighthearted representation of that struggle."

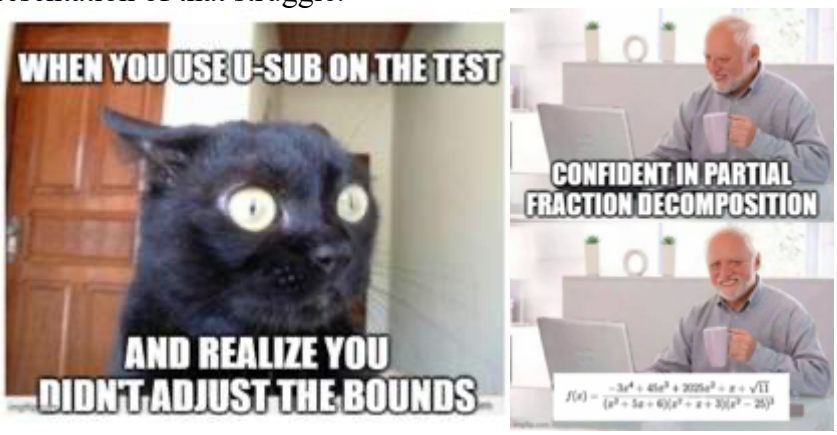

Figure 2. Students' memes illustrating struggles with quizzes and exams: forgetting to adjust bounds on a test (left) and the collective difficulty with the partial fractions problem on an in-class quiz (right).

Reflecting on conceptual connections. Some students used memes as an opportunity to reflect on and articulate conceptual connections between Calculus concepts, including drawing links between material previously learned in Calculus I and new content introduced in Calculus II. One student created a meme contrasting the processes of differentiation (Calculus I) and integration (Calculus II) of the same function, noting that the former was "easy to find," whereas the latter produced a lengthy and complicated formula (Figure 3, left). Another student used their meme to highlight a key distinction within Calculus II itself, focusing on the subtle differences between the proper and improper integrals (Figure 3, right). In their reflection, the student explained that their meme offered a playful way to emphasize the "improperness" of the latter, showing that a seemingly minor change in the integral bounds may result in a significant change in the type of integral – an idea humorously conveyed through the distorted appearance of Winnie the Pooh.

Figure 3. Students' memes illustrating conceptual connections: comparing differentiation and integration of the same function (left) and contrasting proper and improper integrals (right).

Classroom Community and Culture

When completing the Memengful Calculus assignment, many students went beyond interpreting Calculus solely as mathematics learned in this course and created memes that captured various aspects of the classroom community, which they were a part of. This aligns with the idea that a meme is strongly connected to its socio-cultural context (Benoit & Salopek, 2023; Bini et al., 2022), even when the overarching community is a single Calculus II class with its integral elements – students, instructor, rituals, shared meanings, and the power dynamics.

Classroom experiences. Many submissions reflected students' everyday experiences in a Calculus classroom. These experiences are shaped by many factors, such as the students' schedule, seating arrangements, interactions with their peers, inside jokes, and memorable sayings from the instructor. For example, one meme reproduced a recurring phrase the instructor used to close the class (Figure 4, left). As the student explained in their reflection, "Instead of saying just 'any questions?' like a normal teacher, [the instructor] adds a little spice to this saying by asking, 'any questions, comments, complaints, or compliments?" By humorously elevating this classroom ritual, this meme underscored how certain phrases and routines contributed to shaping the unique classroom community.

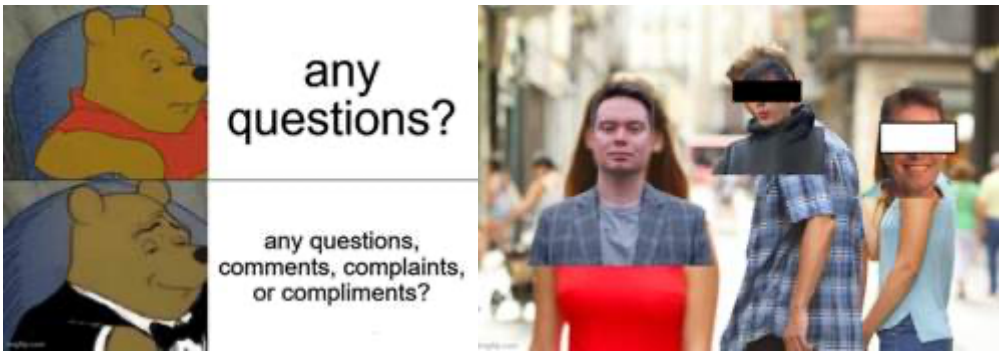

Figure 4. Students' memes illustrating aspects of the classroom experiences: recurring instructor's saying (left) and the student's reluctance to adjust to another instructor's new teaching style (right).

In some submissions, students positioned themselves as integral members and even gatekeepers of this community. The meme presented on the right side of Figure 4 illustrates a student in the role of a "distracted boyfriend" who turns away from his "girlfriend" (a substitute teacher) and directs his attention to "another girl" (regular course instructor). This meme reflects the student's experience of (an award-winning) teacher subbing for the regular instructor for several classes, humorously underscoring the student's loyalty to the familiar classroom culture and reluctance to adjust to a new one. The temporary disruption of the established classroom community thus induced the student's dissatisfaction that turned out to be memorable enough to be immortalized in their meme.

Memes as an opportunity for students to "laugh back." Several students also used pictures of the instructor found on the internet to create their memes. However, in these cases, the instructor's image carried structural meaning (a part of the meme's visual and social context), rather than a mathematical meaning tied to a particular Calculus concept or learning experience.

One student contrasted a German U-boat with the *u*-substitution technique, humorously modifying the famous *Drakeposting meme* by placing the instructor's image on top of Drake's face (Figure 5, left). This meme is typically used to express one's disdain and admiration for certain or different subjects. In the student's submission, the Calculus instructor is humorously positioned as "choosing" the integration technique over a submarine. Another meme depicted the instructor's face with a stern expression, captioned to suggest his reaction to students resisting "to do the hardest integration problem known to man" at the end of a Friday class (Figure 5, center). The student learning experience referenced here is their mental exhaustion that builds up at the end of the week, with the instructor's face serving as a humorous foil for this frustration.

Figure 5. Students' memes utilizing the instructor's image: the instructor choosing u-substitution over a German U-boat (left), depicting the instructor's stern reaction to students resisting "to do the hardest integration problem" at the end of a Friday class (center), and students' productive struggle with irreducible quadratics (right).

Perhaps the quintessential illustration of this theme is presented in the meme on the right side of Figure 5. Similar to one of the cognitive themes described in the previous section, this meme references the students' (productive) struggle with a challenging Calculus concept – integrating irreducible quadratics. However, the context for this meme was shaped by the instructor's picture, which anchored the meme's humor. The student even added a playful nod to the instructor's accent through the tiny caption "I'm smort" embedded in the image. Taken together, these three memes demonstrate how the Memengful Calculus assignment created space for students to "laugh back" at the course authority, humorously re-negotiating classroom power.

Discussion and Conclusion

This study set out to explore how students can use memes to reflect on and communicate their experiences with Calculus and what pedagogical implications this type of assignment might have. Our analysis revealed that memes provided a flexible medium through which students expressed both cognitive and socio-cultural dimensions of learning.

With respect to our first research question, we found that memes allowed students to both articulate their struggles learning Calculus and reflect on conceptual connections across different topics. The former echoes a popular societal perception of mathematics as "nightmarish" and "inherently difficult" (Darragh, 2018), while the latter suggests that the Memengful Calculus assignment encouraged engagement in higher-order cognitive processes, such as creation and metacognition (Bloom, 2010). In this sense, memes functioned not only as humorous artifacts but also as learning tools. This finding contributes to the growing body of research suggesting that memes can promote students' learning across disciplinary domains (e.g., Bradley et al., 2022; Brown, 2020; Kath et al., 2022).

We also found that students reflected on and communicated about the classroom experiences and various aspects of the classroom community and culture. Students' submissions captured each of the Four Frames of Bolman and Deal's (2008) framework. Specifically, memes aligned with the *structural* frame by highlighting students' struggles during course assessments and the necessity to work on a challenging mathematical task at the end of a Friday class. They reflected the *symbolic* frame through inside jokes and the instructor's recurring catchphrases. The *people* frame emerged in memes that incorporated images of the community members, emphasizing the personal and relational aspects of classroom life. Finally, the *power* frame was evident in students "laughing back" at the course authority by using the instructor's images as the memes' contexts. Taken together, these findings further shed light on how memes reflect and contribute to the development of the epistemic culture of overarching communities (e.g., Bini et al., 2022, 2023) – in our case, the classroom community – thus addressing our second research question.

Finally, as became evident in our last theme, this assignment has further pedagogical implications not previously documented in the mathematics education literature. Iloh (2021) discussed the utility of memes in qualitative research, highlighting their potential for building rapport between researchers and participants. Our findings extend this argument to a mathematical classroom, suggesting that meme-based assignments can also foster rapport between instructors and students by creating space for students to "laugh back." In this way, this assignment allowed the instructor to humanize his teaching practice, while the students were empowered to infuse humor into their mathematical learning experiences. Therefore, incorporating student-generated memes in a mathematical course can reduce the traditional power distance, foster a more collaborative learning environment, and support student engagement in a challenging Calculus course.

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Assessment of Student Understanding: Exploring Faculty Optimization Problem Grading

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In this study we examine how faculty members grade optimization problems from calculus. Eight faculty members were interviewed and asked to grade three student solutions for three different optimization problems. In addition, faculty members also discussed their grading philosophies and practices for both optimization problems and in general. We used open thematic analysis to analyze all faculty interviews and two major themes of reflection of understanding and key steps in optimization solutions emerged from the data. These themes that was supported by faculty were further analyzed for the interplay between them. In addition, we delve deeper into reflection of understanding using Skemp's instrumental and relational understanding to establish when faculty assessed good student understanding of optimization problems. Our results provide insight into how mathematics faculty grade student solutions for optimization problems, how they emphasize student understanding, and offer possible insights into the variations in scores we observed in this study.

Keywords: Assessment, Calculus, Optimization, Understanding

Faculty in many mathematics departments are needed to teach calculus courses so they can cover all mathematics courses they offer each semester. As part of their teaching, faculty normally grade homework, quizzes, and exams. Computerized homework that is graded automatically is commonly used, but faculty is still responsible for grading paper and pencil assignments such as quizzes and exams. Grading student solutions is important and crucial as grading communicates instructor's anticipated standards to students (Stiggins & Chappuis, 2005; Yerushalmi et al., 2015), as well as guiding students on how to enhance their work to improve their understanding of the subject material. Black and Wiliam (1998) noted that "feedback on tests, seatwork, and homework should give each pupil guidance on how to improve" (p. 144). Further, effective grading practices can provide feedback that can foster learning, communicate to students what practices and skills help them: learn a discipline, develop problem-solving skills, prepare for formative and summative assessments, and communicate to students about what to focus on to better prepare for future activities (Black and Wiliam, 2009; Marshman et al., 2018; Hattie & Timperley, 2007; Mulliner & Tucker, 2015; Wiliam, 2006). Having consistent grading is also important for ensuring fairness among students. Thus, the feedback and grading provided to students by instructors on their solutions plays an essential role in shaping their learning experiences.

Despite the importance of grading practices of instructors, there have been few studies on the topic. Several prior studies have focused on the evaluations of students' proof in theoretical mathematics classes by professors (e.g., Moore, 2016; Miller et al., 2018) as well as the grading practices of physics professors on students' problem solutions in calculus-based introduction to physics classes (e.g., Docktor et al., 2016; Henderson et al., 2004). None of these studies investigated the grading practices of mathematics faculty in calculus. Thus, our study will specifically investigate the grading practices of mathematics faculty, and in particular, how they

grade calculus students' solutions to optimization problems. The following questions in the context of optimization problems in calculus guide our research: 1) How do mathematics faculty assign points to student solutions?, 2) What features or details do mathematics faculty value when grading?, and 3) How do mathematics faculty assess student understanding when grading?

Literature Review and Theoretical Framework

While studies regarding the grading of student solutions have been on the rise, most prior research lies in the realm of physics (Docktor et al., 2016; Henderson et al., 2004; Marshman et al., 2020). One of the earliest such studies by Henderson et al. (2004) tasked 30 collegiate physics instructors to evaluate two student solutions resulting in the same (correct) answer. This study indicated that instructors were apt to deduct points for flawed intermediate logic with minimal supporting work, while graders may be reluctant to deduct points from a student solution that might be correct. Further, it was noted graders tend to project correct thought processes, indicating that the graders were placing the 'burden of proof' on themselves rather than the student. Of the six graders interviewed in depth in the Henderson et al. (2004) study, only one grader placed the 'burden of proof' on the student and awarded points to a student solution when the student showed explicit evidence of understanding. The other graders placed the 'burden of proof' on themselves and only deducted points when there was clear evidence the student did not understand. Thus, it was observed that clear evidence of understanding was not always needed to award points.

To obtain more consistency in grading solutions, Docktor et al. (2016) developed a problem-solving rubric encompassing five aspects of a desired student solution including descriptions of the problem, mathematical procedures, and appropriate logical progression. Such a rubric provides a solid foundation for other practitioners to implement either holistically or analytically.

In a similar vein, Marshman et al. (2020) implemented a professional development seminar for GTAs enrolled in physics doctoral programs. During their first semester, new GTAs met with each other and the authors of the study to discuss grading strategies and philosophies. Interestingly, the GTAs involved in this seminar believed grading was a formative assessment designed to enhance student learning with feedback. As the seminar progressed, however, it became evident that the GTAs focused on the correctness of a solution rather than treating the assignment as a learning experience. This result corroborates the study by Henderson et al. (2004) who found that graders can experience internal conflict when assigning grades.

Another study conducted by Marshman et al. (2017) implemented a similar professional development seminar as Marshman et al. (2020). The Marshman et al. (2017) study focused on how GTAs developed teaching examples, and if the assistants valued the students implementing the same problem-solving strategies found within the teaching examples when grading student solutions. In addition, they investigated if the GTAs' grading strategies differed between introductory physics and quantum mechanics. They found that the GTAs valued correct answers over explicit problem-solving strategies when grading introductory physics, but the reverse was true for quantum mechanics. Moreover, the GTAs graded the quantum mechanics solutions harsher than the introductory students (Marshman et al., 2017). The GTAs stated that they graded this way because the students in quantum mechanics were more experienced and should know the basic physics concepts. Echoing the findings previously discussed, the GTAs viewed grading as a summative assessment and not a formative teaching tool (Marshman et al., 2017). Therefore, it appears important to investigate if mathematics instructors grade calculus students for correct answers or provide formative feedback.

Other studies researched how professional mathematicians grade students' proof-writing skills. Moore (2016) interviewed four professors and noted remarkable variation in scores contingent upon the perceived student's intent within the proof. The four professors discussed the importance of logical correctness, clarity, and fluency in good proof writing and differed in their evaluations due to seriousness of errors, the students' demonstration of understanding of the proof, the validity of the proof and the associated mathematical concepts. Similarly, Miller et al. (2018) presented nine mathematicians with sample proofs from a transition-to-proof course. This study indicated that professors offer 'the benefit of the doubt' to student proofs that exhibit logical gaps. This 'benefit of the doubt' extended to instances where generic proofs would be accepted as full proofs, and proofs deemed "correct" received less than full credit. Such results indicate inconsistency in how professional mathematicians grade in proof level courses, so it would be prudent to investigate how instructors grade problems in calculus.

We use a theoretical framework of understanding by Skemp (1976), where student understanding is broken down into two categories, 'instrumental' and 'relational'. Instrumental understanding refers to students having certain rules or facts in mind, such as specific formulae or procedures, without having the underlying knowledge or understanding of why these rules work. Conversely, relational understanding refers to a deeper type of understanding, where the underlying connections and rationale behind knowledge is known, allowing students to be able to more easily and appropriately apply their knowledge to new situations. Skemp maintains that students gaining relational understanding is preferred over students only having an instrumental understanding. However, many students only acquire instrumental understanding as many classroom environments do not promote relational understanding. We will use this framework to analyze the participant's grading of the optimization problems and their interpretations of student understanding of these solutions.

Methodology

Participants

For this study, eight mathematics faculty members from a large state university in the United States consented to participate in this study during the Spring of 2024. The participants all had experience teaching calculus for the university that included grading exams and other assignments. The participants were representative of all the levels of tenure-track faculty and the assistant and associate levels for teaching professors. Also, one participant was a postdoctoral researcher with three years' experience teaching calculus, along with additional experience as a graduate student at a large state university. Participants were from a variety of fields in mathematics such as Algebra, Graph Theory, Discrete Mathematics, Partial Differential Equations, and Undergraduate Mathematics Education.

Procedure

The eight participants were interviewed in the spring of 2024. They are listed as F1 to F8 in this study and referred as a neutral gender to mask their identity. The interviews were conducted via Zoom. Each participant was presented with three optimization problems to grade, with three solutions (labeled S1, S2, and S3) per problem. The interviews were conducted in two sessions, each taking approximately an hour, depending on the length it took for the participants to grade the papers. A draft of the interview transcripts was automatically generated via software using the recorded video and then polished up by the researchers after reviewing each recorded interview. The solutions for the problems were generated by the authors and the prior studies in

physics, and the participants were asked to grade them as if they were grading a calculus assignment in the online grading system called Crowdmark. While they were grading the questions, the participants were encouraged to provide justifications on what they were doing, how the grade was determined, and their final score. Additionally, before the participants graded the student solutions to the three optimization problems, they were asked semi-structured open-ended general interview questions about their grading philosophies and practices, as well as specific approaches when grading optimization problems.

Materials

The nine solutions we devised for the interviews consisted of three problems with three solutions S1, S2, and S3 (see Figure 1 for an example solution of the rectangle problem). One of these problems, the Rectangle Problem, is a standard problem asking how to minimize the cost of a fence enclosing a rectangular, partitioned area. The other two optimization problems were the Barrel Problem and the Pipeline Problem, from a previous Calculus exam conducted at the university and are common types of problems in Calculus textbooks. The Barrel problem dealt with optimizing the volume of a cylinder with specific surface area and the pipeline was about building the minimum cost pipeline to a refinery over land and across a river. While the fundamental structure of all three problems is similar, the difficulty in understanding the scenarios and the algebra involved varies between the problems, with the Rectangle Problem being the easiest, followed by the Barrel and Pipeline problems. The problems and their solutions can be shared upon request.

The solutions were designed to highlight the issue of how the faculty members would assign partial credit and motivated by the Physics Education research of how GTAs and faculty grade calculus-based physics problems (Henderson et al., 2004). For the Rectangle and the Pipeline problems, all the final solutions are accurate, however the amount of work shown varies between the solutions, with many of the solutions having what the authors deemed as an insufficient amount of work. The solutions for the Barrel problem on the other hand has a variety of errors and different answers due to those errors. Note that all solutions have approaches which are generally correct. They each find a formula for the quantity to be optimized as a function, compute the derivative, and then determine where the derivative is equal to zero to find the critical points. However, further steps such as verification of the solution and showing appropriate algebra or calculations to justify the work are sometimes brief or omitted. The variety in the amount of work shown and correctness was done to mimic real student work on optimization problems in order to provide the graders with authentic, realistic grading scenarios they are likely to encounter with optimization problems.

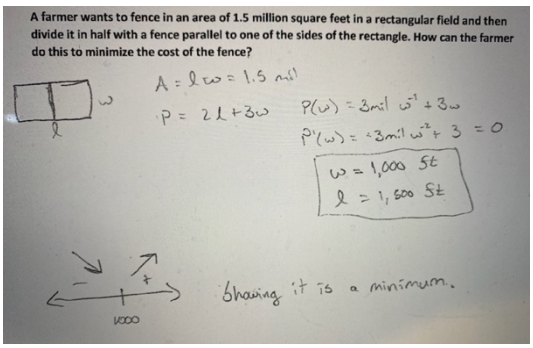

Figure 1. Optimization Solution S1

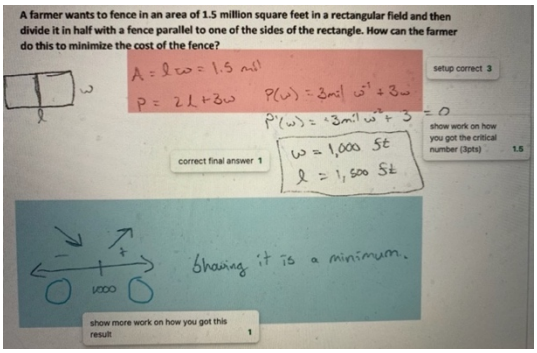

Figure 2. Solution S2 on the rectangle problem

Analysis

The authors engaged in open thematic analysis (Corbin & Strauss, 2014) in the following way. First, all authors individually read over all the transcribed data while identifying and highlighting passages that might be of theoretical interest concerning grading. All authors met to discuss and compare their findings and developed a list of codes for further study. Next, each author reread the transcripts again, searching for each topic of interest; we then put these excerpts into a file of comments related to that code. Through this process, we developed criteria and descriptions of these themes. Finally, the authors individually went back through the data one last time to identify passages that would satisfy the criteria for each of these themes.

Results

Quantitative Data

Table 1 below shows the scores that the eight-mathematics faculty assigned for the three different optimization problems solutions, S1, S2, and S3, on the three optimization problems. In addition, the average, median, and variances for each solution are shown at the bottom of the table. We can see that the largest variances occurred on the rectangle optimization problem with student solutions 1 and 3 and the smallest variances occurred on the barrel and pipeline problems with student solution 3. The rest of the variances were somewhere in between. Furthermore, four of nine solutions typically scored high with an average of over 8, two were more problematic solutions with an average score below 5.5, and the remaining three of nine solutions were somewhere between.

Table 1. Faculty Member Scores for the 3 Optimization Problems and 3 Solutions S1, S2, and S3.

	Question 1 (Rectangle)			Question 2 (Barrel)			Question 3 (Pipeline)		
Grader	S1	S2	S3	S1	S2	S3	S1	S2	S3
F1	4.5	7	6.5	8	9	9.5	4	10	7.5
F2	5	6	8	7	10	8	5	10	8
F3	4	8	7.5	9	9	9	4	10	7
F4	6	9	9	9.25	9.5	9.5	6	10	7.5
F5	4.5	8	9.5	9	10	9	7	8	8
F6	7	7.5	7	6.5	8	8	4	8.5	7
F7	4	7	5	7	8	8	4	7	6
F8	9	10	9	9	10	9	6	10	8
Average	5.5	7.813	7.688	8.094	9.188	8.75	5	9.188	7.375
Median	4.75	7.75	7.75	8.5	9.25	9	4.5	10	7.5
Variance	3.071	1.567	2.281	1.249	0.710	0.429	1.423	1.424	0.482

Qualitative Data

We report on two major themes that emerged from the data: 1) Key steps in Optimization Solutions and 2) Reflection of Understanding. These themes were supported by all participants, and we will show support for the themes with excerpts from the faculty interviews.

Key Steps in an Optimization Solution. All participants talked about key step(s) while grading the optimization problems. While grading the optimization problems, one of the first things that most participants mentioned were checking the key steps of the solution that consisted of constructing the optimizing function, finding the derivative, solving for the critical points, and

verifying which critical point solves the problem for the maximum or minimum. For example, F9 states while they are grading, "there are kind of key steps that I look for along the way" and

> "they're setting up the problem which typically has an optimizing function and a constraint. I don't like the word constraint, but because really it helps us to put the optimizing function into one variable. And then I want to see them take the derivative of the optimizing function, find the critical points, and then I want to see them determine whether or not those critical points really are the things we're looking for."

Therefore, checking key steps served as signposts to quickly assess if the solution is correct.

Some participants went further by combining key steps into chunks. They would look at the chunks as important pieces and would determine the number of points they designated to each chunk depending on their views on the importance of the chunk. For example, F10 comments that *"I solve the problem, show all the steps, and then I just go through and block off each little piece of what they do, and decide… How much does knowing this part account for? How much does knowing this part account for? So, I kind of block it off into pieces*" and F7 also states *"in an optimization problem they set up the function, right? That's one batch of stuff. They take the derivative and find the critical numbers. That's one batch of stuff. They test the critical numbers in the first or second derivative test. That's a batch of stuff, and then they answer the problem. That's a batch of stuff."* This ensures the appropriate allocation of points for important work in the student solutions and assists them when grading. However, not all participants grouped key steps in chunks and just developed their rubrics around the key steps. This is illustrated by F8 with *"[I] break my solution into steps depending on how important it is, and I usually have in mind that this step they should get 2 points. And this step has these points."*

At that point and to varying degrees, the majority would assess the details of the solution to further assess the student solutions and determine how many points they earn on key steps as well as overall. It is during this stage that participants expected the solutions to provide more detail to justify the work (see the next section of reflection of understanding).

Therefore, key steps and key chunks played an important role for participants when building their grading rubrics and while they were grading student solutions for the optimization problems. This ensured each participant graded consistently and fairly on each of the three solutions for the three optimization problems. Although this result was in the context of optimization problems, participants discussed this is how they grade on most topics.

Reflection of Understanding. We now use Skemp's (1976) relational and instrumental understanding to further explore how participants assess whether students understand the mathematics for the optimization solutions. We first discuss participants assessment of instrumental understanding and then move to the assessment of relational understanding.

Reflection of Instrumental Understanding. When analyzing the participant's optimization grading, we found instances where participants assessed the solution for instrumental understanding by not expecting any justification in the solution. They were assessing the solution based on formulas, rules, or facts, without displaying any further underlying understanding.

For example, all participants did not need to see justification for the calculation of the derivative for at least the optimizing function on these three solutions. For example, when F9 is grading the rectangle optimization problem, she states as she reads the solution*, "And then we're taking the derivative, which I think is good." For another example,* F10 states *"I think it can be common for students to just write something down because they know it's supposed to be a minimum, and so they just draw the picture like they know it's supposed to be without doing any actual thinking at all."*

Reflection of Relational Understanding. When analyzing the participant's optimization problem grading, we found many instances where participants assessed specific steps of the solution for relational understanding by expecting justification in that step. That is, they were looking for a reflection of relational understanding in a step that showed a deeper understanding of the rationale and underlying reasons for the mathematics. One prime example of the reflection of relational understanding occurred on the verification key step for the maximum or the minimum. For example, on S2 for the rectangle problem (see Figure 2), F7 states

> "… let's call this the next batch of stuff [boxes verification work in blue] which I think is the most important batch. I still would not call that enough work. If you want it to be a minimum, you know you want it to go down and then up. So, you're not showing me that you actually checked that. You're just showing me that you know what it's supposed to be so, I would imagine that there would need to be like, in my opinion, what I would call enough work, would be put a number over here [draws circles on both sides of sign chart].Put a number over here, you know, like 10 or one, or whatever, and then like 1,500, or whatever over here, and show me that you plugged it back into the first derivative and got this decreasing increasing. So, I would say, show more work on how you got this result."

For another example, F12 wants the hypothetical student to explain why there is a maximum or minimum at the critical point instead of just saying the $V''(r) < 0$, "My concern when there's just so little is that they're just regurgitating a fact. Like, 'Oh, it's less than 0, so that means it's a maximum.' Like, well, do you actually understand why it's a maximum." Therefore, F7, as other participants, expected the solution to display justification for key steps in the solution to show a reflection of relational understanding.

Discussion

In this paper we investigated how faculty assess student understanding on three optimization problems and found three main results. The first result was that for the most part, professors assigned points consistently, except on the rectangle problem. The second result was that professors wanted to see key steps in the solution. When a key step or steps were missing from a solution or there was insufficient work, professors would assess a lack of understanding and deduct points. This parallels the results of Raman (2003) for mathematical proofs where she found that mathematicians valued key ideas in a proof. The key idea is the big picture of why the proof works and can be used to construct the final written proof. Likewise, key steps give an outline or big picture of what needs to be produced to solve an optimization problem.

The third result was that professors wanted to see a reflection of understanding in the student solutions. They interpreted that students did not have a complete understanding when the solution did not include key step(s) or there were errors in between key steps. Prior research in proofs by Miller (2018) and Moore (2016) found a similar result when professors graded mathematical proofs. In contrast, Henderson et al. (2004) found that this was not the case with physics graduate students. This reflection of understanding further evolved into the final result where professors were interested in seeing that students were fluent with the mathematical rules (instrumental understanding) but also wanted to see students providing justification to show they understood the mathematical concepts involved in their solutions (relational understanding).

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Using Expectancy-Value Theory to Explore Motivations in Inquiry-Oriented Linear Algebra

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This study examines undergraduate student motivation in an inquiry-oriented linear algebra course using expectancy-value theory (EVT). While EVT has more frequently been applied in K–12 settings, its use in undergraduate mathematics remains limited. Through interviews with four students, this study explores how components of EVT—expectation for success, attainment value, intrinsic value, utility value, and cost—relate to students' engagement. Findings show that the expectancy-value theory can be helpful in making sense of the motivations of undergraduate students. Students were motivated in different ways, with many citing their key influences as collaboration, real-world applications, and classroom expectations that promote a novel definition of success. Inquiry-oriented instruction appeared to support positive motivational outcomes by normalizing struggle and promoting group-based problem solving. This suggests EVT is a useful framework for understanding motivations in undergraduate mathematics, particularly in the context of inquiry-oriented instruction.

Keywords: motivation, expectancy-value theory, inquiry-oriented instruction

Introduction and Literature Review

Students' motivations have been a topic of mathematics education research for decades, with a particular focus in the last 20 years (Schukajlow et al., 2023). Several studies have been conducted looking at the motivations of primary or secondary school students (e.g., Middleton et al., 2023; Fielding-Wells et al., 2017; Eccles et al., 1993). Middleton and collaborators (2023) analyzed different aspects of mathematics engagement in high school students and found that, related to motivations, social support in class improved student interest in the situation as well as their beliefs in their competence. Of particular relevance to this study, Fielding-Wells and collaborators (2017) explored motivations in an inquiry-based mathematics classroom of 9-10-year-olds using the expectancy-value theoretical framework. They found that this framework was helpful to make sense of student motivation in an inquiry-based classroom, and aspects of this instructional design could be positively influential on student motivation (Fielding-Wells et al., 2017). However, less research has been done studying the motivations of undergraduate students, with or without the use of the expectancy-value theory (e.g., Weber et al., 2020).

One possible explanation for this is that motivations towards mathematics are developed early in life and maintain relative stability throughout (Middleton & Spanias, 1999). However, motivating undergraduate students should not be dismissed. Eccles and collaborators (1993) found that beliefs of confidence lower with age, which may reflect increasing pessimism with age. As a result, it is worthwhile to analyze why older students have lower competence beliefs and what can be done to support their motivation. Additionally, according to Middleton and Spanias (1999), different instructional designs can alter students' motivations, so there may be merit in examining the ways that instructors at an undergraduate level can affect students' motivations in their math courses. As a necessary first step to that, the results of this contributed report will work to identify common motivations that students are expressing from a particular population.

Specifically, I am interested in looking at the motivations of students in an inquiry-oriented linear algebra course, because I hypothesize that this instructional design may have positive

effects on undergraduate student motivation. It has been noted that inquiry-oriented instruction (IOI) is closed related to inquiry-based learning (IBL; Laursen & Rasmussen, 2019), which has been found to have positive student outcomes in many regards: cognitive, affective, and collaborative benefits (Laursen et al., 2014). Because Fielding-Wells and collaborators (2017) found aspects of an IBL environment to have positive outcomes, and IBL has been found to benefit student outcomes (Laursen et al., 2014), I hypothesize that IOI may have positive effects on undergraduate motivations. This paper aims to explore the motivations of undergraduate students in an inquiry-oriented linear algebra course using the expectancy-value theory of achievement motivation.

Theoretical Framework

The expectancy-value theory of achievement motivation, also commonly referred to as just the expectancy-value theory (EVT), is a well-established theory in the field of psychology (Wigfield et al., 2021). This theory was initially developed by inquiring into why women are less likely to pursue STEM-related careers than men (Eccles, 1983). Findings from that large-scale, longitudinal study emphasized that the few sex differences that did emerge indicated that females had a less positive perception of their math ability along with a lower value of the subject than males (Eccles, 1983). EVT declares that individuals make decisions in achievement settings based on (a) their expectations of success on the task and (b) how much they value the task (Wigfield & Eccles, 2000). Mathematics courses are considered achievement settings, because there are opportunities to attain specific goals, such as demonstrating understanding of the content on assessments or passing the course. We will be discussing the constructs here in the order they are listed in Figure 1, which is the order they are presented in Wigfield and Eccles (2000).

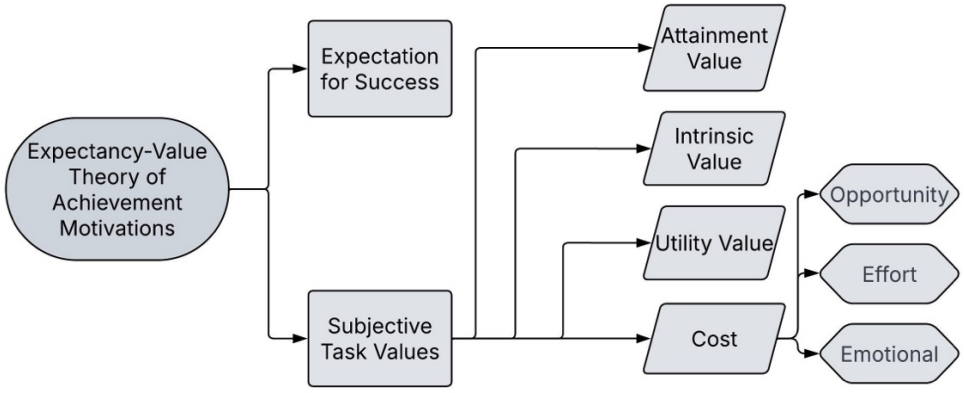

Figure 1. Expectancy-Value Theory of Achievement Motivation Components

The *expectation for success* portion of this theory refers to how much an individual believes that they are able to attain success at a given task (Schukajlow et al., 2023). If a student believes a task is too difficult for them to complete (i.e., they don't think they are smart enough, they don't think they were given enough resources by their instructor, etc.), they will generally be unmotivated to complete that task. Additionally, four types of *subjective task values* are distinguished: attainment value, intrinsic value, utility value, and cost. *Attainment value* is how important doing well on a task is to that individual; this task value is usually tied closely with one's self-concept or identity (Wigfield & Eccles, 2000; Schukajlow et al., 2023). *Intrinsic value* is how interested that individual is in the task or how much they enjoy it (Wigfield & Eccles,

2000). *Utility value* is how much the individual sees the task fitting into their future goals (Wigfield & Eccles, 2000). It is important to note that these task values are subjective, varying by the individual and their experiences. For example, a student who is not particularly interested in mathematics (low intrinsic value) and doesn't care about what grade they get in their math class (low attainment value) can still be motivated to do well in their math course if they view mathematics as something that can help them understand necessary concepts for their job as, say, an electrician (high utility value). *Cost*, which is a factor of decision-making in this theory in line with the subjective task values, is the other end of the scale. Cost may be accounting for how much engaging in the task restricts one's ability to engage in higher valued activities (*opportunity cost*). For example, if I do my homework, that gets in the way of my ability to chat with friends. Cost could also account for how much effort goes into engaging in the activity (*effort cost*), or how much emotional labor goes into the task (*emotional cost*). When people make decisions, they gauge how much it costs and see if that outweighs their values and expectations for success.

I am using this theoretical framework to answer the following research question: *How do students characterize their motivations related to their engagement in an inquiry-oriented linear algebra course?*

Methods

This study investigated the experiences of four undergraduate students from two sections of inquiry-oriented linear algebra at a large public university in the southwest United States. A general day in this course involved the students working on a mathematical task in groups of two to four students, while the instructor and TA circulated the classroom and supported student discussions. The instructor would ask students to share their thought processes with the whole class. The tasks generally had scenarios associated with them and asked students to collaborate with one another and provide reasoning for their answers. An example of the kinds of mathematical tasks provided in this class can be seen in Wawro et al. (2012).

Due to the complexity of students' engagement, multiple data methods were used (as suggested by Middleton et al., 2017): observations, participation surveys (self-report), and interviews. Classroom observations were conducted for the three weeks of class that occurred after the first exam. Additionally, students reported on their participation for one day of each week of the semester by responding to questions in a Google form. One week after the conclusion of the observations, four students participated in an interview with questions asking about their experiences in the class. The interview questions were intended to reveal information about their motivations for engaging with the course material. Note that, for the purpose of this contributed report, only analysis from the interview data will be presented.

To answer the research question, I first familiarized myself with the data by transcribing the interview data and reading through it. Next, I highlighted excerpts based on whether they addressed the students' motivations to engage in a task. Excerpts were identified as *motivations* if they indicated the reasons why someone engaged with the course content (Middleton & Spanias, 1999). From here, I used the constructs from the expectancy-value theory to classify the various motivations as either *expectation for success, attainment value, intrinsic value, utility value,* and *cost* (Wigfield & Eccles, 2000).

Results

In total, 63 excerpts were identified as containing motivations among the four participants. The classifications for these motivations can be seen in Figure 2. Of note, expectation for success

was coded 18 times among three participants. Of the four types of subjective task values, attainment value showed up five times among two participants. Intrinsic value showed up six times among three participants, four of those occurrences were all from one participant. Utility value showed up 22 times among all four participants. Finally, cost was coded 12 times among all participants.

EVT Component	Olivia	Payton	Quincy	Romeo
Expectation for Success	7	1	10	0
Attainment Value	0	1	0	4
Intrinsic Value	1	4	1	0
Utility Value	3	8	8	3
Cost	2	3	6	1

Figure 2. Counts of Student Motivation Excerpts by EVT Component

Expectation for Success

Expectation for success was coded particularly strongly with Olivia and Quincy, in addition to a single time with Payton, with these excerpts generally noting that what counts as 'success' in this class is varied compared to their other classes. For example, Payton compared the differences between the homework expectations and resources provided in this course, stating that, "the homework throws it at you like it expects that you've already learned it, because it just, it presents the problem and you should have the knowledge of the techniques used to solve it, versus the in-class activities are very much presented to us where we're not assumed to have that understanding." Here, Payton is describing that in-class activities provide more support and resources than the homework activities, because for in-class activities, the students are not expected to know how to solve the new material. Overall, Payton demonstrated a high expectation for success in this course, but had a lower expectation for success on the homework because there was not as much support if he got stuck.

Olivia and Quincy also displayed high expectations for success in this course, but associated those expectations with different components of the course than Payton. For example, instead of comparing the homework's expectations to the in-class expectations, Olivia compared this course to her experiences in other courses, stating that "the frustrations I feel in those classes are like [...] 'I plugged in the numbers I'm supposed to and it's not working, so like, what am I doing wrong?' [...] I feel better when my whole team doesn't understand it, or even if they do understand it, I feel better because they're gonna explain it to me." Later, she shared that when she does not understand in this class, she feels as though it is "definitely, not as like personal towards me. [...] We don't really know what we're supposed to be doing yet mathematically, so if you don't understand, like the bigger picture, it's okay." Here, and generally, Olivia understands that there is an expectation for students to struggle initially with concepts in this course, and so even if she does not immediately solve the problem, she still expects to succeed by struggling through the problem with her group. Quincy shares the same kind of opinion, stating that in other classes he gets frustrated and can take it personally when he doesn't understand, but he said he is

able to “kind of just like trust the process with this class.” Each of these students shares that the course expectation of students figuring out the mathematics for themselves in groups helps them to feel less pressure to immediately know how to approach the problem, making the idea of what counts as successful more accessible for students.

Attainment Value

Attainment value showed up for Romeo the most, specifically as he shared about his work ethic and how that shapes his identity. When asked how Romeo pushes through days when he is tired or has low energy levels, he said, “you gotta push yourself to do hard things, you know. That's like my mentality always, you know, just everything's gonna be -- well, there's gonna be challenges, and you're just gonna have to, you know do your best.” In this sense, Romeo would likely have a strong attainment value in any of his courses, as they are all simply “hard things” that he has to push through. Payton shared a bit less explicitly about his work ethic, stating that “whenever I’m trying to learn something new, I always try and get it as much as I can, just through my own understanding” before he uses any other resources such as Google. Generally, attainment value did not show up much in these interviews, which may be because, before conducting the interviews, I was not using the expectancy-value theory to analyze the data, so there were not interview questions aimed at exploring a students’ desire to do well on the in-class activities.

Intrinsic Value

Intrinsic value usually presented as the participant expressing that they enjoyed a specific activity. For three participants in this study, they shared their interest or enjoyment in a specific task in response to the interview question “What is your favorite task we’ve done so far in class?” The reason for the interest was different for each participant, demonstrating some of the different ways that this code can show up for people. Although Romeo did answer this interview question, he did not express particular interest or engagement in the task, attributing his choice of favorite task to how “clear-cut” the solution was for him, so his response was not coded as intrinsic value. Quincy picked his favorite task based on which task provided the best example. Olivia picked her favorite based on what activity she could visualize best. Payton picked his favorite task based on which task introduced new math content. Additionally, for Payton, he expressed interest in the course content without being prompted. He finished his interview saying, “this is probably one of the more enjoyable math courses that I’ve had in a long while [...] it’s a lot of collaborative work too, so it’s very much more social and fun to do which I enjoy.” Payton also shared that he enjoyed teaching or leading his group. Overall, the intrinsic value code was particularly subjective, showing up in a few different ways depending on the participant and depending on the various things each participant enjoys.

Utility Value

For these participants, utility value usually showed up as engaging in a particular way as a means to an end. Payton expressed that he will sometimes take the back seat in his group’s work and work on the problem on his own for the sake of seeing how his groupmates do it and if they find more efficient solving methods. Specifically, Payton said, “I’d like to see how it’s done to see how it compares to how I do it, cuz I am very set in how I solve a lot of these problems and I definitely have done some like inefficient ways of solving them.” Romeo shared the same reasoning, but instead of taking a back seat in his group, he asks his group a lot of questions to gain the perspective of more efficient solving methods. Romeo said, “So to be honest, I'm the

one asking a lot of questions, because you know, everybody has their own process, and I'm trying to understand how they solve it through." For Olivia, she saw the utility value in tasks through the real life applications that these tasks can be used for, when asked to share about her favorite task she said "This whole class feels like just like trying to solve puzzles and so like this was the more helpful puzzle to solve because I'm like, okay, like people do this to actually like... in real life, if that makes sense?" All of Quincy's utility value codes were related to him viewing his engagement with the material as a means to the end of passing quizzes and exams. In one instance, Quincy said "even if I'm focusing for like 20%, I come out of the class ready for the next exam." Overall, utility value showed up as the participants trying to do what helps them learn the material best, as in Payton and Romeo's case, appreciating the real-life applications of the activities, like Olivia's case, or doing what they need to perform on assessments, as in Quincy's case.

Cost

Cost showed up in a few different ways as well. For Payton and Romeo, cost showed up as continuing to work on their board work during whole-class discussions when people are presenting. Payton explained that "my group very often tries to continue working on whatever our brain was still doing." This demonstrated the opportunity cost: fully engaging with the whole class discussion would get in the way of them continuing their work on the board. This opportunity cost also showed up when the participants talked about the hypothetical of people working on an assignment for another class. For example, Quincy said "my kind of reasoning is that maybe, like you got […] something due that day and we just like need to finish that, so […] you kind of set it [the in-class activity] aside because it's like, whatever you're doing at that moment is not gonna matter up until, like the quiz." To Quincy, it is understandable that someone may work on an assignment that they feel is more pressing for another class because whatever is currently being worked on in class will always be less pressing until the assessments.

Another aspect of cost mentioned is the energy levels or 'moods' that the participants were feeling during class. Three of the four participants shared the sentiment that if they are tired, "it's easier to get derailed because I'm not as focused" (excerpt from Payton's interview). Additionally, participants shared that if they are having personal issues, it makes it more challenging for them to engage fully with the in-class activities. I would consider this an effort cost: these students perceive that they do not have the energy to continue engaging because of physical energy levels.

The final way that cost showed up is related to the participant's expectation for success. When Olivia and Quincy felt that even if they fully engaged with the material, they still would not understand it, they would disengage and usually distract themselves with activities on their phones. Quincy said that if he doesn't understand what is happening in class, then:

> I'm just gonna immediately just put my AirPods in like, listen to music, or like, watch a video or like, do something else that, that is well, honestly, that's what I would do now like, even if I have, I told my full attention to this, I'm not gonna understand it. So why, why should I waste my time, or why should I waste my effort..?

These participants did not expect success from engaging with the activity: to them, the cost was too high, and they did not expect to get payoff of understanding.

Discussion

The ways that the expectation for success code presented for these participants was particularly interesting because their perception of what counted as success and the resources available to reach it shifted given the classroom expectations for this course. Generally, being successful in this course became more accessible to these students compared to other courses, because they believed there were ways to be successful in this course outside of completing a problem successfully. I conjecture that this increased accessibility of success may be attributed to the classroom expectations of an inquiry-oriented linear algebra course, but more investigation is required to validate that claim. For example, according to Rasmussen and Kwon (2007) teacher inquiry into student thinking is one of the defining aspects of an inquiry-oriented classroom. One of the participants in this study stated that he felt motivated when the instructor, instead of saying that answers were "bad or wrong," would say "that's one way of looking at it." This student found it motivating that the instructor would interpret students' perspectives instead of dismissing the student contribution because it was incorrect.

Additionally, the fact that this was an inquiry-oriented classroom being studied provided another unique aspect: student reinvention (Kuster et al., 2018). Student reinvention means the students are making sense of the mathematical concepts for themselves, instead of adopting another's reasoning. For an inquiry-oriented classroom, the tasks are designed to support student reinvention by building on student reasoning before formal reasoning is introduced (Kuster et al., 2018). Although these participants never mentioned student reinvention explicitly, Olivia did mention that she thought in group work that "going off of each other's ideas is really cool to see because I'm like, 'Okay, this is what people were doing when they were inventing this math.'" A potential contributor to this shifted expectation for success for these participants is the instructor inquiry and student reinvention, two aspects unique to inquiry-oriented courses. While the results of this study demonstrate that the expectancy-value theory is suitable to make sense of the motivations of undergraduate students in an inquiry-oriented course, it is worthwhile to consider whether this shifted expectation for success can be most directly attributed to the fact that this is an inquiry-oriented course.

Multiples of these components of the expectancy-value theory of achievement motivation (*expectation for success, attainment value, intrinsic value, utility value,* and *cost*) can play a role in any one instance of an individual's decision-making. In this study, we saw instances of students making decisions based on how much they expected to succeed at a task, weighed against how much effort they would need to put in to reach that success. As such, classrooms implementing inquiry-oriented instruction may provide unique environments to explore students' motivations, because the expectations of what is considered success may change for students based on the overall expectations of the class.

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Subjective Combinatorics: A Case of Set Partition Problems

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Research in combinatorics education highlights the importance of examining how students think about counting problems to gain deeper insight into their conceptualizations. Such an examination involves attending to students' interpretations of counting problems, particularly the subjective elements that influence their combinatorial thinking. Subjective combinatorics has recently been introduced to account for these elements in how students interpret counting problems and engage in counting activity. This paper refines the construct and demonstrates its usefulness as a lens for analyzing students' engagement with counting problems through an examination of student work on a set-partition task.

Keywords: combinatorics, conceptual analysis, counting problems, subjective combinatorics.

Enumerative combinatorics, the study of counting problems, is part of many school and undergraduate curricula. Its presence in curricula reflects both its practical relevance in fields such as probability and computer science (e.g., Abrahamson, Janusz, & Wilensky, 2006; Shaughnessy, 1977) and its potential to engage students in critical mathematical thinking (e.g., Kapur, 1970; Martin, 2001; Tucker, 2002). A growing body of research in the mathematics education community has examined how students think about counting problems, identifying common strategies and sources of difficulty (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; English, 1991, 2005; Lockwood, 2013; Lockwood, Swinyard, & Caughman, 2015; Tillema, 2013). Building on this work, it is important to investigate how students interpret counting problems, with particular attention to the subjective elements that influence their combinatorial thinking. To this end, *subjective combinatorics* (Modabbernia, 2025) has been introduced as a lens to examine such subjective interpretations of counting problems, building on ideas developed in the study of *subjective probability* (Chernoff & Zazkis, 2011; Jones, Langrall, & Mooney, 2007). This study situates *subjective combinatorics* within a broader body of mathematics education research and refines the construct by focusing on subjective interpretations of outcomes, illustrating its usefulness through an analysis of student work on a set partition problem.

Brief Background and Contextual Insights

Subjective combinatorics was originally introduced with ideas drawn from subjective probability (Modabbernia, 2025). In this paper, I situate the construct within a broader body of mathematics education research that addresses subjectivity in mathematical reasoning and students' combinatorial thinking. I begin with studies in mathematics education that foreground the interpretive and experiential dimensions of reasoning, followed by research in combinatorics education that has highlighted subjective elements in students' combinatorial thinking.

Mathematical reasoning cannot be separated from students' personal perspectives. As Johnson (1987) observed, "without imagination, we could never reason toward knowledge of reality" (p. ix). This view underscores the centrality of imagination for making sense of experience and highlights how reasoning is inherently interpretive. Speiser and Walter (1996) extend this idea, emphasizing that embodiment and personal experience shape what and how things can be meaningful, and therefore, personal experience is part of how we understand and

use our mathematics. Empirical research further illustrates this interpretive dimension. Tourniaire and Pulos (1985), in their review of proportional reasoning, noted that "studies using problems with different contexts yielded unequal success rates, which suggests that context has an important influence on performance" (p.190). Prior experiences also play a critical role. In the study of students' conceptualizations of calculus, Marrongelle (2004) emphasized that students often drew upon earlier experiences to reason through mathematical tasks. Van den Heuvel-Panhuizen (2005) reinforced the interpretive role of context by demonstrating that students were more willing to generate meaningful responses when problems were situated in contexts they experienced as real. By imagining themselves in such situations, students engaged in sense-making that was directly inspired by the task. At the same time, contexts could also hinder reasoning if they conflicted with students' personal experiences, as in Mack's (1993) finding that a child rejected a pizza-sharing problem because she disliked pizza and would have found the problem more sensible if framed in terms of ice cream. Research in probability has also highlighted subjectivity, where interpretations of outcomes are shaped by personal judgment and belief (Jones et al., 2007). As Batanero et al. (2005) noted, "what is random for one person might be nonrandom for another" (p. 24), reflecting uncertainty specific to each individual. Collectively, these perspectives show that students' interpretations of mathematical problems are shaped by imagination, embodiment, context, and personal experience, providing a foundation for viewing mathematical reasoning as inherently subjective.

These findings, while not situated in combinatorics, emphasize that mathematical reasoning could be subjective. This subjectivity is particularly evident in combinatorics, where many problems are presented through contextual scenarios rather than in purely abstract form. While such presentations enhance accessibility, they also increase the potential for varied interpretations of the problem itself. To consider subjectivity in combinatorics, it is helpful to begin with the way counting problems are stated. A counting problem highlights specific restrictions for representing a set of outcomes and tasks students with determining the cardinality of that set. Prior research in combinatorics education has already drawn attention to interpretive elements in students' reasoning. At the elementary level, English (1993) found that children's understanding of the word "different" shaped the way they constructed outcomes. In her study, "children interpret *different* to mean *different in all ways* and thus see the goal of *all different outfits* as an indication to make each new outfit completely different from the previous outfit(s)" (p. 260). Such findings highlight how students' interpretations of task wording can influence what they count as valid outcomes, underscoring the role of subjectivity in combinatorial reasoning. At the middle-grades level, Fischbein (1987) described intuitions as behaviorally grounded, shaped by prior experiences, and directed toward meeting adaptive demands, even if they sometimes lead to errors. Extending this view to combinatorics, Fischbein and Grossman (1997) showed that students' reasoning was shaped by the interpretive schemata they brought to a task: "Their reactions are different because their intuitions express different schemata, different programs of interpretation related to their age" (p. 28). These findings highlight how students' interpretations of combinatorial problems are grounded in prior experience and the schemata that guide their intuitions, reinforcing the role of subjectivity in their combinatorial reasoning. At the high school level, studies further suggest that students' interpretations of problem statements can strongly influence their combinatorial reasoning. Batanero, et. al. (1997), drawing on Dubois's (1984) classification, showed that students often did not regard structurally isomorphic problems as equivalent when different objects were involved, instead applying different methods depending on the structures they perceived. They also documented how wording could shift

students' reasoning, as in a distribution problem involving four cars and three children where one student reasoned arithmetically: "4/3 = 1; one car for each brother and there is a car left" (p. 191). Such findings illustrate how both the objects embedded in a task and the everyday meanings of words such as "distribute," "divide," or "share" can shape students' subjective interpretations. At the undergraduate level, Antonides and Battista (2022) explained that structuring "is not a form of abstraction, but rather is a process that can be abstracted at various levels and that can build on previous abstractions. Abstraction of structuring leads to mental models of objects, which are integrated sets of abstractions activated to make sense of, reason about, and operate within experiential situations" (p. 798). This perspective aligns with subjectivity, as it highlights how students' reasoning depends on the mental models they construct when interpreting combinatorial tasks.

Taken together, this literature illustrates that students' mathematical reasoning is deeply shaped by subjective dimensions such as imagination, embodiment, context, and personal experience. Within combinatorics specifically, subjectivity emerges through factors including task wording, prior experience, interpretive schemata, and the mental models students construct. These insights highlight subjectivity as a central feature of how students engage with counting problems, providing the foundation for refining subjective combinatorics in the present study. To this end, this study aims to refine the construct of subjective combinatorics and examines its usefulness for analyzing students' engagement with counting problems. In particular, this study attempts to address the following research questions: *What factors contribute to students' subjective interpretations of combinatorics problems? What factors contribute to students' generation of restrictions when solving and evaluating combinatorics problems that are susceptible to subjective consideration?*

Theoretical Framework: Subjective Combinatorics

In an earlier study (Modabbernia, 2025), *subjective combinatorics* was introduced as a construct, viewing combinatorics as shaped by personal judgment and information about outcomes. This perspective highlights how individuals may approach the same counting problem with different assumptions, leading to varied yet reasonable interpretations of sets of outcomes. For instance, in a problem asking for the number of possible outfits from two shirts, five pairs of pants, and three hats, some students may consider an outfit incomplete without a hat, while others may view a shirt and pants as sufficient. In such cases, differences in interpretation lead to different sets of outcomes, and thus to different final answers. In this example, the restrictions could be made explicit by defining an outfit as one hat, one shirt, and one pair of pants, giving the problem a determinate answer. More broadly, while some counting problems are fully specified, others remain open to interpretation; this study focuses on the latter. Such openness can even give rise to more than one plausible set of outcomes, depending on how students interpret what the problem is asking. I refer to this interpretive flexibility as a *multiplicity of plausible sets of outcomes*. This idea supports the broader construct of subjective combinatorics, in which different yet reasonable interpretations lead students to envision different sets of outcomes to count. This interpretive flexibility is evident in a classroom example (not part of the study data), where a student approached the problem: *"In how many ways can a committee of 5 people be selected from a group of 12, if two specific people must be included?"* The student treated the version without the condition as an ordered selection (using P(12, 5)) and the version with the condition as unordered (using C(10, 3)). This shift may reflect the effect of the condition in prompting a different interpretation of what needs to be counted, rather than a mistake. It illustrates how conditions of the task itself can cue a particular structure of the set of outcomes.

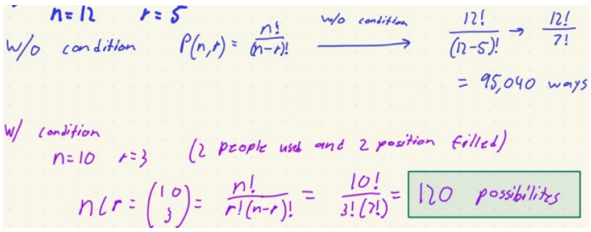

Figure 1. An example drawn from classroom experience (not study data) showing a student navigating different sets of outcomes.

This interpretive variation highlights the role of subjective interpretation not just in the process of counting, but in defining the set of outcomes itself. By situating subjective combinatorics within a broader body of research and highlighting the multiplicity of plausible sets of outcomes, this study refines the construct and extends its usefulness for analyzing students' thinking. Building on this refinement, I now define this construct more precisely.

For problems that allow *multiple plausible sets of outcomes*, *subjective combinatorics* views combinatorics as a matter of belief, shaped by personal judgment and information about outcomes. These are influenced by (1) *student-related factors* such as imagination, embodiment, personal experience, interpretive schemata, and the mental models they construct, and (2) *problem-related factors* such as the context in which the problem is posed and the wording of its conditions. Through this lens, differences in students' reasoning may reflect the interplay of these influences, giving rise to diverse yet reasonable framings of what counts as an outcome.

Method

At the time of data collection, 35 participants were enrolled in a secondary mathematics methods course within a professional development program in mathematics education. All held degrees in mathematics or science, though their academic backgrounds varied. The course, typically completed in the final semester of teacher certification studies, is designed to strengthen mathematical knowledge, foster investigative skills, and explore pedagogical approaches to secondary mathematics, with an emphasis on topics underrepresented in school curricula.

The topic of counting problems was addressed as the final area of exploration. In class sessions, the participants revisited the multiplication and addition principles from their secondary school studies and examined combinations, permutations, the binomial theorem, and combinations with and without repetition. They engaged with a range of counting problems, discussed potential sources of error, and concluded the unit by completing the following task.

You work with a class on counting problems. One of the problems asks students to solve the *Committees Problem*: There are 16 students. How many ways can we organize them in 4 committees of 4 people?
Three students suggest three different answers.

Alex's answer: $\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}$

Ben's answer: $\frac{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}{4!}$

Nikki's answer: $\binom{16}{4}4!\,\binom{12}{4}4!\,\binom{8}{4}4!\,\binom{4}{4}4!$

1) Which answer do you agree with? Explain your reasoning. If you disagree with all, please provide your own answer and justify your reasoning. 2) What do you think was implicitly assumed in each answer? 3) Why do you think the students made these assumptions? The suggested length for responding to each question is about 150 words.

Figure 2.The Task

The *Committees problem* was adapted from a standard combinatorics problem that allows varied interpretations, particularly around grouping. The task encouraged interpretations grounded in students' assumptions about what is being counted. One key source of subjectivity

lies in whether the committees are labeled or unlabeled. Additionally, students may or may not attend to the fact that committees are typically unordered collections of individuals, which affects whether orderings within each group are considered meaningful. The three suggested responses reflect different assumptions about what is being counted. Alex's answer treats the committees as labeled and distinct, assuming the order of group formation matters. Ben's response views them as unlabeled. The inclusion of 4! in the denominator suggests that arrangements of the same four groups represent the same outcome. Nikki's response builds on Alex's by assuming that the order of students within each committee matters, multiplying each binomial coefficient by 4! and treating each group as an ordered list. These responses illustrate how combinatorial reasoning can be shaped by assumptions about grouping and distinguishability. The data consisted of participants' written responses. Instances reflected subjective interpretations served as the unit of analysis. Having repeatedly read the data, I looked for common themes by highlighting noteworthy explanations.

Results

In the following analysis, participants' interpretations of the Committees problem, particularly whether the committees and their members were treated as ordered or unordered, are presented alongside their reasoning. The excerpts illustrate the assumptions participants brought to the task, revealing how they interpreted what needed to be counted. These interpretations played a central role in shaping their combinatorial thinking, influencing how they envisioned the set of outcomes and, in turn, the counting processes they considered appropriate. Among the 35 participants, 9 agreed with Alex's solution, 22 with Ben's, and 2 with Nikki's. One participant supported both Alex and Ben's solutions, and one found all three responses plausible. These interpretations were influenced by a range of factors, including wording, real-world experience, and unconscious implicit ordering—each of which is explored below.

Interpretation 1: The Committees Are Distinct

The role of wording. In counting problems, the language used in the problem statement can strongly shape students' interpretations. Certain words may prompt assumptions about the nature of the outcomes, such as whether elements are distinct or interchangeable, which in turn influences how students conceptualize the set of outcomes. The following excerpt illustrates.

> *Tara:* The committees are likely to have specific roles to fill, so they aren't all the same, if the question wanted the partitioning to be into indistinct groups then it would have used a more general word, like "group" or "team."

Tara's explanation highlights how the specific wording of a counting problem can guide students toward particular interpretations. Her reflection on the difference between terms like "committee," "group," or "team" suggests that certain words may prompt students to interpret the set of outcomes in particular ways. In this case, the word "committee" led Tara to assume that the groups were distinct, revealing how subtle differences in phrasing can shape how students interpret the set of outcomes. Consistent with this interpretation, Tara chose Alex's solution, which reflects the assumption that the committees are distinct.

Real-world experience. Students' interpretations were also shaped by how they connected the counting problem to real-life contexts. The following excerpt illustrates how Tara drew on her everyday experience with committees to interpret the structure of the outcomes.

> *Tara*: A committee generally is made with a specific intent, so creating the 4 committees are likely to have different goals, therefore, it matters which one each student got placed in.

Tara relied on her real-world interpretation of committees, typically tied to distinct roles, to infer that each group served a specific purpose. This led her to treat the committees as distinct and the arrangement as ordered, where assigning students to different committees resulted in different outcomes even if the group compositions remained the same. Tara's reasoning illustrates how real-life associations can shape students' assumptions about what distinctions matter in a counting context and how they conceptualize outcomes. Reflecting this interpretation, Tara selected Alex's solution, which treats all four committees as distinct.

Unconscious implicit ordering. Sequential thinking is a common cognitive pattern in daily life, where actions are broken into manageable steps. In many real-life situations, we approach tasks this way simply because it feels intuitive. When carried into counting problems, this natural tendency can introduce an implicit structure and lead to assumptions about how elements should be selected or grouped. The following excerpt exemplifies how such ordering emerged naturally.

> *Ana*: we want to create 4 groups of 4 students, out of a total of 16. To make it easier, let us do it one at a time: First, we will choose a group of 4 from 16 people, that is, C (16,4). Now, we want another group of 4, but now, out of twelve total people, since four of the previous sixteen have already been chosen. That makes it C (12,4). We have to repeat next step like that for the remaining 8 people to get C (8, 4) and then Finally, we have to create the last group of 4, but now we have only 4 students to choose from. Since the order of choice does not matter, it is rather intuitive that our C (4,4) is equal to 1. Therefore, all we must do is join all those combinations.

Ana's explanation reflects how a step-by-step approach, while intended to simplify the task, can introduce an ordering structure. Although she explicitly stated that the order of choice does not matter, likely referring to the order of students within each group, her sequential construction suggests an implicit assumption of ordering among the groups themselves. Based on this interpretation, Ana supported Alex's solution. This example illustrates how the structure of a counting process can shape the interpretation of the set of outcomes.

Interpretation 2: The Committees Are Indistinct

The role of wording. In some cases, students viewed the committees as indistinct based on how they made sense of the term *committee*. They viewed the term as describing a set of equivalent groupings. This perspective is reflected in the following excerpt.

> *Sarah*: The committees are 4 random groups, so it doesn't matter which group anyone is in.

Sarah's response suggests she treated "committee" as a neutral term, viewing the groups as interchangeable. By referring to them as "random groups," she overlooked any functional distinction, leading her to interpret the committees as indistinct and disregard ordering among them. This interpretation prompted Sarah to select Ben's solution.

Mathematical experience. Some participants approached the problem with expectations shaped by their prior encounters with combinatorics. Their interpretations reflected the distinctions they had experienced in earlier tasks, including how such distinctions were stated. The excerpt below shows how students' mathematical experience can inform their assumptions.

> *Madison*: Because there was no specific indication that the committees are distinct, there was no reason to think that they were. If the question wanted to imply that each committee is distinct then the question would have indicated that in some way.

Madison's reasoning highlights how the absence of explicit wording led to the assumption that no distinction between the committees was intended. Her interpretation was shaped by prior experience with combinatorics tasks and the ways distinctions had been communicated in those

settings. In this way, Madison's explanation illustrates how students draw interpretive meaning not just from what is stated in a problem, but also from what is left unsaid.

Interpretation 3: Committee Members Are Ordered

Real-world experience. Some participants considered not only the distinctness of the committees but also whether members within each committee should be treated as ordered. These interpretations often reflected real-world contexts where roles, positions, or functions within a group are assigned. The following excerpt illustrates how a participant invoked a familiar setting, a sports tournament, to justify why the order of members might matter.

> *Simon:* Imagine we have 16 students and we want to make 4 teams of 4 for a soccer tournament … There might be specific positions we can fill on a soccer team, and if there's a tournament bracket so it might matter what team we end up on.

Simon's response reflects how his familiarity with structured team settings informed his assumptions about the task. By relating the scenario to a soccer tournament, he drew on the idea that individuals in a group may have assigned roles or positions. This connection to real-life experiences introduced distinctions that might not have been explicitly required by the problem but felt natural within the imagined context. Simon's interpretation guided him toward Nikki's solution, which reflects the view that both the committees and their members are ordered.

Interpretation 4: Committee Members Are Unordered

Real-world experience. Some participants drew from real-life contexts to justify that the arrangement of members should not matter. Their reasoning emphasized that a group's identity is determined by its members, not their arrangement. The following excerpt illustrates this view.

> *Sam*: It does not matter whether you think about your committee as Sherry, Joseph and Timmy or as Joseph, Sherry, and Timmy. In either case, the same three people are in the committee. It might also be helpful to contrast this with something like a smartphone passcode (where the order of digits matters), and 1234 is not the same as 4321. One of these will unlock your phone, and the other will not.

Sam emphasizes that a group's composition remains unchanged regardless of member order. Comparing committee formation to a passcode, she noted that in one case, reordering does not affect the group, whereas in another, it fundamentally changes the outcome. This contrast shows how real-world reasoning shaped her interpretation of when order matters.

The interpretations discussed in this section reveal that students' combinatorial reasoning is influenced not only by the explicit conditions of the task but also by subjective assumptions tied to language, context, and structural framing. These factors play a central role in how students construct sets of outcomes and determine appropriate counting strategies.

Conclusion

This study refines the construct of *subjective combinatorics* by emphasizing the *multiplicity of plausible sets of outcomes* and clarifying the roles of *student-related factors* (such as imagination, embodiment, personal experience, and mental models) and *problem-related factors* (such as context and wording). These refinements provide a sharper account of how subjectivity shapes combinatorial reasoning, illustrating that different yet reasonable interpretations of sets of outcomes can emerge from the same task. In doing so, the study demonstrates the usefulness of subjective combinatorics as a lens for analyzing students' engagement with counting problems and points toward future research on how subjectivity shapes mathematical reasoning more broadly.

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STEM Teaching Practices: Before, During, and After Pandemic Response Teaching

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Spring 2020 marked the start of a national pandemic response teaching (PRT) period as colleges and universities responded to the public health emergency and associated upheavals and restrictions. Instructional practices and delivery modalities changed during this period, but there have been lingering questions about any lasting impact on pedagogy. In 2025, we conducted a national survey, following up with instructors of undergraduate chemistry, mathematics, and physics courses who had reported on their teaching in 2019, to explore this question. This paper focuses on questions about modality of instruction, distribution of class time between lecture and more active practices, and a set of other common instructional practices. In general, we find an aggregate return to pre-PRT practices, although there has been uptake of certain auxiliary digital practices (e.g., online office hours).

Keywords: introductory courses; teaching practices; pandemic; instructional change

Instructional practices in introductory STEM courses vary across disciplines, campuses, courses, and individual instructors. Over the last several decades in the US, there has been a push to incorporate research-based instructional strategies (RBIS) in these courses, with a particular emphasis on "active learning," an umbrella term used to describe pedagogical approaches that incorporate doing and discussing mathematics during regular class meetings. Despite some shifts, the majority of introductory STEM students still spend the majority of time in class listening to an instructor lecture or solve problems, something which has been repeatedly documented in large studies across the USA (Apkarian et al., 2021; Dancy et al., 2024; Johnson et al., 2022; Wang et al., 2024; Smith et al., 2021; Stains et al., 2018; Yik et al. 2022b).

Since then, higher education experienced a massive disruption. In particular, the global COVID-19 pandemic caused many institutions to move online in Spring 2020, and to incorporate modified practices due to public health concerns and local policies (Johnson et al., 2020). Across states and institutions, these modifications varied both in their nature and duration, though most colleges and universities returned to "normal" by the end of the 2021-22 academic year. Questions have persisted about any lasting effects of instructors' pandemic response teaching (PRT), particularly in the aggregate (Engelbrecht et al., 2020; 2023). These questions affect both our understanding of the current STEM instructional landscape and our ability to leverage research conducted prior to the pandemic.

This paper is part of an ongoing study aiming to answer those questions with particular attention to practices used in introductory general chemistry, single-variable calculus, and physics at undergraduate institutions. We re-surveyed instructors of these courses who had previously reported on their practices in 2019, giving us a unique ability to report on shifts (or lack thereof) in pedagogy. This paper specifically focuses on two research questions:

1. Has the time distribution of in-class practices, across lecture, individual work, group work, and whole class discussion, changed since 2019?
2. What common practices did instructors use before, during, and after PRT? Which, if any, were more likely to be abandoned, paused, or adopted throughout this period?

Background & Theory

Pandemic Response

On March 11, 2020, after months of monitoring the spread of disease, the World Health Organization officially declared COVID-19 a pandemic (World Health Organization, 2025), sparking international public health policies which included school closures around the world, including most higher education institutions in the United States – at least temporarily (Nurunnabi & Almusharraf, 2020). This generally meant a rapid shift to online remote instruction without significant prior preparation, previous experience, or professional development (Johnson et al., 2020; Kirkman et al., 2021; Skliarova, 2022).

Pedagogy in Spring 2020 is sometimes referred to as *emergency remote teaching* or *emergency response teaching*; we refer to the entire period of Spring 2020 through the end of the 2021-22 academic year as *pandemic response teaching* (PRT), including both the immediate response and more planned instruction as policies shifted and restrictions eased. We note that, while instructors had more time to prepare for teaching under extenuating circumstances for the 2020-21 and 2021-22 academic years, it was broadly understood that the policies in place to ensure public health would be in place for an unknown, but finite, period – as opposed to a permanent adjustment.

Since the shift to PRT, there have been questions about the extent to which those practices will have some lasting impact on college instruction (e.g., Engelbrecht et al., 2020; 2023). Optimists noted many examples of instructors around the world taking the pandemic as an opportunity to examine and reflect upon their teaching practices and experiment with new ideas as alternatives to standard practice became a necessity (e.g., Albano et al., 2020; Gosztonyi, 2021; Maciejewski, 2021; Rapanta et al., 2020) and hoped that change would be widespread, and positive (Engelbrecht et al., 2023). Others urged caution, in particular noting that the emergency-response online teaching enacted in Spring 2020 was distinct in design and implementation from purposefully developed online learning programs (Hodges et al., 2020). There can be no doubt that human experiences through the pandemic impacted individual instructors, but the question of how those experiences impacted teaching practices at scale remains open.

Choosing Practices

One consistent result found across many studies of instructors' practice is that their personal experiences influence their instructional approach (Andrews & Lemon, 2015; Apkarian et al., 2021; Emery et al., 2021; Oleson & Hora, 2014; Yik et al., 2022ab). Thus, we assume that how instructors thought and taught prior to PRT influenced the decisions they made during PRT, and that their experiences during PRT now also influence their current decisions and practices. Figure 1 shows a toy model of this theorized influence: prior practices and beliefs contribute to current practices and beliefs, mediated by PRT experiences. We cannot say, *a priori*, however, in what direction(s) these changes might have occurred.

In higher education contexts, most theories of change focus on guided change (with identifiable change agents and premeditated, though flexible goals) or long-term unguided evolution. The unique conditions of the COVID-19 pandemic with regards to higher education mean that many change theories are not entirely satisfactory in terms of predicting outcomes across the country - hence the need for empirical work such as that presented in this paper.

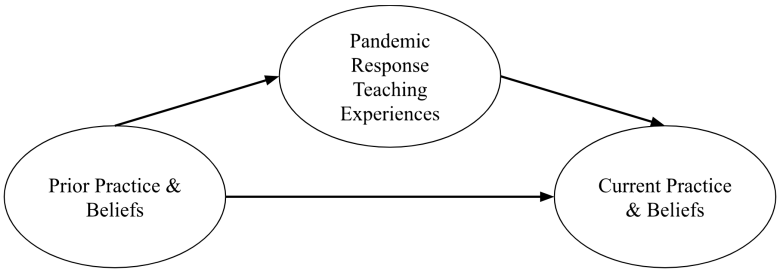

Figure 1. Toy model of theorized influence of prior practice and beliefs, mediated by PRT experiences, on instructors' current practices and beliefs.

The CACAO model (Dormant, 2011; Viskupic et al., 2022) draws on many change theories and is intended to help people plan changes, drawing focus to four elements: the *change* itself, the *change agents* who support the intended *adopters* through the change process, and the *organization* in which they reside. This model outlines five stages of the adoption process for new practices: awareness, curiosity, mental tryout, hands-on tryout, and adoption. Viewed through this lens, we can say that the pandemic forced many instructors to skip directly to the "hands-on tryout" stage, without prior preparation or the support that might be provided by deliberate agents of change. We note that there was not a unitary "change" being tried out, but rather a constellation of strategies, techniques, and technologies being engaged across different and evolving contexts. As such, we explore which aspects (if any) of instructors' PRT were adopted after the crisis had passed and mandates were lifted.

Research Methods

Participants & Data Collection

We leveraged the roster from a previous research study, conducted in 2019, to recruit participants to the study. The initial roster consisted of roughly 17,000 names gathered from publicly available information on university websites and targeted those who had taught introductory chemistry, physics, or single-variable calculus at colleges and universities across the country in the 2017-18 or 2018-19 academic year; 3,679 of these met the desired criteria and participated in the original study.

The entire original roster was contacted again to capture those who taught before *and* after the pandemic-induced disruption c. 2020-2023. To be eligible, instructors must have taught one of the target introductory courses in the 2023-24 and/or 2024-25 academic year. In total, 933 participants provided usable responses to our survey: 339 chemistry, 279 mathematics, and 315 physics; roughly two-thirds of responses in each discipline came from individuals who completed the 2019 survey, allowing for direct comparison. Instructors who participated in both surveys can be linked via an anonymous identifier, allowing for direct comparison.

Data was collected via web-based survey using the Qualtrics platform. The majority of the survey questions were copied from the survey used in 2019 (Apkarian et al., 2021; Yik et al., 2022ab), and asked instructors about their course structure (e.g., class size, classroom setup), department context (e.g., coordination of course content), instructional practices, experiences with research-based instructional strategies, professional context, and personal identity questions. New items were developed to capture instructors' experience during the pandemic response teaching period, including questions about how many terms they taught online, which (if any) RBIS they tried, and the resources and support they had access to in that period.

Data Analysis

This paper reports on three sets of survey items. The first captured instructors' experience teaching undergraduate courses in four different modalities (face-to-face; online synchronous; online asynchronous; hybrid) before, during, and after PRT. The second set asked instructors to estimate the percentage of class time, in a typical week, that students spend (a) listening to the instructor lecture or solve problems; (b) work on tasks individually; (c) work on tasks in small groups; and (d) engage in whole class discussion (these were required to total 100). The third set of questions presented instructors with a list of eight teaching-related practices and asked instructors to indicate if they had used the practice before, during, and/or after the pandemic response period; "never" was also an option, to distinguish between those who skipped the item and those who had never used a practice.

Analyses were conducted on complete cases (by focal item) and so the *N* size varies. We present descriptive statistics leveraging both the earlier data set (collected in fall 2019) and the new data set (collected in spring 2025). We leverage *t*-testing to compare data from 2019 and 2025, both in the aggregate using the entire sample and paired t-tests to directly compare repeat participants and applied a Bonferroni-corrected threshold to account for the increased rate of Type I errors when engaged in multiple hypothesis testing.

Results

Instructional Modality

As expected, instructors in our study were predominantly teaching face-to-face courses until partway through the spring of 2020, when public health led to widespread social distancing policies. In the later part of the spring, online synchronous and online asynchronous were the most common modality in the emergency situation. In the first full academic year of pandemic response teaching (2020-21), with some additional time for preparation but before vaccines were widely available, these four modalities were almost equally in use. The following year (2021-22) saw an increasing return to face-to-face teaching, and by 2025 when the more recent data was collected, the pattern of instruction was back to pre-PRT levels, closely matching reports about teaching modalities early in the Spring of 2020 (see Figure 2).

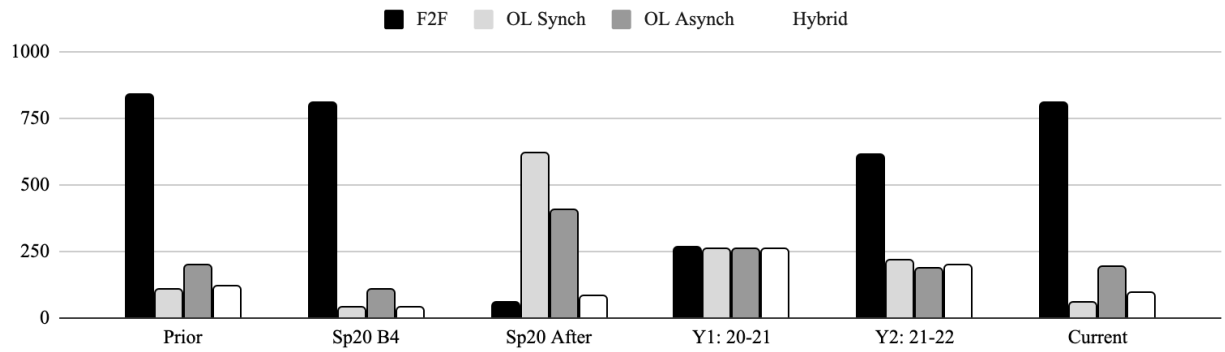

Figure 2. Chart showing number of respondents reporting using each of four modalities, for any course, from before Spring 2020 through Spring 2025 (current).

Distribution of Class Time

One outcome measure used in the first study was the reported percentage of class time, in a typical week, that students spend listening to the instructor lecture or solve problems, as opposed

to working individually, working in small groups, or engaged in whole class discussion. Instructors' averages, disaggregated by discipline, are presented in Table 1.

Table 1. Average reports of the percent of class time students spend in different learning activities.

	Chemistry		Mathematics		Physics	
	2019 *N*=1207	2025 *N*=339	2019 *N*=1293	2025 *N*=279	2019 *N*=1141	2025 *N*=315
Listening	59.01 (24.70)	54.66 (25.98)	56.48 (25.15)	54.36 (25.53)	53.94 (25.59)	53.16 (26.63)
Individual Work	11.69 (11.61)	14.20 (15.37)	11.31 (11.80)	13.04 (13.95)	10.60 (11.68)	11.21 (14.31)
Small Group Work	18.40 (19.43)	20.79 (21.38)	17.14 (17.90)	19.27 (19.10)	22.45 (20.53)	23.99 (21.43)
Discussion	10.89 (11.76)	10.35 (11.37)	15.07 (14.56)	13.33 (14.18)	13.01 (11.64)	11.64 (10.84)

Note: table entries give cell mean (standard deviation). For 2025 data, N=933; for 2019, N=3641.

In general, we see very small changes in the direction of slightly less lecturing and discussion being reported, and slightly more individual and small group work on mathematical tasks. We consider these slight differences to be of no practical significance, well within what we might accept as rounding error on the part of participants. We additionally conducted paired *t*-tests with the 2019 and 2025 data for the 588 individuals who responded to both surveys and filled out this item completely. With an adjusted *p*-value to account for multiple testing, none of the differences were statistically significant.

Note that this is not equivalent to saying that *no* instructors changed their practices over the previous six years. However, those changes average out to zero. As shown in Figure 3, across all three disciplines instructors reported balanced changes toward and away from lecture - both in number of instructors and the magnitude of the reported shifts.

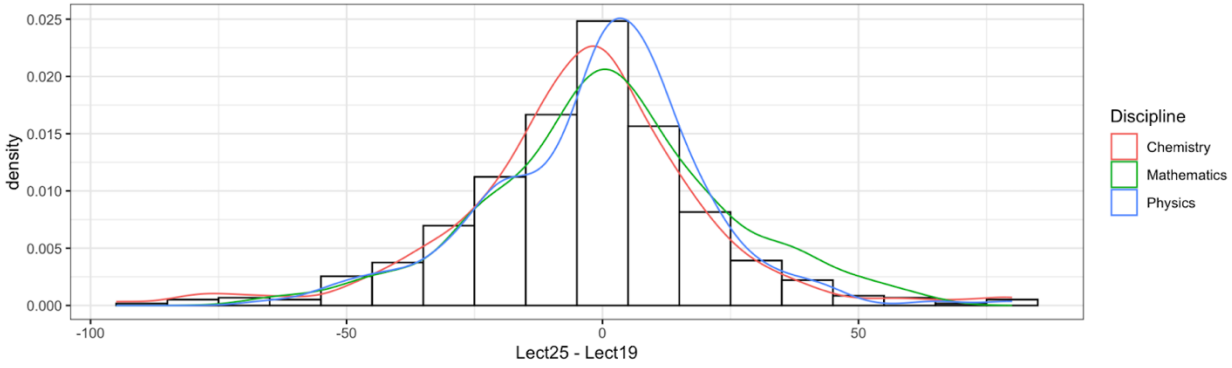

Figure 3. Chart showing density of reported changes in the percentage of class time spent in lecture from 2019 to 2025. Histogram shows aggregate responses, while colored lines show density distribution by discipline.

Usage of Instructional Practices Pre/Post

Looking across all instructors (chemistry, mathematics, and physics) certain patterns emerge from instructors' usage of certain practices before, during, and after the pandemic response teaching period (PRT). Table 2 shows the proportion of respondents who indicated that they had never used a particular practice, and the proportions (of the rest) who indicated each *pattern* of usage for that practice. That is, "only before" refers to those who used a practice before PRT, but not during or after; "during and after" refers to people who did not use the practice before PRT but used it during and after that time. For example, 40% of respondents indicated that they had

never used live polling in their class; of the 60% who *have* used polling, 20% did so only before PRT while 42% used it consistently before, during, and after PRT.

In terms of the CACAO change model, we are particularly interested in practices which instructors started or stopped using during PRT. With regards to the former, the "only during" pattern is indicative of trying out a new practice during PRT and discontinuing usage afterward; the "during and after" pattern is indicative of trying out a new practice and then adopting it. With regards to the latter, the "only before" pattern can be interpreted as trying out *not* doing something (e.g., trying teaching without live polling) and adopting that non-practice afterward; "before and after" discontinues that absence, returning to previously adopted practices. These patterns are the focus of our queries regarding changes to instructional practice which might have been catalyzed by the pandemic response, despite the unplanned nature of these changes.

Table 2. Proportion of instructors who used practices by pattern of usage.

	N_1	Never	N_2	Only Before	Before & During	Only During	During & After	Only After	Before & After	Always
Online office hours	838	-	808	-	-	0.39	**0.42**	-	-	0.15
Recorded lectures	831	0.14	711	-	-	0.30	0.37	-	-	0.26
Supplemental videos	824	0.16	691	-	-	0.15	0.25	-	-	**0.50**
Group work	829	0.14	715	0.08	-	-	-	-	0.29	**0.52**
Think pair share	807	**0.41**	477	0.13	-	-	-	-	0.31	**0.46**
Live polling/clickers	816	**0.40**	489	0.20	0.06	0.12	-	-	0.11	**0.42**
Online exam proctoring	821	0.31	568	-	-	**0.69**	0.14	-	-	0.11
Online HW system	834	0.11	746	-	-	-	0.07	-	-	**0.78**

*Note: Values ≤ 0.05 are omitted for clarity; no cell is truly zero (minimum 2); values ≥ 0.4 are **bold**. Individuals who skipped all possible answers for a particular practice are not included. N_1 refers to the total number of people who responded to the item at all; N_2 refers to the number of people who answered the question and at some point used the practice.*

Fewer than 5% of respondents indicated using any of these practices after PRT without prior usage. Online homework systems were the most consistently used, with 78% of people who ever used such a system reporting that they *always* used such a system. Online exam proctoring, on the other hand, was primarily a PRT-exclusive practice: of the 489 instructors who ever used online exam proctoring, 69% report only using it during PRT, with another 14% reporting that they adopted the practice during PRT and continued to use it afterward.

Practices which were "picked up" during PRT and maintained afterward include online office hours (42% of users), recorded lectures (37%), and supplemental videos (25%). Instructors not in this category were most likely to report using online office hours and recorded lectures exclusively during PRT (39% and 30%, respectively), and to report using supplemental videos throughout (50%).

To summarize the highlights: online/digital resources (office hours, recorded lectures, supplemental videos) were broadly adopted during PRT and maintained afterward; practices related to student-student interaction (group work, think pair share) were consistently used by many instructors, though a sizable group paused their use during PRT; live polling was also used fairly consistently, with a few instructors using it only during PRT and a few pausing their usage in that period; for those who used online exam proctoring, it was primarily consigned to PRT; online homework systems were already broadly in use and were used consistently throughout.

Discussion

We did not identify significant changes in reports of how class time is spent across any of the three disciplines with regards to the four major categories reported: listening to the instructor lecture or solve problems, working individually, working in small groups, engaged in whole class discussion. However, we did identify several changes between 2019 and 2025 in calculus instructors' usage of RBIS. While some change is expected due to an additional five years of teaching experience, we conjecture that some of these changes may have been accelerated by instructors' experiences during the pandemic response period. Note that this discussion is heavily conjectural, and further analysis of the quantitative and qualitative data is needed to support these conjectures.

Increased usage of online office hours, supplemental videos, and recorded lectures starting during PRT and continuing afterward suggests that once instructors had "figured out" these strategies, they maintained their usage. That is, PRT prompted instructors to take certain actions which were relatively straightforward to maintain, and they saw enough value in those practices to continue using them. In comparison, the interactive practices of think-pair-share and group work were used less frequently during PRT than before/after, although some maintained usage throughout. This is perhaps unsurprising, given the logistic difficulties of having students talk in small groups on web-based platforms that were widely in use. That think-pair-share saw a greater drop than group work may again be due to logistical considerations: putting students into breakout rooms to work on a task for 5-10 minutes may have been more manageable than quickly separating students into pairs for just a minute or two.

Online exam proctoring was fraught with considerations of integrity, logistics, privacy, and so on; upon the return to in-person teaching, it is therefore not surprising that it went back to being sparsely used, though it does appear that it had some uptake - perhaps allowing instructors to move examinations out of regular class meetings and save that time for additional instruction. Perhaps more surprising is that there was not an uptick in usage of live polling, since this is a built-in feature of many web-conferencing platforms and might have been easily employed by instructors. Overall, we see no major shifts to in-class instructional practices, although some auxiliary resources were adopted during PRT and remain in usage. This suggests two main conclusions: (1) there has been little lasting pedagogical change since the pandemic response teaching period; and (2) large-scale studies conducted prior to 2020 are likely still valid descriptions of undergraduate STEM instruction.

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Parsing Students' Decision-Making During Analogical Reasoning: A Multi-Constraint Approach

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General theories of analogy dictate that connections can be made between virtually any two domains, thus suggesting an extremely large number of analogies that one might observe. Unsurprisingly, some analogies are of course more useful than others. How then do students come to observe "useful" analogies? Multi-constraint theory is one such avenue for exploring questions such as these. In this contributed report, I adopt a multi-constraint approach to parsing students' analogical reasoning with the goal of uncovering how certain choices are made when developing ring-theoretic analogies to group-theoretic structures. Analysis revealed that the participants exhibited variations in emphasis of analogical constraints during their reasoning in three ways: (a) adhering to basic structure, (b) adhering to conceptual structure, and (c) flouting constraints to advance exploration. Implications for the teaching and learning of abstract algebra and curriculum development are discussed.

Keywords: Abstract algebra, Analogy, Multi-constraint theory, Similarity-based reasoning

What are all the possible similarities and differences might students observe between a group and a ring? One student might focus on simple aspects of their definition, such as the number of operations each structure possesses. Another might focus on deeper features of the objects, such as recognizing both as being general examples of abstract structures that replace the need to focus on specific objects when writing proofs. Yet, it might be strange to hear a student suggest that groups and rings are similar because they are both spelled with letters from the English alphabet, or that they are different because one describes a collection of things, while the other describes an object you might wear on your finger. While such connections may not be considered all that valuable from an educational perspective, some would still count them as examples of analogies. In fact, highly general theories of analogy dictate that connections can be made between any two domains, thus suggesting an extremely large number of analogies that one might observe between any two situations (e.g., Faulkenhainer, Forbus & Gentner, 1983). A question then arises as to why some connections are more commonly observed over others.

To begin answering questions such as these in the context of mathematical thinking, I adopt a prominent theory of analogical reasoning that delineates types of constraints that occur naturally when reasoning by analogy. In this contributed report, I present an analysis of four students' thinking during their construction of (and reflection upon) ring-theoretic analogues to group-theoretic structures. Unlike previous investigations that focused more-so on *what* students created (e.g., Hejný, 2002; Hicks, 2024; Stehlíková & Jirotková, 2002), this report focuses on exploring underlying reasons for *how* certain choices were made over others when reasoning by analogy (e.g., Lee & Sriraman, 2011)

Theoretical Background: Multi-Constraint Theory

The approach to parsing students' thinking presented in this study relies on a theory of analogical reasoning known as *multi-constraint theory* (Holyoak & Thagard, 1989). This theory posits that the development of analogies is inherently guided by sets of constraints that serve as the foundation for coherent analogy formation. While it can be argued that analogies might be generated between any two given situations, it is rarely the case that a person randomly generates

analogies between any two situations as they go about their day-to-day activities. Instead, individuals tend to select particular mappings when generating an analogy. Multi-constraint theory takes as its premise the idea that adherence to a set of constraints leads individuals to develop analogies that satisfy their personal sense of rationality, thus greatly reducing the overall number of analogies that might be created to a smaller subset of "reasonable" analogies as opposed to creating analogies at random. I describe three broad categories of constraints below, all of which serve different purposes in dictating the construction of analogy from an individual's perspective.

Three Types of Constraints

The first constraint is known as *structural consistency*. This constraint suggests that in the course of analogy formation, there is an implicit desire to create, to as great of an extent as possible, a one-to-one mapping between the source and target domains. For example, consider possible analogical comparisons the following statements: (1) *Oliver is taller than Paul*, and (2) *the Sun is more massive than the Earth*. Each of these statements successfully establishes a relation of order between two objects, displaying that there is consistency of a basic structure between the two situations. As such, there is a clear structural mapping between these two domains through a form of proportional reasoning (e.g., Modestou & Gagatsis, 2010).

The second category of constraint is known as *semantic similarity*, which focuses less on the abstract structure that may exist between two domains, and more so on the similarity of the underlying appearance or meanings associated with the two domains. In the previous example comparing the *taller than* relation to the *more massive than* relation, you might notice that the two situations are quite irrelevant to one another despite possessing a common basic structure. Thus, the two statements are low in semantic similarity. In contrast, consider the analogy that underlies the pair of statements (1) *Oliver is taller than Paul* and (2) *William is older than Tim*. In this case, not only is basic structure preserved, but there is also a high semantic similarity. Both situations define relations between people, focusing on properties that are often used to compare and describe people in every-day life. One interesting observation about semantic similarity is that, while it may serve as a constraint to assist with developing analogies in general, it is often argued that a high degree of semantic similarity is less appealing when creating "good" analogies. In fact, preference is often given to analogies that determine similarity between two highly disparate situations that share no surface level similarities (e.g., Gentner, 1983). This concept is ubiquitous to pure mathematical thinking as well, given that isomorphisms only require abstract structure to be similar without any reference to whether the objects themselves within the structures are similar or not (e.g., integers under addition are isomorphic to the free group on one element, yet there is no specific similarity to the elements contained in each group.)

The final constraint is *pragmatic centrality*. This crucial constraint dictates that even when analogies might possess high levels of structural consistency and semantic similarity, an underlying pragmatic purpose must be present for the analogy to be observed. Out of all possible analogies that might be generated, pragmatic centrality serves as the constraint that ensures that analogies are created with purpose. Indeed, pragmatic centrality extends beyond just the analogical creation itself; the constraint of pragmatic centrality is visible when students question the purpose of their choices and reflect upon their overall goal, as well as consider the consequences of their choices. For a summary of the three constraints, see Table 1.

Table 1. Types of Analogical Constraint

Structural Consistency	Adhering to the constraint of purposely generating one-to-one mappings between structures.
Semantic Similarity	Adhering to the constraint of purposely generating analogies between domains with observed similar meanings.
Pragmatic Centrality	Acknowledging underlying purpose of the analogy, either before, during, or after creation.

In this study, I aim to answer the following research question: *In what ways did four students adhere to the three constraints of structural consistency, semantic similarity, and pragmatic centrality while reasoning about ring-theoretic structures by analogy with group-theoretic structures?*

Methods

The data for this study was collected as part of a larger project investigating students' analogical reasoning in abstract algebra. In the Fall of 2019 and the Spring of 2020, rosters were acquired from the abstract algebra courses taught at a large 4-year research university in the southwestern United States. Four students, named Andrew, Brandon, Ellen and Nathan (all pseudonyms) were solicited to be interviewed and participate in a series of 5 clinical task-based interviews (Goldin, 2012). During these interviews, a single camera was positioned toward the student's written work and a lapel mic was given to the student for audio recording. The first interview served to allow students to recall various aspects of group theory and associated concepts and definitions, while the subsequent interviews focused on exploring concepts in ring theory.

The final three interviews all corresponded to the construction of a different ring-theoretic analogy to group-theoretic structures. There were two pertinent goals of these interviews. First was to generate a ring-theoretic analogue to subgroup, homomorphism, and quotient group based on what they knew from group theory by openly responding to the task: *Make a conjecture for a structure in ring theory analogous to _____ in group theory*. Second was to inquire into properties, examples and theorems associated with their newly generated ring-theoretic analogue. During the interviews, the students' understanding of structure was revealed based on what features they attended to while developing their ring-theoretic analogue and was also revealed based on what properties and theorems they decided to investigate following the completion of their analogy.

To analyze the students' interviews, transcripts of the audio were coded alongside the video with respect to the three constraint categories. In particular, the statements made by the students in addition to their written work were scrutinized for any evidence of attention to structural or semantic similarity, as well as any attention to an underlying purpose for their utterances/writing, or considerations of consequences for their construction in the moment (pragmatic centrality). Such utterances were then assigned a code based on the adherence to one or more of the constraints along with descriptions of how the constraint influenced their analogical creation. Following this process, I then identified themes delineating the extent to which each constraint was present across the students' thinking elicited in each interview.

Results

The extent to which each student adhered to the constraints provided insight into their sensitivity to recognizing underlying concepts associated with both group- and ring-theoretic structures, thus reaffirming previous findings related to their sense of analogical structure (Hicks & Flanagan, 2024). But, by attending to the presence of the three constraints in the participants' thinking, a window into their underlying rationale for developing their ring-theoretic analogues was opened, providing further insight into the students' reasons for establishing their analogies. In this section, I present three themes that were identified as possessing particular significance to these constructions: (1) adhering to basic structure, (2) adhering to conceptual structure, and (3) flouting constraints to advance exploration (see Table 2).

Table 2. Observed Themes

Adhering to Basic Structure	Exhibiting adherence to structural consistency and semantic similarity of surface level concepts, such as naming conventions, written symbols, etc.
Adhering to Conceptual Structure	Exhibiting adherence to structural consistency and semantic similarity of deeper concepts, such as mapping the concept of structure-preserving maps from the domain of group theory to the domain of ring theory.
Flouting Constraints	Ignoring one constraint in an effort to freely investigate properties or aspects of an analogous structure.

Adherence to Basic Structure

Perhaps unsurprisingly, most students adhered to a high level of semantic similarity when generating a name for their ring-theoretic analogy for most structures. For example, each student provided the names "subring" and "ring homomorphism" to describe their newly created analogy in comparison to subgroups and group homomorphisms. Across all structures, the students appeared to be strongly adhering to both constraints of structural consistency and semantic similarity, often with little reference to any sort of underlying purpose to the name. For example, when constructing the ring-theoretic analogue to quotient groups, Nathan introduced the concept of a "normal subring" without any suggestion as to what "normal" might mean in the context of ring theory. I refer to the presence of both structural consistency and semantic similarity of surface level concepts as *adherence to basic structure*.

A notable example of adhering to basic structure occurred when Ellen developed her definition for a ring-theoretic analogue to homomorphism. Despite recognizing differences in the structure of group and ring in general, Ellen made an effort to maintain structural consistency and semantic similarity when developing her analogue for homomorphism by combining the homomorphism property into one statement (see Figure 1.) This allowed Ellen to establish a one-to-one relation between the two homomorphism definitions (i.e., each required only one function and one homomorphism property), while also maintaining a similar appearance for both. However, Ellen did not immediately question the underlying meaning of her newly formed homomorphism property that combined the operations into one expression, thus suggesting that pragmatic centrality was absent from her initial creation.

1. **Make a conjecture for a structure in ring theory analogous to homomorphisms in group theory.**

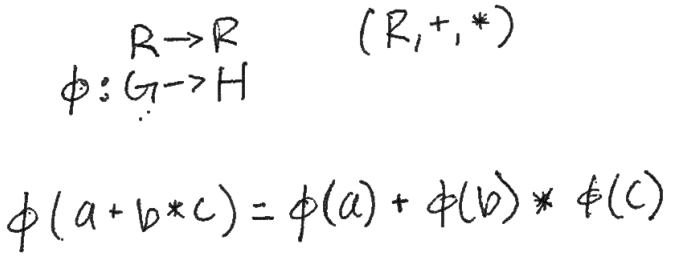

Figure 1. Ellen's initial written response when developing a "ring homomorphism".

Adherence to Conceptual Structure

The presence of pragmatic centrality at the outset of the generation of an analogical structure influenced the manner by which the structure was initiated. In this data, the presence of pragmatic centrality when initiating construction was often indicative of a strong adherence to both structural and semantic similarity, but with significantly different emphasis on *where* the attention was when developing the structure. For example, Ellen attended heavily to the group homomorphism definition when constructing her ring-theoretic analogue, and thus she relied heavily on written, symbolic similarity. In contrast, Andrew adopted a different approach at the beginning. He first established that there would be a ring R with operations + and *, and asserted that a second ring S should exist. However, he immediately began to question what the operations on S would look like:

> *Andrew*: But these (pointing to ring S) don't have to have the same operations, and I don't know what to use.
> *Interviewer*: Can you explain what you mean by that?
> *Andrew*: Well, when we have groups, we can something like this: $\phi(a \cdot b) = \phi(a) * \phi(b)$. If this homomorphism is from G to H, then the dot is the operation on G, and the star is the operation on H. So maybe I can use a dot here (*writing the multiplicative operation for S*), and I can just use a minus for… Yeah, I don't know what to use, unless you have a suggestion.

From this, we can see that Andrew's initial construction of the homomorphism property was guided in part by the constraint of pragmatic centrality; even before writing anything, he was already reflecting on the consequences of working with multiple operations in the context of ring theory.

Compared to Ellen's construction, it would appear that Andrew was perhaps less concerned with structural consistency given that Andrew's definition involved two homomorphism properties rather than one. In terms of establishing structural consistency between the two definitions, Ellen successfully mapped one homomorphism property for groups to one homomorphism property for rings. However, I claim that Andrew still maintained a high level of both structural consistency and semantic similarity, but at a different level than Ellen had previously done. This was present in the following interaction:

> *Interviewer*: Could you tell me what your overall thought process was in making this definition?

Andrew: Yeah, so I figured you need to preserve operations. That's basically how a homomorphism is defined, it's a function that preserves operations. So, if that's how it's defined I just have to preserve operations, but with two different operations now.

Thus, while Andrew's adherence to a written, symbolic structural consistency was not as strong as Ellen's, Andrew displayed an adherence to structural consistency and semantic similarity of a more conceptual structure. Andrew successfully encapsulated the concept of *preserving operations* into one object to be mapped between domains, thus achieving a greater degree of a one-to-one mapping despite having to break structural consistency at the basic level. I refer to this phenomenon as *adherence to conceptual structure.*

Flouting Constraints to Advance Exploration

Adherence to one of basic or conceptual structure was not necessarily the norm for all participants and wasn't even standard for any one particular student across all of their created structures. Indeed, a third phenomenon that occurred was that one of structural consistency or semantic similarity would be flouted in favor of the other, whether at the basic or conceptual level. This often occurred when a student began to question their creation in order to begin exploring alternatives. For example, Nathan initially began describing his concept for ring homomorphism by adhering strongly to symbolic semantic similarity. Unlike Andrew, Nathan's construction was tied to ensuring a similar basic appearance to that of group homomorphism with the added caveat that there were two operations defined on rings. As such, Nathan initially flouted structural consistency in order to accommodate the presence of two operations.

However, after developing a definition for a ring-theoretic analogue to homomorphism (see Figure 2), Nathan began to question his choice. He mused:

> Writing it like this makes sense, because that's how the group is written. But, if I wanted to write two functions, I feel like you would have to separate it, like you would be like, psi maps (R,+) to (S,+'), and then zeta or something maps (R,*) to (S,*')… Now there's two of them.

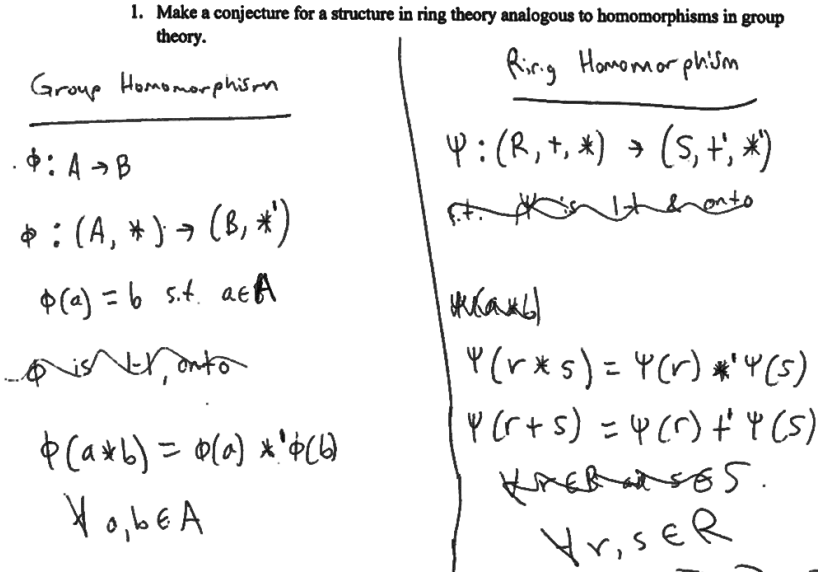

Figure 2. Nathan's written response when developing a "ring homomorphism".

Here, we see that Nathan is now wondering whether there should be two *functions* rather than two homomorphism properties defined on the same function. This investigation led Nathan to ponder whether he could describe both functions in terms of groups (see Figure 3.) In doing so, Nathan purposefully continued to flout structural consistency with the goal of accommodating the presence of two operations for rings.

Unlike Andrew, Nathan never appeared to recognize any one-to-one mapping between the underlying concepts of homomorphism properties for groups and rings during his investigation. Instead, Nathan focused primarily on adhering to a basic semantic similarity, albeit in two forms: (1) establishing similar homomorphism properties for groups and rings, and (2) establishing similar functions defined on the groups and rings where only one operation was mapped at a time.

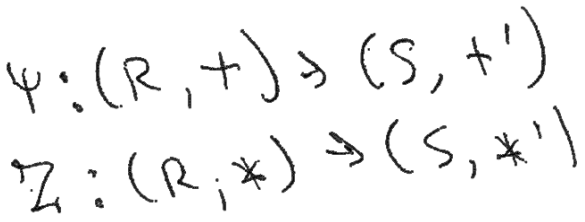

Figure 3. Nathan's proposal to define two functions rather than two homomorphism properties.

Discussion

The results of this analysis affirm a subset of previous findings related to parsing students' analogical reasoning. In particular, multi-constraint theory provided another method for observing that students can engage in a creative process of analogical reasoning in order to actively *invent* new (to them) concepts in abstract algebra (see Hicks, 2023). However, the use of multi-constraint theory to parse students' analogical reasoning revealed new perspectives on understanding *why* students may make certain decisions. Of particular significance is the observation that students may flout constraints in an effort to advance their construction or exploration of a new analogy, and that a stronger adherence to pragmatic centrality may develop over the course of an investigation. This supports the view of analogy as a problem-solving process to sometimes be purposefully implemented (e.g., Carbonell, 1983), rather than only a spontaneous process that occurs only by chance.

I end this paper by offering two suggestions stemming from this analysis that may be beneficial for successfully using analogy in mathematics classrooms. First, teachers may benefit from observing the distinction between adhering to basic and conceptual structures and understanding the value of each. Indeed, while it may appear that adhering to conceptual structure is always the ideal goal, adherence to basic structure also resulted in rich initial investigations and allowed the participants in this study a foothold for furthering their constructions. In contrast, many theories of analogical reasoning devalue those analogies that might adhere too strongly to basic structure (e.g., see Gentner's (1983) concept of *literal similarity*), and typical applications of multi-constraint theory investigated how the constraints supported individuals to avoid creating "bad" analogies.

Second, teachers should encourage their students to push on and question their analogical constructions, even when a correct analogy is given from the beginning. As was the case with Nathan's construction of ring homomorphism, the constraint of pragmatic centrality was not initially present. However, once he began to question his construction, he was able to explore alternatives and consider the deeper meaning of what a ring homomorphism is.

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Measuring Undergraduate STEM Reasoning: Graphical, Covariational and Proportional Skills as Predictors of Course Performance

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This study examined the role of students' graphical, covariational, and proportional (GCP) reasoning skills in predicting performance in introductory STEM courses. Specifically, it investigates whether GCP proficiency predicts overall course performance. Data from 382 students revealed significant course-level differences, $F(2, 379) = 10.27$, $p < .001$. Students in Introductory Chemistry for Engineers scored highest (EMM = 59.0), significantly outperforming peers in Biology (EMM = 49.5) and Chemistry (EMM = 43.9). GCP scores were moderately correlated with course performance ($r = .56$). Mixed-effects modeling confirmed that GCP scores were a strong predictor of final performance ($\beta = 0.025$, $p < .001$), even when accounting for variation across majors. This study has implications for STEM instructors. Assessments should be designed to evaluate students' application of knowledge, problem-solving skills, and ability to formulate coherent arguments. This would give a more comprehensive picture of a student's academic potential that goes beyond memory recall.

Keywords: Graphical Interpretations, Covariational Reasoning, and Proportional Reasoning.

Introduction

Science, Technology, Engineering, and Mathematics (STEM) courses are crucial for addressing the complex problems that plague today's society. They equip students with the knowledge and skills necessary to perform multiple tasks, collect and evaluate evidence, and make informed judgments about information (English, 2016; Topsakal et al., 2022; Whitacre, 2024). Research indicates that STEM graduates make significant contributions to increased economic growth and employment (Bacovic et al., 2022), underscoring their importance. Despite the benefits, success in introductory STEM courses remains a significant challenge, with many students struggling and ultimately leaving STEM majors (Rasmussen et al., 2019; Smith & Funk, 2021). A growing body of research suggests that students' difficulties may be linked to foundational reasoning skills necessary for engaging with STEM content. In particular, graphical interpretation, covariational reasoning, and proportional reasoning are essential for making sense of quantitative relationships, modeling dynamic systems, and applying mathematical concepts in scientific contexts (Altindis et al., 2024; Bowe et al., 2025; Rodriguez & Jones, 2024; Carlson et al., 2002, 2022; Moore, 2025). This study examines whether these forms of reasoning predict success in STEM courses.

Literature Review

Graphical interpretation, covariational reasoning, and proportional reasoning are foundational quantitative practices that underpin success in STEM learning. These forms of reasoning allow students to move beyond rote memorization by supporting their ability to analyze data, model relationships, and draw meaningful conclusions from quantitative information (Altindis, 2025; Emeji et al., 2025). For instance, graphical interpretation enables students to extract and synthesize information from visual representations, a skill necessary in disciplines ranging from biology to engineering (Bowe et al., 2025; Rodriguez & Jones, 2024). Covariational reasoning enables students to track how quantities change in relation to one another (Carlson et al., 2002).

Proportional reasoning enables the recognition and application of multiplicative relationships, which is essential for problem-solving across STEM contexts (Modestou & Gagatsis, 2009). Together, these reasoning practices provide students with the cognitive tools needed to interpret evidence, evaluate claims, and engage in scientific inquiry (Altindis et al., 2024; Altindis & Fonnger, 2025; Carlson et al., 2002, 2022; Rodrigues & Jones, 2024; Moore, 2025).

Graphical interpretation can be defined as a cognitive process through which students actively extract information, recognize patterns, and construct meaning from visual representations (Altindis et al., 2024; Bowe et al., 2025; Curcio, 1987; Glazer, 2011). The process of graph interpretation occurs through three steps: (1) recognizing visual patterns; (3) translating visual patterns into quantitative relationships; and (3) associating the relationships with the referents (Carpenter & Shah, 1998; Shah et al., 1999). While linear graphs are often manageable, interpreting them becomes more challenging as the graphs become more complex. As students struggle with translating nonlinear patterns into quantitative relationships and with identifying how the rate of change varies across different points of a function domain (Altindis, 2025; Altindis & Fonger, 2025; Altindis et al., 2024; Bowe et al., 2025; Carlson et al., 2002; Jones, 2019; Moore & Thompson, 2015; Moore, 2025; Roth & Bowen, 2001).

Covariational reasoning requires a high level of reasoning, as it can enhance students' comprehension of essential mathematical ideas (Thompson & Carlson, 2017). Covariational reasoning is a cognitive process that involves coordinating the values of two changing quantities and considering how they change in respect to one another (Altindis, 2025; Altindis & Fonger, 2025; Altindis et al., 2024; Bowe et al., 2025; Büchter et al., 2024; Carlson et al., 2002). A study by Carlson et al. (2002) revealed that with CR, students were able to create images of a function's dependent variable changing simultaneously with the imagined change of the independent variable, and in certain cases, images of the rate of change for continuous intervals of a function's domain. However, the students' abilities are sometimes misinterpreted by teachers who possess low-level covariational reasoning. When Zeytun et al. (2010) conducted a study on the mathematics teachers' reasoning levels, they found that some teachers have weak covariational reasoning, which leads to poor predictions about a student's ability. The degree to which covariational reasoning is fundamental in educational systems varies among countries (Thompson et al., 2017). The curriculum and Instruction in the United States, as indicated by Thompson and Carlson (2017), are not developing the covariational reasoning abilities of the students, thus resulting in many weaknesses in core mathematical concepts. Proportional reasoning is the ability to understand and work with multiplicative relationships between quantities (Modestou & Gagatsis, 2009). It involves knowing when two situations are proportional and using multiplication to solve problems instead of simple addition. Martínez Ortiz (2015) found that integrating LEGO robotics with mathematics instruction supported students' proportional reasoning by embedding ratio and proportion tasks within engineering design challenges.

Despite their importance, graphical, covariational, and proportional reasoning have often been studied in isolation rather than jointly as an interconnected set of quantitative practices. There is limited understanding of how these reasoning skills interact to shape students' STEM performance and whether they can serve as predictors of success across courses. This study addresses that gap by examining the role of these reasoning constructs in undergraduate STEM achievement, thereby contributing to both theory and practice in STEM education.

Aim of the Study

This study examines undergraduate students' graphical, covariational, and proportional (GCP) reasoning skill,—as measured by the GCP scale—in the context of introductory STEM courses. Specifically, it examines whether students' GCP reasoning proficiency predicts overall course performance. The two research questions guide this work: (1) Do students' GCP scores differ significantly across Introductory Biology, Introductory Chemistry, and Introductory Chemistry for Engineers? (2) To what extent do students' GCP skills predict their final performance in STEM courses?

Method

This study was conducted at a research-intensive university in the northeastern United States. The university is primarily comprised of White students (82%) and has a greater proportion of female students (56%) than male students (44%). Participants (n=382) were recruited from two sections of an introductory biology course required for life science majors, two sections of an introductory chemistry course required for life science and chemistry majors, and one section of an introductory chemistry course required for engineering majors. Data collected included pre- and post-assessments. There was no intervention for these courses; this is an exploratory study.

The GCP assessment scale consisted of 17 multiple-choice items: six testing covariational reasoning skills (Carlson et al., 2015), five focusing solely on graphical reasoning, and six assessing students' proportional reasoning skills. Each item was scored as correct (1) or incorrect (0). A percentage score was calculated for each student to represent overall proficiency in GCP reasoning skills. These students participated in three courses: Introductory Biology, Introductory Chemistry, and Introductory Chemistry for Engineers - taught by five instructors. Sample sizes varied across instructor–course combinations: Introductory Chemistry for Engineers with instructor A (n = 39/132, 30%), Introductory Chemistry with instructor B (n = 168/483, 35%), Introductory Chemistry with instructor C (n = 75/183, 41%), Introductory Biology with instructor D (n = 32/183, 18%), and Introductory Biology with instructor E (n = 68/181, 38%). These differences highlight the uneven distribution of students across courses and instructors. To address these disparities and variations in grading standards, students' final course grades were normalized. Specifically, scores were ranked within each course, converted to percentiles based on their relative standing, and then standardized using a normal distribution (mean = 0, SD = 1). This process ensured fair and consistent comparisons across courses and instructors, allowing student performance to be evaluated on a common scale.

Validity and Reliability

The content validity of the assessment was established through expert review by STEM education researchers in mathematics, biology, chemistry, and physics, who are subject matter specialists who confirmed the relevance, clarity, and representativeness of the items. In addition, several items were adapted and revised from previously validated instruments, including the *Calculus Concept Readiness (CCR)* assessment (Carlson et al., 2015), which was itself developed through research and expert input and has documented evidence of reliability and validity. Construct validity was further examined through Rasch analysis, with all 17 items demonstrating acceptable fit within the threshold of 0.6–1.4, thereby supporting the internal structure of the assessment. Item difficulty values ranged from –1.866 (easiest) to 2.026 (most difficult), reflecting a broad spectrum of student ability levels. The strong negative correlation (r = –0.999) between item difficulty and the proportion of correct responses confirmed the expected

patterns. Reliability was evaluated using multiple indices of internal consistency, as shown in Table 1. All coefficients fell within the acceptable range of 0.70–0.80, indicating that the assessment demonstrates moderate and dependable reliability.

Table 1. Reliability Measures and Scores

Reliability Measures	Reliability Score
WLE (Weighted Likelihood Estimation) Reliability	0.736
EAP (Expected A Posteriori) Reliability	0.76
Cronbach Alpha	0.77

Results

Differences in GCP Scores Across Introductory STEM Courses

Table 2 below shows the differences in GCP scores across STEM courses. At the course level, students in Introductory Chemistry for Engineers achieved the highest GCP reasoning scores (Estimated Marginal means (EMM) = 59.0, SE = 3.23, 95% CI [52.6, 65.3]). Students in Introductory Biology scored moderately, with a combined mean of EMM = 49.5 across two instructors. In contrast, students in Introductory Chemistry had the lowest performance, with a combined mean of EMM = 43.9 across two instructors. These results indicate that students in Introductory Chemistry for Engineers outperformed their peers in both Introductory Biology and Introductory Chemistry on GCP measures.

Table 2. Estimated Marginal Means of GCP Scores by Course

Course	EMM	SE (range across instructors)	95% CI (range)
Introductory Chemistry for Engineers	59.0	3.23	[52.6, 65.3]
Introductory Biology	49.5	2.44–3.56	[42.9, 58.3]
Introductory Chemistry	43.9	1.56–2.31	[39.4, 47.8]

The analysis of variance (ANOVA) confirmed that the effect of course on GCP scores was statistically significant, $F(2, 379) = 10.27$, $p < .001$. ANOVA assumptions were assessed prior to analysis. Levene's test confirmed homogeneity of variance across groups, and the Q-Q plot indicated that residuals were approximately normally distributed, with only minor deviations at the tails. Thus, both assumptions of homogeneity and normality were satisfied. A Tukey post hoc test further confirmed that students in Introductory Chemistry for Engineers scored significantly higher than students in both Introductory Biology and Introductory Chemistry, whereas no significant difference was found between the latter two courses. GCP reasoning scores were moderately correlated with final course performance ($r = .56$), suggesting that stronger reasoning skills were consistently linked to higher achievement.

GCP Reasoning Skills as Predictors of Course Performance

We checked the normality assumption for the final course score and observed clustering in the data. Then, we examine a series of linear mixed-effects models. GCP pre-assessment scores were included as a fixed effect, while course, instructor, and major were initially treated as random effects. We examined the variability of random effects and conducted model comparison tests. As shown in Table 3, the standard deviations of the random effects for Instructor, Majors,

and Course were generally small, with Course exhibiting the smallest values, suggesting a limited contribution to the overall variability. Bootstrap confidence intervals supported this finding, as the standard deviation for Course was nearly zero while Majors displayed slightly more variability. Given these patterns, we further compared models using both the Likelihood Ratio Test (LRT) and the Permutation-Based Test (PBtest). Results indicated that including Majors as a random effect significantly improved model fit (LRT = 4.893, p = 0.027; PBtest = 4.893, p = 0.019). Therefore, the final model retained Majors as a random effect, while Course and Instructor were omitted to achieve a better-fitting model.

Table 3. Random Effects and Model Selection Results

Effect	Std. Dev. (95% CI) – Normal Approximation	Std. Dev. (95% CI) – Bootstrap	Decision
Instructor	0.059 – 0.305	0.069 – 0.359	Omit
Majors	0.000 – 0.223	(–0.044) – 0.108 (p=.027 *)	Retain
Course	0.000 – 0.191	0.000 – 0.000	Omit
Residual	0.681 – 0.790	0.684 – 0.793	–
Intercept	–0.975 – –0.563	(–1.027) – (–0.579)	–
GCP Score	0.021 – 0.029	0.022 – 0.029	Fixed effect retained

**Note: LRT = Likelihood Ratio Test; PBtest gave similar results (stat = 4.893, p = 0.019)*

We examine the extent to which students' GCP skills predicted final performance in STEM courses, while also accounting for variation across majors. As shown in Table 4, the GCP score was a strong and statistically significant predictor of students' normalized final scores. This indicates that for each one-unit increase in GCP scores, students' normalized final scores increased by approximately 0.025 units, holding other factors constant. Thus, GCP skills demonstrated a consistent relationship with STEM course performance.

Table 4. Final Mixed-Effects Model Estimates Predicting Normalized Final Scores

Parameter	Estimate	Std. Error	95% CI (Lower, Upper)	Variance	Std. Dev.	df	t	p
Fixed Effects								
Intercept	–0.769	0.105	[–0.98, –0.57]	–	–	161.91	–7.29	< .001
GCP Score	0.0248	0.0019	[0.021, 0.029]	–	–	367.09	12.94	< .001
Random Effects								
Majors (Intercept)	–	–	[0.07, 0.37]	0.03	0.18	–	–	–
Residual	–	–	[0.68, 0.79]	0.54	0.73	–	–	–

Discussion

This study revealed notable differences in GCP reasoning among introductory STEM courses. Students who enrolled in Introductory Chemistry for Engineers scored significantly higher than students in both Introductory Biology and Introductory Chemistry. These findings suggest that engineering-oriented chemistry courses may place greater emphasis on reasoning practices or attract students with stronger quantitative preparation. Lower performance in

Introductory Chemistry and Biology courses may point to missed opportunities to engage students with reasoning practices as central tools for inquiry. Prior research has indicated that students often struggle with graph interpretation and rate-of-change reasoning when these skills are not explicitly scaffolded (Moore & Thompson, 2015; Roth & Bowen, 2001). Thus, the results highlight the importance of intentional instructional design that develops quantitative reasoning across STEM disciplines, not only in engineering contexts.

We found that GCP skills were a strong predictor of final course performance across all introductory STEM contexts. The final mixed-effects model indicated that for each unit increase in GCP scores, students' normalized final scores increased by approximately 0.025 units, even after accounting for variation across majors. This finding highlights the crucial role of reasoning skills in fostering academic success in STEM. It aligns with earlier work demonstrating that covariational reasoning is critical for understanding functions, modeling phenomena, and solving quantitative problems (Altindis & Fonger, 2025; Altindis, 2025; Altindis et al., 2024; Bowe et al., 2025; Carlson et al., 2002). The moderate correlation between GCP and course performance ($r = .56$) further reinforces this relationship, suggesting that reasoning proficiency can serve as an early diagnostic indicator of students' potential achievement.

The model also revealed that major-level differences contributed modest variability, while course- and instructor-level effects were negligible. This suggests that students' disciplinary orientation and preparation may matter more for performance than differences in course delivery. However, further research is needed to understand why these differences emerge, how majors influence students' reasoning development, and whether specific curricular pathways or instructional approaches account for the observed variability.

Implications

The results carry several implications for STEM education. First, they emphasize the need for instructional strategies that explicitly develop graphical, covariational, and proportional reasoning across disciplines. Biology and chemistry courses, in particular, may benefit from integrating reasoning tasks into laboratory and lecture settings to promote deeper conceptual engagement. Second, the strong predictive role of GCP skills suggests that reasoning-based assessments could serve as early indicators of student achievement, providing opportunities for targeted support. Finally, embedding reasoning tasks in authentic, applied contexts that are similar to those found in engineering courses may help foster quantitative practices across STEM.

Limitations and Future Directions

While this study provides evidence for the importance of GCP reasoning, several limitations should be noted. First, the data were drawn from a single institutional context, which may limit generalizability. Second, although the study controlled for majors, other factors, such as prior mathematical preparation and demographic variables, were not included. Future research should explore these additional influences, as well as examine how instructional practices within biology and chemistry could be redesigned to support reasoning more effectively. Ultimately, longitudinal research is necessary to investigate how GCP skills evolve and whether targeted interventions can bridge performance gaps across courses.

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Mathematical Mnemonics as Tools for Shaping Concept Images in University Mathematics Tutoring

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In mathematics, the teaching and use of "tricks," known formally as mathematical mnemonics, is commonplace. However, little research has focused on this topic, particularly in university mathematics tutoring. In this study, we investigated when and why undergraduate mathematics tutors implemented mathematical mnemonics in their tutoring practice. Video recordings of online tutoring sessions were collected from eight tutors working at an R1 university in the United States. Of the eight, five were observed using mathematical mnemonics in their tutoring practice. Our results show that these tutors used mathematical mnemonics for three reasons: in response to a student committing an error or demonstrating a misconception, as a response that represents how the tutor reasons about the topic, and as part of a predeveloped curriculum script. We argue that each of these reasons is an implicit attempt by the tutor to reshape a student's concept image of the topic under consideration.

Keywords: Tutoring, mathematical mnemonic, concept image

The teaching and use of "tricks" in mathematics education is commonplace. These tricks are formally known as *mathematical mnemonics*. Miller and Obara (2017) defined a *mathematical mnemonic* (referred to herein simply as a *mnemonic*) as "a visual cue or verbal strategy used to aid initial memorisation and recall of a mathematical concept or procedure" (p. 13). While some mnemonics help with recalling definitions and concepts, others help with the application and interpretation of procedures.

The National Council of Teachers of Mathematics documented that, in many mathematics classrooms, "... too much focus is on learning procedures without any connection to meaning, understanding, or the applications that require these procedures" (National Council of Teachers of Mathematics, 2014, p. 3). Indeed, although many mnemonics can help students correctly answer assessment questions, they can obscure the conceptual underpinnings that make them work, reinforcing the common perspective that mathematics is simply a set of rules to be memorized rather than a set of concepts to be understood (Kalder, 2012; Miller & Obara, 2017). While the use of mnemonics can improve student understanding (Greene, 1999; Mastropieri, Scruggs, & Levin, 1987), over-emphasis on such tools can lead to decreased conceptual fluency (Karp, Bush, & Dougherty, 2014; Kilpatrick, Swafford, & Findell, 2001).

Despite this potential pitfall, mnemonics have been shown to produce positive outcomes for conceptual, as well as procedural, learning when they are accompanied or followed by conceptual instruction (NCTM, 2014). Although mnemonics often allow students to succeed in math classes despite lacking the conceptual understanding that makes them work, they can also act as scaffolds for students until they are developmentally prepared to adopt the conceptual understanding (Karp, Buch, & Dougherty, 2014; Manalo, Bunnell, & Stillman, 2000; Miller & Obara, 2017).

In this exploratory study, we observed online university mathematics tutors sharing a variety of mnemonics with the students they were tutoring. The research team's curiosity about the pedagogical value of these mnemonics led to posing the following research question: What are

university mathematics tutors' implicit goals when sharing mnemonics with students during tutoring sessions?

The Importance of Tutoring and the Influence of Tutors

Tutoring brings with it myriad benefits, including improved course grades (Bloom, 1984; Rickard & Mills, 2018), increased mathematical confidence (Lepper & Chabay, 1988), and improved sense of mathematical identity (Bjorkman & Nickerson, 2019), among others. Indeed, Merrill et al. (1995) observed that, while doing problems is the best way to learn, it is essential that this effort is guided such that learners receive positive feedback when they are moving in a productive direction and corrective feedback when making mistakes.

Citing Vygotsky (1978), Mascolo (2009) argued for the importance of the learner's interaction with a more knowledgeable other and that such interactions "pull the less expert partner's functioning to levels that are higher than that which he is capable of sustaining alone" (Mascolo, 2009, p. 5). Although experienced tutors take very deliberate actions during tutoring sessions to provide individualized instruction for their students (Gallagher et al., 2025; Lepper & Woolverton, 2002), tutors also sometimes proceed through a largely preordained set of interactions known as a *curriculum script* (Putnam, 1987). "A curriculum script is an ordered set of goals and actions for teaching a particular topic – the skills and concepts students are expected to learn and activities and strategies for teaching this material. ... [I]t is the teacher's curriculum script, rather than a model of the student based on detailed diagnosis, that determines the sequence of instruction" (Putnam, 1987, p. 39). Tutors, as well as teachers, can use curriculum scripts where they speak and act in a particular sequence that can fail to take the learner's needs into account.

Concept Image

Tall and Vinner (1981) define an individual's *concept image* for a concept as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Although one's concept image for a concept can be rather expansive, only certain parts of an individual's concept image may be evoked when solving problems. Problematically, however, an individual's concept image need not be accurate or even internally consistent, and the portion of one's concept image that is evoked while solving a particular problem may not be helpful for obtaining a solution. These challenges may lead to struggle or even failure during problem solving.

Individualized tutoring often reveals problematic inaccuracies in students' concept images. Fortunately, concept images are malleable and can be updated with further learning (Tall & Vinner, 1981; Vinner, 1983). Our data suggest that the university tutors in our sample used mnemonics to reshape their students' concept images to bring them more in line with their own, which were seemingly more productive.

Theoretical Framing

For this work, we adopted Merrill et al.'s (1995) conceptualization of *tutoring as guided learning by doing*. A significant body of research demonstrates the benefits of active learning (Abell et al., 2018; Bressoud et al., 2015; Deslauriers et al., 2019; Freeman et al., 2014; Prince, 2004; Theobald et al., 2020), in which students engage in idea generation, problem solving, and practice. While this work appropriately places the onus of knowledge creation on students, much of it neglects the importance of having a more knowledgeable other (Vygotsky, 1978) present during this process. Merrill et al. (1995) noted that, although students' active participation in

problem solving is the key to learning, leaving students to practice without the appropriate supervision and support can lead to floundering, confusion, and frustration, which hinders learning. Mascolo (2009) echoed this point, emphasizing the importance of the teacher's role in an active learning environment: "... the idea that students must actively construct their skills and understandings *for themselves* is not the same as suggesting that [they] must actively construct their skills and understandings *by themselves*" (p. 7). While Mascolo's work focused on classroom instruction, Merrill and colleagues' work is focused on tutoring. Still, both agree that the more knowledgeable other guiding the lesson is a critical component of effective instruction.

Tutoring as guided learning by doing (Merrill et al., 1995) emphasizes the importance of the individualized instruction involved in tutoring. Similar to Mascolo's (2009) views on the importance of the teacher, it has been argued that effective tutoring involves the tutor taking an active role in their student's learning. Merrill and colleagues (1995, p. 316) argued that "rather than letting students solve problems on their own, with occasional advice, tutors carefully monitor the problem solving to ensure that it stays on track and to help direct students back toward a productive solution path when needed." Merrill and colleagues referred to this as "guided learning" and pointed out that this practice allows "students to attain the benefits of learning by doing while avoiding some of the costs" (p. 316).

Methods

The current analysis is part of a larger ongoing project studying different aspects of online mathematics tutoring at the university level. The data set consisted of online tutoring sessions focused on topics ranging from College Algebra through Multivariable Calculus that were recorded at an R1 university tutoring center from January 2022 through May 2024. There were eight online tutors who worked 5-10 hours per week. As the research team members were cataloging the mathematical content of these sessions, they made notes of other interactions they found interesting. The data analysis reported here occurred over five months between the research team of co-authors on this paper.

While initially going through the videotaped tutoring interactions, the research team observed several instances where five of the eight tutors shared "tips and tricks" for being successful on a variety of problem types during their tutoring sessions. These tips and tricks met Miller and Obara's (2017) definition of *mnemonics* provided earlier. The research team decided to focus on these instances of mnemonic use for this study.

One researcher reviewed all videos during the timeframe noted above and identified when mnemonics were being used by the tutor. A second researcher then reviewed the videos that the first researcher categorized; there was 100% agreement on the identification of the mnemonics. The whole research team discussed each mnemonic, decided on categorizations, and came to agreement on the attributes of each category. The third researcher then classified each video into the mutually agreed upon three classifications discussed below in results; note that the categories as not mutually exclusive. Thus, a video could have been identified as falling into more than one classification. Finally, the research team reviewed each classification assigned and discussed until consensus was reached.

Results

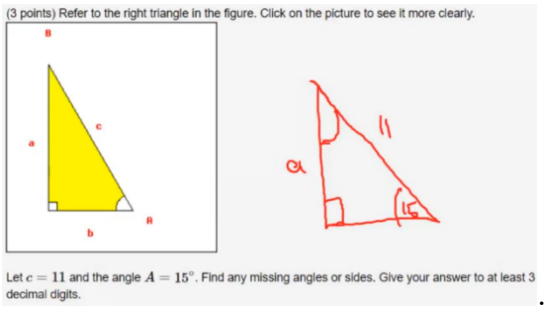
We identified reasons why tutors use mnemonics when tutoring. First, almost all mnemonics were used in response to a student error or misconception. Second, a tutor introduced a mnemonic because it seemed to be how they thought about the topic under consideration. Third,

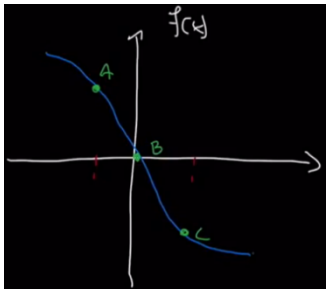
tutors used mnemonics "automatically," even after students had self-corrected and figured out a correct path forward. We now present examples of each of these situations.

Mnemonics as Response to Student Errors and Misconceptions

Mnemonics were almost always used after the tutor saw an error or a misconception. Working with a trigonometry student to find the length of side *a* of a triangle in which the opposite angle and the hypotenuse were known (Figure 1a), Aaron used the mnemonic *SOH-CAH-TOA*. Aaron drew and labeled a picture with the known information and asked the student which trigonometric function should be used to find *a*. When the student suggested cosine, Aaron responded, "I would probably use sine, because remember *SOH-CAH-TOA.* So, sine of 15 degrees would be equal to the opposite side, *a*, divided by the hypotenuse, which is 11."

Another tutor, Bernardo, used a mnemonic for thinking about increasing and decreasing to help a student sketch the graph of a derivative from the graph of a function that was decreasing over an interval from point *A* to point *C* but its concavity changed from concave down to concave up at an inflection point *B* (Figure 1b). The student explained that, starting from the left, the graph of the derivative should be negative and would decrease until point *B* was reached, conflating the notions of *increasing function* and *increasing derivative*, explaining that since the derivative is increasing from point *B* through point *C* and beyond, the graph of the derivative should be drawn in the first quadrant over this interval. The tutor used an analogy of walking on a hill, explaining that in this case, even after passing through point *B*, he would still be "walking downhill" even though "the hill was becoming less steep."





Figures 1a and 1b. Situations prompting mnemonics

Mnemonics as Tutors' Genuine Reasoning

Tutors brought up mnemonics because that is how they, themselves, think about certain problems. More than once, Selena described a mnemonic for determining concavity by drawing eyes on the graph to make happy and sad faces (Figure 2). Selena worked with two students to distinguish between intervals on which a function is increasing and intervals on which the function is concave up. Selena explained, "I'm not gonna lie, I do this really dorky thing whenever I do this – I kinda draw sad and happy faces. And if they're sad, they're concave down. If they're happy, they're concave up." Selena's students found this technique comical, but they immediately began correctly answering questions about concavity by implementing it.

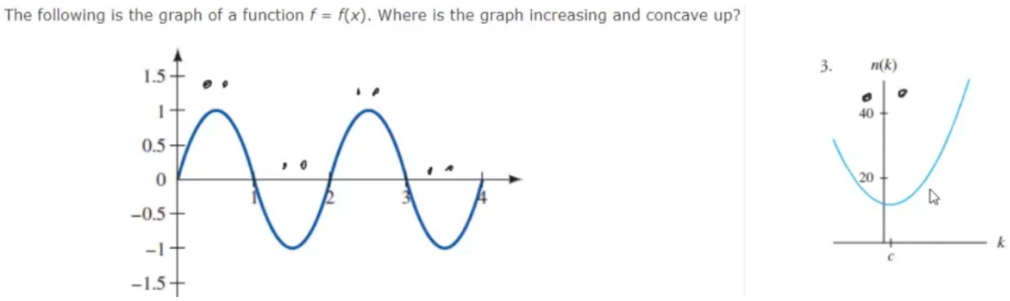


Figure 2. Selena's drawings of eyes to determine concavity of graphs.

Marian seemed to use the mnemonic *rise-over-run* as a command to apply the slope formula. On an algebra review assignment for a calculus class, the student Marian was tutoring had been asked to write equations for two lines (one passing through the origin, and the other with a nonzero *y*-intercept). The student was confused because he had applied the method of dividing a single point's *y*-coordinate by its *x*-coordinate, resulting in the correct slope for the line passing through the origin but the incorrect slope for the other. Marian explained that "You can just find the rise-over-run of that... you just take those two points, and you find their rise-over-run." As she said this, Marian wrote down the algebraic slope formula, that is $(y_2-y_1)/(x_2-x_1)$, and then applied it to the line for which the student had found the correct slope, then used this as a model to repeat the process for the other line. She continued to describe this process as "finding the rise-over-run" throughout the entire tutoring session.

Mnemonics as Curriculum Scripts

Tutors introduced mnemonics even after students detected their own errors and self-corrected, suggesting the mnemonics are part of curriculum scripts the tutors have developed for certain concepts. When asked about a question in which the student needed to determine the sine of an angle whose terminal side lay in the third quadrant, Marian drew and labeled a picture to represent the angle's terminal side and the right triangle it forms with the *x*-axis (Figure 3).

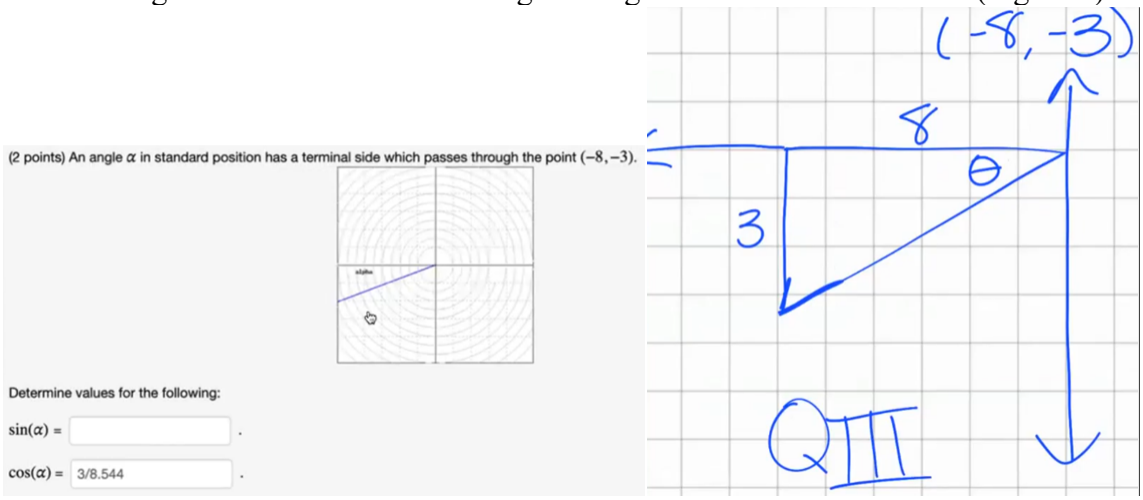


Figure 3. Marian's drawing of an angle with its terminal side in QIII.

As Marian was drawing, her student acknowledged aloud that his mistake was a sign error; he had neglected the fact that the vertical side length should be negative because the terminal side lay in the third quadrant. Marian continued to explain the problem even though the student's

question had been resolved and described the mnemonic *All Students Take Calculus* that she used to identify which trigonometric functions are positive in each quadrant (**A**ll are positive in I, **S**ine is positive in II, **T**angent is positive in III, and **C**osine is positive in IV; see Figure 4).

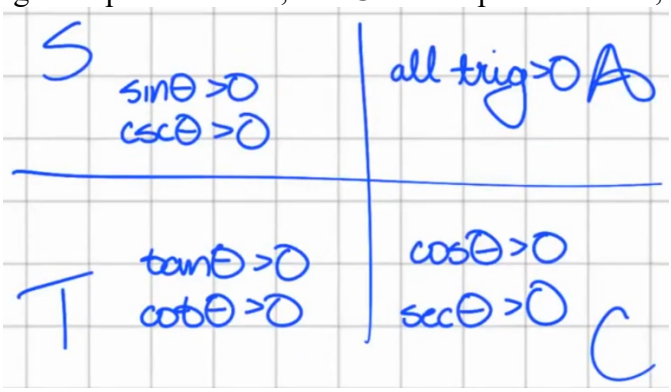


Figure 4. Marian's diagram explaining the All Students Take Calculus mnemonic.

In a session with Carey, a student mistakenly suggested that $a^2 - 4$ should be factored into $(a - 2)(a - 2)$ but quickly corrected himself to $(a - 2)(a + 2)$ after realizing that his initial factoring would yield a positive 4 in the constant term. Although the student had self-corrected, Carey was intent on ensuring the student understood factoring by explaining the "x-method" (Figure 5).

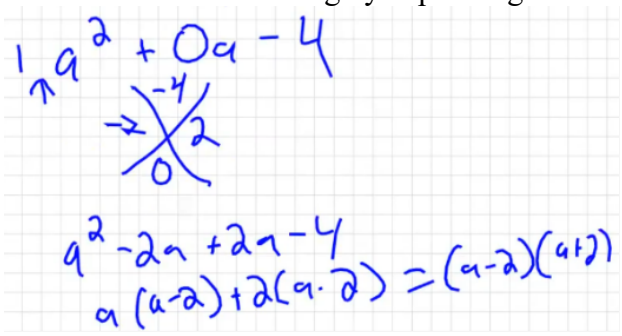


Figure 5. Carey's explanation of the "x-method" used to factor a difference of squares.

Discussion

We found that tutors in our sample used mnemonics for three reasons: in response to the detection of a student error or misconception, because it represented how they think about a concept or problem, and as part of predeveloped curriculum scripts they use for tutoring certain topics. Underlying all of these reasons is an attempt to reshape their student's concept image of the topic under consideration to bring it more in line with their own.

Merrill and colleagues (1995) argued that one of the major benefits of working with a tutor is the tutor's ability to curtail students' unproductive activities. When the students in our data demonstrated a misunderstanding, tutors' uses of mnemonics "direct[ed] students back toward a productive solution path" by providing students with a useful way of conceptualizing the topic, gently suggesting that the student should reconsider their current approach and update their way of thinking to be more in line with how the tutor thinks. Furthermore, as more knowledgeable others (Vygotsky, 1978), tutors modeled modes of thinking that they knew to be reliable and encouraged their students to adopt them, even when students identified and corrected their own mistakes.

Considering the use of a mnemonic as part of a curriculum script, we echo Mascolo's (2009) suggestion that, although students must build their own knowledge, they need not do so on their

own. Curriculum scripts provide a structured framework in which students work on specific kinds of activities designed to help them construct specific knowledge. Although it may not have been necessary for our tutors to describe mnemonics in these situations, as their students had already detected their own errors and self-corrected, they chose to share them to reduce the likelihood of such student mistakes recurring in the future by updating or reinforcing concept images already held by their students.

Implications and Directions for Future Research

This research expands the current body of research related to the use of mathematical mnemonics, specifically by addressing tutors' motivations for teaching mnemonics to the students they assist. Although the students in the sessions we observed seemed to appreciate and derive benefit from the mnemonics tutors showed them, it is essential that the research community ascertains exactly what benefits and drawbacks students might experience from learning and applying such mnemonics. It has been noted that the use of mathematical mnemonics (and, indeed, much of modern mathematics teaching) places a heavy focus on getting the right answers to problems, often at the expense of conceptual understanding (Kalder, 2012; Miller & Obara, 2017; NCTM, 2014). However, it has been suggested that mathematical mnemonics can lead to improved conceptual understanding, either by using them as a scaffold building toward conceptual understanding or by teaching mnemonics and concepts side-by-side (Karp, Buch, & Dougherty, 2014; Manalo, Bunnell, & Stillman, 2000; Miller & Obara, 2017; NCTM, 2014). In our review of the data presented here, we hypothesized that the use of certain mnemonics, such as "rise-over-run" or the use of happy and sad faces to determine concavity, might foster conceptual understanding, while other mnemonics, such as "ILATE" (an abbreviation of *inverse trigonometric*, *logarithmic*, *algebraic*, *trigonometric*, and *exponential* used by Marian but not discussed in this paper) for integration by parts, might obscure or at least not develop concepts. Future studies of mathematical mnemonics need to explore the effects of individual mnemonics on conceptual and procedural fluency among mathematics students at various levels of their mathematical education.

Acknowledgments

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Analyzing a Student's Observations and Conjectures About Markov Chains: Identifying Scheme Elements and Their Instrumentation of Digital Figures

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This study investigates how a first-year undergraduate mathematics student engaged with a set of digital interactive figures to explore properties of stochastic matrices and Markov chains. Using the frameworks of instrumental genesis and Vergnaud's theory of conceptual fields, we analyzed the student's interaction in terms of their observations and conjectures, two key stages in our Observation, Conjecture, Proof, Theorem (OCPT) pedagogical model. Through task-based interview data, we identified elements of the student's emerging conjecturing scheme, including rules-of-action and theorems-in-action that illustrate their process of sensemaking and experimentation. We argue that this scheme development evidences the student's instrumentation of the figures into a functional instrument for mathematical reasoning.

Keywords: Linear algebra, CAS, eigentheory, conjecture, scheme, instrumental genesis

Linear algebra is a foundational discipline within mathematics, underpinning a wide range of theoretical and applied fields. Its concepts and techniques are essential in areas such as data science, computer graphics, quantum mechanics, machine learning, and systems modeling. Both iterations of the Linear Algebra Curriculum Study Group (Carlson et al., 1993; Stewart et al., 2022), convened by prominent linear algebra educators and education researchers, have proposed agendas and goals for the effective and beneficial teaching of linear algebra. Those recommendations include, among others, a need for the study of linear algebra to be integrated with and responsive to digital technology, as well as to be attentive to the formal theoretical training that has traditionally been reserved for advanced courses in linear algebra.

In keeping with these recommendations, we are interested in developing and advancing a pedagogical model of how students, even at the introductory linear algebra level, can engage in meaningful and authentic formal mathematical practices. We have called this model the Observation Conjecture Proof Theorem (OCPT) model, in contrast to the traditional textbook Definition-Lemma-Proof-Theorem-Proof-Corollary (DLPTPC) mode of content delivery, criticized by Uhlig (2002). OCPT seeks to place the mechanisms of discovery, investigation, and reasoning before the presentation of polished textbook theorems. Lin et al. (2012) define the task of mathematics and the sciences as "first to observe phenomena, then to explain them, and finally to predict." Implicit in this definition is the formalizing process by which the observation of a phenomenon or relationship is formalized, conjectured, into a mathematically coherent, provably true or false statement. This process of conjecturing, and its interplay with observation, is nuanced; example generation and experience are powerful and necessary skill to prompt insightful conjectures, even as examples cannot be accepted as sole proof (Lesseig, 2016; Lynch, Lockwood, & Ellis, 2024), except in the case of counterexamples. The sub elements of the conjecturing activity and the role of observation therein have been described as motivating an initial conjecture, boundary testing on the conditions and assumptions, and encountering structures that may lead to a justification (Lannin et al., 2006).

We are currently primarily interested in facilitating and creating digital environments for students to engage in processes of observation and conjecture, which we call interactive figures.

Our previous collection of interactive figures focused on fundamental concepts of linear combinations, determinants, and eigentheory (Peffer, McDonald, & Stewart, 2025). Markov chains are a critically powerful field in linear algebra and have been recognized as a salient and meaningful application by which to explore concepts in linear algebra and specifically eigentheory (e.g., Harsy & Smith, 2024). However, there is little other instructional or theoretical treatment of the teaching of Markov chains in the literature. In this paper, we will discuss a task-based interview conducted using a new pair of interactive figures targeting the properties of stochastic matrices and Markov chains. We will pay attention to the actions and processes of observation and conjecturing enacted by the student, and the underlying mental schemes involved that enable the student to make use of the interactive figures to explore those properties of Markov chains.

Theoretical Perspectives

With an interest in understanding how the interactive figures facilitate students engaging in the mathematical activities of observation and conjecturing, we first turned to the framework of instrumental genesis to describe this interaction. The theory of instrumental genesis has been adopted in mathematics education literature to describe how students use tools and technology as adaptable problem-solving instruments (Artigue, 2002; Guin et al., 2005). A common, prototype application of instrumental genesis is the analysis of students adapting their algebraic and graphical abilities with a graphing calculator (Trouche, 2020), wherein the calculator (or any other tool) known as the artifact acts to reciprocally influence and be influenced by the learner.

Of particular interest to this study is the process of instrumentation, whereby engagement with the artifact in a goal-oriented, problem-solving scenario influences a student to adapt their cognitive, analytical processes to acknowledge and employ the artifact's utility, thus generating the psychological entity of an instrument. In diagnosing and analyzing the process of instrumentation, whose product is a cognitive reorganization of a student's mental models, it is standard to employ schemas to capture this structural adaptation (Turgut & Drijvers, 2021; Gueudet et al., 2022). By identifying behavioral and analytical features of a schema and distinguishing the ways in which a student has adapted or developed a schema to include and employ an artifact, researchers can investigate and qualify the adaptation of the instrument.

Relatedly, Vergnaud (2009) establishes a theory of conceptual fields, described as a class of situations and problems, together with the schemes and individual constructs and employs to address these situations, and the modes of representation that the individual operates and communicates by. While we might envision the model of OCPT as a networking of conceptual fields that coalesce to constitute a mathematician's formal activity, in this paper we will focus specifically upon Vergnaud's compartmentalization of scheme, which dissects schema into four levels of activity, which are described as (p. 88):

- The intentional aspect of schemes involves a goal or several goals that can be developed in subgoals and anticipations.
- The generative aspect of schemes involves rules to generate activity, namely the sequences of actions, information gathering, and controls.
- The epistemic aspect of schemes involves operational invariants, namely concepts-in-action and theorems-in-action. Their main function is to pick up and select the relevant information and infer from it goals and rules.
- The computational aspect involves possibilities of inference. They are essential to understand that thinking is made up of an intense activity of computation, even in

> apparently simple situations; even more in new situations. We need to generate goals, subgoals and rules, also properties and relationships that are not observable.

We wish to employ these components of scheme then to analyze student interactions and their products to describe the forming and adapting schemas at play. As mentioned, the mathematical activity that we are most interested in with these figures is that of observation and making conjectures, and thus we seek to identify the generative rules-of-action that students develop or display in interacting with the figures, and subsequently to interpret the cognitive concepts- and theorems-in-action as they are guiding the student's rules-of-action. In doing so, this scheme documentation should serve to identify the emergent instrumentation of the interactive figures by the student to participate in the described process of observation and conjecture.

As such, our research questions for this study are: How did the student in this study engage with the interactive figures to make sense of conceptual questions regarding Markov chains? What scheme aspects, in the context of observation and conjecture, were displayed?

Method

To collect evidence of scheme relating to observation and conjecture, we designed a qualitative case study consisting of four task-based interviews that were conducted at a public land-grant research university in the USA. Volunteer participants were recruited immediately following their final examination in a first course in linear algebra. These students were all concluding their first year of study, with aspiring majors in mathematics and engineering.

These interviews will constitute data in the first author's PhD thesis, and in this paper, we will discuss one such interview with Alice (pseudonym). At the time of the interview, Alice had just completed her spring semester of her first year of university, intending to study mathematics and having received an A in her first course in linear algebra. The course covered systems of equations, span and linear independence, linear transformations, matrix algebra and invertibility, subspaces of $\mathbb{R}$, determinants, eigentheory, and orthogonality. We chose to present Alice's interview analysis because Alice was entirely unfamiliar with our interactive figures, she had no experience with Markov chains in her linear algebra coursework or other classes, nor had she taken any advanced proof-based mathematics courses.

The materials for the interview included the two digital interactive figures built in Mathematica concerning properties of stochastic matrices and Markov chain iteration. The figures were accompanied by a six-question assignment task mimicking a potential homework. Two questions, Questions 3 and 6, were of particular interest to this study as they asked students to make observations of the properties of stochastic matrices and sequences of Markov chain iterates via the provided interactive figures, though in this paper we will focus exclusively on question 3:

> Question 3: The matrix you have created is called a stochastic matrix: that is, it is a matrix where each column is a probability vector whose entries must be non-negative and sum to 1.0 (probability 100%). Using the first interactive figure, what patterns do you notice about the eigenvalues and eigenvectors of stochastic matrices compared to non-stochastic matrices?

The remaining four questions included conceptual and computational questions involving constructing and analyzing a Markov chain in context of a 2-population dynamic application and

were intended to motivate or build from the student's observations. The assignment also provided relevant working definitions such as for probability vectors, stochastic matrices, and Markov chains. Furthermore, the students were provided with a copy of the course textbook and a calculator for their convenience. The student was prompted throughout the interview to proceed as if they were completing this assignment as regular homework and were thus permitted to use any other standard resource they would consult in their own time, including internet sources.

The data collected for this interview included transcribed audio and video recording of the subject, including screen recording of their work with the interactive figures. As well, the written responses of the student on the assignment document or scratch paper were preserved. At the end of the interview, the students were given a brief exit survey asking what else they might have done in the pursuit of this assignment were it a real assignment, how strongly they believed in the validity of their observations made, and for feedback and user sentiment on the interactive figures. A brief analysis of Alice's observation confidence will be included in the Results section.

In order to target a schematic notion of conjecturing, Alice's interview data was individually analyzed via thematic analysis and open coding (Strauss & Corbin, 1998), by identifying novel propositions or formulas that arose as answers to the above question 3, interpreting their testimony and recorded actions for the generative rules-of-action being followed, and subsequently for the implied cognitive concepts-in-action and theorems-in-action informing these decisions, as prescribed by Vergnaud's theory of conceptual fields. Furthermore, these scheme elements were corroboratively identified as being technology-integrated, in actively involving or relating to the use of interactive figures as a digital artifact. In this way, we may show evidence for the student's incipient instrumentation of the interactive figures involving their schemes of observation and conjecture.

Results and Discussion

Alice's Conjectures of Properties of Stochastic Matrices

We begin by looking at Alice's observations regarding the properties of stochastic matrices via Interactive Figure 1. An example of Interactive Figure 1 as it appeared in the interview (Figure 1), as well as Alice's response to question 3 regarding her observation of properties of stochastic matrices and their powers (Figure 2) are shown below. In the interactive figure, a procedurally generated (stochastic or not) matrix A is provided, as well as its power A^k. For both matrices, the figure displays whether the matrix is stochastic, its eigenvalues, and its eigenvectors. The controls of the figure allow the user to vary the size of A, the exponent k, and to select generating a new stochastic or non-stochastic matrix A.

Alice's response to question 3 contains several conjectures. First, Alice has stated that all stochastic matrices have an eigenvalue equal to 1, though makes no qualification on the other eigenvalues. Next, she has conjectured that the eigenvector associated with eigenvalue 1 has entries that sum to 1, and further remarks that this aligns with a probabilistic interpretation, given that question 3 also defines a probability vector as having non-negative entries that sum to 1. It is worth noting that the figure was specifically configured to display an entry sum of 1, since Mathematica otherwise defaults to normalize eigenvectors with respect to the Euclidean norm. However, Alice does not comment on the potential of scalar multiples of an eigenvector to allow different vector sums for otherwise equivalent eigenvectors.

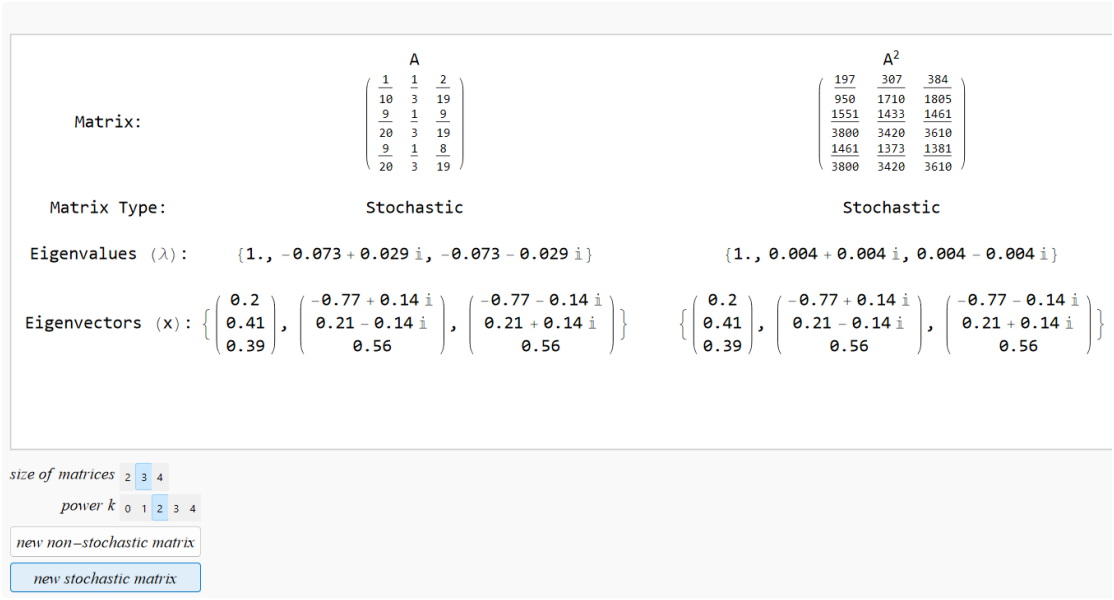

Figure 1. Interactive Figure 1: Properties of stochastic matrices and their powers.

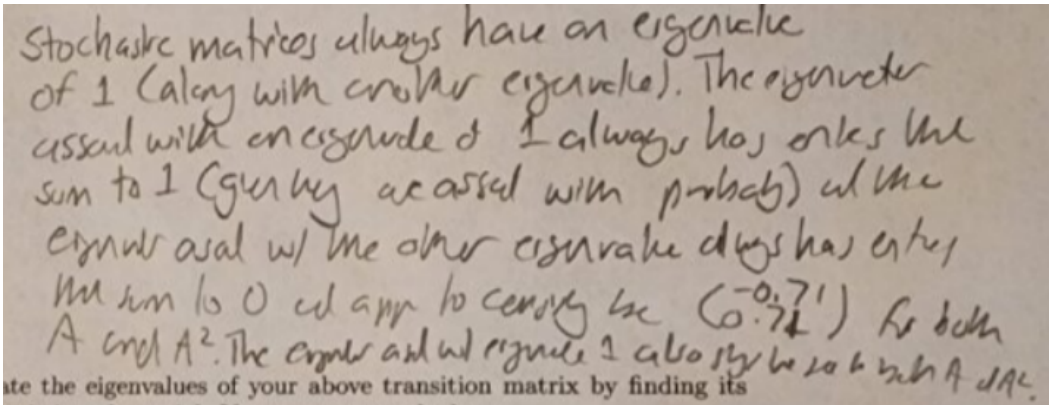

Figure 2. Alice's response to question 3.

While Alice's above response includes other meaningful conjectures, we'll now turn to the interview transcripts to analyze Alice's process of arriving at these ideas.

Alice's Scheme Elements of Observation and Conjecturing

Given free reign over the interview materials, Alice chose to inspect the interactive figures before reading the assignment and the instructions to make mathematical observations via the figures. An excerpt of the interview follows:

Alice: Interesting. These are numbers that don't look very fun to work with. It's probably my first impression. (Alice generates several new stochastic matrices). It's like they all have one. It's my first impression on that.

Interviewer: All have one…?

Alice: As an eigenvalue. (Alice alternates generating several stochastic and non-stochastic matrices). Just interesting for the stochastic matrix.

Immediately from this interaction, we identify several rules-of-action that Alice is employing to inspect the interactive figures to arrive at these mathematical ideas, such as "I look for

convenient or otherwise remarkable phenomena" and "I look at several examples of a situation to draw parallels." From these rules of action, Alice was able to draw forth observations such as the eigenvalue 1 property of stochastic matrices. However, she did not make some less convenient observations such the eigenvalue 1 is the largest eigenvalue of a stochastic matrix, which align her comment that the other eigenvalues are "numbers that don't look very fun to work with."

Thus, it is reasonable that Alice has, by way of the interactive figures, constructed mathematically coherent statements about the properties of an otherwise unfamiliar object, stochastic matrices. However, Alice goes on to display other rules-of-action that display a more complex conjecturing activity. After qualifying her statement about the eigenvalue 1 of a stochastic matrix, Alice alternates generating stochastic and non-stochastic matrices, wherein she does view examples of non-stochastic matrices with leading eigenvalues significantly larger or smaller than 1. We might interpret this activity as following several other rules-of-action such as "I look for conditions under which a conjecture does or does not hold" or "I return to the interactive figures to gather additional evidence in support of a conjecture." From these and the preceding rules-of-actions, we interpret several theorems-in-action, invariant logical propositions that Alice holds to be true and to be motivating her actions:

1. A mathematical conjecture should apply to a specific class of objects (e.g. stochastic matrices).
2. A (universally quantified) mathematical statement should be true in all cases; additional evidence may support a statement but does not prove it.
3. The interactive figures are a source of data and examples of mathematical phenomena.

The first theorem-in-action is evidenced by Alice's intentional distinction between the eigenvalues of stochastic and non-stochastic matrices; her alternating matrix generation implied an experimental phase of cross-examination wherein she became increasingly convinced that the presence of eigenvalue 1 was unique to stochastic matrices. This second theorem-in-action, given by Alice's recurrence of viewing multiple examples even after stating a conjecture, is further supported by her exit survey. Alice was asked about her confidence in her conjectures, including specifically the appearance of 1 as an eigenvalue of stochastic matrices. Alice responded:

> I would say I feel somewhat convinced, like I think experimentally, because I pushed the buttons a lot and I was kind of messing with it for a while. I would say that I think I'm like, at least anecdotally, I feel like that's true, but I would want to have like more explanation of what's happening probably before I'd be fully like, this is definitely what's going on. Like I'd want to go to class and have the teacher be like, this is the result you should be getting, or this is like the reason behind why a stochastic matrix has that result.

This excerpt highlights Alice's cognitive relationship between an observed relationship and the justification of that fact. Even though her response indicates a willing appeal to authority, referring wanting the teacher to tell her "the result [she] should be getting," we still gather that Alice recognizes the observations and examples gathered via the interactive figures as providing compelling anecdotal evidence for the truth of her conjecture. This observed cognition about the role of evidence in conjecture and proof further aligns with the literature on the aspects of conjecturing activity and the role of example generation in forming new ideas.

Finally, in the third listed theorem-in-action, we identify Alice's demonstrated relationship to the interactive figures, namely that of reliable example generation and a source of data for her mathematical experimentation. Beyond Alice not seeking additional sources of stochastic matrices with which to make observations, Alice can be seen to make targeted, intentional

actions via the interactive figures. These include successively generating examples to search for commonalities and patterns and alternating generating stochastic and non-stochastic matrices to test for differences. By these intentional, mathematically reasoned actions taken within the interactive figures, Alice shows evidence of having begun to instrumentate the interactive figures for engaging in the mathematical activity of making observations and forming conjectures.

Alice's Experimentation with the Definition of Stochastic Matrix. Alice had first looked at and begun to hypothesize about stochastic matrices before she read the definition of stochastic matrix provided in question 3. Her first action after reading the definition of stochastic matrix, as well as the sub-definition of probability vector, was to return to earlier matrix computations and visually check if they satisfied those definitions. Thereafter, seemingly primed to search for probability vectors, Alice returned to the interactive figure as instructed by the problem and was quick to make the following new observations:

> These ones are interesting (gesturing to the second eigenvector of A) because it's well these add up to 0 and these (gesturing to the first eigenvector of A and A2) both add up to 1. Yeah, so adding down this representative eigenvector associated with the eigenvalue of 1 sums to 1. And then for the other eigenvalue, the –0.139, the adding up the entry sums to 0... I'm assuming that they [the eigenvector corresponding to eigenvalue 1] because they're associated with probability like the matrix is.

Armed with a new conceptual tool (in this case, that of probability vectors), we may once again describe several rules-of-action Alice is employing, such as "I test a new definition with examples," "I interpret situations in terms of newly acquired language," and "I ascribe meaning to my observations based on definitions and terminology." These rules-of-action, centered on her activity involving a definition that she accepted, mirror earlier rules-of-action she employed when approaching a conjecture that she formed herself. These common features include testing via new examples, including examples that do not fit her conjecture or definition, and collecting her exploratory data from the interactive figures. This example of exploratory action surrounding the definition of stochastic matrices thus aligns with our previous investigation of Alice's conjecturing scheme aspects.

Concluding Remarks

We have observed, in agreement with literature in student conjecturing, several facets of Alice's conjecturing activity, framed as aspects of her observation and conjecturing scheme. These aspects included both generative rules-of-action by which Alice directed her activity, and the interpreted epistemological theorems-in-action that guided her rule system. Furthermore, we made explicit the role of the interactive figure in Alice's pre-proof activity, in part by referencing her direct actions with the figures in her rules-of-action. In doing so, we have evidenced that Alice was successful in some qualities of instrumentating the interactive figures, specifically for the purpose of engaging in mathematical observation and conjecture.

While we have referenced an Observation Conjecture Proof Theorem pedagogical model of promoting student proof activity, the complete explication of this model and the component scheme descriptions of observing, conjecturing, and proving remain a larger goal of the thesis work and beyond. As such, further analysis of this interview data, as well as the remaining interview data, is necessary and planned. Based on this initial analysis, the first and second authors are confident in the figures' ability to encourage productive activity and plan to incorporate them into their teaching.

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Conceptual Blending as a Problem-Solving Tool in Topology

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The topic of topology is an important and unique course in advanced mathematics that carries two simultaneous complexities: logical proof and spatial reasoning. While proof and problem-solving are two areas that are well documented in mathematics education research, topology itself is a fairly rare topic to appear in the literature of mathematics education written in English. To address the gap in the literature, this paper analyzes a study in which two students attempt a topology question in order to understand their problem-solving techniques and how they leverage their various forms of thinking. We networked the theories of Tall's Three Worlds of Mathematical Thinking and Cognitive Blending to describe the participants' behaviors. The results showed that students leveraged a blend of their thinking to progress through the proof.

Keywords: Topology, problem-solving, proof, Tall's Three Worlds, cognitive blending

Introduction

Topology is an important topic in the world of advanced mathematics, with James (2006) claiming, "Topology, for many years, has been one of the most exciting and influential fields of research in modern mathematics. (p 1)" Often said to be "rubber-sheet geometry", undergraduate courses in topology are primarily concerned with the heavily set-theoretical definitions, theorems, and properties. It is a unique course in that context, as it is a heavily proof-focused topic that is supported by a significant amount of spatial reasoning. Even so, there is a lack of educational research on topology. While the topic was used in developmental psychology (Piaget & Inhelder, 1967) and is a field often brought into studies concerning pre-university math education, research that focuses on RUME in topology is scarce among the literature available in English (Vizcaíno et al., 2023).

Literature Review

Educational research in topology is still in its infancy. While researchers have had some interest in topology, the topic is often used as a setting rather than being central to the study. For example, Hicks et al. (2023) attempted to develop a model for "abstraction through analogical reasoning" (p. 807) and used topology as an example to demonstrate part of their model. Additionally, Clark et al.'s (2017) study focused on the incorporation of "primary source projects" into a topology course. The authors' motivation for doing so was based on the idea that original sources for mathematical concepts framed the issues the mathematicians were facing for the students so that they could better understand why the concepts were developed in the first place. Topology was chosen for this study as it was the course the researcher was teaching at the time. On the other hand, some studies have topology near the research interests. Bardelle (2011) explored students' understanding of the topological concepts of finiteness and boundedness with metric spaces, claiming, "everyday use of mathematical terms combined with a lack of coordination of different representations proves to be a hindrance in the conceptualization of basic properties of subsets of Euclidean spaces" (p. 1). Additionally, O'Brien (2023) establishes the connection between the topology of knot theory and tapestry weaving as she claims that those engaging in tapestry weaving internalize and embody diagrammatic instructions in the same way that a mathematician does with topological diagrams.

For papers that are written in English, we have identified three doctoral dissertations in recent years that are principally concerned with education in topology (e.g., Cheshire, 2017; Gallagher, 2020; Berger, 2021). Cheshire (2017) examined the development and adaptations of concepts that students previously encountered as they make sense of the new version of those ideas in the context of topology, primarily with the topic of continuity. In a similar vein, *Reopening the Case of Schema: A Topological Perspective* is a doctoral thesis that seeks to understand and describe students' adaptations to their bodies of knowledge and approaches to problems as they encounter concepts in topology (Berger, 2021). Both studies found that students may struggle with increasingly abstract topics, but they are generally able to adapt to new problems. Another thesis seeks to understand students' use of examples, gestures, and problem-solving strategies to answer questions in topology (Gallagher, 2020). This work demonstrates and describes the ways that students use verbal, gestural, and visual ideas as tools in topology as well as parts of their own thinking. Gallagher found evidence to suggest that it is useful for students to associate their mathematical concepts with diagrams in order to manipulate them and search for important components of the concepts, concluding that teachers should foster students' intuition. All three of these studies referred to the importance of the nature and mechanisms of students' internalization of topological concepts and how they learn to think about them. These three were all empirical studies that included undergraduate students solving problems in topology, and Berger also included graduate students and a research mathematician.

As is evident by the nature of the topic, proof is a fundamental part of topology. To that end, it is necessary to understand the state of the literature on proof in mathematics education research. The study of proofs, proof writing, and the learning of it is now a relatively well-researched area of mathematics education (Mulhuish et al., 2022; Stylianides et al., 2024). According to Selden and Selden (2013), students in higher level math courses have difficulty producing proofs, partly since proofs outside of an intro-to-proofs course have their own content that students must adapt to while trying to structure their proofs. Selden and Selden categorized a written proof into two parts: the formal-rhetorical part and the problem-centered part. The formal-rhetorical part is that which involves the logical structure of the proof. The problem-centered part consists of the content in the proof oriented around the mathematical constructs and concepts at hand. Thus, to understand how students approach proofs, it is necessary to understand problem-solving itself.

Theoretical Framework

The framework for this study is adapted from Schoenfeld's (1985) problem-solving, Tall's (2013) Three Worlds of Mathematical Thinking, and the theory of Cognitive Blending (Fauconnier & Turner, 2002).

Problem-solving in mathematics is a well-researched topic that has had an influence on the educational practices of teachers for many years (Schoenfeld, 1992), and there are four components of problem-solving outlined by Schoenfeld (1985). The notion of *heuristics* is used to describe the problem-solving strategies that students employ, with the idea of *control* referencing how different heuristics and content *knowledge* resources are used. The subtle concept of *belief system*s also plays a role, as Schoenfeld (1985) claimed, "...their beliefs about mathematics –consciously held or not– establish the psychological context within which they do mathematics" (p. 14). Schoenfeld's problem-solving lens provides a context to ask about students' thought processes in topology.

Tall's Three Worlds of Mathematical Thinking describes the embodied, symbolic, and formal worlds as the three forms of thinking in mathematics (Tall, 2013). The embodied world is

that which is influenced by one's perception of the physical world, involving notions such as images, haptics, and motion. The symbolic world involves symbols and operations on them, as often seen in the context of algebra. The formal/axiomatic world describes the thinking dealing with theorems, definitions, and the structured logic-focused concepts of mathematics. This theory enables researchers to describe the different behavior of students as they work through proofs. However, it is worth noting that it is not accurate to the theory to take these worlds as separate, and it is not accurate to suggest that one can move between the worlds of thinking. Instead, the worlds are all descriptions of different manifestations of a given topic as it exists cognitively, and one does not "move" amongst various qualities of a given object. Rather, students can be observed using the different worlds of thinking in conjunction with each other, which demands a theory to describe such a blending of ideas.

The theory of Cognitive (or conceptual) Blending refers to a supposed fundamental cognitive tool of reasoning that involves a person incorporating two worlds of thinking into a single realm to gain further understanding from it (Fauconnier & Turner, 2002). There are two descriptions of a blend: single-scope and double-scope (Bing & Reddish, 2007). Single-scope blending occurs when one world provides framing/information to make sense of the other without any interplay; this is the case when understanding "time flies like an arrow," as one's understanding of arrows is framing a description of time through one's imagination of an arrow in flight. Yet, the phrase does not influence the way one would think of an arrow. Double-scope blending is more advanced, as ideas from both worlds are at play and influencing cognitive creation. It can easily be observed in the context of language that the spoken and written components are distinct yet have great simultaneous influence on each other, creating a double-scope blend when they are used in conjunction.

By networking these theories, we developed a framework that enables a description of how students use different forms of thinking together in their problem-solving by blending them. The research questions to lead this study are:

- What heuristics do students employ as they attempt topology proofs?
- Did blending of the worlds of mathematical thinking occur in students' attempts at solving problems in topology? Was it a useful approach for problem-solving in topology?

Methods

This case study took the form of a single collaborative, structured problem-solving session where two graduate students were presented with three topology problems. The collaborative aspect of the design was intended to encourage the articulation of their thinking as they worked together. The participants of this study were two early graduate students in the researchers' mathematics department. Potential subjects were selected based on their proximity to the beginning of their time in graduate school and their responses to the opportunity. The study involved working on a whiteboard together, and it was audio- and video-recorded. The participants were also asked to construct proofs. The researcher (first author) wrote a problem on the board for the participants to attempt one at a time. Their attempts were examined, though the study was not focused on evaluating the accuracy or validity of their proof or the function of their teamwork abilities; rather, the principal focus was on the nature of their thinking and problem-solving, so they were encouraged to articulate their thoughts. The three topology problems are depicted in Table 1 alongside their motivation for inclusion in the study. These problems were selected to encourage different kinds of reasoning and thinking, thereby gathering

a variety of data from the students. This paper focuses on the students' attempt at the second question. The authors are in the process of analyzing the data from the students' attempt at the other questions for future studies.

Table 1. Interview problems and their reason for selection.

Topology Problem	Intention for Selection of the Problem
1. Provide a metric on (-1,1) that induces the usual topology but is complete as a metric space.	The most direct solution involves making a particular connection between spaces that is often associated with a graph, and the notion of a metric is associated with the idea of distance. Thus, the question is heavily influenced by the embodied world of thinking.
2. Determine whether R imbued with the lower-limit topology is path-connected.	Path-connectedness is typically considered in cases of topologies that can be perceived, yet the space in question is not a manifold that can be imagined easily. This might motivate the students to try to work in the embodied world, yet the proof does not directly follow from that perception.
3. Prove that (0,1) with the usual topology is not compact.	Compactness of a topological space is an abstraction of the notion of compactness in metric spaces, which typically benefits from embodied thinking.

After the interview, a transcript of the conversation was created, and images from the video were used to examine the work that the participants did on the board. This transcript of the data was coded with four themes in mind: Embodied, Symbolic, and Formal worlds of mathematical thinking, as well as Blending. The first three are in line with descriptions from Tall's Three Worlds of Mathematical Thinking, while blending was used to identify moments where the participants were considering different worlds of thinking in conjunction to elucidate a concept or make progress in their proof, according to the theory of cognitive blending. This is distinct from the students bringing up different forms of thinking simultaneously without engaging in any detectable interaction between them. Moments of potential blending were identified when different forms of thinking overlapped or followed each other. NVivo qualitative data analysis software was used to examine and code the video along with its transcript. These codes were analyzed further to determine whether they could be considered evidence of a cognitive blend.

Results and Discussion

The two students began this problem with the correct intuition that the space is not path-connected. They started their approach by defining each term they saw, beginning with their understanding of the lower-limit topology. They handled that concept by writing out the form of the basic open sets (intervals of the form [a, b), with a, b real) as seen in Figure 1. Then, to describe path-connectedness, Student 2 stated, "between any two points we can make a path," and Student 1 replied that he thought a path needed to be "between sets." So, Student 2 drew an image of a path in a space by drawing a circle, labeling points x and y, and then connecting those points with a curving line, then verified that path-connected means "between any two points we can make a path?" Student 1 then affirmed this, and Student 2 drew a second circle as he mentioned the notion of the components of the space. This prompted Student 1 to mention "since

connected implies path-connected…" which brought them to a discussion about the proper direction of the implication between those two properties. Ultimately, they agreed, mentioning the topologist's sine curve as the example, that path-connectedness implies connectedness, with Student 2 then writing on the board "C → PC," the wrong implication. They then considered whether the space is connected, with Student 1 having believed it is. This motivated students to focus on their definition of path connectedness, as they pondered what the definition of a path involves. After they defined a path, they revisited the correlation between connectedness and path-connectedness, realizing the mistake on the board. After correcting it, they realized that if they can find a separation of the space, then the space must not be path-connected. Student 2 suggested they could assume it is and try to arrive at a contradiction, with Student 1 then arguing that they could just consider what sets are open first to avoid that level of work. They began to ponder what the open sets in this space look like, with Student 1 having quickly decided he does not think that the space has a separation. They did not perform any detectable set theory, as they only wrote down the basic open sets before deciding that there was no open partition.

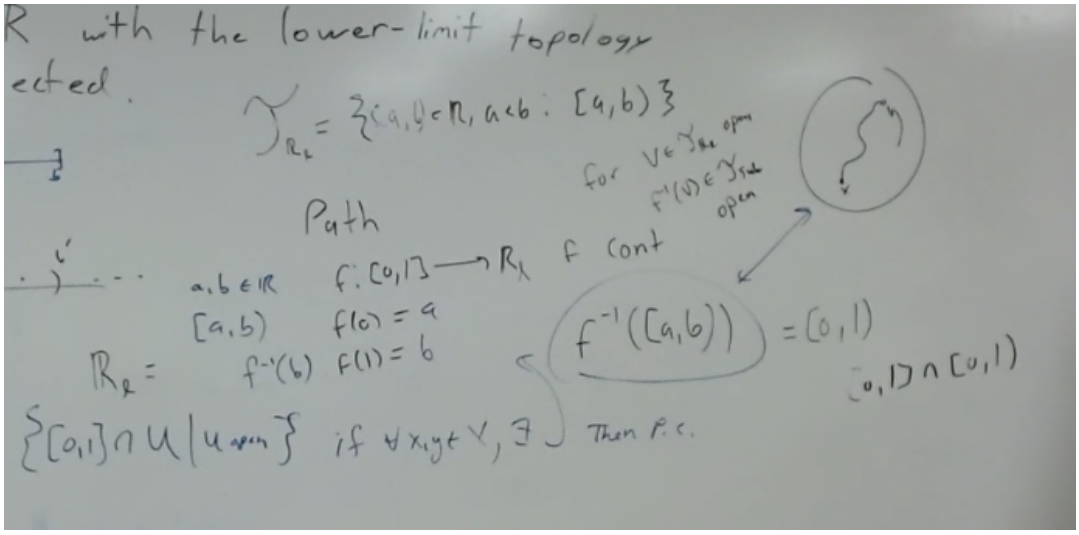

Figure 1. A portion of the students' board work.

The students then engaged in their approach of assuming that the space is path-connected and tried to find a contradiction. Student 2 then stated that he wants to understand what a path would look like between two points "in a basis element" as they decided they need a "continuous function that connects a to b." They began to consider what the preimage under some path of the basic open set [b,a) must "look like", before Student 2 replaced it with the closed set [a,b]. Student 1 then asked the researcher what the definition of a path is, and he asked in response, "How do you think of a path?" Student 1 then gestured a line in the air, then drew the same image that Student 2 drew before, claiming, "it's a continuous sort of connection that exists within the space." The researcher replied that a path is a continuous map from the interval into the space, with Student 2 adding that a space is path-connected if a path exists between "all the points." They then amended their existing definition of a path with quantifiers to extend it into a definition for path connectedness. Student 1 asked what the topology of the interval was, as the researcher affirmed his belief that it has the usual topology, the subspace topology of the real line with the usual topology. The pair thought in silence for a moment, then Student 1 resolved that they wanted to break the continuity of their supposed path. They tried to understand what the preimage of their closed set must look like, with Student 2 having noticed the difficulty in the

question coming from the fact that the path may "backtrack" on itself, in other words, the path need not be injective. This was essentially as far as they were able to take the proof on their own, as the rest of the progress was only made with significant interviewer intervention.

Afterwards, the researcher informed the students that they could find a separation of the space. They began to consider this as Student 1 struggled with the set theory, and the researcher guided them to consider explicitly taking the union of particular sets. This helped them realize that they could construct two open sets that are complements of each other, meaning the space is not connected, implying the space cannot be path-connected. This process of the students' attempt was coded, as Figure 2 depicts the codes as they appear in the transcript, with blue indicating the embodied code, red for symbolic, yellow for formal, and black for identified instances of blending.

Figure 2. The transcript's coding of embodied (blue), symbolic (red), formal (yellow), and blending (black).

Addressing the research question "What heuristics do students employ as they attempt topology proofs?", we note firstly that the participants used a significant amount of formal thinking and reasoning, as seen by the frequency of the yellow blocks in the coding of the transcript in Figure 2. They began Question 2 by defining each property and clarifying the relevant topological space by considering the basic open sets. Additionally, they used axiomatic reasoning with relevant theorems to decide which "path" to follow in their solution, recalling that if a space is path-connected, then it is connected, and applying the contrapositive/contradiction to arrive at their result. There was also an intentionality in what activities they considered productive, as at one moment Student 2 said, "Well… let's just maybe back off from this, since there's not an immediate separation standing out." Thus, the heuristics employed include the following: leveraging logical structures to establish and reframe objectives, considering the difficulty of the possible goals, and, as demonstrated in the next question, cognitive blending of various worlds of thinking as strategies to solve their problems.

Next, we consider the second portion of the research questions, "Did blending of the worlds of mathematical thinking occur in students' attempts at solving problems in topology? Was it a useful approach for problem-solving in topology?" Analyzing this recording yielded various references to Tall's Three Worlds, and some moments suggest a blending occurring between the worlds of thinking. Firstly, the students handled their concept of a path in such a way that exemplifies a cognitive blend. As they began the problem, the first written "description" they provided for path/path-connectedness was a picture meant to represent the concept of a path. This was also later used to resolve a moment of confusion where one student believed that paths needed to map "between sets." While the students initially disagreed about the verbal formal description, they immediately agreed on the picture's validity, and Student 1 used that to rectify his misunderstanding of the formal description. This is significant evidence of Student 1 having used single-scope blending as he interpreted this image into a formal understanding, agreeing that the path connects points, not sets. Also, Student 2 intentionally used this image instead of the formal definition to communicate the definition of a path, even though he had just stated that formal definition. Whether he thought the image was more convincing or it would be familiar to Student 1 in such a way that it would help him make that connection, it is clear that Student 2 has internalized the image in great conjunction with the formal definition of a path. It is as if, internally, the image and the definition are communicating the same idea to him, which supports

the idea of this cognitive blend between the embodied and formal worlds being a familiar tactic to him.

Blending appeared another time regarding Student 1's description of his concept of a path, as he gestured with his hands while stating that it is a continuous map, before drawing the picture on the board depicted in Figure 1 with the two-directional arrow between the image and the definition. Not only does his explanation describe a blend between his embodied gestures, but he also explicitly connected the ideas on the board with an arrow.

Lastly, while the students used the note "P → C" to reference the theorem that path-connectedness implies connectedness (as well as the converse and its shorthand), this is symbolism that has become common to the students to denote the formal theory. There is no evidence they are handling those icons in any way indicative of a symbolic form of thinking (i.e., there was no action taken on the terms). However, their solution to this confusion, referencing the topologist's sine curve, carried some information that enabled them to reconstruct the theorem without performing any formal tasks. This does suggest some form of blending occurred, though whether it is between any worlds of thinking or merely with their recollection of the topic is unclear.

Conclusions

Ultimately, this data provides evidence that blending the three worlds of thinking is a common and useful cognitive tool for students as they work on problems in topology. Blending was detected and analyzed in the context of successful moments of progress through the problem. It could be suggested that these instances of blending constitute comprehensible connections between mathematicians' different forms of thinking that are not only used in the contexts in which they are developed but may also be internalized into the broader cognitive body of the concepts. Additionally, while no blending was mentioned between the formal and symbolic worlds, this is certainly a possibility if one were to handle a problem by using symbolic logic to restructure the problem (e.g., using DeMorgan's law to "distribute/factor" a negation).

While this study presents compelling evidence that helps address the research questions, there are some limitations. For instance, as this study was task-based, we do not have precise descriptions concerning the students' thinking; we relied on the external evidence of their articulations and board-work to make conclusions about what was happening mentally. This could suggest that there is more blending that is not being detected. Additionally, while these students lack the language of our theoretical framework that would have enabled them to describe their own thought processes, asking them to understand the terms and use them causes an additional cognitive burden that could distract the students and form an unnatural environment for their proof attempt. While there is a larger curiosity regarding how students think about topological concepts in general, the typical class structure may not encourage students to think about their own internal conceptualization of those ideas. Instead, it seems that the majority of the mathematics curriculum focuses on developing students' ability to answer questions, which informed the design of this paper's research question. There are currently plans to develop this project further that involve interviews where the students are asked to describe their thought processes as they work and potentially explain their conceptualizations of topological concepts. As part of the first author's dissertation, we are continuing to refine this theoretical framework to develop a model to further guide our analysis of the data.

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An Analysis of Performance, Standard Mastery, and (Re)assessment Behaviors in a Coordinated College Algebra Course

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The purpose of this report is to identify specific content topics and assessment or reassessment behaviors to classify students' performance in a College Algebra course using alternative grading. Our data consist of students' individual standard mastery marks, reassessment frequency, standards reassessed, and final letter grades (which were primarily based on the number of standards mastered in the course). Our analysis involved constructing logistic models and a random forest to identify predictor variables that reduced the Akaike Information Criterion (AIC) and the mean Gini Index. Our results describe two content standards and reassessment behaviors whose attributes were central to our classification models, which can inform the instruction or revision of traditional- and alternative grading-based College Algebra courses.

Keywords: Alternative grading, College Algebra, reassessment, function composition, exponential and logarithmic functions

Alternative grading (AG) practices (e.g., standards-based grading, mastery-based grading, specifications grading, ungrading) are common in secondary (Guskey, 2008) and gaining momentum in undergraduate mathematics education courses (Lewis, 2022). More than 65 years after Bloom's (1968) introduction of learning for mastery, researchers have demonstrated this system's ability to improve both student and teacher perceptions of mathematical learning and coursework. For example, students in AG courses tend to exhibit lower test anxiety (e.g., Harsy et al., 2021; Henriksen et al., 2020; Lewis, 2022), consider assessments to be learning opportunities (Harsy & Hoofnagle, 2020), focus office hours conversations on constructing meaning for math topics instead of answer-getting (Zimmerman, 2020), and provide positive reflections on their experiences in their courses (Collins et al., 2019; Harsy et al., 2021; Henriksen et al., 2020). Instructors of AG courses often report increased feelings of responsibility, coupled with higher expectations for students, which lead to an overall improvement in their attitudes toward teaching their courses (Guskey & Gates, 1986).

Many of the reports cited in the previous paragraph describe (a) implementations of alternative grading, (b) affective advantages to students in courses using alternative grading, or (c) comparisons of students' experiences and perceptions of assessment in traditional and alternative grading courses (Swanson et al., 2025). In contrast, there has been minimal research on the use of student data to identify central topics in AG courses or behaviors that correlate with students' overall mastery of course content. The purpose of this report is to provide insights into such topics and behaviors in the context of a coordinated AG College Algebra course. To this end, we provide the following research questions (RQ) to guide our discussion:

1. Which mathematical topics are most correlated with student's mastery of course content in a College Algebra course utilizing alternative grading?
2. Which student study practices are most correlated with student mastery in such a course?

Theoretical Background and Course Structure

The AG College Algebra course instructors created the course in accordance with the Four Pillars of Alternative Grading (Clark & Talbert, 2023) and Collins et al.'s (2019) characteristics of Mastery-based Testing (MBT). We describe the course composition using the Four Pillars.

Pillar 1: Clearly Defined Standards

Clark and Talbert (2023) do not provide a rigorous definition of the term *"standard,"* leaving the decision on what constitutes acceptable student work open to the instructor's discretion. Still, Collins et al. (2019) state that whatever standards are determined, these standards must be clearly worded and communicated to students on study guides and assessments.

To determine the *standards* for the College Algebra course, instructors implemented a three-step procedure reminiscent of grounded theory (Strauss & Corbin, 1998). First, they analyzed the common final exam and compiled a list of skills they believed students would need to complete the questions (i.e., open coding). Second, they organized the skills into overarching topics that appeared (to them) suitable to serve as units within the course (i.e., axial coding). The instructors utilized the term *standard* to refer to each of the content topics that emerged from the organization process. The final, recurring, stage is to refine the list of standards through implementation iterations until it appears that the standards list encompasses the material conventionally covered in a traditional College Algebra course (i.e., theoretical saturation). The current iteration of the AG College Algebra course includes a list of 18 content standards, some of which are shown in Table 1 below. Students must show proficiency twice on a standard to achieve mastery of that standard. To facilitate this process, the course also includes feedback systems (Pillar 2), marks systems (Pillar 3), and a three-part assessment structure (Pillar 4).

Table 1. A sample of three standards from our AG College Algebra Course

Standard F8	Add, subtract, multiply, divide, compose, and decompose functions given in any of the four representations, and simplify when necessary/possible. Find the domain and range of the combined functions.
Standard EL4	Solve logarithmic and exponential equations. Solutions are either approximate or exact.
Standard EL5	Model and solve applications with exponential functions (compound interest, exponential decay and growth).

Pillar 2: Helpful Feedback

Instructors in the College Algebra course provide targeted, specific feedback on students' assessment and reassessment attempts in the course. The purpose of such feedback is to create a loop through which students know which ideas to revisit before reassessing. Collins et al. (2019) also recommend providing feedback to help students progress toward mastery of course topics, rather than as a penalty, which might disincentivize future attempts at mastery.

Pillar 3: Marks Indicate Progress

Instructors use a binary system to provide marks on students' assessments and reassessments: *pass* and *not pass yet* (Clark & Talbert, 2023). They also condition *passes* on conceptual mastery of a topic and use *not pass yet* marks to facilitate future student learning opportunities of course material, rather than penalizing (often computational) mistakes (Collins et al., 2019).

Instructors use a standardized system, based on the number of standards students master throughout the semester, to assign final letter grades. We show the number of standards corresponding to each letter grade in Table 2 below. The possible addition of a + or – modifier to a student's letter grade is contingent upon engagement activities, such as homework completion and course attendance. Consequently, students' final letter grade in the course is an indication of the amount of AG College Algebra content for which they have exhibited proficiency. When we describe productive predictors for classifying students by letter grade in the results section, we consider our claims to be synonymous with describing specific content or behaviors whose acquisition is associated with students' mastery of AG College Algebra material.

Table 2. The mastery-based grading scheme to determine final letter grades in AG College Algebra

Standards Mastered	0-8	9-11	12-14	14-16	17-18
Final Letter Grade	F	D	C	B	A

Pillar 4: Reattempts Without Penalty

A central component of all alternative grading strategies is allowing students to perform reassessments to demonstrate growth and mastery of course content (Clark & Talbert, 2023). College Algebra instructors provide three opportunity types for students to reassess course standards: (a) required weekly in-class assessments (which contain some instructor-chosen previously assessed standards), (b) optional weekly reassessments of student-chosen standards, and (c) optional reassessment opportunities on the final exam. In this report, we focus a portion of our analysis on reassessment opportunity (b), the optional weekly reassessments.

Methodology

In this section, we describe our data collection and analysis methods. We will use three statistical terms to organize our descriptions. First, we use the term *predictor variable* to refer to student behaviors or standard mastery for which we collected data. We have included a list of some predictor variables in Table 3 below. Second, we use the term *observation* to refer to the unit of analysis in our study, i.e., individual AG College Algebra students. Finally, we use the term *category* to refer to the response variables whose relationship with the predictor variables we attempted to determine (e.g., the final grade in the course).

Data Collection Methods

Population and data sources. The population we chose to study for this report consisted of all students enrolled in an Alternative Grading (AG) College Algebra or College Algebra Corequisite course at a mid-sized regional university in the Pacific Northwest. To construct a sample of students from this population, we collected internal and institutional data for the 134 students enrolled in one of the six Spring 2025 sections of traditional or corequisite AG College Algebra offered at the institution. The first author of this report was an instructor for one of the corequisite courses. We collected data from three sources. First, we imported data from a Qualtrics form used throughout the semester for students to schedule weekly reassessments in the proctor room. The information we imported from this form included students' names, the dates and times of reassessments, and the standards they intended to reassess at each appointment (a maximum of two standards weekly). Second, we collected students' individual standard mastery data from all but one instructor after the semester. Finally, we collected students' final letter grades and demographic data from the institutional research staff.

Predictor variables and sample refinement. In preparation for data analysis, we determined two classes of predictor variables to investigate in relation to students' final letter grade. First, we created three predictors for each standard according to possible mastery level (0 passes-not met; 1 pass-partially met; 2+ passes-fully met). This class of predictors aligned with the intent of RQ1, i.e., determining content concepts with explanatory potential for students' performance (e.g., final grade outcome). Second, we created predictors for behaviors related to the reassessment room, including (a) the total number of reassessment appointments made and (b) the number of standards reassessed throughout the semester. This class of predictors aligned with the intent of RQ2, i.e., determining behaviors that may help explain students' performance.

We used our focal predictors to restrict our initial sample to the set of observations (i.e., students) exhibiting both predictor classes. To this end, we removed 28 students from our initial sample. Of these students, 12 did not have assessment, reassessment, or standard performance data (typically students with early withdrawals), and 16 students were removed due to an instructor not providing standards mastery data. Our final sample consisted of 106 students.

Table 3. A selection of predictor variables guiding our data collection and analysis

F8	Categorical (binary) variable indicating whether the student mastered Standard F8 or not, i.e., achieved two or more passes.
Total.standards.scheduled	Discrete variable describing the number of standards students scheduled for reassessment throughout the semester.

Data Analysis

The second and third authors prepared the data for formal analysis and importation into the R studio platform. This process included (a) assigning identifiers across the disjoint data sources and (b) compiling three distinct spreadsheets into a centralized document containing the 106 observations to import into R studio.

Statistical Techniques. The second author of this paper led the data analysis for this project, which included techniques related to (a) stepwise selection on logistic models using the Akaike Information Criterion (AIC) and (b) creating and measuring the decision trees in a random forest by Gini Index to construct a variable importance plot. We explain these components of our analysis in the paragraphs below.

Statisticians use logistics models to predict the category (e.g., final letter grade) associated with a specific observation (e.g., student with a particular set of standard and reassessment properties). When building a logistic model, a common approach is to use forward or backward step-wise selection. This process involves starting with either no predictor variables in the model (forward selection) or all predictors in the model (backward selection) and then adding (i.e., forward) or removing (i.e., backward) predictor variables to determine how the change improves the explanatory power of the model to classify the observations correctly. A central component of determining a final logistic model is minimizing the AIC value, which reduces the predictors in the model while maintaining explanatory power. Although the forward and backward selection processes can identify different variables as important, any variables that appear in both models provide a strong argument for predictors of high-performing students. Our analysis involved creating both forward and backward selection logistic models to determine important standard mastery (i.e., RQ1) and reassessment behavior (i.e., RQ2) predictors that we could use to categorize our students' final grade performance (i.e., content mastery).

Statisticians use random forest (i.e., a collection of decision trees) as an alternative method to predict the category (i.e., final letter grade) associated with a particular observation (i.e., student performance and behavior). In our instance, we measured the Gini Index to determine the distribution of the predictor variables across the decision trees in the random forest. A low Gini Index (i.e., close to 0) indicates that a particular predictor variable (e.g., mastering Standard F8) produces the cleanest separation of observations into those that correspond or do not correspond with the category (e.g., students who master Standard F8 generally get A grades and those who do not master F8 generally receive lower grades). We used the data from the Gini Index measurements across the random forest to construct a variable importance plot, which shows the predictor variables that contributed the most to improving the tree's predictions throughout the forest. In this plot, the higher the importance score, the more the Gini Index is reduced (i.e., decreased) across all splits in a given decision tree. In other words, a high importance score (e.g., greater than 3) indicates an essential predictor variable for categorizing students' final grades in the College Algebra course. Through our analysis, we identified the two content standards (RQ1) and student behaviors (RQ2) that provided the most explanatory power for the students' final letter grades in the course. We describe these results in the following section.

Results

RQ1: Content Standards Corresponding with Overall Standard Mastery

Our initial logistic models indicated several important predictor variables, both from the forward and backward selections. Among these variables, we found two standards, F8 and EL4, with associated predictors that indicated high explanatory power for students' final letter grades in the course (i.e., mastery of course content; see Table 1 for the text of these standards). For instance, we determined that students' eventual mastery of these standards substantially lowered the AIC across our logistic models. Although only students' reassessment of Standard EL4 was a statistically significant predictor for students' final course performance ($p = 0.004$), we found that manually removing students' mastery of F8 and EL4 from our models led to a drastic increase in the AIC value (e.g., by a factor of 2 to 3 depending on the model). These results indicate that students' mastery of function composition topics (i.e., Standard F8) and ability to solve logarithmic or exponential equations (i.e., Standard EL4) are crucial predictors for students' final letter grades (and subsequent success) in AG College Algebra.

To further investigate the potential importance of standards F8 and EL4 in relation to student success in College Algebra, we created a random forest, measured the corresponding Gini Index, and constructed a variable importance plot. We share a sample decision tree from the random forest and our variable importance plot in Figures 1a and 1b below. We note that the first two branches in the sample decision tree (Figure 1b) are F8 and EL4, and each predictor has a mean decrease in the Gini Index above 5 (Figure 1a), indicating their importance in classification across the decision trees in the random forest. Taken together, the reduction in AIC and the strong decrease in the Gini Index confirm the central nature of standards F8 and EL4 as classification tools for students' letter grades in our course.

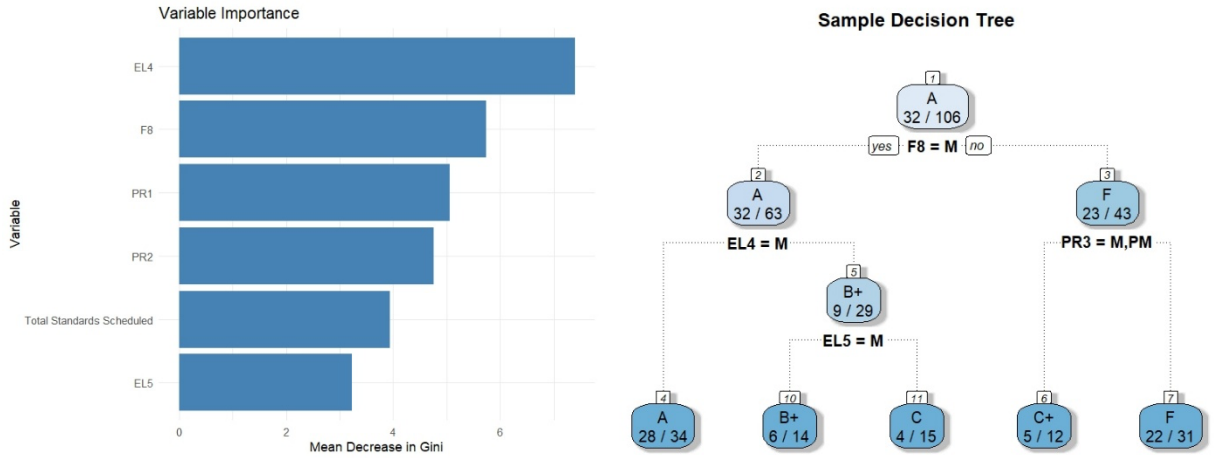

Figure 1. (a-left) A variable importance plot of the largest mean decrease in Gini Index. (b-right) A sample decision tree from the Random Forest method.

RQ2: Students' Assessment Behaviors Corresponding with Overall Standard Mastery

Our results related to RQ1 indicate that the predictors with the most explanatory power related to students' letter grade performance in the AG College Algebra course were content (i.e., standard) related. We anticipated that some students' (re)assessment behaviors might also function as crucial classification tools of students' final letter grades, due to the central nature of reassessment in the AG course. Through our analysis, we identified two predictors that served as useful classification tools. First, we found that the total number of standards that students reassessed over the course of the semester (a) reduced the AIC in both the forward and backward directions of the logistic models and (b) produced a reduction in the Gini Index above 3. Second, we found that the total number of reassessment appointments that students scheduled and subsequently completed also consistently reduced the AIC in logistic models and produced a Gini Index reduction above 3. We note that among our list of predictors, which included students' performance on all 18 standards and various (re)assessment behaviors, students' total number of standards reassessed produced the fifth most significant decrease in the Gini Index, and the number of completed reassessment appointments contributed the twelfth largest decrease. Taken together, these data indicate that the most important student (re)assessment behaviors for classifying students' final letter grades in AG College Algebra involve their reassessment actions. Specifically, students who complete scheduled reassessment appointments and reassess larger numbers of standards tend to achieve higher grades in the course.

Discussion

This report highlights our use of reassessment and student performance data in a coordinated AG College Algebra course to increase our understanding of students' achievement of particular letter grades (i.e., standard mastery) in the course. To this end, we proposed two research questions related to (a) the central course standards and (b) (re)assessment behaviors that could provide explanatory power for students' mastery of the content in the course.

Our data analysis related to RQ1 revealed that students' mastery of function arithmetic and composition (Standard F8), and algebraic fluency with solving exponential and logarithmic equations (Standard EL4) provided a framework for effectively classifying the letter grades of a large number of our students. Our findings reinforce those of prior researchers who claimed that building conceptual meanings for function concepts is not easy for students (Breidenbach et al., 1992; Thompson & Carlson, 2017), whether (a) learning function composition in preparation for

the chain rule in calculus (Clark et al., 1997), (b) building meaning for exponential functions (Ellis et al., 2016), or (c) reasoning about logarithmic functions (Kuper & Carlson, 2020). Additionally, our results extend these findings by showing evidence of the importance of composition, exponential, and logarithmic topics in students' success in a mastery-oriented College Algebra course.

From an instructional perspective, our results related to RQ1 imply that an increased focus, or at least increased attention to these topics, may lead to improved student success in College Algebra courses using alternative grading. Although our study does not include a comparison with students in traditionally-graded (i.e., weighted average) College Algebra or calculus courses, our results suggest that (a) traditional College Algebra course instructors might similarly profit from increased focus on composition, exponentials, and logarithms, and (b) addressing students' notions of function composition, exponential functions, and logarithmic functions may be a productive focus for diagnostic or remedial tools in higher mathematics coursework (e.g., calculus).

Our data analysis related to RQ2 question revealed that students' willingness to schedule and perform reassessments across various standards correlated with higher performance (i.e., increased mastery of content) in the course. Our findings provide empirical evidence that reassessments can be employed as learning opportunities (Collins et al., 2019) and that students who avail themselves of these opportunities often progress toward mastery. While we recognize that our results regarding reassessments may not be fully transferable to College Algebra or other courses using traditional grading systems, we use our results to highlight the possibility that all instructors can modify some (or all) of their assessments to create student learning opportunities (rather than penalizations or unchangeable marks of failure).

We are continuing to collect data on students' standards mastery and (re)assessment practices during the Fall 2025 semester. We anticipate that the analysis of this additional data in the coming months will provide further refinement of the results we presented in this report. Specifically, we plan to test whether our logistic models, in general (and Standards F8 and EL4, and students' reassessment practices in particular), continue to serve as effective classification tools for students' final letter grades. Our eventual results will serve to further the discussion of centrally important topics in College Algebra and the refinement of AG College Algebra courses.

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Snowballing: The Unexplored Issue of Developmental Mathematics and Course Repetition

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In the developmental mathematics education literature, course failure typically leads only to attrition from the course sequence. That is, an overlooked, and not well-understood, experience in developmental mathematics, is course repetition. In this quantitative study, I aimed to investigate the incidence and impact of developmental mathematics course repetition. Specifically, I aimed to document how experiences with course repetition contributed to students' course taking patterns within and beyond developmental mathematics. Findings indicate that the consequences of experiences with developmental mathematics course repetition are significant and far reaching, influencing longer-term academic outcomes including program retention and degree attainment. The need for further investigation, particularly using qualitative methods to understand how and why observed relationships in a student's mathematics/academic trajectory operate and function, is well supported by the quantitative results of this study.

Keywords: Developmental math, course repetition, STEM students, graduation rates

While developmental mathematics (DM) education aims to resolve preparatory disparities and provide a path for degree attainment, research has consistently documented poor academic outcomes among DM students at both two- and four-year institutions (Bahr, 2010; 2012). Short-term outcomes include high rates of course failure and sequence attrition, while longer-term outcomes include longer times to and lower rates of degree attainment (Bailey et al., 2010; Chen, 2016; Fong et al., 2015). Furthermore, despite consistent empirical (e.g., Bailey et al., 2010; Bahr, 2012) and anecdotal (e.g., from instructor practitioners) evidence that DM courses suffer from high rates of failure, the significance and influence of DM course repetition[1] on these outcomes is unaccounted for in the extant DM education literature. In this study, the term *contextual force* refers to a phenomenon that is, in some way, inherent to the learning context, and that indirectly, though importantly, shapes the learning context. Given the high rate of failure documented in DM courses, experiences with course repetition are inherently a part of the DM educational experience. As such, this study asserts that DM course repetition is a contextual force that influences DM education and its outcomes by its direct impact on students and their learning experience.

Literature Review

In Bahr and colleagues' (2017) mapping of students' STEM course taking pathways, reported course outcomes (e.g., pass rate, failure rate, STEM pathway attrition, etc.) only concerned students' performance in their first attempt of a course, explicitly excluding students' performance during subsequent course attempts from analysis and reporting. According to Bicak and colleagues (2023), though more documented in K–12 education research, the rates and implications of course repetition are not well-explored in postsecondary education research. In

[1] *Developmental mathematics course repetition* is defined in this study as the enrollment in a course previously taken because the prior enrollment ended with a failing final grade, withdrawal from the course, or a final grade lower than what is needed to advance to the next course in the sequence.

particular, much of the research investigating DM education is anchored at two-year postsecondary institutions, meaning there is a need for DM education research conducted at four-year postsecondary institutions. Furthermore, while there is significant concern among developmental mathematics researchers regarding DM sequence attrition and completion, few, if any studies, have endeavored to conceptualize or understand the factors (including their functioning and underlying mechanisms) that contribute to attrition or completion. For instance, Bahr (2008) found that the likelihood of successful remediation–operationalized as passing college-level coursework–is inversely related to level of placement, or "depth of remedial need" (p. 444): The lower a student's placement, the lower the likelihood of the student passing college-level coursework. When discussing this finding, Bahr acknowledged that "the mechanisms that underlie the relationships between depth of remedial need and successful remediation are, as of yet unclear" (Bahr, 2008, p. 444). This underscores that the lack of a fundamental understanding of factors that contribute to observed outcomes relating to sequence attrition or completion inhibit the advancement of DM research.

To address this gap in knowledge, this study seeks to answer the following questions:

1. What are the rates and patterns of developmental mathematics course repetition?
    a. What are the mathematics course taking trajectories and longer-term academic outcomes of students who experience developmental mathematics course repetition?

Research question 1a is motivated by Hodara's (2019) finding that DM students are less likely to retain their STEM major than their non-developmental peers. While this finding was not discussed further, program retention is likely related to DM sequence attrition, given the findings regarding gatekeeper course enrollment and completion. As such, this study seeks to investigate how DM course repetition impacts program retention and graduation.

Methods

Data Collection

Data collection occurred at Temple University— a large, four-year, public research university in a large city on the East Coast. I obtained longitudinal student data from the Temple's Office of Institutional Research and Assessment for developmental mathematics course enrollees who entered Temple between the 2013-2014 (13-14) and 2023-2024 (23-24) academic years. The data set included course and program enrollment data, as well as graduation information. Data cleaning yielded an analytic sample of 14,942 students whose enrollment (e.g., campus of enrollment) and academic records (e.g., DM coursework was taken either in-person or virtually during the pandemic) met certain criteria, which ensured uniformity aligned with the scope of this study's investigation. I used subsets of this analytic sample for more precise analyses, which are discussed below.

The Temple mathematics department currently offers three algebra-focused courses—MATH 0701, 0702, and 1021—and one co-requisite laboratory course—MATH 1019—associated with the highest-level algebra course, MATH 1021. That said, given the purpose and functioning of DM education at Temple University—particularly the prior changes to MATH 0701 and the more recent addition of MATH 1019—MATH 0702 Intermediate Algebra and MATH 1021 College Algebra are the DM courses of primary interest in this analysis. Furthermore, this study is interested in the academic trajectory of DM students and their engagement with advanced/college-level mathematics courses, specifically the calculus sequence, which at Temple involves four calculus courses starting with Precalculus. Relatedly, I examined how and when DM students' program of enrollment shifted, if at all, during or after their DM coursework.

In particular, I was interested in DM students who enrolled in a STEM or STEM-adjacent program, referred to as *STEM (adjacent) programs* (e.g., Music Technology or Secondary Education), at some point in their time at Temple because calculus coursework is a requirement only for certain STEM (adjacent) programs.

Data Analysis

To answer research question 1 pertaining to rates and patterns associated with DM course repetition, I established enrollment trends, including most frequent DM course sequences, by analyzing the analytic sample of 14,942 DM students. I then analyzed a more uniform subset of DM students (n=10,482) to establish rates and frequencies of DM course repetition. Chen (2016) found high rates of continued enrollment among remedial students 6 years after initial enrollment, suggesting these students might take longer to complete their degree. As such, to answer research question 1a pertaining to the longer-term academic outcomes of DM students, I analyzed the 5,765 students who enrolled at Temple between the 13-14 and 18-19 academic years because their academic records span at least 6 academic years (Figure 1).

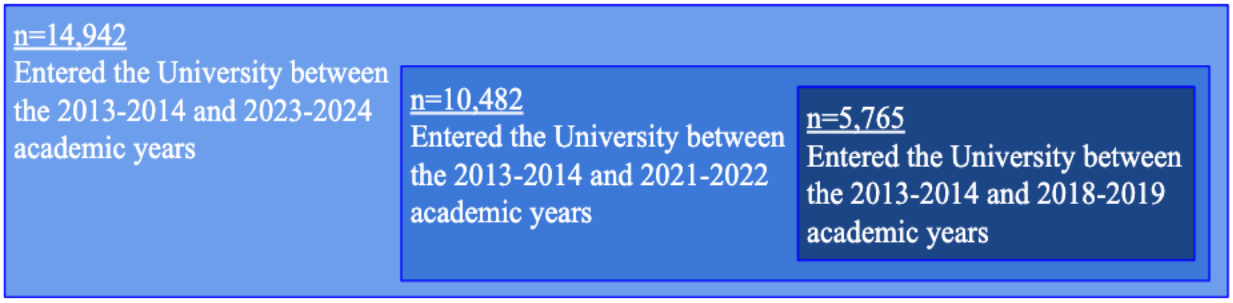

Figure 1. Analytic Samples

Results

Of the 14,942 developmental mathematics students of interest with initial enrollment data (n=14,279), 90.15% took their first DM course during their first year of enrollment, 97.76% within their first two years of enrollment, and 99.36% within their first three years of enrollment. 9,958 students enrolled in 1 DM course, 4,679 students enrolled in 2 DM courses, 302 students enrolled in 3 DM courses, and 3 students enrolled 4 DM courses. 14,932 students enrolled in course sequences of interest (Figure 2); The exclusions involved anomalous course sequences and too few enrolled students for analysis. 25.14% of DM students had to repeat at least one (and up to three) of their DM courses and 20.71% (n=778) of DM repeaters had to repeat one of their DM courses more than once. For instance, a student could take 3 DM courses, repeat two of those courses one time each, and have a DM sequence length[2] of 5 (referred to as a *singular repeater* or *repetition*), or a student could take 1 DM course, repeat the course four times, and have a DM sequence length of 5 (referred to as a *multiple repeater* or *repetition*). Thus, while singular repeaters have longer DM course sequences because of one course repetition per repeated course, multiple repeaters have longer DM course sequences because they repeated a course(s) multiple times. Such instances of multiple repetitions are highlighted in orange in Table 1, which summarizes DM sequence length according to how many DM courses students enrolled in and had to repeat.

[2] *DM sequence length* refers to a student's total number of DM course enrollments during their time at Temple.

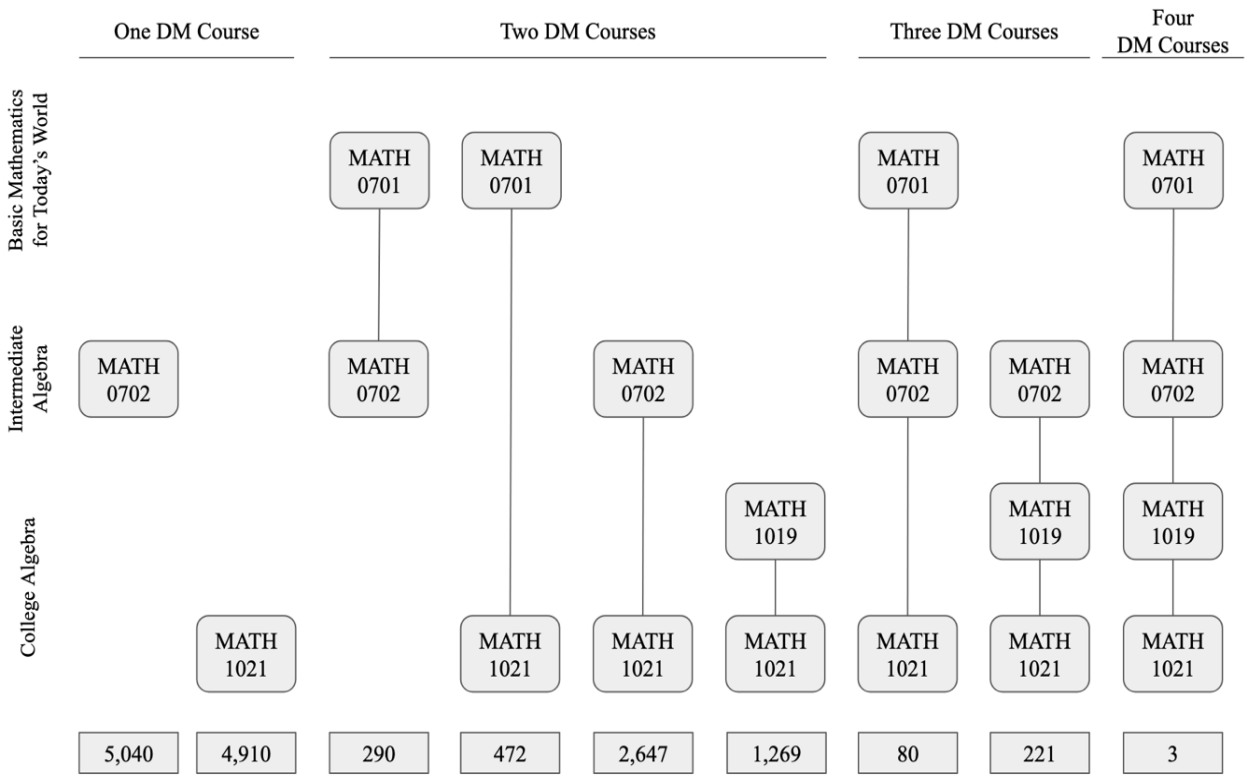

Figure 2. Most Frequent Developmental Mathematics Course Taking Patterns

Table 1. Developmental Mathematics Course Enrollments

Number of Enrolled DM Courses	Number of DM Courses Repeated	DM Sequence Length								**Total**
		1	2	3	4	5	6	7	8	
1	0	7,818	0	0	0	0	0	0	0	**7,818**
1	1	0	1,759	371	9	1	0	0	0	**2,140**
2	0	0	3,251	0	0	0	0	0	0	**3,251**
2	1	0	0	911	246	7	0	0	0	**1,164**
2	2	0	0	0	180	68	16	0	0	**264**
3	0	0	0	115	0	0	0	0	0	**115**
3	1	0	0	0	76	40	0	0	0	**116**
3	2	0	0	0	0	42	14	4	0	**60**
3	3	0	0	0	0	0	9	1	1	**11**
4	0	0	0	0	2	0	0	0	0	**2**
4	1	0	0	0	0	1	0	0	0	**1**
	Total	**7,818**	**5,010**	**1,397**	**513**	**159**	**39**	**5**	**1**	**14,942**

Of the more uniform subset of 10,482 DM students and across all their DM enrollments, 742 students withdrew from at least one of their DM courses, 2,510 students failed at least one of their DM courses, and 326 students experienced both at least one DM course withdrawal and at least one DM course failure. 34.05% of students who experienced both DM course withdrawal

and failure experienced more than one withdrawal and/or failure. Table 2 summarizes the passing rates in MATH 0702 and 1021 based on enrollment attempt. 16.83% of MATH 0702 students (n=5,756) repeated the course at least once and 20.84% of MATH 1021 students (n=6,709) repeated the course at least once. Of the 1,985 students who took both MATH 0702 and 1021, while the rate of MATH 1021 repetition among students with no repetitions of MATH 0702 was 21.91%, the rate of MATH 1021 repetition among students with at least one repetition of MATH 0702 was 45.69%. Of students' final DM course enrollment, 78.63% of students passed, 14.44% failed, and 6.36% withdrew from their final DM enrollment.

Table 2. Final Course Grade by Course and Attempt

	MATH 0702			MATH 1021		
	1st Attempt (n=5,756)	2nd Attempt (n=969)	3rd Attempt (n=146)	1st Attempt (n=6,709)	2nd Attempt (n=1,398)	3rd Attempt (n=271)
Pass [a]	69.96%	57.07%	50.00%	70.52%	63.16%	69.37%
Fail [b]	20.64%	30.65%	34.93%	22.58%	29.83%	25.46%
Withdraw	8.60%	11.87%	13.01%	6.68%	6.44%	4.43%

Note: Course attempts were included if enrollment was greater than 100. [a] Pass = A to C-. [b] Fail = D+ to F.

Of the 5,765 DM students who enrolled Temple between the 13-14 and 18-19 academic years, 3,023 were enrolled in a STEM (adjacent) program during their last DM course enrollment. Additionally, 1,933 of the 13-14 to 18-19 entering students enrolled in at least one calculus sequence course—referred to as *DM-to-calculus enrollees*—following their last DM course enrollment. While 57.07% of STEM (adjacent) students who did not experience DM course repetition became DM-to-calculus enrollees, lower rates of DM singular repeaters and DM multiple repeaters became DM-to-calculus enrollees—27.05% and 23.58%, respectively; Table 3 summarizes the final course grade of DM-to-calculus students' first calculus sequence course differentiated by the extent of repetition in DM coursework. 83.29% (n=1,610) of DM-to-calculus enrollees were enrolled in a STEM (adjacent) program during their last DM course enrollment and 65.09% (n=1,048) of these DM-to-calculus enrollees retained their STEM (adjacent) program enrollment throughout their time at Temple.

Table 3. Final Grade in First Calculus Course Following DM Course Sequence

	No Repetitions (n=2,842)	Singular Repetitions (n=504)	Multiple Repetitions (n=85)
Pass	58.09%	26.79%	14.12%
Fail	28.36%	46.03%	57.65%
Withdraw	12.91%	26.39%	28.24%

Of the 13-14 to 18-19 entering students, 3,804 (65.98%) graduated. The graduation rate among students who repeated 0, 1, or 2 DM courses was 68.12%, 58.97%, and 57.35%, respectively. Table 4 summarizes overall graduation rate by extent of DM repetition and further differentiates graduation rates by time to degree attainment. Of the 1,889 graduates who began their DM coursework enrolled in a STEM (adjacent) program, 47.70% graduated with a STEM (adjacent) degree, which suggests that the rate of STEM (adjacent) program attrition amongst graduates is more than 50%. The students who retained and graduated with a STEM (adjacent)

degree account for 81.54% of students who graduated with a STEM (adjacent) degree, suggesting that relatively few students transfer into STEM (adjacent) programs from non-STEM (adjacent) programs; The STEM (adjacent) program retention rate amongst non-graduates (n=1,189) was 74.10%. The rate of STEM (adjacent) program retention from initial enrollment through graduation varied based on experience with DM course repetition: While the program retention rate was 51.48% among students who did not repeat a DM course, the program retention rate among singular and multiple repeaters was 34.57% and 36.59%, respectively.

Table 4. Graduation Rate and Time to Degree Completion

		Years to Degree Completion				
	Graduation Rate	<4	4	5	6	6+
No Course Repetitions	68.12%	7.29%	57.47%	26.57%	6.69%	1.99%
Singular Repetitions	60.26%	4.43%	42.29%	38.63%	11.45%	3.21%
Multiple Repetitions	52.85%	3.08%	27.69%	41.54%	20.00%	7.69%

Discussion and Conclusion

In working to investigate the rates, trends, and long-term impacts of course repetition in developmental mathematics education, it became quickly evident that the consequences of repetition are numerous and cumulative. Analyses reveal that more than a third of students receive a non-passing final grade at some point in their DM coursework and that more than a fifth of students receive a non-passing grade in their last DM course. Furthermore, while prior research has documented the high failure rates of DM courses (e.g., Bahr, 2012), this study's analyses reveal that DM courses' passing rates decline across enrollment attempts. Taken together, these findings highlight the precarious pathway DM students must traverse to pass or advance further than their DM coursework.

It is clear from this study's results that the impact of DM course repetition extends beyond a student's DM coursework. In particular, the extent of a student's experience with DM course repetition appears to impact further mathematics persistence and performance. While these quantitative analyses do not and cannot provide much insight into how the former finding functions, it is nevertheless a meaningful finding that should motivate future efforts to reduce DM sequence and STEM (adjacent) program attrition. The latter finding that first-calculus course passing rates decrease with experiences with DM course repetition suggests that the efficacy of DM education wanes as students repeat courses. Experience with DM course repetition also appears to impact timely degree attainment, which is a reasonable finding given that multiple repetitions can double or even triple the length of a student's DM course sequence relative to the number of DM courses in which they enroll (e.g., taking 2 DM courses, repeating both twice, and having a DM course sequence length of 6). Furthermore, the findings regarding STEM (adjacent) program attrition and retention through degree attainment—that program retention is low, yet the rate of students transferring into STEM (adjacent) programs is seemingly lower—suggest that addressing the high incidence of DM course repetition could be an effective means of increasing STEM (adjacent) program retention, since there is evidence that DM course repetition is related to post-DM persistence. Specifically, by conducting this study at Temple—a STEM-focused, research institution, where DM coursework is a remedial prerequisite for more advanced mathematics coursework—this study not only contributes to the growing body of DM

education research conducted at four-year institutions, but it also provides insight into how DM education relates to and impacts STEM (adjacent) trajectories.

Implications, Limitations, and Next Steps

This study's findings underscore the significant financial, opportunity, and psychic costs incurred by students when they encounter this common, yet not well-understood, experience within DM education. Specifically, extended course sequences cost students time and money in pursuing their program, degree, and career path. Furthermore, considering the tenuous—though well-studied—relationship students have with mathematics, it is important to consider how students experiencing DM course repetition perceive their learning experience and potential. Thus, a major limitation of this study is that its quantitative methodology provides little insight into how the academic, mental, and tangible costs interact to compound the consequences of DM course repetition. Additionally, unexpected quantitative results made it difficult to conduct certain analyses. For instance, students' course sequences were occasionally regressive (e.g., taking MATH 1021 and then later taking MATH 0701), or students sometimes advanced to higher-level DM courses without having passed the prior prerequisite course. Taken together, this study's findings strongly warrant qualitative investigations of students' experiences with course repetition to understand how and why observed trends and quantitatively inexplicable patterns occur. Furthermore, narrative reflections from DM students who have experienced course repetition could help inform instructional practice, as well. For instance, understanding students' perspectives on their repeated DM course enrollments, both cognitively and affectively, could help instructors differentiate their lessons. Thus, exploring DM course repetition qualitatively will support the investigation of this phenomenon as a contextual force that influences the DM educational environment via its impact on DM students' and their learning experience.

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A Learning Trajectory for Area Under a Rate Graph

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We made a short sequence of interactive tasks in which a student must estimate the total amount of water in a pool, given the water rate (gals/min) at a handful of moments. Through working on these tasks in a small teaching experiment, an undergraduate who had not seen calculus learned to conceptualize the area under a linearly increasing water rate graph as representing the total accumulated volume of water. From this we developed a learning trajectory, highlighting requisite and developing concepts, actions, and teacher-researcher interactions that were key to his learning. His learning trajectory, which entailed imagery of lower and upper Riemann sums, reveals ways to learn key calculus concepts while maintaining the relationship between rate and amount quantities as a central, rather than an easily jettisoned, part of students' understanding of 'area under a curve'.

Keywords: Calculus, Rate of change, Definite Integral, Accumulation

One of the two main questions of calculus can be summarized as the following: "you know how fast a quantity is changing and you want to know how much of it there is" (Thompson et al., 2013, p. 125). Equipping students to conceptualize the accumulation of an amount of a quantity given its varying rate of change has long been discussed as a significant challenge in the teaching and learning of calculus (e.g., Thompson, 1994). In the U.S., students typically emerge from a standard single-variable calculus class quite able to interpret a definite integral as a gestalt area "under" a curve, but in many cases, they struggle to attribute quantitative meaning to it in a context, and often cannot interpret it as a sum (e.g., Fisher, Samuels, & Wangberg, 2016; Jones, 2015; Sealey & Oehrtman, 2005; Wagner 2016). One recommendation for addressing this issue has been to teach the accumulation of an amount with minimal emphasis on 'area under a curve' (e.g., Thompson et al., 2013). Yet areas under curves are also powerful representations for accumulated amounts, and they can reveal important characteristics about how these amounts relate to their rates of change. Our broader goal is to study how to build a student's understanding of an accumulated amount as an area under a curve *through* leveraging their quantitative reasoning in a context. Our hope is that the relationship between rate quantities and amount quantities can thus become a central, rather than an easily discarded, element of their understanding of 'area under a curve'.

In this paper we develop a short learning trajectory of an important piece of this broader understanding, by conducting a short teaching experiment with a college algebra student who had not taken calculus. We chart out how he learned to interpret the area under a *linear* rate graph (flow of water in gals/min) as representing a total accumulated amount (of water in gallons), by engaging with a sequence of interactive tasks. In so doing he developed a method akin to lower and upper Riemann sums. We believe the linear rate graph case is a pivotal one from which a student could generalize this area-represents-amount idea to more general types of rate curves, while staying grounded in quantitative meanings of accumulation and rate.

Background and Theoretical Framing

Accumulating Amounts From Their Rates of Change

Much of the recent research about the teaching and learning of definite integrals focuses on how students learn to interpret integral notation in quantitative contexts (e.g., Greefrath, et al., 2021; articles in *ZDM* special issue: Ely & Jones, 2023). In some approaches to developing quantitatively-grounded interpretations of integrals, such as multiplicatively-based summation (Jones, 2015) and accumulation-from-rate (Thompson, et al., 2013), the integrand—i.e., the "$f(x)$" in $\int_a^b f(x)dx$—is always interpreted as a *rate*. Specifically, $f(x)$ is the rate at which some amount F changes as x changes, and the definite integral measures how much F accumulates over the interval (a, b). For a student using such an interpretation to imagine an integral in terms of area, the area is always then an amount accumulated under a rate graph (e.g., a graph of $y = f(x)$). Thus it is valuable to study how a learner could develop the idea that the area under a rate graph represents an accumulated amount, even independently of any definite integral notation.

Learning Trajectories

Researchers have defined and theorized learning trajectories in a variety of ways. Simon (1995) introduced the construct of a hypothetical learning trajectory as the learning goal, activities, and thinking in which students engage, and Clements and Sarama (2012) depicted it as a conjectured route through a set of tasks designed to engender the mental actions to support a developmental progression of understanding. For the purposes of this study, we follow Ellis et al.'s (2014) characterization of a learning trajectory as a tool that seeks to explain learning over time, in particular, "an account of changes in students' schemes and operations" (p. 2). This follows with Steffe and Olive's (2010) view that a learning trajectory is ultimately a type of model observers construct of a learner's knowledge. Ellis et al. treat a learning trajectory as including a sequence of *conceptual stages* that can account for a learner's observed activity. Each stage is characterized by the *concepts* or understandings that an epistemic student uses at particular moments of their activity (Ellis, 2022). Also included is the context of the student's activity, such as the *tasks* they performed, the *actions and representations* they reasoned with while doing so, their *interactions* with others, and the *teaching actions* that supported their shifts from one stage to the next. In particular, a learning trajectory seeks to account for how specific contextual elements mediated or facilitated the learner's conceptual changes. The learning trajectory distills the student's whole observed activity and its context into the parts that are key for accounting for how they learned.

Methods

We recruited four undergraduate students at a medium-sized university in the Northwest U.S. to work on a calculus-related activity. They were chosen because they were interested and available, and because they were currently enrolled in college algebra and had not taken calculus. We conducted one 2-hour teaching experiment session with each pair. This paper's results are from one of these pairs. One participant had to leave after 15 minutes, so our results are from the other participant, 'Ian' (pseudonym). Participants worked on a sequence of pool-filling tasks on an iPad. The main teacher-researcher, TR1, asked questions along the way to elicit explanations for their actions and comments, but did not explain concepts or help the participants solve the tasks. A second teacher-researcher, TR2, was present for the first part of both sessions also. Participants also used scratch paper, writing implements, calculators, and a large whiteboard.

The sequence of tasks was programmed using Desmos Classroom. Each task had a similar format and layout [Figure 1], including: (a) an animated video of a swimming pool being filled

with water, (b) a table displaying a sample of the water rates (gals/min) at a few moments in time, and (c) a graph of the water rate for the time shown in the animation video. The video had labels showing the rate at which the pool was being filled and the volume in the pool in real time. The number of values shown in the table could be changed.

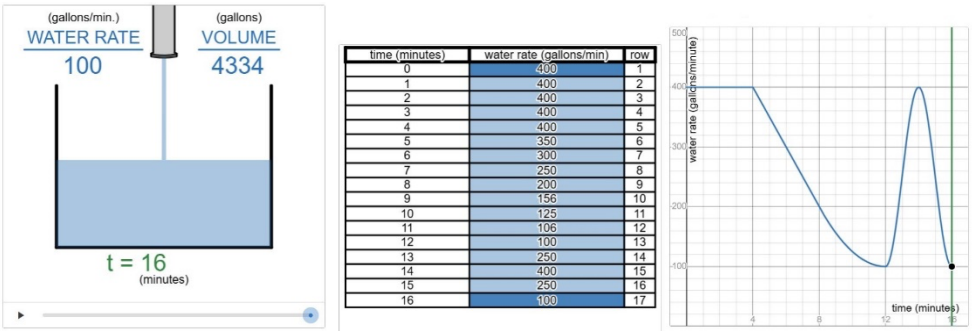

Figure 1. Pool activity, Task 0 (Orientation task): (a) pool animation, (b) rate table, and (c) rate graph

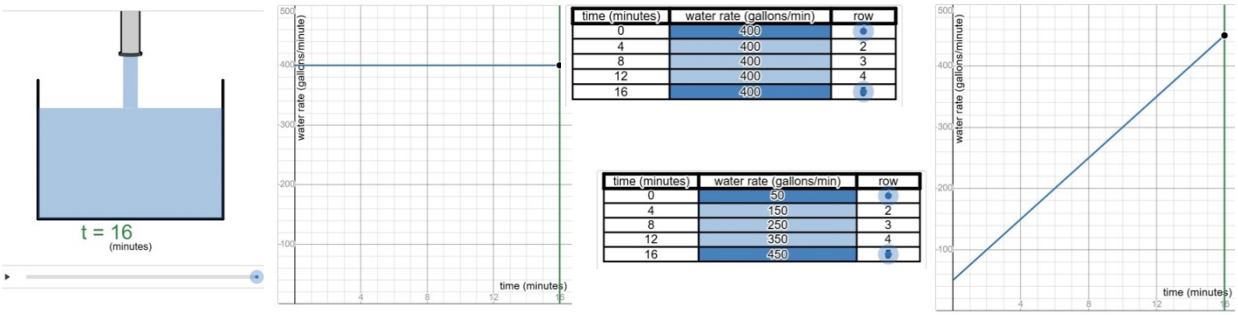

Figure 2. Pool activity, (a) animation, (b) Task 1 table, (c) Task 1 graph, (d) Task 2 table, (e) Task 2 graph

The introductory Task 0 was meant to quickly orient participants to the tools and displays. They were told that the water rate is fluctuating and that the values in the table are as if a picture was being taken of the water rate value at certain moments in time. For Tasks 1 and 2, participants had to extrapolate from just the video and the table the amount of water in the pool at the end of the video ($t = 16$). The water rate and volume readouts, as well as the water rate graph, began as hidden; these were revealed after six guesses or after one relatively close guess. Participants had unlimited guesses and were given general feedback about the accuracy (e.g., "really far below the actual amount"). In Task 1, the water rate was a constant 400 gals/min [Figure 2abc]. In Task 2 the water rate linearly increased from 50 to 450 gals/min [Figure 2ade].

Both sessions were video-recorded, screen-recorded, and transcribed. To analyze the data, we leveraged ongoing and retrospective analysis methods (Steffe & Thompson, 2000) to characterize Ian's changing concepts throughout their sessions. Prior to collecting any data, we formulated a learning goal: the concept that the area under the graph of a rate function measures a total accumulated amount. We developed a hypothetical learning trajectory based on extant research, and modified the tasks. Based on time constraints we narrowed the learning goal to just the area concept for the case of a linearly increasing rate graph, rather than any general rate graph. The hypothetical learning trajectory then served as a source of initial codes for identifying the participants' concepts; we determined as a group whether Ian's task responses provided evidence of a concept from the hypothetical learning trajectory. This initial round of coding also produced emergent codes (Strauss & Corbin, 1990), which we incorporated into a revised

learning trajectory for Ian. Given the challenge of understanding and characterizing a student's mental images, we triangulated evidence from multiple sources to infer Ian's thinking, including his written work, utterances, nonverbal gestures, and shifts in his problem-solving approaches. We then examined the interactions between Ian and the TRs, within the context of the task structure, to identify supports for Ian's emerging concepts.

Results

In this section we summarize Ian's activity and his interactions with the teacher-researchers on Tasks 1 and 2. We analyze the concepts, actions, and attention-directing interactions that we found to be key in his learning of the target concept: *The area under the rate graph measures the total accumulation* of, in this case, the volume of water. We identify two of these concepts that Ian already knew and were requisite to his successful learning through the tasks.

Task 1

When Ian began working on Task 1, he saw that the rate 400 appeared in the table five times, so he calculated the total amount of water to be 5×400 = 2000 gals. He was perplexed when the program told him this answer was 'really far below' the actual amount, and he played back and explored the animation. The TRs then directed his attention to the quantities in the table. TR1 said, "the second column gives the water rate," and Ian replied, "I see, okay." TR2 asked, "In what units?" Then Ian responded, "so we just multiply 400 times 16?" When inputted, the program confirmed this to be 'exactly right.' From that moment on, Ian consistently distinguished rates from amounts and attended to which of these he was representing when graphing or calculating. The primary role of Task 1 was not to enable Ian to significantly develop new concepts, but to invoke his understandings of rate (gals/min) in the context at hand. In particular we note a key concept and the intervention that elicited it in the pool context:

- Key (requisite) concept: *A constant rate times a duration of time gives a total amount accumulated over that time interval.*
- A key attention-directing intervention: TRs pointing out the quantity type, and units, of the outputs in the (rate) table.

Task 2

Developing and graphing an underestimate (lower staircase). In Task 2, the water rate linearly increases over the 16 seconds. Initially the table shows the water rate values at every 4 minutes [Figure 2ad], and the graph is not visible. Ian used these values to estimate the total amount of water as 50(4) + 150(4) + 250(4) + 350(4) = 3200 gals. He pointed out to TR1 that he did not need to use the last value on the table (450 gal/min when $t = 16$). The feedback from the program was that this was 'pretty far below' the actual amount of water. Ian said, "I was thinking just like steps from the way it appears," and sketched Figure 3a on paper.

At this point, Ian did several things that, in our analysis, were not crucial to the learning trajectory. He tried several ways to compensate for the lacking amount, trying to incorporate the final rate value 450 (at $t = 16$) somehow in the calculation. He made the program generate more rate values for smaller increments of time and noticed that he could do the same type of calculation. He said that "the more [steps] you have, the more accurate it is"—a valuable insight, but not key in his learning trajectory. He did use these values eventually to find an equation for the rate curve, initially trying a quadratic curve but ending up with the correct linear formula. He did not use the formula however, and after trying another several estimates, the program told him

the actual correct amount (4000 gals) and revealed the actual rate graph. This did not seem to surprise him, since he had already figured out the equation of that curve and knew it to be linear.

TR1 decided it would be beneficial to have as a referent on the whiteboard a large graph akin to what Ian sketched on his paper [Figure 3a]. He drew and labelled axes [Figure 3b] and asked Ian to draw a graph of his stairs and asked him how many he wanted to use. Ian said, "there's a number that's quite accurate while still not being massive." He drew his graph with four stairs [Figure 3c], and said, "the staircase I was kind've thinking would look something like that." Ian immediately added, "you just kind've imagine having either a line like that [straight line in Figure 3d] or in between the stairs that averages it [new line in Figure 3e]."

Figure 3a-e. A sequence of rate-vs-time graphs produced by Ian on Task 2.

These actions provided Ian with representations that were crucial to reason with. Ian's ability to represent and interpret his graph indicates that he already had the following complex understanding, and the following key actions enabled him to employ for Task 2:

- Key (requisite) concept: *A rate graph is a continuous record of how a quantity's rate of change varied over time, and each point on it is a multiplicative object* (Saldanha & Thompson, 1998) *representing the rate of flow of water at that moment in time.*
- Key actions: (a) estimating the total amount using constant rates over sub-intervals, and (b) graphing the corresponding (under-)estimating rate function as a 'lower staircase' on the same graph as the actual rate function [Figure 3d].

We briefly note that the other pair of participants we studied were unable to complete Task 2 because they could not consistently employ this concept.

Interpreting the rate graph between under- and over-estimate staircases. Ian noted that his 'lower staircase' underestimates the actual amount of water, and then he drew an 'upper staircase' [Figure 4a] overestimating the actual amount by, as he said, putting the excess "growth" at the beginning rather than at the end of the interval. He added, "This volume [tracing the larger rectangular path we have marked in Figure 4b with a solid line] is bigger than this volume [tracing the smaller rectangular path we have marked in Figure 4b with a dashed line]." Here Ian had begun to use the following new concept:

- Key (new) concept: *The area of a rectangular region under a constant rate graph* (where the rate is in gal/min) *measures a volume* (in gal).

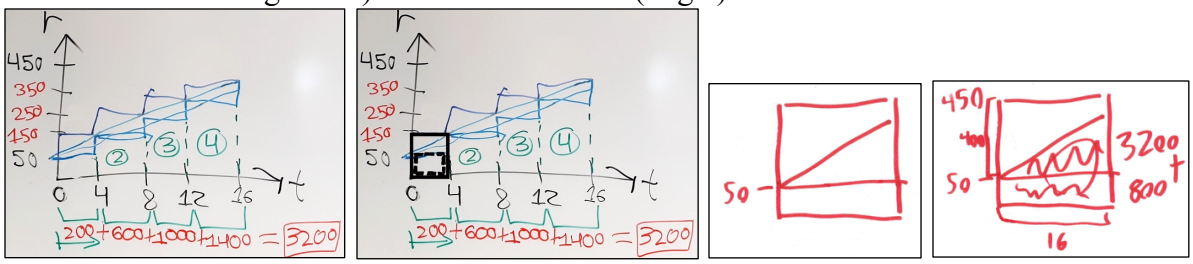

Figure 4a-d. Rate graphs and diagrams Ian used when reasoning with areas on Task 2.

Looking at the graph [Figure 4a], Ian then said there is something between the two staircases that was "more accurate, I guess, to the growth." TR1 asked where in between, and Ian replied,

"exactly in between, right?" Although he didn't say it at that moment, later Ian confirmed that this was when he strongly suspected the actual amount would be exactly halfway between his over- and under-estimates, once he saw the graph that displayed all three together (Figure 4a).

Ian then calculated that "exactly in-between" value. He determined the over-estimate amount from the upper staircase (4800), found the difference between that and the lower staircase underestimate to be 1600, took half of that (800), and added this to under-estimate. He got 4000, and since he already knew this to be the exact actual amount, he said, "Okay, so that method does work." His calculation, confirmed that the together with the earlier empirical correct answer feedback from the program, confirmed his geometrical inference from the graph.

When asked why, Ian explained by appealing to the geometry of how the rate graph cut in half the regions between the two staircases: "If you just draw any rectangle and have a triangle in it, it's half. So, I just thought since all of those are squares, it's just half. So, I just tried finding what half would be for that." Here Ian used the following new concept, which was enabled by the following key action:

- Key (new) concept: *The areas between the lower and upper staircases represent the difference in volume between the under- and over-estimates*.
- Key action: Generating the upper staircase overestimate on the same graph as the lower staircase and the actual linearly increasing rate graph.

Area under a linear rate graph. We interpret that Ian's combination of realizations enabled the following type of reasoning. Seeing that the actual volume is exactly halfway between the under- and over-estimates, and also that the region under the actual rate graph exactly splits the difference between the regions under the two staircases, together with his recent concept that the area in a rectangular region represents a volume, Ian was able to infer the following:

- Key (target) concept: *The area under the linear rate graph represents the actual exact water volume*.

This was confirmed by Ian's subsequent activity. TR1 asked if his idea allowed him to use a simpler way to find a volume just by considering the (rate) graph. Ian said, "If it's a straight line then yes," and he drew a simplified geometric diagram with just one large triangle atop a rectangle of width 16 [Figure 4c]. But then he said "No, that's wrong. I'm thinking of it as, that's rate of growth, not that it starts at 50 filled up already. Never mind. Discount that idea." This comment suggests that he was thinking through which aspects of his diagram represented amounts and which represented rates. With TR1's encouragement, Ian continued his geometric calculation [Figure 4d], getting the same answer of 4000 (referred to without units). He said, "that's interesting that it's as simple as that." "That you can just calculate the area and find the… Yeah. Right. Huh. That's pretty neat." This was an empirical confirmation for Ian of his new concept. The fact that he calculated the area of the entire trapezoidal region on its own indicates that he viewed the region as an object whose area measures the water amount, rather than just as a representational tool that suggests how to calculate that amount.

Role of Task 2 elements in Ian's learning trajectory. We have already described the role some key actions and interventions had in Ian's learning trajectory. We also noticed the role that several features of Task 2 had in this trajectory. Task 2 initially provided Ian only with the pool video and the table of linearly changing rate values, and not a rate graph. It required Ian to estimate an amount value, then gave him feedback about that value, and it let him do this six times before revealing the actual volume value and the rate graph. The limited information in the initial table afforded Ian's development of an underestimate of the true amount, since those values naturally appear as 'steps' in the table. The task led Ian to change the number of values in

the table, which allowed him to develop more accurate estimates of the rate. The delayed revelation of the actual water amount was important, because it gave him an amount value that he used to empirically confirm his developing concepts about areas as amounts. The delayed revelation of the rate graph may have been less important in Ian's case, since he happened to have already figured out the linear function that fit the table data, and he probably could have graphed it on his own if asked. Finally, we conjecture that Task 2's interactive structure in the first place made his staircases fundamentally situated within his active process of estimating the true amount, rather than simply becoming quantitatively untethered geometric objects.

Discussion

Through these interactive pool-filling tasks, Ian learned that the area under the linear rate graph represents the actual exact water volume. In so doing he used his own method of under- and over-estimating an actual accumulated amount by using lower and upper Riemann sums. Although we did not study what further learning was enabled by Ian's new concepts, here we discuss two promising affordances of his concepts. These affordances help justify why this pool filling task sequence could be more beneficial than another slightly more efficient sequence.

After all, to learn the target concept in just the linear rate case, it would likely be quicker for a learner to average out the increasing water rate over the whole interval, imagining that a pool filling constantly at that rate would end up with the same amount. Graphing could show this averaging geometrically, so they could imagine the areas serving as proxies for the amounts. In comparison with that hypothetical route, Ian's learning trajectory positions him much better to generalize his concept in calculus, because he has developed the key imagery used in calculus to conceptualize an integral as a limit of upper and lower Riemann sums.

A second affordance is that, unlike the students in studies we mentioned earlier, Ian's new area-represents-amount concept is conceptually anchored in the quantitative relationship between rates and amounts, and is not just something he noticed empirically about areas of regions. Yet there was an important part of Ian's learning trajectory that was empirical. In Task 2 he was told the actual amount accumulated in the pool. He noticed this value was halfway between his under- and over-estimates, just as the area under the rate graph is exactly halfway between the areas of the lower and upper staircases. Did he just *associate* the area with the amount, or did he develop any sense of necessity about why the area must measure that amount?

It is difficult to say from our data. Nonetheless we wonder exactly how much necessity is readily available for any learner in the process of constructing the area-represents-amount concept, through any learning trajectory. For instance, another path might allow a learner to more explicitly reason with 'canceling' units: gals/min × min = gals. But does that carry much more sense of necessity for them? Izsák (2025) proposes an approach that grounds quantitative area reasoning with integrals in a particular measurement interpretation of multiplication. This makes the units canceling feel much less like a magic trick, yet in that approach there is still fundamentally an association made between a unit square and a unit of the accumulating quantity (e.g., volume). With this in mind, our study highlights a question for further study: Is some degree of empirical association unavoidable when initially learning the concept that areas under rate graphs represent accumulated amounts? Is the concept's security based on how the learner experiences these areas as increasingly reliable tools for interpreting amounts?

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GTA Perceptions of Precalculus Students' Graphing via Covariational Reasoning

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The success of any educational change requires the support of instructors, and at the introductory undergraduate mathematics level, up to 30% of those courses are taught by graduate teaching assistants (GTAs). Research has shown the prevalence and importance of covariational reasoning at the precalculus/calculus level, and this study examines how GTAs interpret and respond to students' attempts at graphing via covariational reasoning. Through engaging in a version of Carlson et al.'s (2002) Bottle Problem, and then watching and responding to two precalculus students' video responses to the problem, 11 GTAs had the opportunity to express their teacher decision making around student thinking. I analyzed their responses using Herbst et al.'s (2023) categories of perception teachers use when considering framed student contributions: normativity, serviceability, and responsiveness. The results indicate that GTAs primarily focus on serviceable and non-normative aspects of students' work. I detail these results and discuss implications for GTA training.

Keywords: GTA training, covariational reasoning, instructor decision making

Graduate Teaching Assistants (GTAs) are often each responsible for teaching hundreds of undergraduate students over the course of their graduate program, particularly in introductory level courses. Building programs and compiling resources support these instructors, but understanding how these GTAs think about and respond to their students' thinking is both a critical and needed component to ensure that these resources meet GTAs' needs. In this study, I focus on GTA's perceptions of undergraduate precalculus students' work on a graphing task – an adaptation of Carlson et al.'s (2002) *Bottle Problem*. GTAs received student work with both quantitative and nonquantitative attempts at covariational reasoning. I analyzed 11 GTAs' responses to the work using Herbst et al.'s (2023) categories of perception, a framework enabling a focus on GTA's attention to the normativity, serviceability, and responsiveness of the student work. Analysis was guided by the research question: What do GTAs perceive in undergraduate precalculus students' work on a task aimed to support graphing via covariational reasoning?

Literature Review & Theoretical Frameworks

In this section, I discuss literature linking covariational reasoning and graphing, literature linking instructional noticing and decision making, and I describe the state of GTA training in the U.S. The first two include descriptions of the theoretical frameworks used in this study: covariational reasoning (Carlson et al., 2002) and categories of perception (Herbst et al., 2023).

Covariational Reasoning and Graphing

Covariational reasoning – reasoning how two quantities change together – is a construct built from research on students' quantitative reasoning – reasoning about and operations with measurable attributes (Carlson et al., 2002; Thompson & Carlson, 2017). Researchers have demonstrated the importance of students' covariational reasoning to support productive meanings for a variety of topics in K-12 and beyond. Relevant to this study its importance in precalculus meanings for rate of change relevant to graphing linear, quadratic, and exponential functions (e.g., Ellis et al., 2015; Johnson, 2015; Moore, 2014; Tasova, 2021; Yu, 2022).

The *Bottle Problem* (Carlson et al., 2002), in particular, was the impetus for a covariational reasoning framework; it involved students reasoning about the relationship between the height and volume of water in a spherical bottle as water was being poured into it. Their study distinguished students' mental actions based on how students described quantities changing. For example, *directional covariational reasoning* involves students describing one quantity's values increasing/decreasing/staying constant as another quantity's values are increasing/decreasing, and *amounts of change covariational reasoning* involves students describing changes in a quantity's values as increasing/decreasing/staying constant as another quantity's values changes by equal amounts. Students can use graphs to represent these relationships and use covariational reasoning to justify their graph by attending to (changes in) quantities' magnitudes. For example, for a linear relationship, students can justify a Cartesian graph of a straight line as representing a linear relationship between quantities by highlighting equal magnitude changes in one quantity and identifying corresponding equal magnitude changes in the second quantity.

As researchers have attended to students' covariational reasoning with graphs, they have uncovered just how difficult this reasoning can be. For example, students may struggle to conceptualize changing quantities as varying *smoothly*, reasoning instead with *chunks* (Castillo-Garsow, 2013). Reasoning with *chunks* can lead to attempts at *amounts of change* covariational reasoning that consider intervals of change that are too big (e.g., the relationship between quantities may shift from an increasing to a decreasing rate of change within the chosen interval) (e.g., Stevens et al., 2023). Students may also have meanings for graphs that make it impossible for them to represent dynamic relationships traced out from right-to-left or straight up and down (e.g., the graph would no longer represent a function), even if doing so would appropriately represent the relationship they conceive in their context (Moore et al., 2019).

Research on students' graphing has also identified several non-quantitative ways of reasoning. *Iconic translations* (Monk, 1992) and *thematic associations* (Thompson, 2015), for example, involve students attending to shapes and phenomenon to justify shapes in their graphs. For example, a student might conclude a graph representing a rider going around a Ferris wheel is curved because a Ferris wheel is circular, regardless of the quantities under consideration. Or a student might focus on the phenomenon that a rider traveling at a decreasing speed results in a concave down graph, even if the graph is relating two different distances (rather than distance-time graph). Called pseudo-analytical behavior (Vinner, 1997), these types of reasoning are classified when particular mental actions are indicated without providing adequate evidence they "possessed an understanding that supported that behavior" (Carlson et al., 2002, p. 358).

Altogether, this research indicates a wide variety of students' reasoning with graphs. This study aims to begin researching what GTAs, not researchers, attend to when interpreting some of the same student work that inspired these researchers.

Instructor Noticing & Decision Making

What instructors do with their knowledge of student thinking depends on what instructors notice and what their instruction makes available to be noticed. Several researchers have attended to mathematics teacher noticing (see an overview in Sherin et al., 2011), but in this article, I adopt a framework designed to categorize teachers noticing around student contributions to a problem-based task. Herbst et al. (2023) describe three categories of perception: serviceability, normativity, and responsiveness. These categories, which also rely on the idea of framing, serve as the analysis framework for this study (heretofore SNR framework).

Herbst et al. (2023) used a task-based geometry lesson as the context for their categories' examples. This context is important in determining how the other categories are interpreted. That

is, framing is "the organization of events and individual experiences according to certain principles" (p. 403), and thus identifying an event as "task-based", "geometry", and a "lesson" results in certain expectations. Herbst (2006) has developed the notion of instructional situations to describe how teachers frame students' work on problems (e.g. "construction" and "proof" for Geometry). Thus, if a teacher frames a task-based geometry lesson as involving construction, based on their expectations of what that entails, it presents the teacher with opportunities to anticipate and observe students' actions on the task, and moreover, opportunities to scaffold students' work accordingly. With this lens in mind, the following constructs are the SNR categories of perception available for a teacher to notice student contributions.

Normativity. Based on how a problem has been framed, a student's contribution is deemed as more or less normative based on the extent to which it complies with or breeches what the teacher deems to be appropriate within the frame of an instructional situation. Thus, while a correct solution to a problem would certainly be seen as normative, a student who has an incorrect solution to a "construction" problem but who used construction tools (e.g., a compass) would still have a more normative contribution than a student who sketched a diagram correctly.

Serviceability. Serviceability attends to the extent to which a student's contribution fits within the teacher's instructional goal. For example, a student with a more serviceable solution may not have a complete solution or may have overgeneralized an observation, but they contributed an observation that is relevant to the teacher's instructional goal for the lesson. In contrast, a less serviceable contribution from a student could be one in which a student provided a method to solve a task that is not the same as the method included in the instructional goal.

Responsiveness. Responsiveness attends to the extent to which a student contribution relates to the question asked in the problem. For instance, the aforementioned student who sketched a diagram correctly responded to the question asked, whereas students who change the givens or do not entirely address the question (e.g., are missing components of the solution) would be providing contributions that are less responsive.

As mentioned, Herbst et al. (2023) framework detail these constructs using examples from teachers discussing student contributions for a task-based lesson involving construction in a high school geometry class. A teacher does not necessarily perceive any one of these during a given lesson. In this study, the focus will be on GTAs' perceptions when responding to student contributions of a task-based problem for precalculus students involving covariational reasoning and graphing.

Instructor (GTA) Training

Up to 30% of introductory undergraduate mathematics courses are taught by graduate teaching assistants (GTAs). Over the course of their graduate student career, GTAs may be responsible for instructing hundreds of undergraduate students. Training for these GTAs is often less extensive than traditional K-12 instructor training programs, and moreover, faculty members responsible for developing instructional materials for GTA training often find it challenging to locate instructional materials. Nevertheless, Kung and Speer (2009) have reported that GTA instructors want to place an emphasis on building GTA's skills in probing and using student thinking. Recent efforts by the *Mathematical Association of America* (MAA), such as the *CoMInDS* resource suite and the *Instructional Practices Guide* (Abell et al., 2022) have begun to address the need for instructional materials for college instructor/GTA training. Both also provide insights into how to support understanding and using student thinking in teaching, but more work is needed to understand how GTAs currently think about student thinking and how their classroom context influences their noticing and decision making.

Methods

In this section, I describe the population and context of this study, the process of adapting the *Bottle Problem Task* into a GTA training task, and the analyses I conducted.

Methodology

This study required two phases of data collection: precalculus student responses to the Bottle Problem and GTA responses to those students' responses. First, the *Bottle Problem* was adapted into a task given to 39 students in an Applied Precalculus class at a mid-size university in the Northeast. This task included students submitting video responses to the task. From the 33 students who consented to share their classroom data, we de-identified their videos and created transcripts of each of the students' responses. These responses were collected and analyzed for correctness and level of covariational reasoning. A subset of two student videos and transcripts were then provided a group of 11 graduate students who were enrolled in a one-credit hour seminar on teaching undergraduate mathematics. These 11 graduate students were in their first or second year of teaching, and some of these graduate students either had or were currently teaching different sections of the same Applied Precalculus course. These graduate students submitted their own video responses to (a) the same adapted Bottle Problem task as the precalculus students and (b) how they would respond to the two students from the subset of student videos and transcripts. All GTAs in the course consented to sharing their classroom data.

Task Design

In this section, I describe the process of adapting a research task using for assessing students' covariational reasoning to a classroom task and then again to a GTA training task (Figure 1).

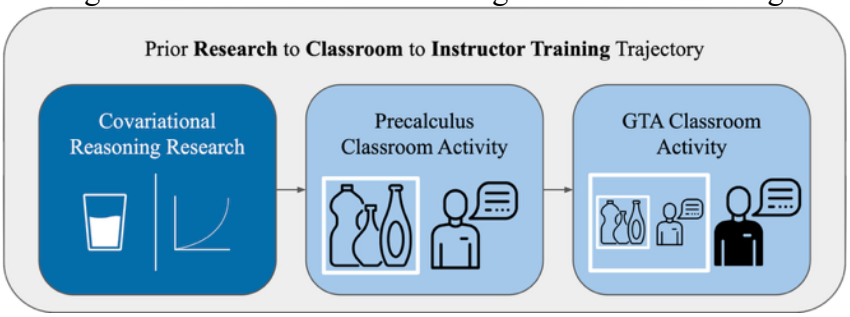

Figure 1. Transition from research to classroom to GTA training.

Adapted Bottle Problem Task. The original *Bottle Problem* Task asked students to graph the relationship between the height and volume of water poured into a spherical bottle. In Moore's *Advancing Reasoning Project*, his team adapted this problem to be used in a classroom activity for preservice secondary mathematics teachers by incorporating several differently shaped bottles. I adapted this version of the task into two parts. In Part I, students worked together in groups in class to get accustomed to creating graphs/bottles and to introduce the language of covariational reasoning to them. In Part II, students completed the activities outside of class and submitted 3-5 minute video responses of their solutions. Each part had different bottles/graphs, and in Part II, students had the opportunity to make their own bottles/graphs and needed specifically to recognize descriptions of quadratic and exponential relationships. In total, students completed 8 bottle-graph pairs.

GTA Training Task. The *Adapted Bottle Problem* video responses were analyzed according to their correctness and covariational reasoning (see details in Analysis section). For the GTA training material, the goal was to identify two videos for the GTA students to respond to. The

learning goals of the GTA activity was to (a) introduce them to procedural vs. conceptual thinking (b) introduce them to covariational reasoning and (c) respond to student thinking by attending to students' covariational reasoning. To do so, the GTAs first read an introduction to procedural and conceptual thinking, including an introduction to Carlson's Precalculus Concept Assessment (Carlson et al., 2010) and potential responses to the *Bottle Problem* (Abell et al., 2024). The GTAs were then given the same *Adapted Bottle Problem* task, including submitting their own 3-5 minute video of their reasoning.

Then, the GTAs watched the two selected videos and received the transcripts of the two precalculus students explaining their reasoning on Part II of the *Adapted Bottle Problem* task. One student exhibited *amounts of change* reasoning considering too large of intervals of change and another student used *thematic associations* with "speed." Neither exhibited entirely correct solutions. Kala's work, though incorrect, used amounts of change (serviceable), constructed expected representations (normative), and carefully color-coded and followed directions (framing/responsiveness). In contrast, Leonhard relied on language around speed to explain his reasoning (non-serviceable), constructed more than one graph for a unique bottle (non-normative), and did not follow instructions to color-code or only explained a subset of his bottles/graphs (framing/responsiveness). The GTAs then submitted a second 3-5 minute video on how they would respond to those students.

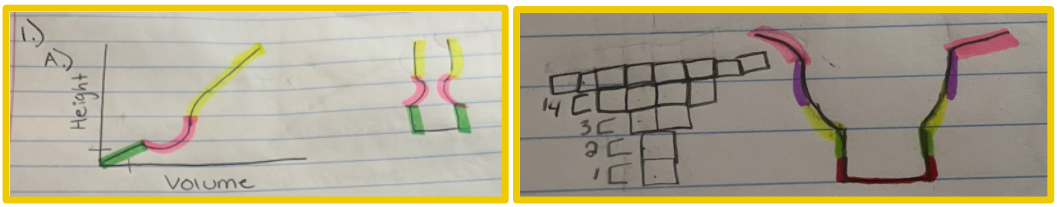

Figure 2. Sample of Kala's work on the Adapted Bottle Problem.

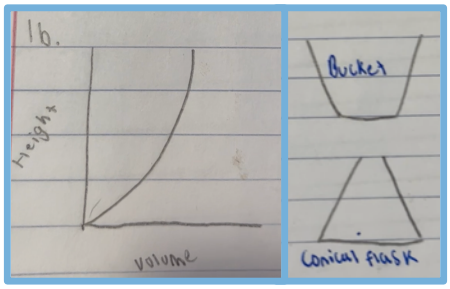

Figure 3. Sample of Leonhard's work on the Adapted Bottle Problem.

Analysis

The goal of the analysis was to attend to the GTA's responses to students' covariational reasoning. To do so, I engaged in a two-part analysis: on GTA responses to the *Adapted Bottle Problem* and on GTA responses to student solutions of the problem. In this manuscript, I focus on the latter part. In this part, I used the SNR framework from Herbst et al. (2023) to code the GTA responses to the student work. I used transcripts of the videos and referenced accompanying student work to code GTA utterances/pointers.

In particular, I created codes for each of the categories of perception. *Correctness* was coded based on the GTA's perception of a student's final graph shape. *Framing* was coded based on the GTA's perception of the prompt. *Normativity* was coded based on GTA's perception of student's production of reasonable responses. *Serviceability* was coded based on GTA's perception of a

student's covariational reasoning and/or discussing mathematical relationships, with specific codes pointing to *directional* and *amounts of change* covariational reasoning. However, because these GTAs were new to the construct of covariational reasoning and Kala's amounts of change reasoning was not entirely correct, to code for serviceability, it was only required for the GTAs to attend to quantities students identified and how those quantities were changing together. *Responsiveness* was coded based on GTA's perception of a student's response as a complete response to the problem statement, including color coding.

Moreover, the SNR framework enables the negation of each of these codes (non-normative, for example). Particularly given the open nature of the task, the line between the two can be difficult to define objectively, and so determinations were based on GTA's reactions. For example, one GTA may consider a graph correct as long as the general shape is appropriate (e.g., curved vs. linear segments) whereas another GTA might consider a graph incorrect if the relative steepness of lines in a graph are not considered. Thus, to code for students' responses to student work, an initial code of serviceability, normativity, responsiveness, and correctness was coded. Then, if the GTA's reaction to the work was positive (e.g., praising) the work based on that category, the code remained as is; if the GTA's reaction to the work was negative, the code was negated (e.g., correctness to incorrectness).

Results

In this section, I detail the results of the analysis on GTA responses to Kala and Leonhard's student work. The overall distribution of GTA comments are in Figure 4 below. These results indicate that GTAs largely attended to the serviceable and non-normative aspects of the student work. In the remainder of this section, I detail some of the findings and note what particular aspects of the student work the GTAs perceived.

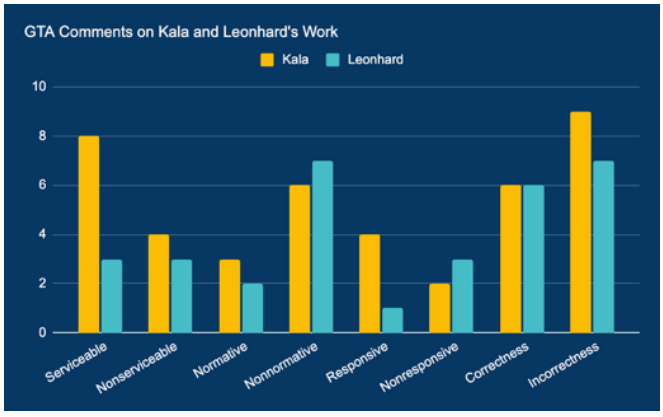

Figure 4. GTA perceptions of Applied Precalculus students' responses to the Adapted Bottle Problem.

GTA Responses to the *Adapted Bottle Problem* and Framing

Although not prompted, many (5) GTAs recognized how difficult the problem itself was, sympathizing with the students. For instance, one GTA remarked that the problems are "not like super intuitive at first glance". In particular, for one of the problems, a GTA remarked they would have preferred the axes to be switched because as is, it "kinda throws off a bit based on conventional [axes]." Two instructors noted that the framing of the problem would require additional instructor support. However, in general, GTAs still had a positive response to the problem, with one stating, "I'm all for it. I think it's good to keep them on their toes."

GTA Responses to Kala's Work

For Kala's work, one of the key perceptions from GTAs were that the work was non-normative. Four GTAs commented on the pink "dip" in Kala's graph in Figure 4 (left). One stated that Kala "dipped the pink part down and said that it decreases but she wasn't fully aware of the fact that she was talking about the decrease in the rate of change." The same GTA called the shape "kinda weird because that wouldn't happen on any of these graphs." These comments attend to the non-normative nature of Kala's solutions, specifically incorporating Kala's covariational reasoning in her description of Kala's contribution.

The second key perception from GTAs was that the work was serviceable. Seven GTAs praised Kala's attempts to use "blocks" (Kala's language) to support her reasoning (see Figure 4 right). One GTA stated "the way she approached it with the squares does demonstrate that she understands the concepts behind the assignment." Another GTA stated that she saw that Kala used this approach "so she can get an idea" of the problem. These comments attend to the serviceability of Kala's approach, specifically using the blocks to support not only constructing the graph but justifying her graph via amounts of change covariational reasoning.

Lastly, and somewhat surprisingly, three GTAs praised Kala's use of color-coding. It remains unclear whether or not the teachers saw color-coding as part of the framing, since Leonhard did not color code and the two GTAs specifically noted a need for him to do so were part of the three that praised Kala's color-coding.

Leonhard's Work

For Leonhard's work, the key perception from GTAs was that the work was non-serviceable and incorrect. One GTA thought Leonhard "did not seem to have as much of an understanding of rate of change" and just "throwing out a few terms." Another GTA noted that Leonhard himself remarked on wishing he had numbers, and stated "since he's given the relationship, you can certainly get numerical values like Kala did." Only three GTA specifically referenced the "faster/slower" speed language (and two praised it). These comments attend to the non-serviceable ways in which Leonhard justified his actions. Overall, GTAs noted some pseudo-analytic actions Leonhard took (even if viewed positively), but not thematic associations.

The second key perception from GTAs was that Leonhard's work was nonnormative. Six students commented that Leonhard produced *two* different bottles for the same graph (Figure 3), though one GTA praised it. As one GTA discussed, "he had a couple options for the bottle. He did say that both of them worked even though both of them were quite different." That is, the two bottles Leonhard produced represent two different rates of change between height and volume, making it impossible for Leonhard to have an appropriate solution. These comments attend to the nonnormative (and also incorrect and nonresponsive) nature of Leonhard's work.

Discussion

Using the SNR and covariational reasoning frameworks, this study showed both that GTA's perceptions span the categories of normativity, serviceability and responsiveness, and that these 11 GTAs primarily focused on the serviceable and non-normative aspects of undergraduate precalculus work on a graphing problem via covariational reasoning. Looking at GTA's comments more closely, it seemed that, even with little introduction to the constructs of covariational reasoning, GTAs were able to perceive and attend to students' covariational reasoning and pseudo-analytic actions in this task. Further research can look at the influence of the task including an instructional goal of covariational reasoning in GTA's perceptions and also the impact additional GTA training in covariational reasoning has on GTA perceptions.

Acknowledgments

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Qualitative Analysis of Student Responses to Metacognition Prompts in College Algebra

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Students in a college algebra course completed reflective metacognition journal assignments about their studying and performance in the course. This paper is a thematic analysis of the qualitative data collected from those journal assignments and surveys about the effects of the journal assignments on their studying and performance in the course. We investigate how the journal assignments affected the study strategies they used in college algebra and what students learned about themselves as learners through these reflective assignments. Findings indicate that metacognition journals were effective in supporting students in reflecting and changing their study habits. Qualitative analysis is used to describe students' reflection and changes to study habits after engaging with journals embedded in a mathematics course.

Keywords: College Algebra, Metacognition, Developmental Math, Qualitative Analysis

The first year in university can be challenging for students academically. First year courses traditionally have higher rates of student failure or withdrawal and have a direct impact on student retention and degree completion overall (Flanders, 2017). Developmental mathematics courses like college algebra, which often do not serve to meet a degree requirement but rather prepare students for further university level mathematics courses, are no different (Tunstall, 2018). To support students in successful completion of college algebra researchers and instructors at several institutions have worked to redesign the course to include more active learning methods, revised curriculum, and innovative pedagogical approaches like modeling-based learning (Tunstall, 2018; González-Muñiz et al., 2012). While these reforms show promise, there is still work to do in supporting students in understanding mathematical concepts during a critical course in their first year at university.

One approach to improving student learning in mathematics is through supporting self-directed learning via metacognition. Metacognition is defined as "thinking about one's own thinking" or awareness or analysis of one's own learning or thinking processes (Tanner, 2017; Flavell, 1979). Although several frameworks exist to understand the process of metacognition, we conceptualize metacognition to include the skills of goal setting or planning, monitoring progress, and evaluating performance (Lovett, 2011; Tanner, 2017). While we know that acquiring and practicing these skills are critical to student performance in mathematics (Verschaffel et al., 2017), many students enter their first year of university without direct instruction or experience with planning, monitoring, and evaluating their mathematics learning. Within STEM, one approach to improving student learning is through incorporating metacognitive instruction alongside academic content within the course itself (Lovett, 2011; Tanner, 2017; Colrthorpe et al., 2019). These metacognitive supports may include reflection on learning through exam wrappers (Lovett, 2011), discussion prompts (Colthorpe et al., 2019), and questionnaires (Lucangeli, et al, 2019). These approaches are appealing to educators since "... metacognition continues to be a predictor of mathematics performance" (Desoete & De Craene, 2019, p. 556) and their implementation are a brief and manageable addition to a course for both instructors and students (LaCaille et al., 2019).

In this study metacognition journal assignments were embedded throughout a college algebra course. Students completed seven journals and a post survey regarding their experience with the journals. In this paper we analyze responses from one journal and the post survey. As we examined these student responses, we sought to answer the following research questions.

1. [RQ1] Did students report changing their study habits due to metacognition journal assignments? If so, how?
2. [RQ2] How do students define a productive study session? What can we learn from this about their study habits in college algebra?
3. [RQ3] What did students learn about themselves through metacognitive journaling in college algebra?

Methods

Metacognitive Intervention

The college algebra course was incrementally redesigned between 2020 and 2022. One of the key changes was the addition of metacognition journals to the course. These metacognition journals were graded assignments; students received full credit once they completed a journal assignment. There were seven metacognition journals throughout the semester and together made up 4.2% of the final grade. As part of the course redesign process, students were given surveys at the beginning (pre-surveys) and end of the semester (post surveys) to assess the effectiveness of the course changes. Beginning in Fall 2022, students who completed the surveys were asked to consent in a research study that would allow their survey responses, their course grades, and coursework to be analyzed. (They did receive bonus points for completing the pre and post surveys.) In this paper, we are analyzing and discussing responses of study participants on one of the metacognition journal assignments and questions from the post survey. This study was conducted over three semesters: Fall 2022, Fall 2023, and Spring 2024.

Data Collection

In Fall 2022, 473 of 754 students enrolled in the course (62.7%) participated in the study. The Fall 2023 semester had 538 of 672 students enrolled (80.1%) participating in the study. Enrollment in this course is much lower in the spring semesters. In Spring 2024, 107 of the 133 enrolled students (79.2%) participated in the study. Thus, the study had 1,118 participants.

The post survey asked students specific questions about the metacognition journals. One question asked students which of the metacognition journals was the most useful for them. According to this question, Journal 2 was the most helpful, and it is for that reason that we chose to analyze students' responses from that journal assignment. We also analyzed responses from two questions about metacognition journals from the post survey.

Coding and Analysis

Journal responses were coded through an iterative and collaborative process with three researchers. An inductive and comparative method (Mirriam & Tisdale, 2016) was used to develop themes from the data. During the first round, open coding (Saldana, 2016) was used by two researchers to develop initial codes. The researchers then refined those codes, coming to an initial agreement on definitions. These codes were tested on a sample of student responses and researchers met again to check coding agreement and utility of codes. During this second round of coding, definitions were refined and additional codes added when needed. A third and fourth round of sample coding was performed until agreement between researchers was reached on all

codes. The remaining student responses were then coded by a single researcher. After all responses were coded, researchers engaged in thematic analysis (Glesne, 2018) of student responses. Themes were developed by researchers through analysis of patterns within the coded data.

Data and Results

Journal 2

During the second and third week of the semester, students were asked to log when they studied for their college algebra course. Between the two weeks, students were given information about spaced practice and how it is shown to be more effective than studying all at once. After two weeks, students completed the metacognition journal assignment. We are investigating the open-ended responses students submitted on that assignment about how the classify productive study sessions and the changes about their study habits.

Productive study session. Students responded to the prompt "How do you define a 'productive' study session?" We created fifteen codes that were grouped into five themes.

1. Environment. This theme includes whether people studied alone or with others and the level of distractions within the environment.
2. Outcomes. This theme is characterized by what students finished in the study session; for example, did they learned something new, complete a task, or work efficiently?
3. Planning. This theme includes codes scheduling time to study or making lists of tasks to complete during their study sessions.
4. Self/Affective. This theme includes codes that describe students' ability to focus during a study session or feeling confident after the study session.
5. Study Habits. This theme is characterized by the types of study methods students used during a study session, e.g. explaining concepts to others or reworking examples.

The 1,116 responses produced 2,091 codes. To avoid double counting responses that contain two codes from the same theme, we are reporting the percentage of codes instead of percentage of responses. Figure 1 shows the percentage of codes for each of the above themes.

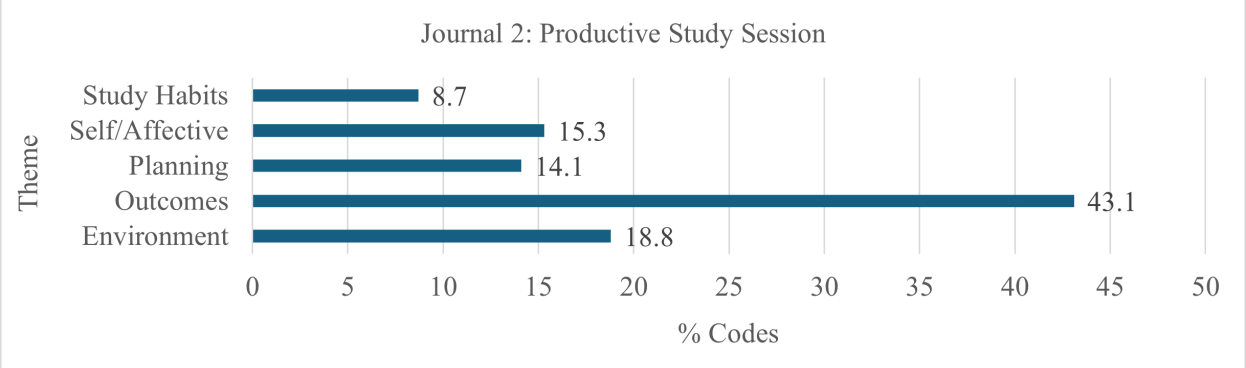

Figure 1. Theme vs. % Codes from Journal 2: Productive Study Session.

Study changes. Students responded to the prompt "Reflect on what we discussed in class about metacognition and what you discovered in the last couple of weeks about how you study and learn effectively. What changes will you make to your study strategies? What study strategies will you continue?"

We created twenty codes and categorized the codes into four themes.

1. Environment. This theme includes codes related to where students will study, how they will reduce distractions, and whether they will study alone or with a group.
2. Evaluating. The only code in this theme is "evaluating/reflecting" and was used when a student mentioned reflecting periodically on their study skills to see if they want to make additional changes.
3. Planning. Codes in this theme relate to students planning when they would have a study session, including if they intend to do spaced practice, or planning what topics they will cover in a study session.
4. Study Habits. This theme is characterized by study practices or habits students will use in the course. Habits range from general best practices like attending class to specific study skills like rereading notes or self-quizzing.

There were 1,117 responses which produced 2,182 codes. The percentage of codes for each theme is displayed in Figure 2.

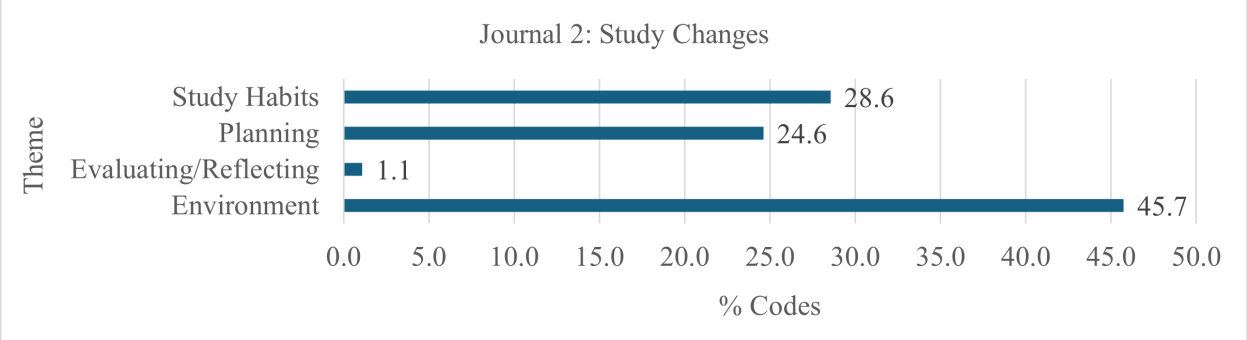

Figure 2. Theme vs. % Codes from Journal 2: Study Changes.

End of Semester Survey

At the end of the semester, students were invited to complete a post survey that included questions about their self-efficacy, their mathematical anxiety, and general course questions. The questions on self-efficacy and mathematical anxiety were direct parallels to questions asked during the pre-survey to track their progress over the semester. This line of inquiry is being explored within the same data (Preschel, 2025) but is outside the scope of this paper. The post survey included multiple choice and open-ended questions about students' experience with and the effects of metacognition journals.

Self as learner. We coded student responses to the prompt, "What (if anything) did you learn about yourself as a student through these metacognition journal assignments?" We generated a total of nineteen codes and grouped them into five themes.

1. Planning. This included codes where the student mentioned future-oriented practices like setting goals for a study session or a course.
2. Self/Affective. This theme included codes that referenced internal aspects of the student. Examples include mentioning how well they focus or that they learned something about how to motivate themselves.
3. Study Habits. This theme included codes that referenced habits, tools, or actions that would aid them in study, (e.g., using outside resources, or trying a new study skill.)
4. Evaluating/Reflecting. This theme included codes that showed students learned either to diagnose which material they needed to study or to reflect on how their studying was going.

5. Did not learn anything. This theme was a singular code where students mentioned that the journals did not provide them with anything.

This prompt provided us with 1118 responses with 1020 codes. See Figure 3 for the percentage of codes in each theme.

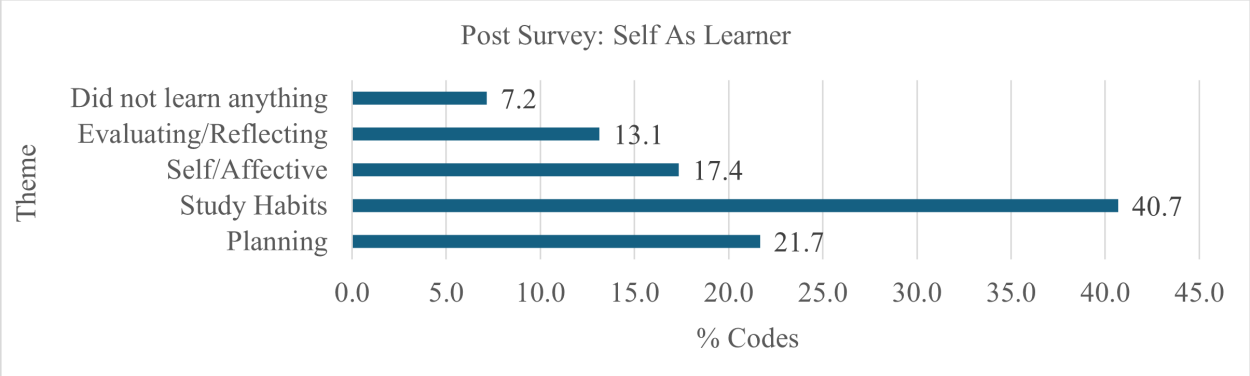

Figure 3. Theme vs. % Codes for Post Survey: Self as Learner.

Study changes. Students were asked to "Describe any changes you made to your study habits because of the metacognition journal assignments." A total of twenty-seven codes were sorted into five themes.

1. Environment. This included codes related to external factors to the student and the environment they studied in.
2. Planning. This included codes where the student mentioned future-oriented practices that do not occur within the moment of studying.
3. Study Practices. This included codes that pertained to actions that occurred while studying or events/actions that were considered studying.
4. Self/Affective. This theme included codes that referenced internal aspects of the student. Examples include increased confidence or activities to reduce stress.
5. Study Habits – General. This theme was for codes that gave vague statements about study habits that were not as specific as other themes (e.g., "No change" or "I changed my habits").

This prompt provided us with 1118 responses with 1101 codes; the percentage of codes in each of the five themes is in Figure 4.

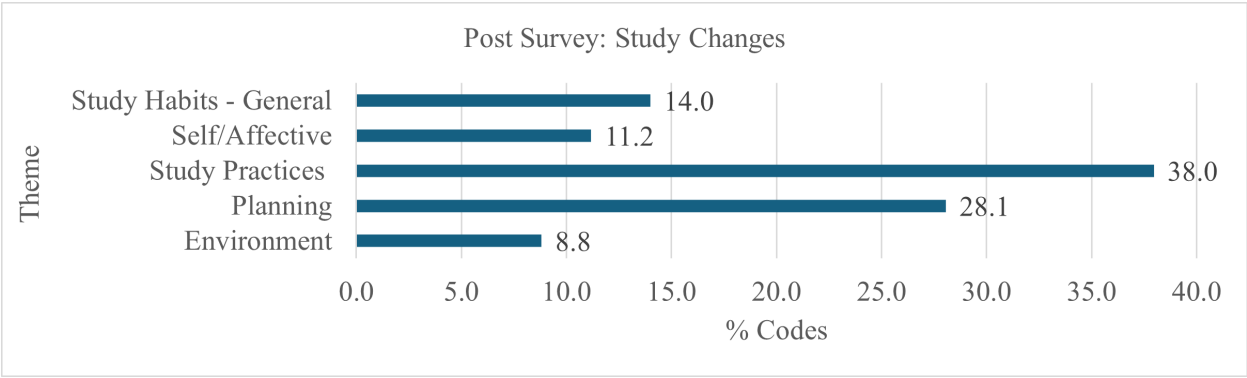

Figure 4. Theme vs. % Codes for Post Survey: Study Changes.

Discussion

Journal 2

In the discussion that follows, percentages reported for themes will indicate the percentage of total codes for the prompt, while percentages for reported codes will correspond to the percentage of student responses with that code.

Responses to the Journal 2 Productive Study prompt, which give insight into RQ2, fell into five thematic categories: Environment, Outcomes, Planning, Self/Affective, and Study Habits. The most prominent theme was "Outcomes" at 43% of coded entries. (See Figure 1.) The highest occurring codes in this theme were "Complete a Task" (40%) and "Understanding/Learning" (37%). While these two codes may provide students with a sense of productivity in their study session, we questioned if these measures indeed align with student learning. The least frequent theme, "Study Habits" (9%), included study habits that would give more concrete measure of student learning, including the codes "Retrieval Practice" (4%) and "Explains Concepts to Others" (2%). These practices would provide students with more direct measure of their learning but were much less frequently reported by students as part of a high-quality study session. Related, the theme "Self-Affective" (15%) reveals that students often rely on "Feeling Confident/Good" (8% of responses) or being able to "Focus" (20% of responses) as an indicator of a productive study session. As we know novice students frequently overestimate their skill or feel overconfident, this may or may not be associated with increased learning. Further, "Focus," without attention to high-impact study strategies, like reading over notes versus retrieval practice, may or may not align with learning. Lastly, we see that students often look to their "Environment" with 32% of students reporting "Limiting Distractions" as how they measured a productive study session.

For the Journal 2 Study Changes prompt, students described changes they would make to their studying after monitoring their habits for 2 weeks. Since this journal was assigned at the beginning of the semester, the responses give us a baseline measure for RQ1. The most frequent change reported by students was the theme "Environment" (46%) which included the codes "Remove Distractions" (35.5%) and Location (34.7%). Students understood the importance of the environment in relation to their studying after reflecting on their study sessions. Furthermore, students perceived the study environment as a significant barrier to their studying, and they planned to improve it by intentionally limiting distractions or moving to a location more conducive to studying. The second most frequent theme was "Study Habits" (28.6%) which included codes related to study habits students noted were useful in their studying including "Increasing Study Time" (8.8%), "Working/Practice" on problems (8.6%), "Taking Breaks" (6.7%), completing "Assignments" (8.3%), and "Office Hours" (6.1%). While adoption of these practices is positive, high impact study habits like "Spaced Practice" and "Retrieval Practice," were less frequently reported by students. Finally, "Planning" (25%) was another common theme within student responses. Students reported planning the time that they studied (19.2%), spacing out the timing instead of cramming (21.6%), and planning what they worked on during a session (2.1%) were all changes they would implement after monitoring their study sessions. Responses to the Post Survey Study Changes prompt indicate how successful students were in continuing these habits.

Post Survey

As noted in the Results, the most prominent theme for the Post Survey Self as Learner prompt was "Study Habits" with 40.7% of coded entries. This theme included responses where

students mentioned they "Learned to Study Effectively" (14.3%) or referred to their "Study Habits" (14.7%) or "Study Frequency" (4.5%). These findings address the second part of RQ1 ("If so, how [did their study habits change]?") by showing that students described their changes with increasing specificity. Many codes within this theme also carried an evaluative component. For example, "Time Management" (7.8%) included responses where students said they were good at time management or needed to improve. This not only helps to answer RQ3, but it shows students are identifying their strengths and weaknesses regarding study skills. While responses to the Post Survey Self as Learner prompt highlight students' broad perceptions about their study skills, the second prompt from the post survey offers a more detailed view of the specific changes students reported making.

The codes for the Post Survey Study Changes prompt provide insight into how students' study practices varied and what outcomes emerged. The three most frequent codes, each clearly separated from the rest, were "Increase Study Time" (11.2%), "Studied Earlier for Exams" (10.9%), and "Spaced Practice" (10.4%). These fall into two different themes: Planning ("Spaced Practice" and "Studied Earlier for Exams") and Study Practices ("Increase Study Time"). Overall, the themes "Study Practices" (38%) and "Planning" (28.1%) were the most common themes. This suggests that students not only adjusted the concrete ways they studied but also developed stronger planning strategies to support those habits.

In relation to RQ1, the Post Survey themes and codes added both detail and precision, while opening avenues for future inquiry beyond the scope of this paper. For example, future analysis could examine student outcomes associated with codes, explore connections across codes, prompts, and themes, or trace temporal patterns of change across the semester. Future work could investigate how to strengthen support for students. For example, a support that was implemented in this course (but its study was outside the scope of this paper) was Metacognition Mentors. Additional considerations could include adapting prompts and support to align more closely with the study habits revealed in this project.

Conclusion

Together, these findings suggest that the metacognition journals effectively introduced students to metacognitive practices and encouraged reflection on those practices. At a base level, 93.5% of students stated they learned something about themselves as learners through the metacognition journals. The responses show that students are beginning to think about what helps them learn best. However, students may not be good at evaluating how well they truly understand a concept. For example, students measured productivity in a study session by focusing more on finishing tasks, their environment, and how they felt while studying. Not surprisingly, when students planned on making changes, they primarily focused on removing distractions with a secondary focus on the study habits they implemented. At the end of the course, when students reflected on the study habits they changed, they were most likely to mention more time studying, study earlier for exams, and include more spaced practice. The frequent mention of spaced practice shows growth in students' awareness about high-impact study practices. Their end of course reflections also indicate that students knew what appropriate study habits for themselves would be, even if they did not meet their own expectations. Thus, it is important for instructors to discuss and model effective study practices and encourage students to continue to reflect and try new strategies.

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Students' Mathematical Reasoning About the Period of the Sum of Trigonometric Functions

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This study investigates students' mathematical reasoning when determining the period of the sum of two cosine functions through exploratory teaching interviews. Two undergraduate students engaged with Desmos applets to explore the sum of two cosine functions. The analysis revealed output-oriented reasoning that focuses on identifying input values where function output repeats, and alignment-oriented reasoning that focuses on where the period of the sum function aligns with the periods of individual trigonometric functions to determine the period of a sum of two trigonometric functions.

Keywords: trigonometric functions, periodic functions, mathematical reasoning

Introduction

The periodic nature of trigonometric functions is often highlighted in mathematics education (Weber, 2005; Ortega, 2017; Brown & Rice, 2011), although relatively few studies examine how students understand the periodic behavior of these functions. This paper presents findings from exploratory teaching interviews (Steffe & Thompson, 2000) aimed at exploring students' reasoning about the sum of two cosine functions. The study is driven by an interest in how students develop foundational knowledge relevant to Fourier series. Fourier analysis is an interdisciplinary topic connecting mathematics, physics, and engineering education. Although the Fourier series is a significantly advanced mathematical topic, it is often missing from standard undergraduate curricula. When learning Fourier series, the importance of periodic functions is recognized as central (Leith, 2024; Mays and Loverude, 2019; Jia et al., 2011). Fourier series breaks down complex periodic functions into sums of sine and cosine functions, which are used to analyze, design, and optimize models in physics and engineering. The initial phase of the study suggested the need to investigate students' progression of thinking as they learn about the periodic behavior of sums of sine or cosine functions. Therefore, this paper addresses the research question: *What mathematical reasoning do students use to determine the period of the sum of two cosine functions?*

Literature Review

While many studies (Blackett & Tall, 1991; Doerr, 1996; Brown, 2005; Weber, 2005; Steckroth, 2007; K.A.Thompson, 2007; Moore, 2014) have investigated various aspects of trigonometric functions, their periodic behavior remains underexplored. Educators and practitioners (Miller, 2001; Martinez Ortega, Preciado Babb & Velasco, 2017) emphasize innovative teaching strategies incorporating physical movement to deepen students' understanding of periodic functions. Stupel (2012) advocated for connecting the periodic properties of trigonometric functions with fundamental number theory concepts, particularly through dynamic visualization tools like GeoGebra that enable students to explore how changes in parameters of a function affect the pattern and periodicity. Studies by Buendía and Cordero (2005) and Shama (1998) explored how students understand the periodic nature of functions and periodicity in general. The findings suggest a disconnect between recognizing periodic behavior in situations and relating this periodic nature to a formal mathematical context. Recent research (Khan, 2025) examined an undergraduate engineering student's conception of periodic functions,

finding that students conceptualize a function's period as the distance between two distinguishable points, such as maxima, where output values repeat, and patterns repeat through the domain. However, despite this saturated work on periodic function understanding, there remains a significant gap in research investigating how students develop mathematical reasoning about the periodic nature of the trigonometric sum, particularly using graphing technology like Desmos.

Theoretical Perspective

Constructivism serves as the foundational learning theory for this research. According to constructivist theory, knowledge is not passively received but actively constructed by an individual. Resnick and Glaser (2016) describe constructivism as a framework in which learners build understanding by integrating new information with their existing knowledge. This paper explores how students develop mathematical reasoning about the periodic nature of the sum of two cosine functions. The exploratory teaching interview (Steffe & Thompson, 2000) rooted in constructivist principles, provided the necessary framework for this investigation. This approach allowed the design of strategic perturbations through researcher prompts and tasks that reveal students' evolving conceptions as they work through the tasks. Khan (2025) reported a student's thinking of periodic functions. The findings enabled the formulation of a hypothetical framework (Table 1) describing students' reasoning about periodic functions and guided the methodological approach of this study.

Table 1. A hypothetical framework of students' thinking of periodic functions.

Setting/Context	Aspects of Concept Image
General conception of periodic functions	A student thinks that a periodic function has repeating and unchanged pattern in a graph.
	A student thinks that a periodic function has same output values after a certain interval.
	A student thinks that a periodic function has repetitive output after a period throughout the domain.
To determine whether a function is periodic	A student thinks that a periodic function has same output values for two reference points and there is an unchanged pattern within these two points. And the pattern keeps repeating throughout the domain.
	A student thinks about an unchanged pattern as a chunk that is repeated throughout the domain.
	A student thinks about a function being periodic in terms of function formula. If there is a horizontal shift P, then $f(x+P)=f(x)$.
Understanding of 'a period' of periodic functions	A student thinks about a period as the distance between two maximum or minimum points of the function.
	A student thinks about a period as the interval for which the function's graph remains unchanged throughout the domain.
	A student thinks about a period as a constant value p. For a horizontal shift p, the function produces same output.

Methodology

This study employs exploratory teaching interviews with two participants to investigate how students use their conception of the period of a trigonometric function in developing their understanding of the period of the sum of two trigonometric functions.

Two volunteer participants, Eloise and EL (pseudonyms), were recruited from a pool of students who participated in another interview that explored their conception of periodic functions. Both participants had prior exposure to trigonometric concepts and functions.

Data were collected through interviews conducted via the Zoom video conferencing platform. The interviews were scheduled as one-on-one sessions and video recorded in full to capture both the participants' verbal explanations and on-screen interactions. During the interviews, participants engaged with a dynamic task using Desmos, a web-based graphing calculator.

In this interview, the participants worked on four tasks that were designed to explore the progression of their thinking about the period of a function that represents a sum of two or more trigonometric functions. In this paper, we report on Task B, where participants were presented with two cosine functions, $\cos(mx)$ and $\cos(nx)$ where m and n were integer parameters ranging from 1 to 10. In addition, a summation function $h(x) = \cos(mx) + \cos(nx)$ was defined. Using the Desmos applet, the participants were able to graph either the individual functions or the combined function h. The Desmos environment provided interactive sliders for the parameters m and n, allowing the participants to dynamically adjust the values and observe the resulting changes in the graph(s). The following figure is a snapshot of the Desmos applet.

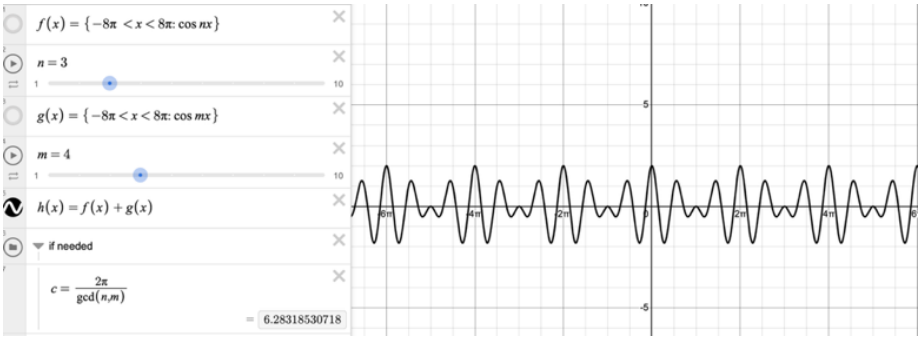

Figure 1. A view of Task B in Desmos

To analyze the data, the qualitative data analysis method (Saldaña, 2021) was implemented. I first transcribed using Adobe Premiere Pro and then manually corrected auto-transcription errors. I read through the data and named codes based on themes. I labeled the data into discrete parts based on the codes I created. Next, I compared similar events in the data and collated excerpts that were labeled with a particular theme. Then, I drew connections between the codes to find how the codes can be grouped into categories to answer the research question.

Results

In this section, I report on Eloise and EL's evolving thinking process in exploring the period of $h(x) = \cos(mx) + \cos(nx)$. Before task B, Eloise and EL were introduced to Task A, where they explored the periods for the functions $y = \sin(3x)$ and $y = \cos(5x)$. First, Eloise determined the distance of a pattern that repeats within a graph of these trigonometric functions with different frequencies, then she tried to choose input values for which the output values

repeat. She mentioned the numerical difference between these input values as the period of the functions. EL, on the other hand, approached the same task differently. She used proportional reasoning to rationalize that the period of $\sin(3x)$ would be $2\pi/3$. First, she pointed towards the length of one period of $2\pi/3$, then mentioned that as she knew the period of the sine function is 2π, the period of $\sin(3x)$ would be $\frac{2\pi}{3}$ because if she changes the input for the function within the length of 2π, there are three times as many periods for $\sin(3x)$ compared to $\sin(x)$. The interviewer then asked the participants to think about the period of $y = \sin(3x) + \cos(5x)$. Eloise imagined a graph with some oscillations but was unsure about whether it was periodic. However, after the interviewer presented the graph of this sum function, Eloise again looked for the distance between input values for which the output values are the same. In response to the same question without any probing, EL initially imagined the sum of $\sin(3x)$ and $\cos(5x)$ as a periodic function with a period that would be common for both functions. In her words-

> "If the period of A ($\sin(3x)$) is $2\pi/3$ and the period of B ($\cos(5x)$) is $2\pi/5$, then the period of the combined one will have some integer times $2\pi/3$ and also some possibly different integer times $2\pi/5$. And from what I know $\frac{1}{3}$ and $\frac{1}{5}$, these are not multiples of each other, and their least common multiple, I think, is 1. So, I would say the period would be 2π".

Task B is designed to explore the reasoning students' use in determining the period of the sum of two trigonometric functions. Eloise and EL were presented with the Desmos applet, and they initially explored the applet on their own. After interacting with the applet, Eloise initially perceived that as m and n gets larger, the graph of $h(x) = \cos(mx) + \cos(nx)$ may have a similar pattern to $\cos(x)$ with many peaks in between intervals. While exploring the applet, EL noticed that a regular cosine function with a frequency m makes the period smaller as the value of m gets larger. She also mentioned that the period of the sum of two cosine functions with different frequencies depends on the multiple of the periods of each regular cosine function with frequencies m and n.

Next, in this results section, I will present Eloise and EL's thinking to determine the periods of the function $h(x) = cosmx + cosnx\ where\ 1 \leq m \leq 10, 1 \leq n \leq 10$ for different combinations of m and n.

For Cases $n \neq m$, With no Common Multiple

The interviewer asked Eloise to list the periods of the function h, for different combinations of (n, m) as n, m takes integer values between 1 and 10. Eloise relied on the graph of h to see for which x values the output repeats, focusing on the peak-to-peak distance of the interval. She used this reasoning to find periods of h for combinations like $(1,2), (2,3), (1,10)$. She thought that for (n, m) combinations that $n \neq m$ have no multiple in common, the function repeats every 2π interval. EL, in contrast, for the same situation imagined that the function h will start repeating at 2π, as each $\cos(mx)$ and $\cos(nx)$ need to meet at a place where the period of h is a multiple of both the periods of $\cos(mx)$ and $\cos(nx)$. When (n, m) are different with no multiple in common other than 1, the period of the sum would be 2π. Although Eloise relied on the function formulas, EL conceived the period of h as the distance between points where the individual periods of $\cos(mx)$ and $\cos(nx)$ '*meet*' or coincide.

For Cases Where $n = m$

When $n = m$, EL continued to demonstrate her focus on underlying relationship between the period of h with n and m. However, Eloise initially imagined the period of h would be 2π. To verify her assumption, she utilized the graph of h, and noticed that the period is π. She paused to reconsider and reflected on what the output of h looks like for $n = m$. Eloise realized that if $n = m = 2$, then $h(x) = 2\cos(2x)$, and the period would be half of $2\cos(x)$ which is π. For $n = m = 3$, Eloise focused on the pattern to determine the period of h. She thought $2\cos(x)$ as the base function and compared the period of $2\cos(3x)$ with $2\cos(x)$. She mentioned that the period of $2\cos(3x)$ would be $1/3$ of the period of $2\cos(x)$. At this point, Eloise started recognizing the multiplicative aspect of the pattern of change in the period of h with the changes in n and m. The following figure shows that she wrote a hypothesis. She thinks that when $n = m$, the period of h would be $2\pi/n$.

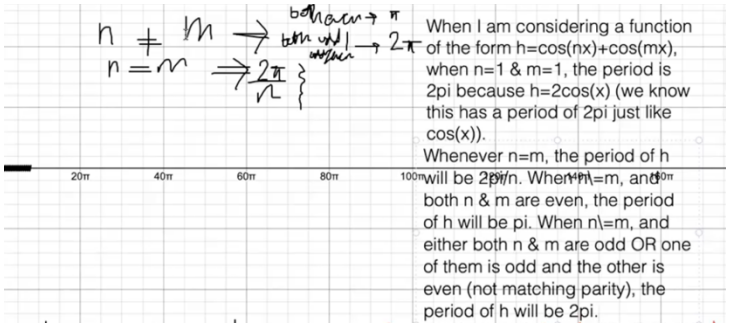

Figure 2. Eloise's initial hypothesis for the period of h, when n=m and $n \neq m$

For Cases $n \neq m$, With Common Multiple

After finding some periods of h for different combinations of (n, m), Eloise also came up with a hypothesis that for cases when $n \neq m$ and n, m are both even, the period of h would be π. When $n \neq m$, and either both n and m are odd, or one of them is odd and the other is even, the period of h would be 2π.The interviewer then offered Eloise to consider periods of h for combinations like $(6,4), (8,4), (3,9)$. Her hypothesis failed when she determined the period $\frac{\pi}{2}$ from the graph of $h(x) = \cos(8x) + \cos(4x)$. Using the graph Eloise demonstrated that the period of a function is the distance of an interval between two repeated outputs of the function for $(n, m) = (8,4)$. She looked for the inputs of h for which she could get the repeated output values. For instance, she thought that if the input for $h(x) = \cos(8x) + \cos(4x)$ is $\pi/2$, then $\cos\left(8 * \frac{\pi}{2}\right)$ and $\cos\left(4 * \frac{\pi}{2}\right)$ both have even integer multiples of π, and the output is 2, the same as the output of $h(0)$. She rechecked the process with an input $\frac{\pi}{4}$, but realized that in this scenario $\cos(4 * \frac{\pi}{4})$ had an odd integer multiple of π and did not return the same output value as $h(0)$. She concluded that as she started with an initial input 0 of $h(x)$, for another input $\pi/2$ of $h(x)$, there exists the smallest interval for which the output of $h(x)$ repeated. Throughout the interview, to determine any period of the sum of trigonometric functions, Eloise strictly focused on getting repeated outputs. She assumed the periods of h are related to (n, m) being the same, even, or odd, and she focused on verifying her assumptions by finding the smallest intervals between two same outputs. Then the interviewer probed her by asking whether the period of h is related to having any common factor between n and m. She quickly comprehended that the period of h would be 2π divided by the greatest common divisor of n and m.

For cases when n and m are multiples of the other, EL continued to compare the periods of $\cos(mx)$ and $\cos(nx)$ individually with the period of $h(x)$. She was thinking about the ratio of 2π and the common multiple of n and m to be the period of h. She thought that if $\cos(2x)$ has a period of π, and $\cos(4x)$ has a period of $\pi/2$; then the period of $h(x) = \cos(2x) + \cos(4x)$ would be the π as there are two of $\pi/2$'s in π. EL also graphed $\cos(2x)$, $\cos(4x)$, and $h(x)$ together and showed that at $x = 0$ and $x = \pi$, all these functions have maximum values and '*they meet at these points.*' Figure 3. below shows that EL marked the outputs for $\cos(2x)$, $\cos(4x)$, and $h(x)$ when the inputs are 0 and π.

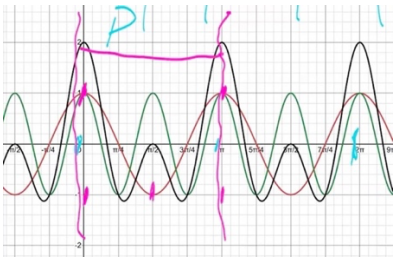

Figure 3. EL shows the periods for the sum function h when n=2, m=4

At this point, EL generalized that the period of $\cos(nx)$ would be $P_1 = \frac{1}{n} * 2\pi$, and the period of $\cos(mx)$ would be $P_2 = \frac{1}{m} * 2\pi$. She thought that the least common multiple of $\frac{1}{n}$ and $\frac{1}{m}$ plays a role in determining the period of h. The interviewer then asked EL to consider $n = 6$ and $n = 9$ to determine the period of h. EL found this combination unique as these are not multiples of each other, but in her words, *'they do combine'*. The following excerpt suggests that EL was referring to the least common multiple in determining the period of h.

Interviewer: What do you mean by they combine?

EL: Do they combine? We have $\frac{1}{6}$ and $\frac{1}{9}$…. So if we have something divided by 6 and we have some other thing divided by 9 (she wrote $\frac{x}{6} = \frac{y}{9}$). Do these ever line up? I think if we do $\frac{3}{9}$ and $\frac{2}{6}$, these are the same. So then, I think this (the period) will be $2\pi/3$

Interviewer: What exactly did you try to get from writing $\frac{2}{6} = \frac{3}{9}$?

EL: I was thinking if we have $\frac{x}{6}$ as some number of periods of the first function, and $\frac{y}{9}$ as some number of periods of the second function. And we want them both to line up at 2π. So if we find two numbers x and y that make this work, then we get the period of the larger function.

Interviewer: Okay.

EL: And then I found 2 and 3 that made this ($\frac{2}{6} = \frac{3}{9} = \frac{1}{3}$) work, and we get $\frac{2\pi}{3}$.

EL was thinking about the least common multiple of n and m as the input within the interval $(0, 2\pi)$ where the output starts repeating for all three functions, $\cos(mx)$, $\cos(nx)$, and h. As she exhibited her thinking of the least common multiple of (n, m) playing a role in determining the period of h for different combinations of n and m, the interviewer asked whether she could generalize her thinking about the period of the sum of two cosine functions. EL spontaneously mentioned that it would be related to the greatest common divisor of n and m. She thinks about the least common multiple of $(\frac{1}{n}, \frac{1}{m})$ as the same as the reciprocal of greatest common divisor of

(n, m). Figure 4. shows EL's formula to determine the period of the sum of two cosine functions.

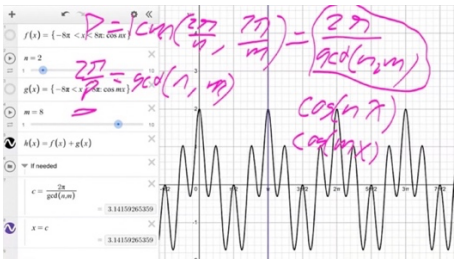

Figure 4. EL's formula to determine the period of h.

Discussion and Limitations

The results of the study suggest two distinct types of reasoning that students use to determine the period of the sum of two cosine functions. The constructs *output-oriented reasoning*, and *alignment-oriented reasoning* represent different ways students utilized their existing mathematical knowledge to make sense of the period of a trigonometric sum.

Eloise's approach of *output-oriented reasoning* focused on interpreting visible and numerical results from the graph. She identified periodic nature of the function by seeking repeating, unchanged patterns and verified the period by finding the smallest input interval producing the same output. Her hypothesis relied on characteristics like evenness or oddness of parameters, verified with graphs rather than focusing on the foundational number theory concepts. This approach is effective in most cases as it aligns with her conceptions of periodic functions but required support to understand the role of the common factor of the parameters n and m in generalization of the period of h.

EL's *alignment-oriented reasoning* centers around the understanding of the multiplicative and proportional relation between frequency and period, recognizing that for $\cos(nx)$ the period is $2\pi/n$. This understanding allowed her to see that for larger frequencies, the trigonometric functions produce smaller periods. She also recognized that the periods of individual functions relate multiplicatively when combined and that their frequencies coordinate to form an aligned period. She thought about how individual cycles 'line up' or 'meet', seeking points where the periods of $\cos(nx)$ and $\cos(mx)$ align with the period of the sum of these functions. EL consistently demonstrated the proportional aspect while comparing the periods. She thinks that if $\sin(x)$ has period of 2π, then $\sin(3x)$ has period $2\pi/3$ because there are "three times as many periods" in the same interval. Using the multiplicative and proportional aspect with the alignment of the periods in her reasoning, she was able to generalize a rule for the sum's period, linking it to the greatest common divisor of the frequencies of the individual cosine functions. This reasoning reflects a robust concept of periodicity, allowing EL to derive a generalizable formula of the sum of two cosine functions.

The distinction between *output-oriented* and *alignment-oriented* reasoning has key pedagogical implications. Both forms of reasoning reflect active engagement in learning; *output-oriented* reasoning emphasizes concrete, visual evidence, whereas *alignment-oriented* reasoning fosters the ability to generalize the foundational mathematical idea. A limitation of this study is that the generalization of the period of h as presented does not hold mathematically when either n or m is an irrational number. Although the interview protocol included tasks designed to explore students' understanding of such cases, the results of these explorations are not included in this paper.

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Teaching Logic for Social Justice: How Students' Activity Identifying Mathematical Assumptions Can Support Their Reasoning Around Prisons and Abolition

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Mathematics and mathematics education are implicated in broader societal inequities. These inequities can manifest in racialized and gendered classroom experiences and are perpetuated by the very practices students are encouraged to employ. While some scholarship in undergraduate mathematics describes possibilities for using mathematics toward social justice, this area has been largely limited to algebra, statistics and calculus, leaving proof unexplored despite its critical role within K-16 mathematics. The aim of this study was to develop an instructional theory for how students can productively discuss and debate sociopolitical issues through practicing competencies from proof-based mathematics. I report data from the first cycle of a constructivist teaching experiment with two undergraduates designed to help them use logic to support their sensemaking about prisons and abolition. Ultimately, students' successful activity attending to definitions and identifying assumptions in a geometric context supported their ability to identify commonplace assumptions about prisons.

Keywords: teaching math for social justice, logic, equitable instruction

Introduction

Undergraduate mathematics has long upheld and perpetuated societal injustices (Adiredja & Andrews-Larson, 2017). Some scholars argue that the very practices students learn in their math courses can amplify existing societal inequities (e.g. B. Martin, 1997). In particular, the disciplinary values of decontextualization and objectivity can permit mathematicians to draw harmful conclusions while receiving little scrutiny (Leyva et al., 2022; D. B. Martin, 2009). When mathematicians *decontextualize* data, they disregard social context and can draw conclusions based on their subjective choices of which variables to foreground or background. For example, 'achievement gap' scholarship uses statistics to draw deficit-based conclusions about Black and Latin*[1] students without attending to important factors such as school funding or racialized tracking patterns (Gutiérrez, 2008). Thus, a finding that reflects deeply entrenched societal injustices was misinterpreted as an indication that historically marginalized students in mathematics are less capable. Furthermore, the belief that mathematics is *objective* adds weight and influence to these inaccurate and biased conclusions (D. B. Martin, 2009). There is seemingly no room to argue with an objective finding, especially if its subjective nature is hidden in plain sight.

Just as mathematics can be used as a tool to support inequities, mathematics can also be used as a tool to fight them. Prior work describes possibilities to teach students to analyze, critique, and change societal injustices using mathematics (e.g. Gutstein, 2003). However, this work relies on statistics and computation (Harper, 2019), drawing connections between math and social justice that are not generalizable to more abstract areas of mathematics, such as proof. This is notable as proof plays a critical role throughout K-16 mathematics. In fact, the National Council of Teachers of Mathematics (2000) lists reasoning and proof as one of its five process standards,

[1] Salinas (2020) promotes the use of the term *Latin** to refer to Latinx/a/o people as a collective. Salinas advises reserving the term *Latinx* for gender-expansive *Latin** individuals.

arguing that K-12 instruction should enable students to "recognize reasoning and proof as *fundamental aspects* of mathematics [emphasis added]." I propose, however, that proof can be modified to honor the goals of teaching math for social justice and subvert the math disciplinary values of objectivity and decontextualization.

Advanced undergraduate mathematics serves as an important setting for teaching logic and proof for social justice. All secondary math teachers are required to take courses in proof, and these courses are where teachers form their opinions about proof-based mathematics (Knuth, 2002). However, these courses are currently taught in a manner that amplifies the disciplinary values of objectivity and decontextualization (Leyva, McNeil, et al., 2022). These instructors then bring a perspective of proof as decontextualized and objective into their K-12 classrooms. In the realm of proof, mathematical arguments rely on axioms, the taken-for-granted assumptions and logical starting points in a mathematical theory. Further results are then deductive consequences of these axioms. Traditionally, axioms were intended to reflect self-evident or empirical truths, and thus, proof was considered a means for drawing conclusions about objective reality (Dutilh Novaes, 2020). Unlike traditional axiomatics, however, modern axiomatics is agnostic towards whether axioms or the theorems that rely on them are true in an absolute sense (Dawkins, 2014). Theorems instead hold a conditional status, where the emphasis is placed instead on assuming the necessary axioms and constructing a valid proof (Dutilh Novaes, 2020). Thus, the modern perspective towards axioms subverts the problematic view of proof-based mathematics as objective, without recourse for critique. I also argue that the attention to axioms, or mathematical assumptions, can enable students to identify assumptions underlying their sociopolitical beliefs. This is noteworthy as K-12 teachers are explicitly expected to help students use and attend to mathematical assumptions (CCSSM, 2010). Yet, these approaches to applying logic and proof to students' sociopolitical views have been unexplored in the literature despite the impact they would have on K-12 math instruction.

This study's research goal was to develop an instructional theory for how students can productively discuss and debate sociopolitical issues through practicing competencies from proof-based mathematics. The instructional theory was developed via two cycles of constructivist teaching experiments (Steffe & Thompson, 2000), each with a pair of undergraduate math students. Students' activity identifying and relaxing mathematical assumptions, or axioms, in a geometric setting was designed to parallel their activity identifying and relaxing assumptions about prisons and prison abolition. The goal of this study was to develop a theory for how advanced mathematics can support students in discussing social justice issues. Accordingly, this study addresses the following research question: In what ways can students' reasoning about logical assumptions in advanced mathematics support their reasoning about assumptions in a sociopolitical setting?

Literature Review

I describe prior work that links math to social justice and scholarship that describes the role of sociopolitical assumptions, or logics, in oppressive systems. Teaching Math for Social Justice (TMSJ) encompasses a range of curricular interventions that enable students to both "read" and "write" the world using mathematics (e.g. Gutstein, 2003). These studies have been successful in increasing interest in mathematics (Rubel et al., 2016; Turner & Front Strawhun, 2013), providing opportunities for students to use mathematics to critique oppressive systems (Gutstein, 2003; Kokka, 2019; Rubel et al., 2016; Turner & Front Strawhun, 2013), and creating opportunities for students to engage in social change (Stocker, 2012; Turner & Front Strawhun, 2013). However, TMSJ implementations can also be constrained in their ability to sufficiently

cover either the social justice or the mathematics content. In some cases, lessons are entirely divided into two distinct halves covering mathematics or social justice content (e.g. Bartell, 2013). In other cases, eliciting understandings of the mathematics and of the political topics encompassed conflicting pedagogical goals. For instance, Gutstein (2003) occasionally had to tell his students how to find the answer or substantially simplify the mathematics. He also struggled to balance providing sufficient historical context with giving students opportunities to explore the mathematics. These trade-offs can be inherent to the added complexity that real world data sets can have relative to the nicer quantities students are used to working with in traditional school mathematics.

Harper (2019) describes Gutstein's difficulties negotiating conflicting disciplinary commitments as a theme that extends across TMSJ. Furthermore, many TMSJ scholars rely on data science to connect mathematics to social justice issues (e.g. Gutstein, 2003; Rubel et al., 2016; see Simic-Muller, 2019, for an example in the undergraduate context), which is notable as data science occupies only one category of the mathematics students are expected to learn in school. In fact, Harper (2019) comments that TMSJ work has generally struggled to connect social justice content to mathematics content standards at the secondary level. It is thus unclear to what extent TMSJ is even possible in more abstract areas of mathematics, including proof-based mathematics.

Prior work also describes the role that *math logics,* or the assumptions, norms and values shaping how society views mathematics (Battey & Marshall, 2023), play in perpetuating inequities in mathematics education. The notion of *logics* (Acker, 1990)*,* or organizing assumptions that justify and uphold certain systems, is widely used throughout scholarship that seeks to critique different ideological instantiations of injustice. For example, Leyva, McNeil, et al. (2021) draw connections between racialized and gendered events in an undergraduate calculus class and their associated math logics. They described how the logic that calculus should operate as a weed-out course inspired an instructor to tell his students to drop his course if they were struggling. Participants in this study indicated that this seemingly neutral event was racialized and gendered. Leyva et al. argue that when left unaddressed, math logics can continue to preserve inequities in math classrooms. This study makes use of the analogous roles of axioms in mathematical arguments and assumptions or logics in (political) arguments. By teaching students to analyze assumptions, or logics, across disciplines, this study addresses a limitation of prior TMSJ work by unifying the social justice and mathematical goals. Students' cross-disciplinary reasoning about geometric assumptions and carceral logics will serve as a guide for how proof can help students question oppressive logics more broadly.

Theory

Abolitionist theory framed the design of students' sociopolitical activities and served as a means for analyzing students' assumptions about prisons. Abolitionists contend that abolishing prisons requires dismantling the structures and ideologies, or pro-prison assumptions, that support them while building systems that address the problems prisons claim to solve. I describe a key construct from abolition, carceral logics (Gruen & Marceau, 2022), that played a central role in the study design. *Carceral logics* account for the everyday, taken for granted understandings people have about prisons and punishment (Gruen & Marceau, 2022). I anticipated that participants would initially rely on carceral logics without realizing they were doing so. In fact, even movements in support of prison reform can employ carceral logics. For example, some advocacy for removing the death penalty has relied on the assumption that a life sentence provides a satisfactory alternative (Davis, 2003). In this case, reformists inadvertently

evoke the carceral logic that incarceration is a sufficient solution to crime or violence. The goal of the study was not for the participants to take on abolitionist views or discard their initial perspectives but rather to identify and question carceral logics.

Methods

Data Collection

Data collection for this study consisted of two phases, both taking place over 10 weeks, consisting of an entrance interview, 7 teaching experiment sessions, and an exit interview. The data corpus includes interview transcripts, artifacts (student work), and video data from the teaching experiments. I recruited two participants per teaching experiment cycle. The present analysis covers data from the first of the two teaching experiments. My participants were a South Asian man (P1) and a Latina woman (P2), both of whom had taken multiple proof-based courses.

The goal of these experiments was to develop a theory of how students may use competencies from advanced undergraduate mathematics to support their discussions of prisons and abolition. To enable students to practice these competencies, I constructed a series of exercises that were designed for them to practice specific mathematical skills in a geometric setting then apply them to analyzing arguments about prisons and abolition. I used conjecture mapping (Sandoval, 2014) to develop hypotheses for how these exercises would enable students to engage in the three desired learning outcomes: identifying underlying assumptions of claims, relaxing these assumptions, and imagining different worlds or systems. This analysis focuses on the first learning outcome. These learning goals were participatory as the intention was for students to engage in authentic and productive discussion rather than perform well on a post-test. The starting set of conjectures were preliminary in nature, with the expectation that they would be incorrect in important ways.

As a facilitator, I generally guided students towards accurate mathematical interpretations and toward identifying the underlying commitments behind their political beliefs. However, I avoided imposing my own political beliefs onto the students. I encouraged participants to relax or question the assumptions they held *to a degree.* I did not ask participants to relax or replace assumptions that were fundamental to their identity or that involved the respect and inclusion of historically marginalized groups.

Analysis

I engaged in two types of analysis: ongoing and retrospective. Between sessions, I engaged in an *ongoing analysis.* I took note of whether and to what extent students identified or relaxed the assumptions included in the given session. I also identified instances in which the cross-disciplinary connections were successful or unsuccessful and developed preliminary conjectures regarding the relationship between students' reasoning across disciplines and the type of mathematical activity that could support student reasoning. I adjusted the exercises based on these revised conjectures. The ongoing analysis yielded tentative conjectures that account for how students would respond.

The retrospective analysis commenced after each experiment had concluded and involved a preliminary analysis to update the conjecture map. I identified episodes from the experiment that suggested the students successfully identified an axiom or a political assumption. I then worked backwards to identify mediating processes and conjectures that account for their successful activity. I coded for mediating processes, both those that were originally conjectured and ones that were unanticipated.

Results

The following excerpts are from Session 4 of the teaching experiment, which was the last session focused on having participants identify axioms and assumptions. Specifically, participants were given multiple examples of circles in non-Euclidean geometries and asked to identify what kinds of assumptions about circles these non-Euclidean circles brought up. They were then asked to critique a series of claims about prisons, with the goal that the participants would identify specific key carceral logics.

In the first half of the session, participants were asked to produce a definition of a circle and based off this definition, draw a circle in a series of non-Euclidean geometries. The participants first drew a circle in the Taxicab Geometry, a modification of the Euclidean geometry with a different distance metric. The taxicab circle resembles a Euclidean square and I had anticipated that participants would be hesitant to accept it as a circle. Ultimately, both participants accepted it. They were then asked to construct a circle in the Gap Geometry, a modified version of Euclidean geometry with a deleted strip, which produced a discontinuous, unbounded curve. Both participants rejected this curve as a circle and appealed to a definition of a circle that included the stipulation that circles were "closed and continuous:"

Author: And so for the 'closed and continuous,' is this something that to your knowledge, is canonically part of the definition of a circle? Or is this something that you've added in now that you've noticed this, and you're like, "that shouldn't be a circle."

Participant 1: I think it's always been part of the definition, but we exclude it because it would be superfluous in the Euclidean geometry.

Participant 2: Yeah, I think I'm assuming that it's in the definition. But I don't think I've had a definition that wasn't exclusive to the Euclidean geometry.

Participant 1: All the circles in Euclidean geometry are closed and continuous, we just don't say it because we just assume that it is.

Author: Out of curiosity, what's the difference between this and the not-round [taxicab] circle we had? Because it's also true that all of our circles in Euclidean are round, and we don't say that in the definition, but that's something that we're used to. What's the difference between a circle being round versus circle being continuous? Is there one that seems more wrong than the other?

Participant 2: I guess that we're holding the implication that it has to be closed and we're seeing that as more of a priority, carrying over the definitions in the different geometries.

Participant 1: I agree, I think that makes sense.

Both P1 and P2 acknowledged that they had never encountered a definition of a circle that included the stipulations "closed" and "bounded;" however, they maintained that these were both implied in the definition and were only omitted because they had previously only worked in Euclidean space. Interestingly, when I asked them to distinguish between their receptiveness towards accepting the Taxicab circle and their reluctance to accept the Gap Geometry circle, P2 suggested that it came down to a subjective decision. Namely, P2 saw the properties of being closed and continuous as "more of a priority." While both participants were unwilling to accept the Gap Geometry circle, they were consistent in their application of definitions. They agreed that the Gap Geometry circle would indeed count as a circle if "closed" was removed from the definition of a circle.

In the second half of the session, I asked the participants to assess various commonplace stances towards prisons. After introducing them to each claim, I asked them if they had any clarifying questions about it, without explicitly asking them to attend to definitions in particular:

> *Author*: So our first claim is that "prisons keep communities safe by incapacitating people who commit crime." So what are some initial questions you have about this perspective? This can include clarifying questions or things you'd want to know to assess the validity of the statement.
>
> *Participant 1*: Oh, well, I think we need some definitions. What do you mean by 'incapacitating people,' what do you mean by 'communities,' and also what do you mean by 'safe?' So for 'incapacitating people,' does that mean just sequestering them from society? Does that mean rehabilitating people? Does that mean keeping their freedoms at a minimum, so they have a much smaller change to commit the crime again in their current state? Which definition of 'incapacitating' are we meaning? Communities... does that mean the non-prison population, does that mean any susceptible parts of society we need to keep from harm's way? What do we mean by 'safe'? Are we just getting rid of crime altogether by sequestering those who've committed crime? Are we just reducing the amount of danger as opposed to getting rid of danger altogether? Are you saying there is no chance prisoners could harm society if they're put into prison? Are you saying there's no chance of them getting out or there's no chance of them still committing crimes while in prison?"

P1 immediately identified the need to define the terms: communities, safe, and incapacitation, and subsequently outlined various logical consequences for utilizing certain definitions (e.g. "Are you saying there is no chance prisoners could harm society if they're put into prison?"). When later asked to identify assumptions of the same claim, P1 again evoked the definition of 'safe:'

> You're assuming the prison has no adverse effects on the community. For example, if you have a lot of well-known members of this society in prison, how would that affect the community? Yes, they are safe from the crimes, but does that mean they are safe? Does that mean they are doing okay? Does that mean they are handling... Let's say there is a group of 10 family members, 9 go to prison. What's happening to the other one? Are they safe from crime? Would they have been safer with those family members on the other side? There are many effects of imprisoning people committing crime that extend beyond making the community safe. So that needs to be included in your assumption, that you are making it safe. Because 'safe' implies 'better.' Those might not be intertwined in this definition.

Ultimately, P1's attention to definitions when considering various definitions of circles extended to considering definitions related to prisons. Considering definitions ("those might not be intertwined in this definition") then became a resource that enabled him to identify a carceral logic, that prisons have no harmful effects on communities.

Discussion and Implications

Ultimately, the participants' activity developing a general definition for a circle outside of Euclidean space, and observing the consequences of this definition, primed P1 to spontaneously consider definitions when discussing assumptions about crime. While by the end of the experiment, he ultimately argued in favor of using prisons in some cases, P1 was successfully able to both identify and question a key carceral logic, namely that prisons have no adverse effects on communities. He also hinted at a few other commonplace assumptions: that

incarcerated people are not part of any community, that people are necessarily safer when their community members are incarcerated. The identification of these assumptions was borne out of an attention to definitions, first in a geometric setting, and later in a setting that discussed prisons.

Participants' activity with definitions in some ways reinforced and in other ways subverted what McNeill et al.(2022) refer to as the objectivity discourse, or the idea that mathematics is inherently objective. In their geometric activity, both participants maintained that circles needed to be closed and continuous, that this had always been part of the definition but had simply been omitted because it was "superfluous" in Euclidean geometry. This contrasted with their earlier acceptance that circles did not have to be round and that their circle construction in the taxicab metric counted as a circle. While P2 seemed to suggest that her own subjectivity was playing a role, both participants characterized their definition of a circle as standard. In contrast, P1 spoke directly to the subjective nature of how someone might define the terms: incapacitation, safety, or communities and the different downstream implications of some of these choices. While neither participant explicitly evoked nor refuted the discourse that mathematics is neutral (B. Martin, 1997; McNeill et al., 2022), their activity did subvert it by illustrating the very consequential differences that arise from how concepts like 'community' or 'safety' are defined. Of course, while it was a goal of this study for students to use skills from advanced mathematics to reason about a social justice issue, these activities were not explicitly designed with the intention of subverting specific math logics. Future work may attend more directly to the possibilities for advanced mathematics to help students subvert their problematic assumptions about mathematics.

Finally, this study builds on prior work describing TMSJ implementations. In contrast with prior TMSJ work, this study provides a potential example of how mathematics can aid in students' *interpretation* of sociopolitical issues. Much of the prior work in teaching math for social justice involved analyzing data or mathematical phenomena and bringing in an external political perspective to serve as a means for interpretation. This is necessary because data can be interpreted in myriad ways depending on the assumptions and commitments people are bringing in. In those cases, it is the interpretation that influences the data; the mathematics itself has little bearing on the development of a political interpretation. Indeed, this created conflict between the mathematical content students needed for a given lesson and the important social justice background that was required (e.g. Gutstein, 2003). However, in this teaching sequence, the attention to and manipulation of definitions was used to support students in questioning commonplace perspectives and interpretations of prisons and incarceration. Future work may consider other avenues, beyond analyzing definitions and assumptions, through which proof and logic can support students' reasoning about social justice topics.

Limitations

As a teaching experiment with four participants (two in this report), it was not a goal of this study to generalize from these specific participants' starting points, trajectories, or reasoning. Additionally, as a series of activities that took place outside of a classroom, the specific context of this learning environment differed substantially from that of a typical undergraduate class. Thus, while this study may be viewed as a starting point for the *possibilities* of integrating advanced mathematics and social justice mathematics, there are real barriers and limitations specific to the classroom setting that did not apply to this study and would need to be studied in future work.

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Factors Shaping Community College Math Major Motivations and Persistence in Mathematics

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Given declining numbers of undergraduate students choosing to major in mathematics, research in undergraduate mathematics education seeking to understand the motivations and perceptions of math majors is more important than ever before. Of particular interest in this report are community college mathematics majors—they make up a significant proportion of math majors in any given year and yet existing research on recruitment and retention in STEM has been conducted almost exclusively at four-year colleges. We report findings from in-depth interviews with eight community college math majors regarding their motivations to major in mathematics. We then examine the ways in which such motivations interact with institutional and departmental structures traditionally noted to influence persistence in STEM majors. We discuss implications for improving math major retention and advancing the field's understanding of the influence of collegiate context on perceptions of mathematical culture.

Keywords: community college, math major, motivations

The declining number of undergraduate students choosing to major in mathematics—and ultimately graduating with a mathematics degree—has been cause for increasing concern both within U.S. mathematics departments and worldwide (Hall et al., 2022; Musu-Gillette et al., 2015; Seymour & Hewitt, 1997; Seymour, Hunter & Weston, 2019). However, most research collapses math majors within the umbrella of "STEM" or "math and science majors" (see Seymour, Hunter & Weston, 2019 for an overview) and thus is too broad to describe challenges specific to the mathematics major. Some research on STEM majors investigates the influence of high school mathematics achievement (e.g. Wang, 2013; Trusty, 2002) or specific first or second-year mathematics courses on student persistence (e.g. Ellis, Fosdick & Rasmussen, 2016); however, such information is likely not enough to understand phenomenon regarding recruitment and retention of mathematics majors specifically (Hubbard, 2023) given that they interact far more frequently with mathematics departments compared to other STEM majors.

Moreover, the majority of research on recruitment and retention in STEM fields has been almost exclusively conducted at four-year colleges. Little is known about community college STEM majors, and even less about community college mathematics majors (Mesa, Wladis & Watkins, 2014). This is problematic given that almost half of all mathematics majors in a given year are enrolled in community colleges (NCES, 2020), and roughly half of all graduates from four-year colleges at started at a community college (Foley, Milan & Hamrick, 2021). Thus, community colleges are a critical yet understudied part of the math major pipeline.

Of particular interest to recruitment and retention of community college math majors are students' motivations for majoring in mathematics, and how these motivations are influenced by institutional experiences (Wang, 2013). In this report, we asked:

1. How do community college math majors describe their motivations for majoring in math?

2. In what ways are institutional and departmental structures in alignment or tension with these motivations?

The data from this report comes from eight in-depth interviews with students majoring in math as of summer 2024 and enrolled in a two-year college within the City University of New York (CUNY) system. This data is part of a larger mixed-methods project that seeks to identify factors that impact community college math major recruitment and retention, and to create the first retention model for community college mathematics majors. In answering the research questions noted above, we seek to contextualize the field's understanding of the motivations of math majors for pursuing mathematics, and to provide hypotheses regarding the mechanisms by which such motivations might interact with larger structural factors within a math department.

Literature Review

Tinto's model of student departure (Tinto, 1987) is arguably the most influential existing model of college student retention in the literature. Tinto posited that family background, pre-college schooling, and individual student attributes influence student persistence via two "integration" variables which impact a student's decision whether or not to drop out. These variables are 1) academic integration, often measured by course grades; and 2) social integration, often measured by frequency of interaction with other students and faculty. However, subsequent literature has critiqued the relevance of Tinto's model for community college students and other "non-traditional" students (Bean & Metzner, 1985; Ashar & Skenes, 1993; Borglum & Kubala, 2000; Karp, Hughes & O'Gara, 2010; Stuart, Rios-Aguilar & Deli-Amen, 2014).

Bean and Metzner (Bean & Metzner, 1985) developed a model for retention of "non-traditional" adult learners (either over 24 years old, a commuter student, and/or a part-time student) that primarily focused on an institution's academic utility rather than its social environment. The authors contend that social integration is significantly less influential for such students, given that they likely enter higher education with their own support systems and social networks. Their model of student attrition contains three main variable categories: environmental, academic, and background, and these variables then influence academic and psychological outcomes, which in turn determine a student's decision to persist. While this model may be more relevant for groups of community college mathematics majors, it still is not designed to examine the particular interactions between students and mathematics departments and courses that are likely to be relevant to improving math major retention specifically.

Given the limitations of both existing models described above, we focus our review of the literature on variables relevant to retention of mathematics majors. Literature specific to retaining mathematics majors is slim, so we supplement the current review with literature on the more general category of STEM major retention. Within this literature, there are several key factors noted to influence STEM major retention. Most of this work has been conducted in four-year contexts (Wang, 2021), which focuses on factors such as grades and disposition towards mathematics in high school (Musu-Gillette et al., 2015; Shaw & Barbuti, 2010; Wang, 2013; Trusty, 2002; Evans, Chen & Hudes, 2020) and perceptions of mathematics self-efficacy or self-concept once taking college courses (Ellis, Fosdick, & Rasmussen, 2016; Wang, 2013). Though we do not attempt a comprehensive overview of all relevant factors here, we see three overarching categories from existing literature as relevant to our core constructs: sense of belonging and social integration (Rainey et al., 2018; Tinto, 1987; Rodriguez, & Blaney, 2021; Good, Rattan & Dweck, 2012); engagement with undergraduate research opportunities (Russel, Hancock & McCullough, 2007; Eagan et al., 2013; Seymour et al., 2004; Hunter, Laursen &

Seymour, 2007); and mentorship (Apriceno, Levy & London, 2020; Atkins et al., 2020; Chelberg & Bosman, 2019).

In general, the presence of each of these three factors has been associated with increased persistence in STEM degree attainment, while their absence contributes to attrition. Research in the area of sense of belonging notes how both institutional cultures and student-to-student interactions can influence students' perception that they fit in or belong within their chosen STEM major. The underrepresentation of women and students of color in STEM, in particular, can reduce their sense of belonging which can be improved or exacerbated by the institutional and pedagogical environments at the college (Rainey et al., 2018; Rodriguez & Blaney, 2021; Good, Rattan & Dweck, 2012).

Research on participation in undergraduate research experiences (UREs) and engaging with faculty mentorship has also consistently reported positive impacts on STEM major retention and subsequent interest in STEM careers (Seymour et al., 2004; Eagan et al., 2013; Apriceno, Levy & London, 2020; Atkins et al., 2020; Chelberg & Bosman, 2019). For example, Russell, Hancock & McCullough (2007) surveyed over 4000 STEM majors concerning their experience with UREs associated with the National Science Foundation and found that participation in such programs was associated with increased interest in STEM careers and graduate school. Participation in UREs and involvement with mentorship are also likely to contribute to students' sense of belonging. For example, Hunter et al. (2007) reported that UREs fostered a sense of belonging to students' desired scientific discipline and helped them to "[feel] like a scientist," (p. 53). However, given that the majority of existing research was conducted with STEM majors in four-year institutions, it remains an open question the extent to which community college math majors may perceive such resources as useful or accessible within their institutional context. The current study aimed to investigate exactly this question.

Theoretical Framework

To structure our investigation of community college math majors' motivations and the interaction of those motivations with institutional and departmental features, we turned to Critical Bifocality as it has been conceptualized and operationalized for use in mathematics education research (Chen & Horn, 2022). Traditionally, Critical Bifocality is a theoretical perspective that posits that investigations into social processes of marginalization or oppression necessitate a bifocal approach-focusing on the relationship between individual agency as well as structural features of environment. While we do not claim that the experience of being a math major in a community college context must necessarily entail marginalization or oppression, community college students are more likely compared to four-year college students to belong to groups not historically represented in higher education (Goldrick-Rab, 2007) and community colleges (and the students who attend them) have traditionally been stigmatized within higher education (Gauthier, 2020). Combined with the fact that we were interested in examining interactions between individual motivations and institutional/departmental structures, this led us to believe that utilizing the lens of Critical Bifocality would allow us to attend to the most relevant constructs for our student population.

Chen and Horn (2022) provide four principles that characterize research in mathematics education that take such an approach. Although all four principles were instrumental in informing the larger project, for the sake of space we focus the current discussion on two of the principles that were most useful for informing the data discussed in this report. They are, "(1) analyzing interactions between structures and lives; while (2) recognizing the pervasiveness of oppressive structures," (p. 802). Here, we report findings that speak to both of these principles—

individual student motivations for majoring in mathematics and influences from interactions with departmental and institutional structures.

Methods

In summer 2024, we invited all current math majors enrolled in any community college within the City University of New York (CUNY) system to participate in a paid one-hour interview about their experiences in the major. Eight agreed to participate; they varied on a range of characteristics including age, whether they were enrolled part time (PT) or full time (FT), and whether they had ever transferred from another college prior to the interview (see Table 1).

Table 1. Descriptive Characteristics of Interview Participants

Participant ID	Age	Gender	Last math course(s) completed prior to interview	Enrollment intensity	History of transfer
1001	44	Man	Calculus I	PT	yes
1002	29	Man	Calculus III, abstract algebra, differential equations	FT	yes
1003	24	Woman	Calculus I	FT	yes
1004	26	Woman	Precalculus	PT	yes
1005	22	Man	Calculus I	FT	yes
1006	25	Man	Abstract algebra	PT	yes
1007	21	Man	Calculus III, linear algebra, abstract algebra	FT	no
1008	23	Man	Calculus I	FT	no

Each interview was guided by a semi-structured protocol that consisted of the following topics: rationale behind major choice; course-taking (including placement); perceptions of quality of instruction in college math courses; influences of external life factors such as work, family, and health/disability; and influences of institutional culture, structures and resources on persistence within the major. The data for the current analysis is drawn from questions regarding students' motivations to major in mathematics as well as a segment of the interview in which students were specifically asked to comment on their use and perceptions of institutional features commonly noted as influencing persistence in STEM majors in the broader literature (undergraduate research opportunities, mentorship, and community involvement). Influences from students' environmental life factors were also important, but are beyond the scope of this report.

Analysis

Each interview was recorded and transcribed, then coded in several stages using thematic content analysis (Braun & Clarke, 2006). First, five researchers read all transcripts and met to generate common emergent themes. For example, at this stage researchers agreed that there was an important theme regarding students' motivation to major in mathematics and the notion of "real world applicability"; however, the exact definition of "real world" was not yet precisely specified. Then, two researchers took the generative initial themes, reviewed and refined them, and developed a more nuanced set of subcodes. For example, some students were motivated to major in in mathematics due to advancements that such a degree would provide for them in their current or former job. Still others noted that their motivation stemmed from a broader

appreciation for the relevance of mathematics in explaining phenomenon occurring outside of college classrooms. This was carried out in a number of cyclical rounds until consensus was reached between the two coders about the subcodes and the definitions of the overarching themes. Then the final list of themes was shared with a third researcher (another of the original five) for further discussion and refinement, leading to final thematic analysis results presented here.

Results

One prominent theme that emerged from the student interviews was that most community college math majors in the sample were interested in mathematics primarily for its utility, often favoring applied over pure mathematics. Seven of the eight participants shared that they felt that majoring in mathematics would be beneficial because it would facilitate advancement in their current careers (1001, 1003, 1004, 1005) or prepare them for success after transferring to a bachelor's or graduate degree program (1002, 1005, 1007). In contrast, one participant (1006) reported choosing the mathematics major because they aspired to become a mathematician. Overall, participants expressed greater appreciation for applied or "practical" aspects of mathematics than for purely theoretical ones.

However, students frequently felt conflict between their orientation around the practical aspects of mathematics and their experiences of the structure and culture of mathematics departments and classes, where they felt that pure mathematics was the prevailing norm. Two subthemes emerged within this broader theme: one concerning how students experienced this mismatch within their mathematics courses, and another addressing how the mismatch was perceived in extracurricular mathematics contexts, including math clubs, research opportunities, and internships.

Conflict Between Students' Motivation and Course Structures and Culture

1005 illustrated this conflict when describing what an ideal experience would be like within the math major, stating that most of what he would learn would be about, "not just solving maths, they will be able to teach you how to apply this mathematics into the work field...Because like I said, learning how to solve [for] x and y is not enough...I want to know how to use this stuff out there [in the workforce]." In this way, 1005 contrasts purely theoretical aspects ("just solving maths") with those practical aspects that would support him in his transition into the workforce.

Although career relevance is a motivation that has been reported by retention literature focusing on non-traditional students (e.g. Bean & Metzner, 1985), participants occasionally provided evidence that they viewed this common practice-orientated motivation for majoring in math as contrasting with the motivations of a "typical" mathematics major or of the mathematics instructors they have interacted with. This contrast was well-illustrated by 1002, who was set to graduate with their associate's degree in mathematics after the semester in which they were interviewed. When asked about what kind of mathematical activities they enjoyed, they responded, "There are people who love math ... I am not one of them, for sure... The people who love math, who are deep into the most abstract imaginary concepts that are applicable maybe in like 500 years or something… that's just not me." Although 1002 went on to say they do like mathematics, they saw this form of liking as outside of the norm for the motivations of a typical mathematics major because it does not privilege abstract ways of thinking.

1007, who was also set to graduate with their associate's degree in mathematics that semester, expressed similar views. They stated, "I guess I'm not really like those kind of

mathematicians where they...see this simple thing and then they'll just devote their couple years to that thing. And in the end, it's just for curiosity... I actually wanted to see something, how I can use math in the real world." Here, 1007 draws a contrast between himself and "those kinds of mathematicians" in which his interests in more broad applications of mathematics are seen as distinct from topics that would be of interest within a dissertation or academic research article. These findings both align with existing literature (which have noted the importance of real-world relevance and applicability to community college students) and add nuance specific to mathematics majors. In particular, students appeared to make sense of their belonging within a departmental culture they perceived as privileging pure and theoretical mathematics over practice-oriented, applied mathematics. This perception, in turn, seemed to undermine students' sense of belonging in the major. Whether and to what extent such departmental cultures contribute to students' decisions to leave the major remains an open question that we focus on in future phases of this project.

Conflict Between Students' Motivation and the Structure and Culture of Extracurricular Mathematics Activities

Beyond their coursework, many students noted that extracurricular mathematics opportunities also conflicted with their mathematical interests, and subsequently influenced their sense of belonging in the major. Among the seven who reported motivations for majoring in mathematics relating to practicality or their interest in applied fields, it was common for participants to report a feeling of misalignment with institutional resources such as internships, or departmental resources such as social math clubs and undergraduate research projects. For instance, both 1007 and 1003 reported having an initial interest in joining a social math club, but each mentioned a hesitation in participating because they felt these clubs were intended more for students interested in pure mathematics. For 1003, the computer science club felt "more accessible" than the math club, and so she chose to "attack [math] from the lens of computer science" instead. Likewise, 1007 expressed reservations about the mathematics faculty member leading the math club as someone who graduated from an elite institution of higher education with a mathematics degree. He felt that he was more easily able to communicate with a different math club facilitator, who, "might not be the most intelligent or like most successful who graduated from, I don't know, MIT, Caltech, NYU, but the way they communicate with students, the way they engage with students are the best...I guess those kind of professors at math clubs, I think those would be really enjoyable." In this way, we see students' perceptions of their own motivations interact with this structural feature of their mathematics department. In this case, these motivations are perceived as being in conflict with their perceptions of these opportunities, which thus influences students' sense of belonging and subsequently their participation.

In contrast, 1006, who was motivated to major in math in order to become a mathematician, utilized a variety of departmental resources, including his institution's math club and a summer research experience for undergraduates (REU). 1006 reported engaging in these opportunities due to the connection he felt to the other students and faculty members that participated in them. He perceived these individuals as those who, like himself, "also want to talk about [math] problems for the sake of just talking about the problems instead of just going to get the grade." He explained further, "you find people like that sometimes in your classes as well. So, Math Team, certain classmates in my classes who also actually enjoy what they're doing make me feel like I belong." 1006 described math team as "where the mathematician math majors go" in contrast to math majors interested in other applied disciplines. In this way, 1006 saw his motivations as in alignment with his perceptions of these departmental resources, which

further supported his perception of belongingness within the major. We see such findings as in alignment with other work demonstrating the potential for mathematics departments to reinforce exclusionary assumptions of not only who is welcome in mathematical spaces, but what kind of mathematics is welcomed into those spaces (Rodd & Bartholomew, 2006; Battey & Leyva, 2016).

Conclusions and Implications

The broader project from which these data were drawn aimed to generate hypotheses about ways that existing retention literature may fail to fully capture the experiences of community college mathematics majors. Our first finding indicated that students were frequently interested in mathematics because of its perceived utility and applicability to applied problems. Students described this practicality in multiple ways, including the utility of mathematics for career goals or future fields of study, as well as the applied, problem-solving aspects of mathematics courses themselves. In particular, students often viewed this preference for utility as in conflict with a dominant mathematics higher education culture that they perceived as prioritizing pure mathematics, done for its own sake, as a core ideal. This feeling of conflict with mathematical norms appeared to reduce many community college mathematics majors' sense of belonging.

This sense of conflict and misfit occurred both within students' mathematics courses, and within extracurricular mathematics activities offered by the mathematics department, such as the math club, the math team, and undergraduate research opportunities.

These findings illustrate how students' motivations for majoring in mathematics may interact with structural features of mathematics departments in ways that generate either conflict or alignment. This highlights a potential mechanism through which college mathematics culture may play a critical role in shaping community college mathematics majors' sense of belonging, both within the major and the broader mathematical community. We view the cases presented here as an example of Critical Bifocality, illustrating the relationship between individual experiences and departmental structures that may be uniquely intensified in the community college context.

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Using Mathematical Knowledge When Learning to Program: The Case of Jack

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Students seeking careers in STEM will increasingly be required to develop proficiency with computational tools. As such, many students will be at least incentivized, if not required, to learn to program early on in their undergraduate careers. We hypothesize that, though many opportunities exist in which students can learn programming skills, providing undergraduate students with opportunities to program in the context of mathematics courses will be advantageous for students. We report on interviews conducted with students as they learned to program in the MATLAB language, undergraduate mathematics problems. We discuss how a student's mathematical knowledge mediated his programming practices.

Keywords: Undergraduate Mathematics, Programming, Computation, Digital Tools

There have been several recent calls for investigations into the role of digital experiences, including programming and computational activity, at the undergraduate level. In a recent special issue in the International Journal of Research in Undergraduate Mathematics Education, guest editors noted that "further research on digital experiences at the university level is definitely needed" (Geraniou et al., 2024, p. 688). They included "computational thinking and programming instructional design and policies" (p. 688) in university mathematics in a list of topics to be investigated. In addition, Lockwood and Morken (2021) explicitly call for "more studies in undergraduate mathematics education that explicitly explore relationships between computing and the teaching and learning of mathematics" (p. 404), saying that "much more needs to be investigated about the role of computing within RUME, including identifying and exploring potential benefits, affordances, challenges, and problematic issues" (p. 405).

In response to these calls, we conducted a teaching experiment (Steffe & Thompson, 2000) in which the solving of undergraduate mathematics problems was leveraged as a context in which the students might learn to program. We briefly report on initial instances of a student leveraging mathematical knowledge to mediate his programming activity. That is, we are interested in addressing the following research question: *In what ways might students' mathematical knowledge mediate their early engagement in machine-based computing?*

Literature Review

In mathematics education, there is a rich history of exploring the role of the computer in supporting students' learning. Papert (1980) argued that the computer provided opportunities for enriching students' mathematical thinking and activity. Many studies have featured students' use of programming to solve problems in various mathematical contexts including elementary topics (e.g., Benton et al., 2018; DeJarnette, 2019), and specific topics in undergraduate settings (e.g., Buteau et al., 2020; Lockwood & De Chenne, 2020). Our work continues in this vein, focusing on the domain of first year undergraduate mathematics.

Researchers have also focused on the intersection of computational thinking and domain-specific thinking. For instance, multiple studies (Lockwood & De Chenne, 2020; 2021; Medova and Certkova, 2021) reported on affordances of writing and running code for students' combinatorial thinking. In line with these studies, there is a growing literature base exploring

computing and computer programming in STEM and mathematics education (e.g., Caballero et al., 2019; Weintrop et al., 2016; Benton et al., 2018; Sinclair & Patterson, 2018).

We situate our study within this body of work and specifically examine novice coders' experiences learning to code while solving problems in mathematics. That is, "programming" as a practice is applicable to a wide variety of domains beyond computation in STEM fields, and one can learn to "program" in a variety of contexts. We are primarily interested in STEM-intending students initially learning to program in the context of mathematical problem solving. We hypothesize that such contexts can be leveraged effectively to provide students with an increasingly important skillset for the modern workforce.

Theoretical Framing

Our research question focuses on the interactions between students' mathematical knowledge and their engagement in the activity of machine-based computing (Lockwood & Morken, 2021), particularly as they learn to code for the first time. As such, we theoretically ground both the activity of programming and also mathematical knowledge.

Machine-based Computing

While framings of computer-adjacent activities such as programming, computing, and algorithmic thinking abound, we are primarily interested in early coders' experiences and activity when learning to utilize computational software to solve math and science problems. We thus focus our account of programming within the broader frame of machine-based computing. Lockwood and Morken (2021) characterize machine-based computing as "…the practice of developing and precisely articulating algorithms that may be run on a machine" (Lockwood & Morken, 2021, p. 405). Programming is an example of machine-based computing as it involves writing structured syntax that makes calls to underlying algorithms (e.g. via variables and functions) to be run in sequence through an interpreter to enact a computation and produce a result.

Importantly for our purposes, the activity of programming involves: a) writing, organizing and structuring specific syntax to be read by the interpreter, and b) making sense of the products of the computations in reference to the task or problem to be solved. These components align with Weintrop et al.'s (2016) characterization of programming, a computational thinking practice that "consists of understanding and modifying programs written by others, as well as composing new programs" (p. 139). We view the "understanding", "modifying" and "composing" aspects of Weintrop et al's characterization as features of programming on which we wish to focus.

Mathematical Knowledge

We employ a radical constructivist (Glasersfeld, 1995; Piaget, 2013) perspective to characterize students' understandings and infer aspects of their cognition by observing their utterances and mathematical activity. We conceptualize students' development of understandings in terms of their assimilations to schemes (Thompson et al., 2014) and draw from the fundamental constructs of accommodation and abstraction as the underlying mechanisms of learning and knowing. While we only have space to report on broad themes observed across many instances of students' uses of programming to solve mathematics problems, our built models (see the Methodology section) of the students' mathematical understandings that inform these broader themes are rooted in the fundamental constructs of radical constructivism.

Relevant to our analyses are Thompson and colleagues' (2014) notion of meaning and understanding. An understanding is a "cognitive state resulting from an assimilation" and a

thinker's associated meaning is the "space of implications existing at the moment of understanding" (p. 13). Importantly, as we discuss in the Methods section below, the students were leveraging programming to solve problems in mathematics content for which they had already received instruction. As such, the students were drawing from "stable" mathematical meanings (p. 13), that is, meanings for which the related understanding resulted from assimilation to a scheme. This contrasts "in the moment" understandings (p. 13) which still result from assimilation but may be fleeting because of memory constraints or are only momentarily attained in the process of accommodation. Thus, we both examine the salient aspects of the students' stable mathematical understandings being drawn from in the moments of programming, and we also note inferable features of the students' in-the-moment understandings of the logical mechanisms of computation and the workings of the computational interpreter that produced the outputs they observed.

Methods

We employed Steffe and Thompson's (2000) teaching experiment methodology to examine salient aspects of the students' learning of programming. For this report we focus on the results of individual teaching experiments with students enrolled in mathematics courses at a university in the United States. The teaching experiments entailed 90-minute sessions in which a student utilized MATLAB (Mathworks, 2024) live scripts[1] to solve mathematics problems.

Because of spacing constraints, we report on multiple interviews with a single student, Jack (pseudonym). Jack was in an engineering program and had previously taken the calculus sequence. The selection criterion for the interviews was that the students did not have (or had little) prior experience with programming and were willing to discuss their thoughts as they engaged with the live script.

The interviews took place via Zoom. Participants shared their screens while working in the live script, and the sessions were recorded using the built-in recording feature on Zoom. Participants read (within the livescript) provided textual, diagrammatic, and applet-based teaching elements to learn the syntax, structure, and formatting of code to perform computations in MATLAB. Throughout each live script were frequent goal-orienting prompts to enter code in cells to practice running code to solve problems. Students were free to ask the interviewer clarifying questions and were often prompted to anticipate what they expected the code to produce before running the MATLAB interpreter.

In line with the "responsive and intuitive interaction[s]" (Steffe & Thompson, 2000) common to teaching experiments, the interviewer frequently asked hypothesis-confirming questions while and after students produced and ran code, thusly exploring the participants' reasoning through interaction in the sessions. If the students, after long periods of trial and error, professed to be stuck, the interviewer would aid by providing a hint or parts of the code or syntax.

In retrospective conceptual analysis (Steffe & Thompson, 2000; Thompson, 2008), the research team reviewed the records to build models of the students' thinking using theoretical constructs to explain the observed mathematical activity. The researchers drew from foundational constructs of radical constructivism to infer features of the students' mathematical knowledge being activated and drawn from as they programmed. For this report, the research

[1] Many computational environments have accompanied "notebook" features where code can be structured and formatted alongside text and figures, can be organized into sections, and the output of the code can be viewed within the notebook. Jupyter notebooks are the well-known notebooks for Python. MATLAB's version of this is called a "live script".

team attended to Jack's evolving programming activity, searching for instances where his mathematical knowledge mediated his writing, running, and reviewing of code based on outputs given within the live script. We constructed narratives of episodes in which there were clear or important interactions between Jack's mathematical activity and his programming activity. The episode reported on is exemplary of larger themes taken from the data narratives.

Results

We present an episode featuring Jack's construction of for loops while solving kinematics-related problems. This episode features both smaller-scale and larger-scale instances of Jack' use of mathematical knowledge while engaged in programming activity.

"Distance is velocity times time"

In a multi-session activity centered on loops, Jack was tasked with approximating the distance traveled by car when given velocity and timestamp[2] measurements throughout a car's journey. The velocity information was stored in a 1x8 array called "velocities" and the timestamps corresponding to the velocities were stored in a 1x8 array called "times".

The two tasks in this section were to: 1) create a "distances" array with a for loop and use it to approximate the length of the windy road, and 2) create an "accelerations" array giving the average acceleration of the car over each time interval of travel.

Figure 1 gives Jack's first attempted loop[3] to create and use the distances array:

```
velocities=[13 9 10 4.5 2.2 13 6.7 5.4]
times=[0 120 375 447 807 867 959 1079]
%v=d/t
%d=vt
distances=[]
for i=1:8
    distances=velocities.*times
end
sum(distances)
```

```
distances = []
   ⋮
distances = 10^4 × 0 0.1080 0.3750 0.2011 ...
   ⋮
ans = 3.2140 e + 04
```

Figure 1. Jack's initial loop to produce a distances array.

Jack noted that this didn't seem correct because the loop gave the same output each time. He then was prompted to explain the output and said the following:

> *J:* So we're doing 13*0, 9*120, 10*375 … so on and so forth. So it's outputting that [i.e. the array output is performing those multiplications.] and then sum(distances) is adding that up.

Jack then noted that the loop was not needed to create the distances array via the calculation distances=velocities.*times, and that the loop was simply performing that calculation multiple times over. Jack then spent some time attempting to alter his code to progressively construct the distances array within the loop, eventually attempting the following:

[2] By "timestamp" we mean the time values at which the corresponding velocity measurements were observed.

[3] In the prior live script, Jack had performed arithmetic on arrays, including using the element-to-element array operation ".*", which multiplied the entries of arrays that share indices.

Figure 2. Jack's second attempt creating the distances array.

Jack quickly abandoned this approach. While moving his cursor progressively over the entries in the "times" array he described that for each index i, the loop "… multiplied [the velocities array] by [the time at index] 1, 2, 3, 4, … across times.". We note that Jack was interpreting the structure of the given outputs MATLAB by performing mental arithmetic. Also, implicit in this process was Jack drawing from an in-the-moment understanding of the ordered operations imposed by the MATLAB interpreter that would produce the given outputs. Jack then leveraged this in-the-moment understanding to re-structure his code in a way that reflects the ordered operations to produce the outputs he was expecting.

After another similar attempt, Jack articulated that his loop needed to, at each step, multiply the numbers in the velocities and times arrays that shared an index. After receiving help recalling previously learned syntax, Jack produced a for loop that achieved his stated goals. Importantly, this approach does not accurately approximate the length of the windy road, which was the goal of the task. Jack noted that his results did not match the provided final answer, but he did not act on this discrepancy nor attempt to revisit his code, being confident that the code had achieved his intended calculation. The interviewer chose to move on to the accelerations portion of the task, waiting to see if this discrepancy would surface again.

Jack was then asked to summarize his work on the distance approximation task and write down a formula that would calculate the distance for an arbitrary velocity v and time data t. Jack wrote "D=sum[v1*t1+v2*t2+…]" and said:

> *J:* Distance equals the sum of velocity 1 times time 1 plus velocity 2 times time 2 and so on and so forth … through the whole data set.

We note now (and will elaborate later) that Jack's stated distance calculation matches the calculations that Jack intended to (and eventually did) create with a loop. Jack's distance approximation entailed multiplying the velocity data by their corresponding timestamps, expanding the standard model $d = v * t$ to be enacted with velocity-timestamp pairings.

"Acceleration is velocity divided by time"

Moving on, Jack employed a very similar strategy to generate an array that would calculate the approximate accelerations of the car over the intervals of time, saying "Same thing as before … just tit for tat position 1 divided by position 1 in the array, and then 2 and 3 and so forth." Running the interpreter, Jack realized that there was an error somewhere in his code (Figure 3), as the first entry in the accelerations array was Inf (for "infinity").

> *J:* That can't be right … So we have 0 on the first value of "times", so it should be undefined but we're getting infinity for some reason? … So, I mean that's our starting point so our acceleration is 0, I mean but we have a starting velocity … so really we shouldn't be counting that first value I'm thinking. … So I mean acceleration is the velocity minus the original velocity over the delta of time … if we're starting at 13,

that's our velocity original, and then t is still just 0 … so a starting acceleration of 0 so that makes sense. … we're staying at a constant speed.

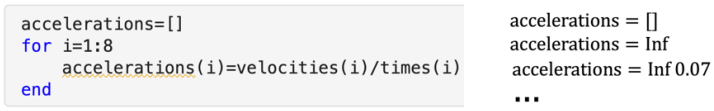

Figure 3. Jack's first attempt creating an accelerations array.

Here, we see Jack attempting to justify the "Inf" result by conceptualizing the start point of the data, as if the car had started without acceleration. Jack had suggested that we could ignore the first entry in accelerations, and the entries afterward would capture the "time delta" accurately and so the rest of the array was correct. Following up, the interviewer asked:

I: Now you just mentioned that for acceleration you want the delta of the velocities over the delta of the times, and so I'm wondering if in your mind the deltas of the velocities is just the same as the velocity itself in the time slot.
B: Yeah I mean acceleration is just velocity over time at that point in time. I guess we could put it that way.

After some time exploring what he meant by "delta" and returning to some of the calculations to examine whether his code was measuring the "delta"s, Jack eventually produced the following loop to calculate the accelerations.

Figure 4. Jack's final accelerations loop

We see two different and important instances of Jack's mathematical knowledge mediating his programming practice. First, as Jack built the distances array, he was able to envision element-to-element multiplication operations occurring in multiple structural configurations and make inferences about the MATLAB interpreter and the logic behind the outputs he was seeing. Jack was not only able to parse when an array resulted from one-to-one multiplication across two arrays but also was able to infer the impact of his syntax on the order imposed by the loop.

On a larger scale, Jack's first chosen model for approximating traveled distance was an expansion of the basic model d=v*t to be enacted on each velocity-time pairing in the dataset. This expansion was similar to those of problems in prior lessons of the teaching experiment, where a major theme was the gained ability from MATLAB to make many calculations at once. Supported by Jack's multiple statements such as "distance is velocity times time" and "acceleration is velocity divided by time" and his work on other similar modeling problems, we infer that this notion of making many calculations at once was an underlying goal taken on by Jack that mediated not only his writing of the code, but his interpretation of the validity of the code output. This differs from other possible goals he might have taken on, such as conceptualizing the construction of quantities (Thompson, 1990) necessary for the solution of the problem and then attempting to enact the quantitative operations programmatically, Jack was primarily concerned with making repeated calculations in the form of loops or array arithmetic.

Moreover, Jack required explicit guidance from the interviewer to become aware that his acceleration loop captured point-by-point division but did not capture his stated "delta"s necessary for typical rate of change formulas.

Discussion and Conclusion

There are two primary takeaways from these results and analyses, both related to the ways that Jack's mathematical understandings mediated his programming activity. The first is that, in these early instances of coding, Jack was developing and leveraging in-the-moment understandings how the program interpreter read his written code. That is, particularly apparent in his for loop refinement, Jack was generating and testing hypotheses about the relationships between their structure of the code and the structure of the resulting output. More than just parsing syntax and debugging errors, there was a cognitive element of inference on Jack's part where he made conjectures about how the code interpreter must be using his written code to produce outputs structured in the ways he observed.

Unlike other contexts in which the results of coding might be interpreted, to parse and interpret the structure of his outputs Jack was engaging in mathematical thinking not unlike the thinking involved in other tasks in the undergraduate curriculum. Students attempting to parse, understand, and leverage formulas such as the distance formula $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ or sums such as Riemann Sums, partial sums of series, or infinite series similarly have to generate an understanding of the structure and mechanics of the formulas, but also develop an understanding of the appropriate and expected result of the calculation. Unlike these more traditional instances of envisioned sequences of operations, in computational software the results of many sub-stages of the process are viewable, interpretable, and thus able to provide more figurative material from which a student might mediate their programming or mathematical practice.

Contrastingly, the instances reported on in this study were of basic arithmetic operations. One might imagine similar tasks where the sub-stages are themselves complex and would not bring the same affordance to a novice programmer. Finally, these episodes also highlight potentially important considerations for task design in math-based introductions to programming and computation. Jack's successful structuring and writing of the loops were in some ways independent of the problem's quantitative reasoning requirement for the successful approximation of distance and accelerations. That is, the programming-specific goal of the task was to give students opportunities to flexibly navigate the subtleties of loop construction to produce variables of different types. We saw that Jack was able to achieve this goal even though he did not attend to the quantities in a way that would accurately measure the desired approximation. Rather than making recommendations one way or another, we simply note that future research should further investigate the complexities of the requirements on students to leverage particular understandings or ways of thinking as a precursor to successful program-writing. We consider that one eventual goal would be for students to develop enough flexibility and familiarity with coding to then enable computation to be leveraged as a tool for effective quantification and mathematical reasoning.

Acknowledgements

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Setting and Sustaining Intentions: Two Stories of MGTAs' Growth as Instructors

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Mathematics graduate teaching assistants (MGTAs) often teach or support undergraduate courses and can have a significant impact on learners' experiences in mathematics. However, professional development (PD) programs rarely provide MGTAs with supportive, extended opportunities to learn about, enact, reflect on, and grow in evidence-based teaching practices. We designed a multi-year PD program for MGTAs focused on engaging students in mathematics and enacting equity-focused practices. We analyzed multiple sources of data to investigate shifts in MGTAs' teaching practices using principles of the PD. Applying the Theory of Planned Behavior, we describe the growth of two participating MGTAs. Both participants set and sustained intentions, modified their behaviors to align with their intentions to support engaging group work and empower women, respectively, and experienced positive shifts in their attitude and perceived behavioral control. Key implications include the importance of opportunities for MGTAs to reflect on their intentions and teaching practices in long-term PD.

Keywords: mathematics graduate teaching assistants, equity, active learning, teaching practices, theory of planned behavior

Mathematics departments in the United States rely heavily on mathematics graduate students (also known as mathematics graduate teaching assistants or MGTAs) to teach or to support large lecture-based courses such as Pre-Calculus and Calculus. MGTAs might lead problem-solving sessions associated with lecture courses or be assigned to teach a course as the lead instructor. In some doctorate-granting institutions, MGTAs teach up to 68% of Calculus I courses (Selinski & Milbourne, 2015). Yet, MGTAs rarely receive substantive preparation for teaching mathematics (Deshler et al., 2015; Ellis, 2014; Speer et al., 2017). Professional Development (PD) programs that exist do not typically extend beyond the MGTAs' first year in graduate school (Bookman & Braley, 2022; Deshler et al., 2015; Ellis, 2014; Harris et al., 2009; Kung & Speer, 2009). Preparation for a broad scope of evidence-based teaching practices (e.g., active learning, supporting student engagement in mathematical practices, facilitating collaboration) is crucial as many MGTAs represent future faculty and play an important role in teaching practices at post-secondary institutions (Ellis, 2014; Golbeck et al., 2016; Speer et al., 2017).

The proliferation of PD programs for MGTAs, although slow-going over the past fifty years, has more recently gained speed, as researchers have begun focusing on MGTAs' development as teachers (e.g., Beisiegel et al., 2019; Belnap, 2005; Speer, 2001). In a study of mathematics graduate programs in the U.S., Braley et al. (2023) observed that a greater number of graduate programs now offer some form of PD, such as an orientation to the department and teaching duties (79%) and/or a term-long teaching seminar (75%) as well as occasional workshops (73%). As the field has turned its attention to issues of equity (Brown, 2018), we see a similar focus in PD programs for MGTAs, with 65% of programs now offering some PD about equity (Braley et al., 2023, p. 1046). However, little is known about the equity-focused content in these programs.

Engaging in equitable mathematics teaching practices means accounting for students' identities in the classroom (Gutiérrez, 2009). For instance, a wealth of research demonstrates that active learning teaching practices do not inherently support female students in collegiate mathematics courses (Johnson, 2024). Rather, equitable instructors offer deliberate opportunities

for women to explain and justify their mathematics solutions uninterrupted (Johnson, 2024). While some PD has supported MGTAs in increasing their use of equity-focused teaching practices (Coles & Piccolo, 2025; Pilgrim et al., 2025) few studies have investigated MGTAs' general understanding of teaching practices (Beisiegel, 2017; Miller et al., 2018; Speer, 2001; Speer et al., 2010) and, more specifically, how PD programs can help MGTAs learn about and enact teaching practices that contribute to improved student outcomes.

Theoretical Framework: Theory of Planned Behavior

Despite MGTAs' impact on learners, only a handful of studies have investigated their understanding of and growth in teaching practices. As one example, Miller (2018) conducted a case study of experienced MGTAs, most of whom had experienced a year-long PD program focused on active learning and were teaching a Pre-Calculus course for the third time. While the MGTAs were provided lesson guides that emphasized active learning, Miller found that they often reverted back to lecturing when students were not engaging in group work. Another example (Beisiegel et al., 2019) investigated MGTAs' growth in their knowledge of teaching over the course of five years. They found that novice MGTAs were often unable to consider teaching practices that support active engaged learning as they needed time to adjust to routines of teaching (e.g., lesson planning) before adopting practices they had not yet experienced.

Our research investigates MGTA growth through the lens of Theory of Planned Behavior (Ajzen, 1991), which focuses on *behavioral intention*, *attitude toward the behavior*, *subjective norms*, and *perceived behavioral control* (see Fig. 1). In the context of the PD program, *behaviors* refer to the teaching practices that MGTAs use in the classroom and *intention* is an MGTA's plan to use certain teaching practices. While intention directly impacts resulting behavior, an MGTA's perception of their ability to employ teaching practices influences their pedagogical choices (i.e., their perceived behavioral control).

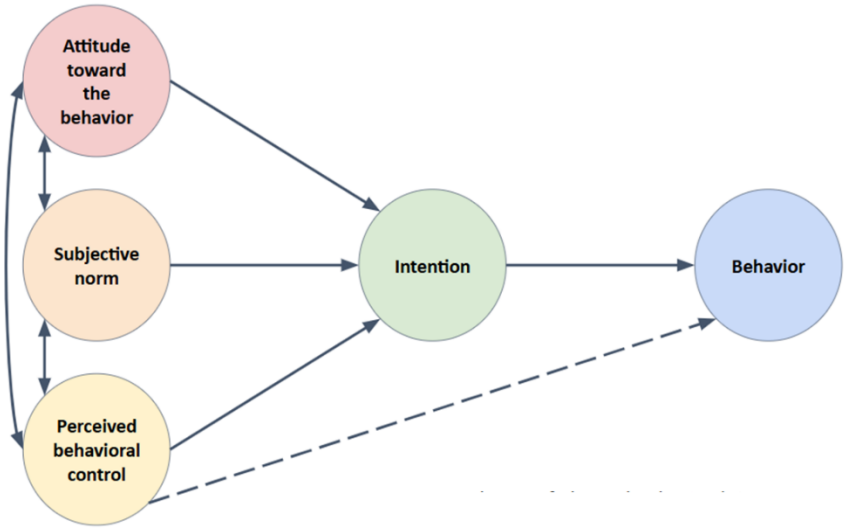

Figure 1. Theory of Planned Behavior (Ajzen, 1991, p. 182)

We implemented a PD program to support MGTAs in engaging students and enacting equity-centered teaching practices. In this report, we focus on two participants because of their commitment to setting and sustaining intentions in terms of active learning and equity. Our research question is: What change in behaviors, attitudes, and perceived behavioral control do MGTAs who set and sustain intentions in teaching experience during a two-year PD program?

Context of the Study and Methods

ELITE PD consists of five elements. In the first year of the graduate program, ELITE PD offers a three-day fall orientation as new MGTAs arrive at the university (topics include: the department and MGTAs' roles, community building with their peers, develop introductory knowledge of teaching practices focused on active group learning and dynamics) and a fall graduate teaching seminar (GTS) that is focused on their first term as an MGTA (topics include: more information about the department, the university's teaching support resources, classroom management, and how to implement small-scale instructional practices with attention to equity and active learning). The second year of the ELITE PD program offers a fall-term seminar Introduction to Active Learning and Equity (IALE; MGTAs build upon their understanding of and facility with active learning and equitable teaching practices) and a winter-term seminar Advanced Active Learning and Equity (AALE; MGTAs engage with specific research-based pedagogical approaches as well as equity research associated with teaching). We expect MGTAs to complete these seminars in the second year of their graduate programs, but some choose to focus on passing qualifying exams before taking IALE and AALE. The fifth element of ELITE PD has the MGTAs who have completed the first four elements become teaching mentors to new MGTAs. During each year of the ELITE PD program, we interview participants twice (fall and spring), video record them teaching three times (exceptions to this are when the MGTA is on a research fellowship or has an assignment as a grader for a course), and also survey their students. In IALE, the MGTAs complete journal reflections that are also used to understand their growth.

We explore the growth of two participants, Ayla (pseudonym), in incorporating group work practices, and Athena (pseudonym) in empowering women in the classroom. We chose to report on Ayla's growth because of her choice to repeatedly reflect on a salient struggle with engaging students in group work, as well as perseverance in their efforts to modify their teaching practices. We chose to report on Athena's growth because of her awakened intention to empower women and their notable shifts in teaching practices to realize this intention.

For Ayla, we analyzed five classroom observations (O2-6) and post-observation debriefs (D2-6), two interviews (I1-2), five reflection journals (R1-5), and an exit ticket from the IALE (ET). Note that each of O2-6 and D2-6 occurred once per term from winter of their first year in ELITE PD to spring of their second year, I1 occurred during GTS, R1-5 and ET occurred during IALE, and I2 occurred during the AALE. For Athena, we analyzed three classroom observations (O1-3) and post-observation debriefs (D1-3), and two interviews (I1-2). Note that each of O1-3 and D1-3 occurred once per term from fall of their first year in ELITE PD to spring of their first year. I1 occurred during IALE and I2 occurred at the end of the spring term. We analyzed these data by identifying each MGTA's intentions, and then identifying the components of the Theory of Planned Behavior relevant to their intentions.

Findings: Ayla's Growth in Supporting Group Work

In the first year of ELITE PD, Ayla's teaching practices centered on answering the questions of students with raised hands, without specifically instructing students on how to work on mathematics, e.g., in groups, pairs, or individually (O2, O3). Primarily, Ayla's intention for teaching was to "just let them work and then answer their questions when they pop up" (D3). However, Ayla's intentions for students to collaborate and talk with each other also started to percolate in her first year. She stated that her plans were "to try to make sure every student feels heard…and then try to encourage them to ask their group mates, if possible" (D2).

During IALE, Ayla's desire for students to work in groups increased; she wanted to "inspire collaboration" (R4) and get them "talking and participating" (R5) but she faced too many

challenges to change her practices the ways she wanted. First, Ayla expressed that "forcing students to work in groups feels uncomfortable" (R1). Second, Ayla felt pressure to avoid asking students to work in groups because of perceived resistance to changing norms mid-term: "I have already set the norm [to not do group work], so I am not sure how the students will react to it" (R1). Third, Ayla indicated a perceived lack of behavioral control, stating that it's "hard to get them to work in groups" (D4). At the same time, Ayla managed to make incremental progress towards her goal by asking every pair or group of students to check each other's answers (O4).

By AALE, and in the subsequent term, Ayla had experienced shifts in attitude, subjective norms, perceived behavioral control, and she began to incorporate co-constructing and revisiting group work norms (O5, O6). Ayla felt "empowered to try new things with students" (Fall ET), believed that she encouraged students to get "into groups pretty easily" and have "lots of conversations" (D6), and exuded confidence in describing her new teaching practice:

> My favorite one was the norms… like, 'these are the norms' and 'are there any other norms that you as students wanna add?'...it is one of the big reasons why I went from having almost no group work in the last term to really great and productive group work this term. (I2)

Table 1 shows an overview of Ayla's shifts towards supporting group work throughout years one and two of ELITE PD, which she ultimately credited for much of her growth:

> I've taken a lot of things from our seminar and tried to incorporate them into my classes…at the beginning of the term, I did group work norms, and I did it in the first class…and then I did it again when we came back from the snow day. Because I was like, 'I wanna remind everyone of the norms that we agreed on.' And I really think that helped develop equitable work and group activities. Because part of my norms were 'no one's done until everyone's done', making sure people don't get left behind...there's no dumb questions…And I think just doing that at the beginning of the term, really helped develop my, like, facilitating the group work that's been going on in my class. (D5)

Table 1. Ayla's Growth in Supporting Group Work

	Year 1 of ELITE PD			Year 2 of ELITE PD		
	GTS			IALE	ALE	
	Fall	Winter	Spring	Fall	Winter	Spring
Attitude				Uncomfortable with forcing group work (R1)	Empowered to try new things (Fall ET)	Frustrated but confident (O6)
Subjective Norm	Students may resist group work given that no group work norms have been established (R1)				Students easily work in groups when norms are set (D6)	
Perceived Behavioral Control				Hard to get students to work in groups (D4)	Confident in supporting productive group work (I2)	
Intention	Wants students to collaborate and talk with each other					
Behavior		Not promoting group work (O2, O3)		Asks students to check other's answers (O4)	Co-constructs and revisits group work norms (O5, O6)	

Findings: Athena's Growth in Empowering Women

After the first interview with Athena, it was clear that she was sharply aware and growing in awareness of gender (in)equity. When asked how she first learned about gender equity, she immediately responded that she learned about it "because I'm female, and the science field is dominated by men" and reflected on how her unique background and experiences moving to the U.S. for graduate school made her realize that women deserve more:

> I come from [Anonymized Country], so the women just have less, you know. So there's… not too many expectations. Not until I came to the US, and I'm like, 'Okay, equity is something to think about.'…it has become important to me, especially as I go through a lot of stress right now in my PhD…it wouldn't be fair to, um, just not get the same treatment a man would get, or the same recognition… (I1)

Athena's lived experiences seemed to motivate her awakened intention to empower women in mathematics. However, her shift to enacting behaviors in alignment with her intention was not immediate. Rather, her shift followed deliberate reflection on her teaching. When asked what she learned about her teaching from the first observation, she said, "I really approach this teaching thing in a very different way, in a wrong way…I don't even go to learn. I go to do my work and just, just get home….I'm not focusing on learning…that's not great" (D1).

Although Athena may not have focused on learning from her teaching that day, she reflected on her teaching during the post-observation debrief interview. When asked about unforeseen challenges with regard to equity and inclusivity, Athena's reflection included perceived behavioral control related to opportunities for women to present on mathematics:

> One of the group's that's only girls…they're like, 'Oh, is it okay to present something like that?'...I think they had the harder problem…I was very like, 'Okay, I get it. It's a hard problem. I'll do it on the board for everybody'…I did not have a problem with it, because they were just unlucky that day. (D1)

Athena's reflection appeared to prompt changes in her teaching practices, because in the next two terms Athena adjusted the ways she enacted group work to provide more opportunities for women to present mathematics. In the fall, Athena had permitted groups to self-select individuals to go up to the board and present solutions, which did not guarantee that women would present solutions (O1). But in the winter, Athena specifically encouraged women to talk about mathematics with the whole class listening (O2). For instance, during the second observation, Athena persisted in encouraging a woman to answer a question despite this student's shyness and lack of confidence, and despite seeing another raised hand from a man sitting in the front of the classroom (who would later present his work at the board):

> *Athena*: How are we doing on 1C? Did we get an answer? Um, does anyone have the answer for 1C, or…? [Athena nods head and looks with a smile to a woman near the back] What's your answer? [Student laughs a bit nervously] It could be right, you know? [Student(s) mumbling] Huh?
>
> *Student (woman)*: We did it but we're not feeling, like, confident.
>
> *Athena*: [Man near front raises hand, and Athena looks his direction, then looks back to woman, raises eyebrows in earnest, and gently waves her hand downwards towards the man, which prompts him to lower his hand. Athena continues to make eye contact with the woman.] One sec… What answer did you get?

Athena additionally offered opportunities for women to present mathematics at the board. She later reflected on this choice, revoicing her intention: "the girls should also get involved…not

just the guys" (D2), and expressing a happy attitude as she excitedly relived the moment when one of the women got an answer right:

> The guys were coming to the board to share their answers, and I just offered a girl, and she got the right answer. But she was so shy, you know…I gave her the morale and the motivation she needed to go to the board. Because she got it right, you know?…the worst part was that she got it first before the guys…they're not shy, you know…Yes, I felt good about it and she did feel good. I noticed she was happy. I looked at her, you know, because she felt like, I don't know why. I just felt she was happy, and I liked it. (D2)

By the spring term, Athena had modified presentation norms so that the individual(s) presenting within a group would rotate from one class to the next (D3). By making these changes, Athena ensured that women in groups of men would also have the chance to present their work, and furthermore, that every student would have a chance to talk about mathematics. Additionally, Athena continued to verbally encourage women presenters. For instance, Athena encouraged two women to discuss their solution at the board, saying, "you can teach us" (O3).

During the third debrief, when asked what went well in terms of equity, Athena first described how she ensured that everyone would get a chance to present and how she scaffolded solutions with some women to prepare them to present their work. She said, "if I did not do that they might have not been able to present their work on the board, and they might have just been left out" (D3). The interviewer then asked a follow-up question, saying, "I remember from the last time I did a debrief interview with you, you mentioned having the girls and women in your class feel really included. Do you have thoughts on how that's going this term?" (D3). In response, Athena again articulated her intention to provide opportunities for women and expressed confidence in doing so:

> I pretty much have learned how to navigate that…it's no longer a problem for me. I just treat everyone equally. I also make the ladies present their work, even if they are in, like, a group of men, for example. Like, it's okay if you, if there are, like, four guys and two girls. It's not okay for just the guys to present. The girls can also present the answer. You know what I mean? Like, on the board, not just the guys (D3).

Along with Athena's shifts in behaviors and perceived behavioral control, Athena's attitude towards teaching more generally shifted in positive directions over the course of the year. By the third debrief interview, Athena described enjoying and looking forward to teaching, saying she was "beginning to enjoy the process," "meeting my students makes me happy," and "I fell in love with a pattern I've been using, and that's been working" (D3). Similar to Ayla, Athena credits ELITE PD for her growth in enacting equitable teaching practices:

> Before I joined ELITE PD program, I was not very aware of inclusivity and equity, like, I just designed, I just did my job, you know what I mean? I just showed up to class to do my job, irrespective of how the student felt, or, like, I did not think about 'Oh, I should think about my students' feelings, or who they are.' You know what I mean, I just did my job. But having been part of ELITE PD, I'm able to understand that every student is different, and so are their needs. (I2)

Table 2 shows an overview of Athena's growth in empowering women in her classroom in year one of ELITE PD.

Table 2. Athena's Growth in Empowering Women

Year 1 of ELITE PD			
	IALE	AALE	
	Fall	Winter	Spring
Attitude	Tired after teaching, generally (D1, D3)	Likes to see shy women happy they are right (D2)	Excited for teaching and students, generally (D3)
Subjective Norm		Men always raise hands or present solutions at the board (D2, D3, I2)	
Perceived Behavioral Control	Challenging to support women in presenting difficult solutions (D1)		Confident in supporting women to present solutions (D3)
Intention	Empowering women in mathematics		
Behavior	Groups self-select student presenters (O1, D1)	Chooses women to talk and present when raising hands (O2, D2)	Rotating who presents within groups (D3) Preparing women to present (O3, D3)

Discussion

The Theory of Planned Behavior (Ajzen, 1991) helped us to observe shifts in MGTAs' attitudes, thoughts, intentions, and behaviors. Regularly interviewing and inquiring into each MGTA's experiences provided us with opportunities to follow their narrative threads–the major changes in various components of the theoretical framework, and the reflections and classroom experiences that instigated those changes. Ayla's attitude of discomfort with group work shifted, as she maintained her belief in its benefits for students, and persevered in her intention to enact, and ultimate enactment of, engaging group work strategies. Further shifts of her perceptions of students' perceptions of group work (subjective norms), as well as increased confidence (perceived behavioral control), influenced Ayla to grow in enacting group work strategies in subsequent courses. In contrast, Athena began her year in the program with a strong desire to empower women, which was sparked in terms of her intentions for teaching in the fall. Having set her intentions on empowering women by providing them with opportunities to talk about mathematics, Athena enacted new teaching practices in the winter and sustained her intention with further modification to her behaviors in the spring. Together, Ayla and Athena's stories of growth suggest implications for the potential benefits of reflection and extended PD.

The lessons we learned from analyzing Ayla and Athena's experiences and observing their growth have implications for other PD providers. In particular, regular observations and post-observation debriefs have the potential to sustain MGTAs' attention to their teaching. In addition, we find it noteworthy that reflection journals in IALE and observation debriefs provided opportunities for MGTAs to consider their values and intentions for teaching and whether or not their teaching practices are in alignment with those intentions. Specifically, reflection on challenges and successes with teaching supported both Ayla and Athena to consider ways to develop engaging and equitable teaching practices. This finding aligns with other research on the benefits of deliberate reflection for MGTAs and instructors of mathematics, more generally (e.g., Quaisley, 2024; Rogers, 2014). Future analyses will investigate patterns of growth across additional participants.

Acknowledgments

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Student Experience With Collaborative Teamwork in Post-Secondary Discrete Mathematics

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As part of a project developing and researching the use of team-worthy lessons in discrete mathematics in two- and four-year colleges, we conducted a study of three instructors and their 105 students. Classroom observations indicated that the use of team-worthy lessons changed the structure of classroom activity from primarily didactic lecture to at least 75% student-centered activity (e.g., group work, student presentation). However, a change in observed structure is only evidence of an affordance, not of students' experience of it or how it might have been productive for students. After each team-worthy lesson, students completed surveys about what went well (and not). Analysis of student responses used a framework based on social learning theory. Here we present results about students' self-reported experiences in terms of agency, authority, creativity, and self-efficacy.

Keywords: discrete mathematics, professional development, team-worthy lessons

Background

For many students, discrete mathematics is a first exposure to formal logical reasoning, proofs, and proving. Undergraduates entering discrete mathematics have many experiences with procedural problem solving and often attribute the authority for sense-making to textbook, instructor, or software (Herbel-Eisenmann, 2007). Like calculus, discrete mathematics shares a history of enormous disparities in outcomes correlated to demographic variables (Campisi et al., 2024). Research in undergraduate mathematics education, notably in calculus, has indicated that disparities can be addressed by linked changes in student materials and instructional methods (Bressoud & Rasmussen, 2015). This literature has also indicated that group-worthy learning opportunities for students combined with use of socio-culturally informed instructional approaches can build positive relationships among students, between students and instructor (and institution), and foster engagement in and outside of class (Archie et al., 2022; Laursen et al., 2014; Leyva et al., 2022; Yoshinobu et al., 2023).

The research reported here comes out of a multi-year project leveraging lessons learned from calculus reform to develop and research use of team-worthy activities in discrete mathematics (Discrete Mathematics Project, 2026). The project's research aims are to (a) explore how use of team-worthy lessons to replace lecture perturbs/enhances student opportunities for engaging in the intellectual work of discrete mathematics and (b) refine professional learning materials and processes to help faculty incorporate student-collaborative methods into their teaching. This report focuses on a subset of the data collected in the larger project to explore the research question: *What evidence is there from classroom observation and student survey that the use of team-worthy lessons creates opportunities for students to develop self-efficacy, authority, agency, and creativity, as components of their learning experience?*

Theoretical Perspective

Our approach was anchored in social cognitive theory and its attention to self-efficacy, agency, and authority (Bandura, 1997). The conceptual framing also drew on educational equity (Gutiérrez, 2009) and mathematical creativity (Regier & Savic, 2018). The framework we used included five categories (Carmody, 2024):

- **self-efficacy** (self-perception of one's own abilities for a given task),
- **authority** (perception about a source of information that has power for validation, verification, or legitimation of a student's or a group's work in mathematics; can be ascribed to a live human in the room or to others (e.g., in a textbook))
- **agency** (self–perception that is both personal and an aspect of interplay between teacher and student regarding who the agent of action is in doing mathematics),
- **mathematical creativity** (components in the process of intellectual work in mathematics),
- **equity** (in the narrow context of the meeting of student needs for access to ideas and identity development in the classroom, more on this below).

Agency involves both a willingness to act and a readiness to act (Bandura, 1997). Creativity, the use of imagination or inventiveness, influences what a learner may act on (and how they will take action). Agency is a co-requisite for experimentation related to how and what to act on (Regier & Savic, 2018). Beyond creativity and the ideas in social cognitive theory (self-efficacy, authority, agency), are the self-conceptions that make up mathematical identity and contribute to who students see themselves to be as learners and doers of mathematics. The framework offered by Gutiérrez (2009) involves interactions among mathematical identity, power, access, and achievement as aspects of educational equity. Gutiérrez noted, "equity is ultimately about the distribution of power, power in the classroom, power in future schooling, power in one's everyday life, and power in a global society" (p. 5). Note that we do not attempt to define equity here. Rather, we give background for what shaped the narrow use of equity concepts in the study in our attempt to document student experience.

When analyzing what made successful schools successful, Gutiérrez (2012, p. 26) found "a key feature of this successful context lay in teachers placing student needs, not just mathematics, at the center of their work." Whose needs are met and who decides they have been met depend on context. Some needs are established by and for an institution by majority society expectations (as defined in the curriculum and requirements for graduation or certification) and other needs are established by students themselves (e.g., needs behind why they chose to gain an education, seek certification, take a particular course). Needs being met for students is, therefore, part of a personal experience, established by the student's own background and supported by the institution in its practices, environment, and standards — including what happens at the intermediate level of the classroom. Thus, the analysis for this report centers on the key feature of the fulfillment of a student's asserted educational needs. Unlike Gutiérrez, who considered schools in the context of society, the focus here is students in the context of the classroom. Equity is far more than meeting a student's needs; it includes how this fulfillment is achieved. This report is not an attempt to address the broader issues of educational equity but to exemplify a means of documenting evidence for the particular aspect of equity related to students' perceptions of their needs being met and the openings offered to students to build personal agency, authority, self-efficacy, and mathematical creativity in the classroom setting.

Methods

Setting and Participants

The data for this study were collected from multiple visits to discrete mathematics classes at two- and four-year colleges. Conducted with appropriate institutional review board approval, data gathering and analysis included three instructors (pseudonymously: Antilles, Peter, and Miles) and the 105 students enrolled in their classes. During the study semester, the three

instructors themselves engaged in a short-course for professional learning about the student-centered methods for which the lessons were designed. The short-course focused particularly on noticing student thinking and strategies for orchestrating classroom discussion. Instructors had access to both a user guide giving an overview of how to teach with the activity-based lessons and detailed instructor roadmaps for each lesson. Instructional resources included:

- basic principles of team-worthy tasks and role-taking by students in team groups (Facilitator, Reporter, Documenter, Questioner)
- guidance on how to form, structure, and manage student teams
- lesson materials (e.g., student handouts) and recommendations for lesson preparation
- suggested questions to pose during instruction
- commentary on common student difficulties and strategies for addressing them
- written-out sample solutions for each task, including optional assessment items.

All three campuses were designated as Hispanic serving. Attendance in the on-campus classes varied, with an average of 80% of enrolled students present for observed classes. We visited at least two different team-worthy lesson uses for each of the instructors, each lasting 60 to 75 minutes (4 for Miles, 3 for Antilles, and 2 for Peter) in addition to at least 150 minutes of typical classroom teaching for each instructor (i.e., not using team-worthy lessons). Instructors interspersed team-based lessons with typical lecture across the semester, at least 1 lesson per month. Students completed a short survey after each team-worthy lesson (average $\leq$ 10 minutes). The descriptive statistics presented here were the result of quantitative analysis of observation protocol details across all observed classes, summary statistics from selected survey responses, and relative frequencies from qualitative coding of open-ended responses.

Observation

The observation method used for documenting classroom activity was based on the *Toolkit for Assessing Mathematics Instruction - Observation Protocol* (TAMI-OP; Weston et al., 2023). The TAMI-OP tool divides a classroom experience into two-minute intervals where checkmarks for a variety of actions by instructors and by students are recorded in a spreadsheet. Once documented, the actions can be grouped into four profiles based on the predominant actions that were checked: interactive lecture, group work, student presentation, or didactic lecture. An "interactive lecture" environment is teacher-centered yet includes questions and answers being exchanged between the teacher and one or more students in a whole class setting whereas the "didactic lecture" environment is teacher-centered and has little or no expectation of feedback from students to teacher or vice versa. The other two profiles are made up, respectively, of actions involved in "group work" (e.g., students work together in small groups, the instructor moves through the room guiding group activity) and "student presentations" (e.g., a student presents on their own or as a reporter for a group).

Survey

Two open-ended items on the survey asked students to:

O1. Share at least one aspect of your experience today with math teamwork that NEEDS IMPROVEMENT

O2. Share at least one aspect of your experience today with math teamwork that was POSITIVE.

Four Likert scale items on the survey asked students to rate aspects of their class session experience (see Table 1). The first two prompts are proxy measures for student sense of belonging and efficacy for engaging in the face of the challenge of working with others. The pacing questions are a proxy for identifying access to intellectual work experienced by students.

Table 1. Likert-like Rating Items on the Survey

Question	Scale
L1. Overall, how would you rate your experience of class today?	1=Satisfying to 5=Not Satisfying
L2. Overall, how would you rate your experiences today in working with others?	1=Fantastic to 5=Terrible
L3. How would you rate the pace of today's activities?	1=Too Slow to 5=Too Fast
L4. Thinking about all class meetings, including today, how would you rate the pace of the class so far?	1=Too Slow to 5=Too Fast

Qualitative Coding

Inductive analysis of students' responses began with the framework. Through three rounds of coding (open, thematic, categorical), the definitions of each subarea were refined. Distinguishing evidence of authority and of agency in student reporting was challenging. A student's reported actions might arise from being an agent in mathematics or might be more connected to their own sense of authority (e.g., validating or verifying their own legitimacy). A further complication, in a group setting, is that students participating in discussion may show agency or authority together (e.g., as a group collectively) and individually. In fact, the authority of the group may yield to any one person. Thus, for the purposes of this report, authority was considered a personal exercise. Consequently, in coding student's responses we distinguished between agency (acting on one's own work) and authority (verifying the legitimacy of one's or others' work).

Results

The first two rating questions were about overall experience and experience working with others. These were proxy measures for student sense of belonging and for engaging in working with others. For both items, the vast majority of responses were in the top two positive categories, with 98% in the top three (i.e., fewer than 2% of student responses in negative categories, see Figures 1 and 2).

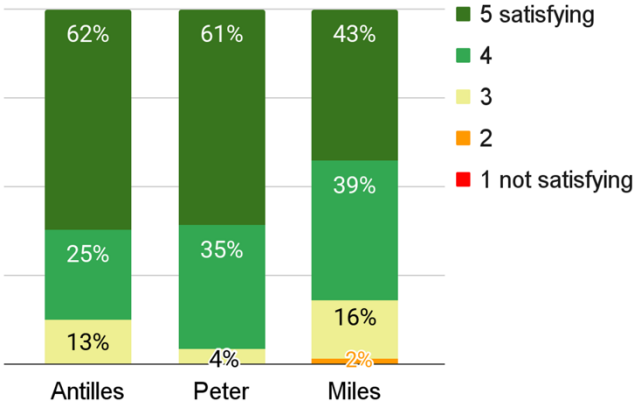

Figure 1. Student rating of overall experience today.

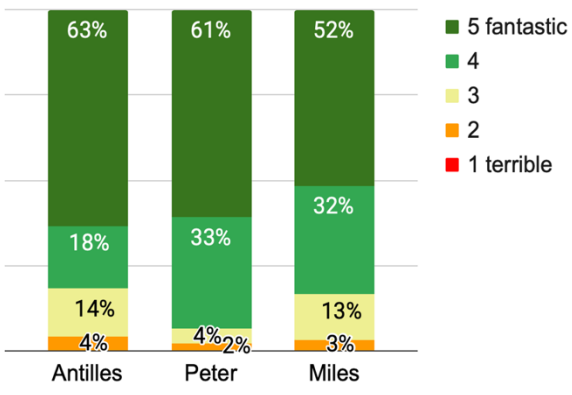

Figure 2. Student rating of experience working with others on mathematics today.

The questions about pace of the day's team-worthy lesson activities and for the class so far, were a proxy for identifying the access to intellectual work experienced by students. If pacing is about right (in the middle), that suggests students felt the work was accessible. For both items, most responses were that pacing was reasonable (not too fast, not too slow, see Figures 3 and 4).

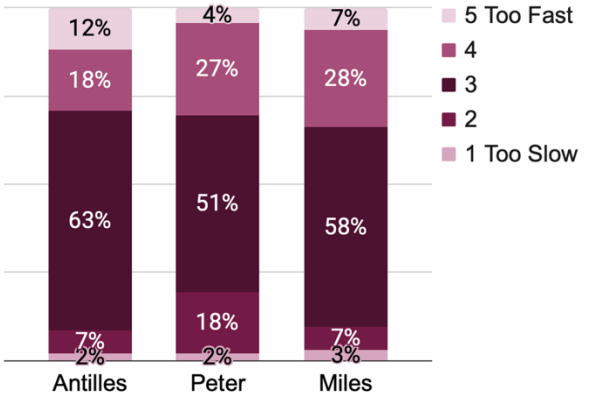

Figure 3. Student rating of pace of activity.

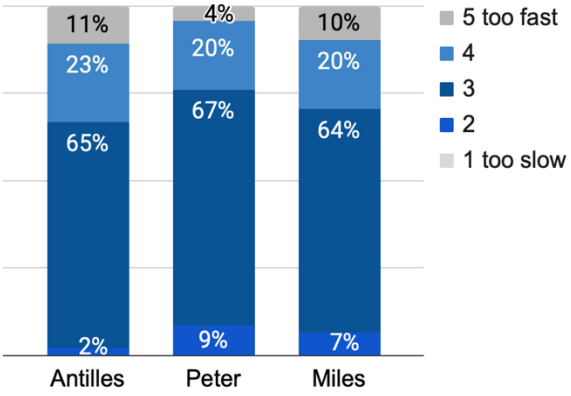

Figure 4. Student rating of pace of class so far.

Analysis of observation data indicated that the use of team-worthy lessons changed the structure of classroom activity from primarily didactic lecture (see Figure 5) to at least 75% student-centered activity (e.g., group work, student presentation; see Figure 6). Figures 5 and 6 suggest that the professional learning opportunities offered as support for instructor use of the materials was sufficient for a change in classroom practice when lessons were substituted for typical practice. However, a change in observed structure is only evidence of an **affordance**, not of students' experience (i.e., if and how the much larger proportion of group work and student presentation time was productive for students' engagement or learning). The examination and coding of written self-reports from the student surveys shed light on what went well (and not).

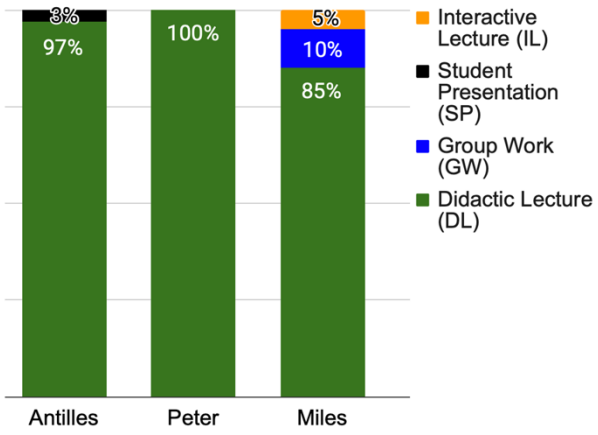

Figure 5. Distribution of instructional formats during typical classes.

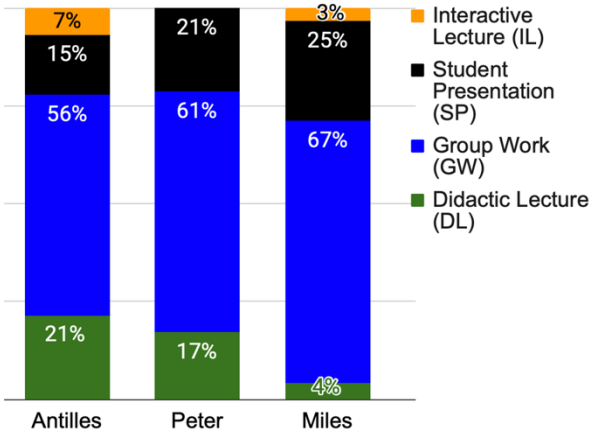

Figure 6. Distribution of instructional formats during team-worthy lessons.

The distribution of coded categories in responses to the question about what was positive differed across instructors (Figure 7, next page). For this report, only one primary code was assigned to each student's response, thus the columns sum to 100%. The code of *creativity* was the primary code in fewer than 6% of responses for any instructor. Similarly, the equity-related

category of *no unmet need* was explicit in only a few responses in each class. Students commented on a positive experience of *agency* by remarking on their pleasure with exploring mathematical ideas and engaging with teammates. A primary code of *authority* arose when a student reported satisfaction with their own or their group's competence or legitimacy in mathematical reasoning. In about 10% of responses in each class, students reported a positive experience of calibration of *self-efficacy*, in understanding the mathematics in the lesson. Finally, the most notable positive experience for many students was that they *worked together* in coming to a deeper understanding of the mathematics at hand (more on this new category, not in the original framework, below).

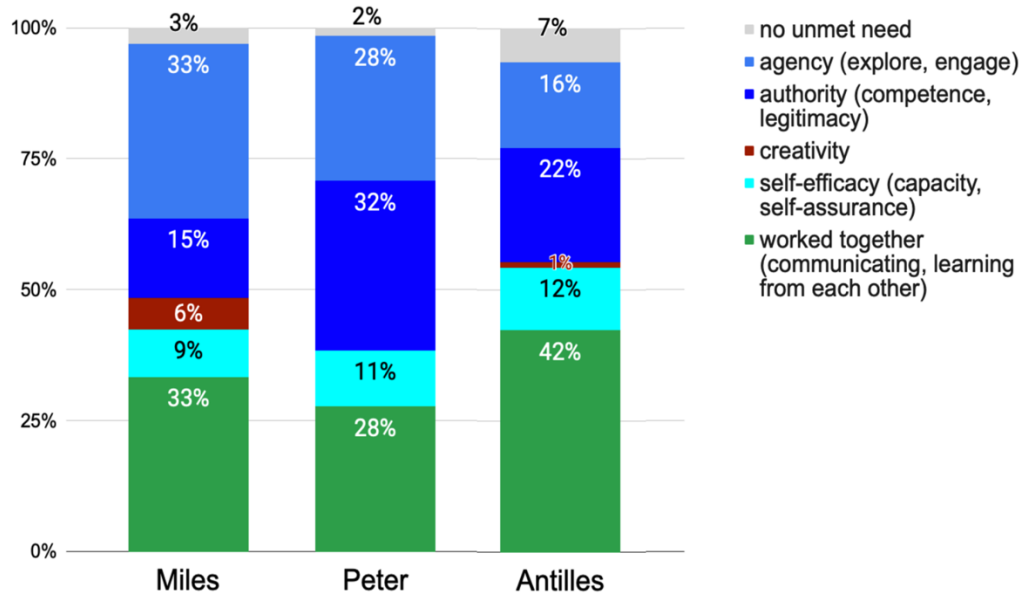

Figure 7. Distribution of percentage of coded student response across all activities, by instructor for "Share at least one aspect about your experience today with math teamwork that was POSITIVE."

Comments about what needed to improve to meet students' felt needs also varied by instructor (Figure 8, next page). As with the positive responses, each student comment got one primary code. Students sought better *access to tools* (e.g., whiteboard pens) or *directions* (e.g., details from the instructor about what was expected for success). Students also reported a feeling of not being clear on if/how they might have *choice* or make *decisions* related to questioning others within their group or on other teams. For each instructor, nearly 20% of students had concerns about how to behave to engage in *collaboration* so that all in the team were involved in interpretation, problem-solving, and/or communication of ideas. The phrase *no suggestions* indicates students commented that all was well or left the answer blank.

Noteworthy in Figure 8 is the smaller proportions of Miles' students who indicated a need for improvement in *choice/decision* or *access to tools/directions*. Miles regularly referred to group roles and reinforced students enacting those roles when he visited groups. The other two instructors used the roles mostly for the launch (facilitator come get supplies) and wrap-up (reporter share group's conclusions). Miles encouraged students to attend to their roles and be purposeful in participating in the team roles (e.g., during a visit to a group Miles sometimes suggested a Questioner visit other groups to ask about ideas).

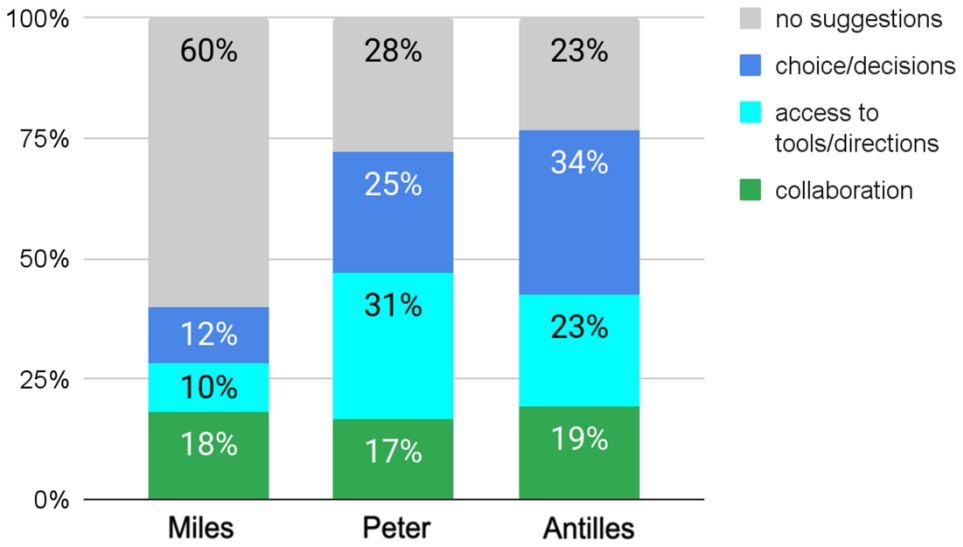

Figure 8. Distribution of codes for student post-lesson survey comments about what needed to improve.

Students generally enjoyed the teamwork and felt it helped them grow as users and doers of mathematics. At the same time, student survey responses suggested that more scaffolds for instructors to implement and orchestrate structures within and across teams for working together well would make student collaboration more effective.

Discussion

In addressing the research question, the findings do indicate there is evidence from classroom observation and student survey that the use of team-worthy lessons creates opportunities for students to develop and demonstrate self-efficacy, authority, agency, and creativity, as positive aspects of their learning experiences. For the "needs improvement" question, most students in Miles' class indicated no suggestions while Antilles' and Peter's students highlighted needs for clearer instructions, improved participation, and better collaboration. For the "positive" question, collaboration emerged most frequently of all positive characteristics noted by students, with at least 28% for each instructor. In future work we plan to add this category to the analysis framework. The distributions in Figures 7 and 8 reflect students' reports of their needs, both met and unmet, for creating and sustaining team-based learning.

Though the extent to which those experiences were equitable was not the focus of the student survey, there seems to be evidence that student needs for engaging in the intellectual work of discrete mathematics were being met in many ways. The majority of students indicated collaboration as an aspect they deemed positive. Yet, collaboration was also the area they felt needed the most improvement! A clear next step is revising the instructor short-course to share with instructors the results of this study and to build up supports for instructor learning in the areas students indicated need improvement.

A next research area is: What (re)balancing of instructor skills is needed to realize the potential of the (re)balancing of activity structures afforded by the lessons? Future work can examine how instructional skills, combined with the affordances of lesson content, contribute to the development of agency, self-efficacy, authority and additional aspects of equity (e.g., belonging) in mathematical sense-making among discrete mathematics students. Research on impact in terms of student mathematical knowledge growth is part of future work.

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Beyond a Single Name: PVEST and Intersectionality in Multilingual Women's Identity Formation in a U.S. Mathematics Ph.D. Program

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The experiences of multilingual women in U.S. mathematics Ph.D. programs are shaped by intersections of language, race, gender, and nationality. This narrative inquiry examines the identity formation of three women from Korea, Colombia, and Mexico, employing a combined framework of intersectionality and Phenomenological Variant of Ecological Systems Theory (PVEST). Findings identify three distinct pathways: the carryover of prior gendered invalidation, leading to impostor syndrome despite high commitment; context-dependent coping across bilingual ecologies; and the consolidation of an advocacy identity in a more protective environment. Theoretically, the study extends PVEST to Asian and Latin populations and clarifies a dual-function coping mechanism where strategies can be both adaptive and costly. Findings call for strengthening protective factors and identity-inclusive instructional designs in mathematics education is necessary to support this growing student population.

Keywords: Multilingual, PVEST, Intersectionality, Narrative Inquiry

Multilingual students constitute a significant portion of post-secondary mathematics classrooms (Data USA, n.d.). Recent research has begun to examine their experiences, although much of this valuable work has focused on multilingual students under a single name (e.g., Hwang et al., 2022; Rios, 2022, 2024). While this research has provided important insights, there remains an opportunity to explore how these multilingual students enter classrooms with diverse backgrounds and multiple social identities–including but not limited to race, gender, and nationality–that intersect in complicated ways (Hwang, 2023; Leyva & Joseph, 2023; Rios, 2022, 2024). These identities do not operate independently. They interact with one another and with students' multilingual status to create distinct lived experiences (Hwang, 2023). Although analyses examining single identity categories are informative, they may not fully capture the specific challenges that emerge from these intersections (Hwang, 2023; Leyva & Joseph, 2023; Rios, 2024). That is, there remains potential to further understand how students interpret experiences, navigate challenges, and develop their identities in mathematics education.

This study builds on existing work by examining the lived experiences of three multilingual women in a mathematics Ph.D. program: Helena, Jihyun, and Maria (pseudonyms). To analyze these stories, I adopt narrative inquiry, which focuses on how individuals make sense of their lived experiences (Clandinin & Connelly, 2000; Connelly & Clandinin, 1990; Polkinghorne, 1995). Within this approach, I employ a theoretical framework combining intersectionality (Crenshaw, 1991) and Phenomenological Variant of Ecological Systems Theory (PVEST; Spencer et al., 1997; Spencer, 2006). In this study, intersectionality is used as a lens to analyze how students' multiple social identities create unique structural contexts (Crenshaw, 1991; Gholson & Martin, 2019; Graham, 2022; Leyva, 2016; Leyva et al., 2022). PVEST then provides a framework to analyze how each student interprets and reacts to their experiences within that context, leading to the formation of their emergent identities (Spencer et al., 1997; Spencer, 2006; Velez & Spencer, 2018). Following the PVEST framework, I view identity as dynamic and contextually situated, emerging through the navigation of challenges arising from one's intersectional positions (Spencer, 2006; Velez & Spencer, 2018).

This study addresses the following research questions:

1. How do intersecting identities (language, race, gender, nationality) shape each student's lived experiences and stressors?
2. How does the five-stage PVEST model explain the processes through which each student copes with stress to form an emergent identity?
3. What do the narratives of these three students reveal about their life-stage specific outcomes?

Theoretical Framework

Intersectionality

I use intersectionality to analyze the structural context that shapes each student's experience. Emerging from Black feminist scholarship, intersectionality explains how interlocking systems of power–such as racism, sexism, and nationalism–create unique contexts of oppression and privilege (Crenshaw, 1991; Leyva, 2016). This lens is critical for this study because it moves beyond analyzing single identity categories in isolation and instead focuses on how their convergence shapes a person's lived reality.

For the multilingual women in this study, their positioning in U.S. higher education often frames them through a deficit lens as 'non-native English speakers.' I intentionally use the term 'multilingual' to recognize the linguistic resources these students possess, but it is the societal label of 'non-native' that often forms part of their structural reality that privileges English. This label intersects with their identities as women and as Latina or Asian, creating a unique structural context where linguistic ability can be conflated with intellectual competence (Leyva & Joseph, 2023). Intersectionality provides a tool for examining how these overlapping systems shape students' experiences.

Table 1. The Five Stages of the PVEST Model (Spencer, 2006).

Stage	Explanation
Net Vulnerability	The balance or imbalance of risks (e.g., social stereotypes, discrimination) and protective factors (e.g., educated family, role models)
Net Stress Engagement	The balance or imbalance between stressors (i.e., actual experiences of challenge) and available supports
Reactive Coping Methods	Problem-solving strategies an individual uses to manage the stress they are experiencing. These can be adaptive or maladaptive.
Emergent Identities	The stabilization of repeatedly used coping strategies into a predictable, yet fluid, pattern of response that forms an aspect of one's identity–how individuals perceive themselves within their context.
Life-Stage Specific Coping Outcomes	Resulting behaviors and outcomes of an emergent identity, which can be productive (e.g., academic achievement, well-being) or unproductive (e.g., psychological distress, dropout)

Phenomenological Variant of Ecological Systems Theory (PVEST)

To analyze how students navigate their experiences shaped by intersectionality, I use PVEST (Spencer et al., 1997; Spencer, 2006; Velez & Spencer, 2018). PVEST models how individuals form identities through a cyclical process that begins with *Net Vulnerability*, and is followed by *Net Stress Engagement, Reactive Coping Methods, Emergent Identities*, and *Life-Stage Specific Coping Outcomes*. Table 1 summarizes each stage.

These stages are not linear but cyclical. One's subjective interpretation of a challenge shapes their reactive coping method. When the person feels a particular coping method is producing desirable results, that coping method stabilizes into an emergent identity. This identity then leads to life-stage specific coping outcomes that create a feedback loop, altering the students' initial net-vulnerability for the next cycle of stress engagement (Spencer, 2006). In this study, the model allows me to trace how each student's net vulnerability translates into their emergent identity.

Method

In this study, I employ a qualitative research design using narrative inquiry, a methodology that focuses on understanding one's lived experiences through what they tell (Clandinin & Connelly, 2000; Connelly & Clandinin, 1990; Polkinghorne, 1995). This approach was chosen because it allows for an in-depth exploration of how participants make sense of their lives and construct their identities over time. The focus on subjective, lived experience aligns with the phenomenological perspective of PVEST, which argues that an individual's perception of events is central to their development. In this study, narrative inquiry provides the method to access and understand these phenomenological processes.

Researcher Positionality

As the researcher, I share several social identities with the participants. I am a Korean woman who received her K-16 education in Korea and is now a multilingual international student in a U.S. mathematics education Ph.D. program. My own experiences with linguistic challenges and feelings of incompetence have shaped my interest in this topic. This shared positionality helped build rapport and trust with the participants, which allowed for more open sharing of their stories. At the same time, I was mindful of my tendency to attribute my own challenges to personal deficits. Throughout the research process, I engaged in reflexivity to avoid imposing this deficit perspective on the participants' experiences and instead sought to understand the broader context of their stories.

Participants and Data Collection

The participants for this paper–Maria, Helena, and Jihyun–were first-year Ph.D. students in mathematics at a large Midwestern university in the U.S. All are multilingual international students whose K-12 education was primarily not in English. Maria is from Mexico, Helena is from Colombia, and Jihyun is from Korea. All names are pseudonyms.

I collected data in three stages to capture each participant's experiences: Participants first wrote an autobiography, "Me and Math: My relationship with mathematics up to now," about their relationship with mathematics based on the idea from Di Martino & Zan (2010). After that, I conducted one 60–90 minute semi-structured interview with each participant to explore events from their autobiography in more detail and to ask further questions. Lastly, a series of follow-up interviews was conducted via email or shared documents, allowing participants time to reflect on their responses.

Data Analysis Procedures

The data analysis was conducted in two main phases: narrative construction and theoretical cross-case analysis, respectively, narrative analysis and analysis of narrative according to Polkinghorne (1995). I first analyzed the collected data to construct a first-person narrative for each participant. This process involved coding for specific experiences, identifying themes, and organizing them into a plot. Each constructed narrative was then sent to the participant for member checking to ensure the representation was fair and accurate. In the second phase, the finalized narratives of Helena, Maria, and Jihyun were treated as the data for a cross-case analysis. I applied the theoretical frameworks of Intersectionality and PVEST to this data. First, I used intersectionality to analyze how each participant's intersecting social identities shaped their experiences. Then, I used the PVEST 5-stage model to analyze the process of identity formation for each student, tracing how their different backgrounds led to different vulnerabilities, stressors, coping methods, emergent identities, and life-stage specific coping outcomes.

Findings

This section presents the analysis of each participant's narrative. The order of cases presented here demonstrates three key mechanisms through which contexts influence emergent identity: how past environments become internalized and continue to shape coping methods, how individuals navigate multiple environmental demands, and how environment transitions can catalyze emergent identities.

Jihyun: "I decided to study science harder and show that I could do better than these boys"

Jihyun's story shows how gendered invalidation from earlier schooling can continue to shape her academic experience. As a Korean woman in a U.S. mathematics Ph.D. program, her current academic experience is shaped by a worry about being perceived as incompetent, which is rooted in sexist treatment from her past educational experience in Korea.

Her net vulnerability combines significant risks from past trauma with current pressures from the model minority myth demanding mathematical excellence for Asians. The following is one of the quotes from the narrative that shows her experience of sexism.

> .. [a] teacher in that class said something like, "A girl makes the water cloudy [translated from Korean saying]," which is similar to "A rotten apple (=a girl = me) spoils the whole barrel" in English. I decided to study science harder and show that I could do better than these boys.

Jihyun described that such experiences contributed to her perfectionism. In addition to those risks, she felt pressured to meet the standard based on the model minority myth: "Sometimes I feel I am representing Koreans… If I do something wrong… I am afraid they would think every Korean is neither good at mathematics nor nice." At the same time, she wanted to contest the common bias that "Asians are not good at English." These pressures are internalized and shape her net stress engagement and reactive coping methods.

After moving to the U.S., she faced a different environment where lectures were rapid in English and on-the-spot participation and responses were normative. Combined with the vulnerability that she had, in response, she monitored and censored herself (e.g., "I need to think at least three times before I verbally say something" or "I try to erase my Korean accent"). At the same time, she found some strategies, such as relying on written materials rather than verbal conversations.

These repetitive coping methods have been stabilized as a commitment-centered identity (i.e., "Mathematics is my life. Mathematics is who I am.") while retaining imposter syndrome.

As life-stage outcomes, she retained high achievement as a trade-off for greater cognitive load and self-doubt. However, there was an important advance: she discussed her imposter syndrome and concerns about English with her advisor, and the advisor agreed to provide some support, which would increase protections and hopefully reconfigure net vulnerability in the next cycle.

Helena: "Don't get me wrong. I am glad that I speak English, but Spanish is my language."

Helena's story illustrates context-dependent coping across multilingual ecologies: how she adapts by strategically using Spanish as the language of her inner world and community, while treating English mainly as a professional tool. She is a Colombian Ph.D. student in mathematics in the U.S. Helena's net vulnerability is relatively low due to strong protective factors, including a supportive cohort, Spanish-speaking Latina roommates, and a Catholic community, which buffers the linguistic risks she faces as a multilingual student. Her net stress engagement is situational: stress spikes when comprehension depends on real-time oral English, such as fast speech, unfamiliar accents, multi-party talk, which makes it more challenging; but in mathematics, it is buffered by written channels, such as notation, boardwork, and slides.

Helena employs two distinct reactive coping methods depending on the context. Within her cultural community, she finds emotional regulation and strength, stating, "having Latinas around me and speaking Spanish with them is important to me." In academic settings, she uses a different but equally adaptive strategy to manage cognitive load and stress: choosing to be quieter. Her statement, "I tend to be quieter in English than in Spanish," reflects a strategic choice to conserve energy and avoid the high stakes of real-time verbal processing in a second language. The stabilization of these context-dependent coping methods results in an emergent identity that is confident and leading in Spanish-speaking contexts but more reserved in English-speaking academic ones.

Her life-stage specific coping outcomes are currently productive, marked by successful adaptation to her first year. However, a potential challenge arises from the interplay between her adaptive coping (being quieter in English) and future professional demands. Her anxiety about teaching–"I am scared that I am not explaining myself well"–reveals a potential unproductive outcome. The issue is not that her coping strategy is deficient, but that a strategy that is highly adaptive for learning may not be sufficient for the demands of teaching, highlighting a tension she must navigate in the next cycle.

Maria: "I know the answer, but why I don't have the courage to raise my hand and say it?"

Maria's narrative illustrates how a supportive new environment can facilitate the transformation of an identity shaped by past intellectual suppression, as evidenced by her experiences in Mexico and the U.S. As a Mexican woman in a U.S. Ph.D. program, she works to overcome earlier invalidation coming from her gender and redefine her place in mathematics.

Maria carried a risk of gendered invalidation in her undergraduate studies in Mexico. In Mexico, as the only woman in the cohort, her contributions were often ignored (e.g., "Maria got a good grade because the professor just gave it to her," or "Maria did well on the exam because her exam was easier than ours"). The silence of peers, as she noted, "none of the other classmates said anything," reinforced the isolation. In response, she developed a reactive coping method that was twofold: she used silence as a protective strategy while internally reframing her struggle into a purpose. She came to see her persistence in mathematics as necessary "for a woman to be represented in this profession," believing she had been "breaking the stereotype and making it possible for another woman" in her hometown. This sense of purpose became an emergent identity that she carried into her next academic environment.

When she moved to the U.S., the new environment with a cohort of six other women served as a protective factor. While this supportive community reduced her net vulnerability, she still faced academic stressors. Despite her fluency in English, the rapid pace and accents of U.S. classroom communication were challenging. In response, she developed coping methods such as relying on mathematical symbols and written work rather than verbal participation. Despite some challenges, the supportive social context provided the safety needed for the emergent identity she had begun to build in Mexico, as an advocate for women in mathematics, to be validated and reinforced. This validation led to productive life-stage-specific coping outcomes. She successfully passed her qualifying exams and now feels a sense of belonging. This success could serve as a new protective factor, reducing her net vulnerability for the next cycle.

Across these three cases, this paper presented how past environments, present supports, and transitional contexts interact with students' identities in distinct ways. Together, these narratives demonstrate the mechanisms of PVEST in practice–how vulnerabilities are internalized, how coping strategies stabilize, and how emergent identities take shape.

Discussion

This study analyzed the narratives of three multilingual women in a mathematics Ph.D. program to understand how their social identities intersectionally shape their emergent identities in the U.S., a different context from their home countries. Across cases, real-time English discourse demands, such as pace, accent/intonation, and multi-party talk, were common, yet their meaning and impact differed: Jihyun carries forward legacies of gendered invalidation under a model-minority myth, increasing her worries about evaluation; Helena navigates a split present–socially well supported while academic English remains demanding; Maria's move into a more protective environment enables the consolidation of a purpose to represent women in mathematics.

PVEST clarifies how these vulnerabilities become integrated into identity through a cycle of stress, engagement, developing coping methods, and stabilization of those coping methods. Through their coping methods, Jihyun developed an identity committed to mathematics with imposter syndrome, Helena developed an identity responsive to bilingual contexts, and Maria developed an identity as an advocate for women in mathematics. The density and fit of protections influenced these trajectories towards identities and more or less productive life-stage specific outcomes. For instance, Maria's supportive cohort reinforced her advocacy, while targeted supports for Helena and Jihyun showed the potential to reconfigure their net vulnerability for current or future challenges. Across these cases, protective factors were powerful enough to change the direction of each student's academic and personal journey.

This study's findings make four theoretical contributions to the literature on the identity development of multilingual students. First, this study provides an empirical demonstration of PVEST's applicability to Asian and Latin students in mathematics education, extending the framework beyond its primary application. While PVEST offers a lens for understanding identity formation in context (Spencer et al., 1997; Spencer, 2006), its use in mathematics education has predominantly focused on the experiences of the Black community (e.g., Graham, 2022; Gholson & Martin, 2019). This study contributes to the understanding of the framework's usefulness for other marginalized populations, illustrating how the interaction of risks, supports, and coping mechanisms also shapes the identity processes of multilingual Asian and Latin women.

Second, this research provides an empirical demonstration of intersectionality and PVEST. In this study, intersectionality explains what constitutes each student's unique net vulnerability (Velez & Spencer, 2018). It reveals that the experience of multilingual students is not

monolithic, but rather a specific structural position created by the intersection of language, gender, race, and national origin. PVEST, in turn, explains how students navigate these structural realities (Velez & Spencer, 2018). It traces the process through which their specific vulnerabilities are linked to reactive coping methods–such as Jihyun's perfectionism, Helena's strategic silence in English, and Maria's reframing of challenges–which then stabilize into their Emergent Identities (Spencer et al., 1997; Spencer, 2006).

Third, this research empirically clarifies the complex mechanisms underlying core PVEST concepts. While PVEST is a non-deterministic framework that anticipates diverse and nuanced outcomes (Spencer, 2006), Jihyun's case provides an example of this principle by demonstrating a clear dual-function mechanism. Her perfectionism is adaptive, producing the desired productive outcome of high academic achievement. At the same time, the same strategy can be mentally draining, causing negative effects like self-censorship and imposter syndrome. By explaining this complex process in detail, this research extends beyond theory to provide an empirical account of how a single coping method can simultaneously lead to both positive and negative outcomes.

Finally, this study contributes to the understanding of the role of ecological transitions in identity development. Maria's case shows that a supportive new environment does not just create a new identity. Instead, it can serve as a space to validate and reintegrate a resilient identity that was already developed in response to a previous hostile environment. This finding highlights the interplay between individual agency–Maria's prior efforts to reframe her struggle into a mission–and structural support. It suggests that environments serve as catalysts that can unlock and reinforce the potential of an identity already in progress.

The findings from this study call for attending to students' complicated cognitive and affective efforts and challenges that are not visible behind high achievement. Additionally, it is essential to acknowledge students' plural ecologies, recognizing that they may have developed different identities across various contexts. This implies that mathematics educators must devise strategies to help students integrate their positive identities into the mathematics classroom. To achieve this goal, it is important to enhance protective factors, such as having mentors and peers from similar backgrounds, and provide opportunities for developing and utilizing various coping methods, including instructional designs that adjust pacing, promote multi-channel communication, and structure equitable participation (Hwang et al., 2022; Rios, 2022, 2024).

Conclusions

This study analyzed the identity formation of three multilingual women in graduate mathematics by integrating intersectionality and PVEST. The findings traced three distinct pathways: the carry-over of past hostility (Jihyun), the navigation of a split ecology across social and academic contexts (Helena), and the consolidation of identity through a protective transition (Maria). Theoretically, the study extends PVEST's application to Asian and Latin women in mathematics education, specifies the dual-function nature of coping mechanisms, and highlights the role of ecological transitions in reinforcing identities.

As this narrative inquiry captures a snapshot of their lived experiences at the time of data collection, their identities are likely to have continued evolving in subsequent PVEST cycles. This temporal limitation, along with the researcher's own positionality as a Korean multilingual woman, shaped the scope and interpretation of the findings. Recognizing these aspects, future research should test the mechanisms identified here across multiple sites and over longer time scales.

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Setting Up Group Tasks: Rights and Duties of Instructors and Students

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In this contributed report, we share results from a study focused on the verbal communication acts of four undergraduate mathematics instructors while they are setting up group tasks. All four instructors teach at the same large eastern university and are experienced with active learning instructional practices. We found that two of the four instructors engaged in numerous verbal communication acts that assigned rights and duties to themselves as instructors and to their students. The other two instructors assigned only a few rights and duties to themselves or to students. We describe patterns in instructors' assignment of rights and duties, as well as how instructors' verbal communication acts shaped the classroom learning environment. Finally, we suggest how verbal communication acts focused on assigning rights and duties may relate to opportunities to learn and propose considerations for supporting students' participation in group work in active learning undergraduate mathematics classrooms.

Keywords: group work, opportunities to learn, positioning, participation, communication

Research on undergraduate mathematics classrooms has increasingly emphasized active learning and the use of group work to foster student engagement and deeper understanding. While numerous studies demonstrate that active and inquiry-based learning can improve learning outcomes (Laursen & Rasmussen, 2019; Freeman et al., 2014), the dynamics of how group work is set up by instructors remain underexplored. Work in this area is especially important since scholars note that group work does not automatically ensure a rich participation of all students in mathematical discussion; rather, instructor practices play a critical role in shaping the quality and distribution of learning opportunities (Cohen, 1994; Tanner, 2013).

To address this gap, we leverage positioning theory as an analytical tool, contextualized within Esmonde's (2009) theoretical framework for Opportunities to Learn (OtL). Esmonde's OtL framework identifies four interrelated dimensions of mathematics learning in groups: participation, social ecologies, communication, and identity development (detailed in the following section). In positioning theory, storylines, speech acts (or communication acts), and positioning (or rights and duties) form a triad in which the three constructs are mutually interdependent (Harré, 2012; Herbel-Eisenmann et al, 2016). This study is grounded in Esmonde's OtL framework and informed by positioning theory; we leverage two closely connected components of positioning theory to support our data analysis: communication acts, and rights and duties (Drageset & Ell, 2024). We identify instances of instructor communication acts, specifically verbal communication acts, in which they assign rights and duties either to themselves or to their students.

Recent scholarship has highlighted specific instructional practices related to each of the four dimensions of Esmonde's OtL framework that broaden access to OtL in group settings. Instructor facilitation practices such as assigning roles or explicitly inviting all members to contribute have been shown to promote more balanced participation (Webb et al., 2014). Related to social ecologies, patterns of social interaction combine with classroom layout, group size, and

the design of collaborative tasks to affect who has access to shared resources and whose ideas are visible (Horn, 2012). Regarding communication, researchers emphasize the importance of instructors explicitly framing discourse norms – for example, requiring explanation, justification, or collaborative sense-making rather than answer sharing (Yackel & Cobb, 1996; Yackel & Rasmussen, 2002). Finally, for identity development, instructor beliefs and positioning moves are critical: affirming students' competence, validating diverse approaches, and distributing authority have been linked to stronger identity alignment with mathematics (Hand, 2012; Nasir, 2002).

While this body of work demonstrates the importance of instructional practices, little research has examined how instructors set up group tasks in undergraduate mathematics classrooms – the period when expectations, norms, and participation structures are first introduced. Task setup is a critical moment when access to OtL can be expanded or constrained, yet it remains underexplored. In this report we share findings from our study investigating the verbal communication acts instructors use when setting up group work. Specifically, we ask the following research questions: (a) What rights and duties do undergraduate mathematics instructors assign to themselves and/or their students during verbal communication acts while setting up group tasks? And (b) What patterns are identifiable across instructors in their assignment of rights and duties during group work setup?

In our discussion section, we consider how instructors' verbal communication acts that assign rights and duties may shape students' opportunities to learn, and we suggest how this study contributes to a deeper understanding of how active learning environments are enacted. We also highlight practical considerations for supporting effective group work in undergraduate mathematics classrooms to increase students' participation in mathematical discussions.

Literature and Theoretical Framing

As mentioned above, Esmonde's (2009) Opportunities to Learn (OtL) framework identifies four interrelated dimensions that structure access to mathematics learning in groups. Participation concerns who is invited to engage and how access to problem-solving is organized. Social ecologies emphasize the role of cultural norms and patterns of social interaction along with physical arrangements, group composition, and resource distribution in how the group organizes their relationships. Communication captures the expectations and norms for mathematical discourse, such as justification, explanation, and turn-taking. And identity development attends to how learners are positioned – as capable contributors or marginalized participants – through instructor and peer interactions. Together, these four dimensions provide a framework for examining what students are asked to do, how they are enabled or constrained in doing so and ultimately, the opportunities available to students to take up more central roles or positions as doers of mathematics.

For this study, we employ positioning theory, which includes the interrelated constructs of storylines, rights and duties, and communication acts. Rights and duties refer to the positions individuals and groups perceive themselves and others to hold in interaction (Drageset & Ell, 2024). In relation to Esmonde's OtL framework, they are enacted and negotiated through communication, and they shape who speaks, who listens, and how contributions are taken up. These dynamics influence participation (the ways individuals are invited to or restricted from engaging), social ecologies (patterns of authority and interaction), and identity development (how competence is recognized or marginalized). Rights and duties are not fixed; they evolve as participants interact, challenge, or reinforce each other's positions. Rights and duties are deeply intertwined with the communicative processes that structure opportunities to learn.

Communication acts (including verbal and gestures), aligned with the communication construct of the OtL framework, work to uphold or shift storylines (described next) and confirm or introduce new rights and duties. We use the construct of communication acts in this study with a specific focus on verbal communication acts that serve to name rights and duties of instructors and students. The third component of positioning theory, storylines, are "the ongoing repertoires that are already shared culturally or they can be invented as participants interact" (Herbel-Eisenmann et al., 2016, p. 188). In relation to Esmonde's OtL framework, storylines contain aspects of participation (how people participate), social ecologies (patterns of social interaction) and identity development (who is seen as more or less competent). Participants may have different storylines, even within the same class or group, and storylines can evolve as participants see new ways to interact or engage with each other. Analyses focused on non-verbal communication acts and storylines, while potentially useful for future studies using this data, are beyond the scope of this report.

Methods

The data for this study was collected at a large, research-intensive, urban university in the eastern United States. We video recorded the first session of four classes in two different sequences of precalculus courses. Students in both courses are mainly freshman STEM students. The demographic makeup of these courses in terms of race/ethnicity and gender reflects the diversity of the campus (62% identified themselves as Female students, over 17% as Black/African American, 9% as Hispanic/Latinx, 13% as Asian, and 44% as White students, in a recent report). Between 10 to 15 instructors (mostly non-tenured) usually teach these courses, often in more than one section. The primary data for this study consists of four video-recorded episodes of four of these instructors introducing a group task in class. Each episode begins at the moment the instructor first references the task and ends when students transition into working together on the task. In addition to these video recordings, supporting data includes task artifacts (e.g., worksheets, slides, board captures) and contextual information (class size, course type, and institution). All instructors provided informed consent, and student images were blurred in all visual material.

We define the episode (the full instructor-led setup of the group activity) as our primary unit of analysis. Within each episode, we segment the interaction into verbal communication acts which are bounded events in which the instructor carried out a single type of action (e.g., framing the task, explaining roles, presenting norms for talk). Instructors' verbal communication acts serve as our coding unit. Most instructional moves during the setup of group work involved the instructor prompting tasks, encouraging students to do work, reflecting on their own learning, and justifying class structure and organization. We categorized these verbal communication acts in one of four categories: instructor's rights, instructor's duties, students' rights and students' duties. We defined rights as the entitlements of participants and duties as the obligations or tasks that are assigned to participants. Two coders initially coded select verbal communication acts together to develop a coding scheme and then coded the remaining data independently. Finally, the two coders came together to gain consensus on the codes, and to select representative quotes for sharing in the results.

Results

Our coding revealed numerous instances of instructors engaging in verbal communication acts in which they assigned rights and duties while setting up group tasks. Table 1 provides the number of times we coded each participating instructor's verbal communication acts as assigning

rights and duties. Note that all names used are pseudonyms. Table 2 provides examples of the types of verbal communication acts that were coded as teacher rights, teacher duties, student rights, and student duties. In the next section, we will present data analysis from quantitative and qualitative perspectives.

Table 1. Instances of instructor verbal communication acts in which they assign rights and duties.

Distribution of Assignment of Rights and Duties in Speech Acts				
Pseudonym	Teacher Duty	Teacher Right	Student Duty	Student Right
Barb	6	6	7	6
Celia	6	3	2	4
Harry	2	2	3	2
Donna	2	1	0	3

Table 2. Examples of the type of instructor communication acts in which they assign rights and duties.

Examples of Communication Acts that Assign Rights and Duties			
Teacher Duties	Teacher Rights	Student Duties	Student Rights
Facilitate student introductions	Assign students to groups	Be open to learning math in a new way	Have good mathematical learning experiences
Caring for the success and well-being of students	Select students to present work to the class	Communicate with group mates	Receive assistance with difficulties
Minimize lecture	Solicit knowledge from student groups	Join the group the instructor assigns them to	Be accepted into a community within the classroom

Analysis

The results presented in the previous section suggest that instructors' strategies for setting up group tasks vary widely. In this section, we present analyses from both quantitative and qualitative perspectives to more fully elaborate on these variations. First, we share two different perspectives for making sense of these results from a numerical perspective. We then provide descriptions of each of the four instructors' strategies for setting up group work to add context to the numerical findings.

Making Sense of the Data from a Numerical Perspective

These results reveal a variety of ways instructors set up group tasks, which can be understood through two different ways of viewing these numbers. First, instructors' use of verbal communication acts focused on assigning rights and duties differed in absolute quantity. While Barb and Celia assigned rights and duties relative to engaging in group work extensively,

Harry's and Donna's verbal communication acts while setting up group tasks included far fewer instances of assigning rights and duties.

Viewing these results from a different perspective reveals that instructors' verbal communication acts also varied in the relative number of instances of assigning rights and duties, as well as whether they assigned those rights or duties to themselves or their students. Barb and Harry assigned rights and duties to themselves as instructors, and to students, in ways that were roughly balanced across the two distinct roles (instructor vs student). In contrast, Celia and Donna assigned rights and duties in ways that were imbalanced. Celia and Donna both assigned more teacher duties and student rights, and fewer instances of instructor rights and student duties. We observed a balanced assignment of rights and duties in the verbal communication acts of one instructor who assigned numerous rights and duties (Barb), and a second instructor (Harry) who assigned relatively few rights and duties. Correspondingly, an imbalanced assignment of rights and duties occurred in the communication acts of one instructor, who assigned many rights and duties (Celia), and another instructor, who assigned relatively few rights and duties (Donna).

Making Sense of the Data From a Qualitative Perspective

Of the four teachers in this sample, Barb's verbal communication acts related to assigning rights and duties were the most numerous. Barb also provided the most explicit expectations for herself and her students in relation to group tasks. She positioned both herself and her students as a classroom community, providing explicit time for students to introduce themselves to others nearby and share contact information with each other, establishing chosen "math families" with whom they would work in class. She set specific duties for students with respect to these math families, saying that she "expect[s] that you guys are [talking] with the person next to you or some part of your math family multiple times in the class" and that students "don't always have to get it right…but you have to be doing as much as you can to learn from those around you." She sets herself as having the authority to monitor the status of group conversations, repeatedly commenting to specific students and the class that they should be communicating more (e.g., "this is awfully quiet for people who are working together" and "You have to talk with people. I'm sorry.") While she sets herself an authority to monitor and structure group conversations, she explains that this group format is motivated by students' right to have an engaging mathematical learning experience (e.g., "I don't want you guys to be working by yourself because it's so boring") and learn mathematics in an optimal manner, explaining that lecture is "not the best way to learn math, the best way to learn math is being interactive with it." Overall, Barb positions students as belonging to "math families," with certain rights and duties associated with belonging to such a group, and herself as a director and supporter of these groups.

Similar to Barb, Celia also engaged in numerous verbal communication acts focused on assigning rights and duties. And Celia also explicitly oriented her class around group work, explaining that she would occasionally provide "mini-lectures" to introduce concepts, but that the remainder of class time would be devoted to students "talking to each other, working on problems, and, of course, asking questions." During this work time, she stated her expectation that students should be "polite" to one another and that "if someone asks a question and you know the answer to it, it's good practice for you to explain it to somebody else." She also emphasizes her role in managing these group discussions, saying "I'll walk around and if I [think] someone's hanging back or not trying to engage, I'm going to try to encourage you to jump in and have that conversation." She encourages students to share their authentic thoughts about mathematics because "if I see a lot of people get it wrong, that's a nice clue to me that we

need to slow down and talk about something more." Thus, Celia positions those in her classroom similarly to those of Barb's but places a stronger emphasis on student questions.

Harry exhibited fewer instances of verbal communication acts focused on rights and duties but also structured his class around group work. Harry set less direct expectations for himself and students than Barb and Celia. Interestingly, he was the only instructor in our sample to assign students to groups, rather than letting students choose their own groups. He did this by handing each student a playing card as they walked in, then grouping based on kinds of cards (e.g., all four kings together, all four sixes together). While Barb and Celia repeatedly stated and reiterated specific expectations for the group work environment of their classrooms, Harry briefly explained to students that "you should first ask the other people in your group if they can help you out and if everybody's stuck, then just let me know and then I'll come by and help out." While this dynamic of consulting group members and the instructor about mathematical questions is similar to what Barb and Celia describe for their classrooms, Harry does not present a specific social framework around ideas of comfort or community.

Donna had the fewest instances of verbal communication acts focused on rights and duties, and, unlike the other instructors, Donna did not explicitly mention group work as a component of the course. However, she did position herself similarly to other instructors, describing students' right to ask questions and her duty to answer student questions. For example, she stated, "it's because you asked a question that all of a sudden things made sense. That's how this class works. All right, give me more questions." Notably, she emphasized that students in her classroom should be able to ask questions without shame, telling students that they shouldn't "be embarrassed about asking me questions" and that "you're going to make mistakes. Congratulations. You're human … So, I don't want you feeling bad if you made a mistake. I'm not going to think any worse of you." Thus, while Donna did not directly mention group work, she did express the value of asking questions, much like the other instructors.

It is also worth noting some general observations from across all four instructors' classrooms that may differ from active learning classrooms more generally. From a positioning standpoint (Herbel-Eisenmann et al., 2016), the students in these active learning classes mostly accepted the duties assigned to them. Barb sometimes pushed students to talk together to exchange emails/phone numbers, but otherwise, there were no instances of students resisting or speaking out against the active learning environment. And, while students have a right to learn, the instructors never suggested that students had a right to choose the learning environment. All of these courses are designed with active learning in mind, and students do not have another option. It is therefore an implied duty to adapt to the expectations of the classroom. Barb points out that high school classes might still have lecture models, and students will have to make the adjustment (a student duty) if that was their experience.

Discussion: Implications, Questions, and Future Work

In this study, we have examined several distinct ways in which mathematics instructors structure group work in pre-calculus courses. Notably, we found that, although all instructors involved in the study were experienced university mathematics teachers who self-identified as having strong commitments to implementing active learning practices, they varied in the extent to which they explicitly positioned themselves and students in group work settings. While some instructors (Barb and Celia) laid out specific ways in which students should interact with one another and the instructor, others (Harry and Donna) provided less direct guidelines for classroom interactions.

In active learning classrooms, communication and interaction are central and impactful aspects of students' learning experiences. Thus, it is crucial to have effective methods of revealing and understanding the vast differences in learning environments and experiences that result from different patterns of communication and interaction across active learning classrooms. Using communication acts, rights and duties provides one way of understanding important differences between classrooms that might otherwise seem quite similar.

As mentioned earlier, in the literature and theoretical framework section, rights and duties refer to the positions individuals and groups perceive themselves and others to hold in interaction (Drageset & Ell, 2024). They are enacted and negotiated through communication, and they shape who speaks, who listens, and how contributions are taken up. Rights and duties, as dynamic indicators of positionality, provide a way of understanding the OtL available to any individual student in classrooms that are all using active learning instructional practices, and that might seem similar in many ways. Leveraging positioning theory enables us to see differences in students' access to OtL, along with the degree to which students are more or less able to shift from marginalized or peripheral roles into more central roles of participation.

It is important to note that, while we have presented counts of rights and duties in this study separately for instructors and students, there may be a connection between instructor duties and student rights. If instructors have a duty to provide certain actions or resources to students, then correspondingly, students have a right to access those instructional actions or resources. There may also be a connection between instructor rights and student duties, as evidenced by some paired examples of the types of rights and duties instructors assigned. For example, if an instructor has a right to select students to present their work to the class, students may have a corresponding duty to present that work. Finally, there may be some rights and duties that exist independently. For example, a student duty that is not in direct relation to an instructor's right is the duty to help peers when they have questions. We aim to explore these connections in future work, especially in longitudinal data obtained from these courses.

Looking forward, we aim to further explicate the connection between these positioning acts and various aspects of the OtL framework. We are especially interested in questions such as: to what extent does the quantity or balance of verbal communication acts related to rights and duties correspond to OtL? How are these dimensions related and what are the implications? Additionally, we aim to incorporate student communication acts into our analysis and explore whether and how students take up, reject, or negotiate the positionings laid out by instructors in the initial class sessions, with particular attention toward how these roles influence students' access to mathematics content.

We conclude by highlighting the importance of instructors supporting effective group work in undergraduate mathematics classrooms by explicitly sharing or developing norms for participation, by naming rights and duties that position students as knowledgeable and capable doers of mathematics, and by opening the possibility for students to enact rights and duties in new and evolving ways that serve to increase their participation in mathematical discussions.

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Experiences of Students with Dyslexia in Undergraduate Mathematics: Cognitive Challenges and Navigational Strategies

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How do undergraduate students with reading difficulties engage in mathematics – a symbolically dense, text-heavy subject? This study analyzes reflections from seven undergraduate STEM students with dyslexia on their approaches to mathematical learning and problem-solving. Our findings indicate that students experienced multiple dyslexia-related cognitive challenges, including processing, organizing, and manipulating conceptual information. To combat these challenges, students reported strategies to structure information visually and minimize the cognitive burden of the tasks. These results offer an initial account of how students with dyslexia approach and experience undergraduate mathematics.

Keywords: dyslexia, working memory, cognition, visualization, student experiences

Dyslexia can be characterized by measurable difficulties with phonological processing, affecting reading and writing skills, and is often unexpected in comparison to other academic abilities (International Dyslexia Association, 2025; Lyon et al., 2003). Dyslexia is estimated to affect between three and seven percent of the population (Wagner et al., 2020), and comorbidity with other neurodevelopmental differences is high (Moll et al., 2020). In the past two decades, enrollment of post-secondary students with specific learning differences has increased dramatically, with dyslexic students constituting the largest subgroup (Richardson, 2021).

Despite this prevalence, little is known about how students with dyslexia understand, learn, and do undergraduate mathematics. Research on undergraduate students with dyslexia (USDs) tends to focus on study skills (e.g., Andreassen et al., 2017; Mortimore & Crozier, 2006), general academic challenges and strategies (e.g., Knight et al., 2024; MacCullagh et al., 2016; Richardson, 2021), or identity and perception (e.g., Griffin & Pollak, 2009; Pino & Mortari, 2014). Language-based difficulties (such as reading, memorization, note-taking, organizing work, and communication) and cognitive difficulties (such as concentration, executive functioning, retaining the meaning of texts, and processing sequential information) are also explored in the literature (Klein, 1993; Mortimore & Crozier, 2006; Riddick et al., 1997; Smith-Spark et al., 2016). These cognitive difficulties largely relate to working memory (WM) – a mental workspace of limited capacity for the temporary storage and manipulation of information (Baddeley, 2000; Baddeley & Hitch; 1974) – sometimes used as a framework for understanding specific learning differences (e.g., Jeffries & Everatt, 2004; Swanson, 2020; Swanson & Siegel, 2001).

For mathematics specifically, the discussion often focuses on foundational or lower-level mathematical issues faced by USDs, such as problems recognizing symbols, notation, or "following a sequence of steps or actions" (Searle & Sivalingam, 2004, p. 5). Developing learning support resources and recommendations for USDs has also been discussed extensively (Perkin & Croft, 2007; Trott, 2003a, 2003b). Our study highlights, in their own words, the cognitive challenges and navigational strategies USDs report when engaging in higher-level mathematical tasks (such as proof production and comprehension) and when reflecting on undergraduate mathematics learning. In doing so, we expand and develop the findings of Perkin's engineering case studies (Perkin, 2004). As Lambert and Harriss (2022) did for research

mathematicians with dyslexia, we aim to develop an emergent account of USDs' mathematical experiences from an 'insider perspective'. Our primary research questions are:

RQ1: *What cognitive challenges do USDs experience in mathematics courses?*

RQ2: *What strategies do they implement to navigate these cognitive challenges?*

Theoretical Framework

We frame our study within the theory of neurodiversity, which conceptualizes neurological differences (and dyslexia in particular) as natural and valuable variations in human cognition. Originally developed in the autistic community (Robertson & Ne'eman, 2008) and recently adopted in studies on dyslexia (e.g., Lambert & Harriss, 2022), neurodiversity posits that cognitive differences are a natural part of human diversity and can carry both affordances and challenges. This perspective stands in contrast to the traditional *medical model* (also called the *deficit model*) of disability, which depicts dyslexia primarily in terms of individual impairments.

Through this theoretical lens, we view challenges that USDs experience in undergraduate courses not as evidence of deficiency, but as reflections of how students with dyslexia interact with cognitively demanding mathematical tasks and learning environments, which oftentimes privilege particular forms of information processing. At the same time, this framework highlights the strategies that USDs develop to navigate these challenges as distinctive strengths they bring to undergraduate mathematics. We follow an insider perspective (Lambert & Harriss, 2022; Lambert et al., 2019) by studying USDs' own reflections on their perceived challenges and navigating strategies, positioning their voices as central to understanding both barriers and affordances in mathematical learning.

Methods

The data reported here are part of a larger study aimed at understanding how USDs use hand gestures during cognitively demanding mathematical tasks. We recruited seven participants who were professionally diagnosed with dyslexia, representing a range of STEM majors but all enrolled in undergraduate mathematics courses: Aoife, Aran, Caitlin, Eibhinn, Maeve, Sean, and Siobhan (all pseudonyms).

Each student completed an online survey, then participated in a task-based interview (Interview #1) and a stimulated recall interview (SRI; Interview #2). The survey asked participants to (1) describe their experiences learning mathematics at university, including classroom interactions and assessments, (2) explain how dyslexia had impacted their learning of mathematics across school and university contexts, and (3) identify strategies or coping mechanisms they use for managing mathematical tasks. These survey responses provided descriptive context for each participant and were used to facilitate discussion in Interview #2.

The task-based interview invited students to solve mathematical problems while verbalizing their reasoning, which provided insight into the cognitive and linguistic processes they engaged in while problem-solving. The tasks reported on in this study are summarized in Table 1, below. Because of the design of the larger study, participants were not given access to a pencil and paper while working on these tasks. The SRIs followed several days after Interview #1; During the SRIs, students viewed selected video clips of their own problem-solving and reflected on the challenges they faced and strategies they employed. Building on these reflections, the interviewer also asked participants about dyslexia-related difficulties they experienced when navigating undergraduate mathematics courses, drawing on their earlier survey responses.

Table 1. Summary of Interview #1 tasks.

Task name and source	Task description
Triangle Task (Nathan et al., 2014)	True or False? The sum of the lengths of two sides of a triangle is always greater than the length of the third side. Provide proof for your claim.
Gear Task (Nathan et al., 2014)	An unknown number of gears is connected in a chain. You know what direction the first gear turns, how could you figure out what direction the last gear turns? Provide a justification as to why your answer is true.
Two-Color Theorem (Kokushkin, 2024)	Read and understand the proof of the statement: "$n \geq 1$ circles are given in the plane. They divide the plane into regions. It is possible to color the plane using two colors, so that no two regions with a common boundary line are assigned the same color."

All interviews were audio-recorded and transcribed verbatim. Survey responses and interview transcripts were analyzed using the constant comparative method (Glaser & Strauss, 1967). We first coded our data to identify specific challenges and navigating strategies. These codes were further combined into categories, which served as the foundation for latent themes that captured shared experiences across participants (Braun & Clarke, 2006). To enhance trustworthiness, two researchers independently coded an initial subset of transcripts, resolved discrepancies through discussion, and applied the refined scheme to the full dataset. In the excerpts presented here, emphasis added by the authors is indicated through underlining.

Results

We report on four themes that provide insight into our research questions. We remind the reader that in our experimental setting, participants were required to complete all activities mentally. This experience was used to prompt students to reflect on their daily struggles processing (i.e., understanding, structuring, and manipulating) mathematical information. We do not claim these challenges and strategies are unique to USDs; however, we believe it is important to recognize how such challenges are encountered and addressed by these students.

Challenges

Many of the challenges our participants reported can be understood as a difficulty (A) with organizing and structuring written and verbal information, and (B) with holding and manipulating information in their WM.

Organizing and structuring information. Our participants reported persistent obstacles to engaging in undergraduate mathematics, centering on difficulties in internally (re)organizing written and verbal information. Describing the experiment conditions, Aran found it "hindering" not being able to write, as writing "helps [him] keep all [his] thoughts in place." Siobhan found it notably difficult to keep her train of thought organized and "in line" when answering mathematical questions. Similarly, Sean found it "much harder" to tackle the mathematical problems verbally, as "[his] ideas get jammed with one another, and [he] lose[s] [his] train of thought." He further elaborated: "When I'm thinking it in my head…it's all just black and white. So in other words, you know, it all just gets kind of mixed up within one another."

To articulate their difficulties in processing written and verbal content, participants often used the language of "visualization" – forming mental images to help recognize and structure relevant details into a coherent whole. Sean expressed a need to be provided "some sort of diagram or something that's not just fully word based" – because "if there's nothing to visualize, you have to be your own person to visualize it." Eibhinn found it challenging "trying to picture

the situation from what [she] was reading" while Caitlin similarly remarked: "It's very hard when I can't draw it out, because I feel like I'm a very visual…I need to see how it's actually happening." When asked "what was difficult about these tasks?", Siobhan indicated it was:

> "…hard to like, visualize and not being able to write anything down, it's just like coming up with what to say just off the tip of the tongue, usually I have to like, write down my thoughts, or, like, organize them in a way that's sometimes hard to do that on my head."

Along similar lines, participants sometimes expressed frustration or dissatisfaction when they were unable to clearly 'see' how information is assembled: Siobhan shared how "textbooks… sometimes give an image, but it's all combined. Or if they're, like, telling you to graph things, it's the whole graph at the very end" – not showing "the process being done" from the presented information. Likewise, Sean expressed dislike "when someone gives [him] just a big paragraph of an essay for a math problem, because [he] can't visualize that."

WM challenges. Participants also reported difficulties with activities that placed demands on their ability to simultaneously hold and manipulate information – highlighting WM as a potential source of struggle. For example, reviewing the gear task, Sean commented:

> "I couldn't really see how it would turn, because I could imagine, you know, the gears all lined up, right, but seeing them all go like, in motion with one another, that's a little too much that I can't think in my head."

While Maeve described visualization as "really easy" for her – even saying "I can take things in my mind and twist them around…like, on SolidWorks" – she struggled when trying to coordinate several objects simultaneously, an activity that places greater demands on her WM:

> "…I can never visualize that [interlocking gears turning], like, I can never figure out which way it's going…I always get mixed up with the way that the arrows are spinning… when [the gears], like, attach, in my mind, they're going the same direction, even though I know they're not."

In the context of the Two-Color theorem, Eibhinn struggled and ultimately failed to grasp the inductive step in the proof. In recounting her struggle, she distinguished between the "easy to picture" static arrangement of k colored circles (the inductive hypothesis), and the "hard to picture" inductive step, which involves adding one more circle and recoloring. Failing to mentally operate on her concept image in the inductive step suggests a difficulty in navigating the cognitive load for the task, and an overtaxing of her WM. On a more general level, Sean explained he keeps a mental "whiteboard" of task-relevant information, that he updates while working through a problem:

> "I would have, like a whiteboard of a thing in my mind, and I'm just drawing this out and just combining one with another and seeing where that goes…for me, I really just need to visualize it. And could be because I can't read well and…it takes me a longer time to comprehend the writing in general."

Sean is aware this strategy has its limitations, adding "the hard thing about it [is] that there's only so much room you could have on there, and when you start throwing at me more and more stuff, I might lose track of certain things." Our participants' reflections suggest that it was not only the conceptual content, but the cognitive load in simultaneously visualizing and manipulating task-relevant elements, which they felt posed a significant challenge to them.

Strategies

Participants conveyed a number of 'coping mechanisms' or 'strategies' to offset the impact of dyslexia on their mathematical learning. We report on these strategies below, organized to complement the challenges identified in the previous section.

Organizing and structuring information. A core tool for structuring thoughts and answers (mentioned by most of the participants) was writing. Aran explained that "writing all these things down, helps you remember them, and helps [him] keep all [his] thoughts in place." Similarly, Sean expressed his desire for "just a little piece of paper or something to write it all out, or, you know, put one idea with another" as this would have aided him to "answer the question more fluently." Writing also featured in the routine some students were taught to help them tackle problems: Maeve reported she systematically identifies her "given, assume, and find" in a question, to ensure she does not overlook important information.

The repetition of information through writing also gave students confidence they had correctly understood information from the presented text. Maeve, referencing exam questions, explained: "I'll write *m* is equal to one, even though it's in the question. Just having it out there, like is, like, just reassurance that I know *m* is equal to one." Echoing this, Aran described how writing gives him confidence, sharing: "I know I already got this in the problem. I trust myself. I wrote this things down [sic]." For Interview #1, he reported using his hands "as [his] pencil" to mentally underline text and highlight important information. Aran felt that "it's not permanent, but it would help [him] understand, at least at that point, what…[he's] trying to understand."

Students used their hands not only as tools to focus their attention, but also to organize abstract aspects of their mathematical work. While discussing the triangle task, Aran explained he gestured to give himself "an image of what [he] was trying to do" and to "visualize what's happening" – to understand the task and to marshal his approach to its solution. To keep her thoughts "in line", Siobhan pointed to parts of the table before her, left to right, as she solved the gear task. She clarified she was "categorizing" her thoughts, and giving herself a visual representation of spatially organized "thought process[es]" to structure her verbal answer. Eibhinn and Maeve also used gestures to facilitate communication with the Interviewer, with Maeve clarifying "[she] was trying to draw with [her] hands that time, not for [her] sake, but to convey what was happening in [her] brain."

All of our participants identified visualization as a key strategy for attempting the mathematics tasks – with Sean stating "visualizing it has always been [his] best friend in trying to do any math problem". In her SRI, Aoife suggested that, to her, the underlying meaning of mathematical information is expressed visually, and not symbolically:

> "I think it just makes a lot more sense that way, as an image. Like, in my mind, I can kind of think about it that way a lot better than like the numbers or like the math or equation type thing like that…I can solve it, but it doesn't really mean that much to me. But I think I like images and pictures, I think that's how I think."

Navigating cognitive load. Our participants described a range of strategies to help them retain and manipulate information in their WM. These included strategies of selective attention while encoding information, active rehearsal for retention, and mechanisms for offloading information. In what follows, we elaborate on these strategies for navigating cognitive load.

Given our participants' persistent challenge in processing written text, selective attention emerged as one of the key encoding strategies. Broadly, this strategy can be summarized as deliberate actions for reducing redundant processing by focusing on the most relevant or the most important pieces of information. Several participants described underlining key pieces of text in problem statements or coloring mathematical expressions to minimize overlooking critical details. Sean, for example, shared: "When I take down my notes, especially for math, I like to color things in…when I'm thinking it in my head…it's all just black and white."

Some students described strategies for reformatting written information in a more cognitively accessible way, which, as they reported, was especially helpful in mathematics. Maeve, for instance, sometimes replaced technical terminology with simpler terms: "By switching the element with an easier-to-read element, I'm able to focus more on the math than the reading of the problem, while still understanding the context." Similarly, Siobhan noted that symbolic expressions were easier to process when visually rearranged: "When you're doing a division problem, if it's like, *p* minus one divided by *n* minus one, I would understand it better if I just saw it written out vertically, not just horizontally on the page."

Another common strategy students described involved re-reading written text, or 'rehearsing' information. These practices were referenced as an efficient way to retain information, especially when writing is not available. Eibhinn reported using noise-dampening headphones during exams, as they allow her to "hear [herself] say the stuff a little bit more" clearly in her head. Aran shared he needed to re-read "[j]ust to get the information out" because he would "forget it…not understand it fully, or mess it up a little bit." Sean re-read text to help him connect information at the end of a paragraph to information he was given at the beginning, explaining:

> "I always try to relate this new information to what we talked about before, so that that original information I got in the beginning doesn't get lost. So I'm always trying to go back and just trying to fill in these gaps that the problem is slowly giving me."

Most participants described strategies for offloading encoded information to alleviate demands on WM. For example, Aran explained that when working on a mathematical problem, he typically writes down everything he considers "useful" before attempting a solution. In a similar vein, Aoife emphasized the importance of recording information, as "making it out of [her] head and on the page" prevents her from becoming overwhelmed and "mixed up." Several participants reflected on how gestures also supported their WM when writing was restricted. During his SRI, Aran explained that hand motions helped him to make progress on the gear task, as "moving [his] hands helps [him] actually move the gears in [his] head," and allows him to "remember, in a way, what's happening…keep [his] thoughts into [his] hands and [his] motions." To help her stay engaged and retain information, Caitlin noted:

> "A lot of times, I space out when someone's talking to me…I think drawing and hand movements really helped me stay engaged. If, like, I can move my hands or picture it a little bit, I can commit a little bit more to memory and then be able to see it."

Discussion and Conclusion

All of our participants identified challenges related to mathematical learning, commonly associated with dyslexia – including reading text, writing and understanding symbols, and taking notes in class (MacCullagh et al., 2016; Perkin, 2004; Searle & Sivalingam, 2004). Together, our findings suggest that a larger obstacle to undergraduate mathematics for these students may be in place before any reasoning or calculation is undertaken: A barrier in translating mathematical text to mathematical meaning, structure, and action.

Mathematical texts possess features that can prove challenging for students to navigate, such as complex sentence structures, precise definitions, specialized terminology and symbols, and a high density of concepts (Shepherd et al., 2009). For students with dyslexia, these features may compound existing difficulties with processing information from text (Perkin & Croft, 2007). Mortimore and Crozier (2006, p. 240) found that USDs disproportionately struggled with "organizing essays" compared to their non-dyslexic peers, suggesting that dyslexia exacerbates the challenge of organizational or language-heavy exercises for these students. Our findings suggest that these organizational difficulties extend to processing and structuring mathematical

information presented in text, and producing a linear, coherent narrative in the way of a solution to a problem. This complements the reported challenges of research mathematicians with dyslexia in "translating mathematical thinking into text" (Lambert & Harriss, 2022, p. 99), as well as those of one postgraduate engineering student with dyslexia, who encountered problems "structuring written work" and "deciding what to write" (Perkin, 2004, p. 7).

To combat their difficulties, many of our participants identified *writing* as a way of (visually) structuring their thoughts, giving them confidence and assurance in the information they had read. Students also used their hands to support problem-solving activities and to communicate their solutions, sometimes relying on their gestures to supplement key aspects of their answers. Reframing mathematics as a visual activity was standard across our participants, echoing Bacon and Handeley's remark that "a preference for reasoning strategies based on visual processes may be a robust feature of the dyslexic cognitive profile" (Bacon & Handeley, 2014, p. 337). An inclination for 'visual thinking' has been anecdotally noted by a number of authors (e.g., Griffin & Pollak, 2009; Lambert & Harriss, 2022; Perkin & Croft, 2007; Trott, 2003a, 2003b) and is repeatedly among the guidance given for supporting USDs (Trott, 2003a, 2003b). We found our participants identified strongly with this characterization, and often that mathematical meaning-making was synonymous with 'visualization' in their accounts.

Participants' accounts also indicated that the cognitive load in simultaneously remembering and manipulating task-relevant elements imposed a substantial challenge (cf. Sweller, 1988, 2010). Swanson and Siegel (2001) argue that specific learning differences – including dyslexia – reflect "deficits" in WM, related to the phonological loop and central executive of the Baddeley-Hitch model of WM (Baddeley, 2000; Baddeley & Hitch, 1974). Bacon and Handeley (2014, p. 341) refine this view, asserting that dyslexic students' reasoning is "disproportionately" negatively affected when they are required to remember visual patterns – proposing that the tasks of remembering and manipulating information are "competing for the same visual cognitive resource" and that disadvantage arises only under conditions of high WM load. To compensate, our participants described strategies such as minimizing cognitive demands (via selective attention or offloading) and rehearsing information to preserve it in WM. *Gesture* also appeared throughout the participants' recollections of undergraduate mathematics, serving as a means of visualization, communication, and offloading (Risko & Gilbert, 2016; Kokushkin, 2024). In the context of high school students, Hord and colleagues note visual representations (including gestures) "may provide scaffolds for students with learning disabilities related to difficulties with working memory...to facilitate concentration on processing, storing, and integrating information" (Hord et al., 2016, p. 190). Future work of the authors will focus on the role of gesture and embodiment in mathematics learning for USDs.

We caution against drawing broad conclusions on dyslexic undergraduate mathematics education, due to the small sample size of our study and its focus on personal experiences. We also do not claim these challenges and compensatory measures are unique to students with dyslexia, or any specific learning difference. Dyslexia, however, introduces its own layer of complexity, and thus may disproportionally disadvantage undergraduate students on certain tasks. We believe these experiences of USDs, reported in their own words, offer a unique, rich, and informative perspective on the strategies and challenges they face when engaging in undergraduate mathematics learning.

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Association for Women in Mathematics as a Place for Reimagining Gender Dynamics in Graduate Mathematics

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This study examines gender-inclusive practices within a graduate chapter of the Association for Women in Mathematics (AWM). Using an ethnographic approach, I explore how AWM navigated their inclusive practices. Results revealed that inclusion was viewed as both a logistical necessity and a strategic move toward equity, fostering community building and identifying male allies. Additionally, tensions such as maintaining women-centered spaces and replicating gendered labor divisions also arose. AWM's efforts reflect both an aspirational reimagining of gender dynamics and the persistent realities of societal gender inequities. These findings contribute to understanding how gender-inclusive social structures can both challenge and reproduce dominant gender norms.

Keywords: graduate mathematics education, gender, student organizations

Despite improved representation at the undergraduate level, women are still underrepresented in doctoral mathematics, with only 28% of doctoral degrees awarded to women in 2018 (National Science Foundation, 2021). Beyond graduate studies, the lack of representation for women extends to academic positions, with only 24% of women occupying tenure or tenure-track positions and 11% occupying full professor positions in mathematics (American Mathematical Society, 2019). Additionally, while women who have earned PhDs in mathematics in the US have slightly lower initial unemployment rates than men, their first post-PhD jobs are less likely than men's to be at top universities, and the percentage of new women PhDs who were employed at bachelor's-only departments is much higher than for men. This discrepancy occurs despite the prestige of their PhD-granting department (Vitulli, 2018).

In addition to the inherent inequity in the underrepresentation of women in advanced mathematics, underrepresentation has been shown to be detrimental to women's experiences at the postsecondary and graduate level. Women have experienced overt prejudice (Herzig, 2010), the impact of negative stereotypes (both gendered and racialized), and experiences of invisibility (Herzig, 2002b). Furthermore, women have suffered damaging impacts because of negative gender stereotypes (Good et al., 2012), experienced a lack of belonging, reported insufficient or negative advising experiences (Herzig, 2002a, 2002b, 2004), and had to adapt their identities to minimize their gender expressions (Herzig, 2010; Rodd & Bartholomew, 2006).

Although the experiential reality of being a woman in advanced mathematics includes negative factors, much of the research in this area has positioned them as outsiders or "'other' but with a right to be there" (Rodd & Bartholomew, p.42). This focus on women's negative experiences in doctoral mathematics, with little understanding of women's strengths and agency, perpetuates deficit narratives. Furthermore, some theoretical framings imply these deficits lie within the students themselves, overlooking sociopolitical themes in women's experiences (Damarin, 2008).

In addition to limited research examining the experiences of women pursuing doctoral degrees in mathematics through a critical lens, there is a dearth of literature that works to understand supports for women in advanced mathematics. Extending the research, through a critical lens, of women graduate students' experiences and their supports, this study examines the

social practices of a graduate chapter of the Association for Women in Mathematics (AWM). To investigate these areas, I answer the research question: *How does the discourse of inclusion mediate the social activity of AWM?*

Theoretical Framework

Theoretically much of the literature (e.g., Herzig 2002a, 2002b) has relied on assumptions that position women as needing to move toward an existing, static culture, framing their experiences in doctoral mathematics as an inability to integrate. To reframe women's experiences in a way that highlights their agency and impact on their local departmental culture, I leverage the figured worlds framework, with a particular emphasis on authoritative discourse (Holland et al., 1998) and refine this lens through the construct of prefiguration (Curnow, 2022).

Figured Worlds

Holland et al. (1998) define a figured world as "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). Many studies have leveraged figured worlds as contexts with their own histories and cultures that construct identities as individuals participate in them (e.g., Solomon et al., 2010; Voigt et al., 2020). However, Holland et al. (1998) emphasize that identities are co-developed *with* figured worlds. That is, identities are formed in the process of participating in the activities organized by figured worlds, *and* figured worlds are produced and reproduced through the work of participants. As such I leverage the figured worlds framework to understand how individuals, through their participation in AWM, contribute to their figured world of PhD mathematics.

Authoritative discourse. Holland et al. (1998) propose discourses as significant in mediating the construction of figured worlds. Discourses "capture the ways in which people recognize socially and historically significant identities…through well-integrated combinations of language, actions, interactions, objects, tools, technologies, beliefs, and values" (Gee, 2015, p. 1). Additionally, Holland et al. (1998) include the construct of *authoritative discourse* in their descriptions of figured worlds, which are discourses that are often accepted without question. Authoritative discourse "demands that we acknowledge it, that we make it our own" (Bakhtin, 1981, p. 342). Individuals may reject or attempt to modify an authoritative discourse, but they cannot simply ignore them. The authoritative discourse of *inclusion* has been identified as significant in education (Dune, 2009).

Inclusion "signifies a right to membership and participation, often viewed as a marker of political correctness and sensitivity…" and has been identified as an authoritative discourse in education (Dunne, 2009, p. 305). However, the unquestioned implementation of inclusive practices may lead to unintended consequences (Dunne, 2009). This includes limitations in discussing differences and relinquishment of responsibility for meaningful change by treating inclusion as a terminal act. As such, critics of unqualified inclusion encourage one to address the nuances of inclusion and answer the question "who is served by inclusion?"

While authoritative discourse via the figured worlds allows one to account for the impact of systemic and social power structures, there is a need to understand agency at the group level. One way to frame the activity of the AWM group that emphasizes their activity as one of empowerment and choice, is through the lens of *prefiguration*.

Prefiguration

Prefiguration refers to the idealized social practices of a group that reflect the aspirational or desired outcomes of the group in alignment with their political goals (Uttamchandani, 2020). Prefiguration as a theoretical lens affords the identification of "...the emergent, ongoing development of practices and philosophies of a collective…" (Curnow, 2022, p. 556). Because AWM is an inherently political organization, with the official goal of "promoting equitable opportunity and treatment of women and others of marginalized genders and gender identities across the mathematical sciences," prefiguration allows the politicized practices of the group to be foregrounded. Lastly, using a prefigurative lens gives prominence to the agency demonstrated by AWM, by highlighting their activity as an enactment of political resistance.

Methodology

The overall methodology in answering the research question - *How does the discourse of inclusion mediate the social activity of AWM?*- was an ethnographic approach. This approach includes identifying a culture-sharing group, selecting themes to study about the group, gathering information in the group context, generating an overall interpretation of the group, and presenting patterns of the group (Cresswell & Poth, 2016). A graduate chapter of AWM was selected as the culture-sharing group, allowing the investigation of the intersection of gender and mathematics.

Context and Participants

Southwest University (SU) is a large public university in the southwest United States and is classified as a very high research activity institution. The mathematics department at SU has approximately 200 students enrolled in their doctoral program, with about 15% of students identifying as women. There are approximately 60 faculty, a minority of which are women.

Participants included five women enrolled, or recently enrolled, in the doctoral mathematics program at SU. The women were selected using a stratified purposeful sampling (Patton, 1990) with respect to years' experience in the program. This included one woman enrolled in coursework, two women in early research, one woman in advanced research, and one woman taking a leave of absence after completing coursework. Additional sampling criteria included participation in the local graduate chapter of the AWM.

The SU graduate chapter of AWM was reestablished in Fall 2019, after a ten-year period of inactivity. At the time of the study, AWM was starting its fourth year after this period of dormancy. Since coming out of dormancy, the group had initiated a variety of activities including weekly social events known as 'Teas,' a mentorship program, a faculty interview series, and most recently an industry panel showcasing women working professionally in STEM.

Data Corpus

Participant data included a mathematics autobiography, one episodic narrative interview (Mueller, 2019), six weeks of diary entries (Cresswell & Poth, 2016), one future-focused diary entry, and one semi-structured interview per participant (Bernard, 1998). There was also one focus group interview with all five participants (Morgan, 1997).

Observation data included two quarters (16 weeks) of observations and field notes of the AWM activities. This primarily included weekly Teas and Executive Council meetings for a total of 24 observation hours. For each observation I gathered information about the space, purpose, people, and dialogue. Additionally, the observation data included one recorded conversation with

the current AWM president about her perspective, as a representative of the group, on the inclusion of men in their events.

Analysis

In conducting this ethnography, my goal was to understand the figured world of AWM at SU. In support of this goal, I refined the ethnographic approach through thematic analysis to capture more fine-grained themes (Braun & Clarke, 2006). Analysis included reduction of the participant data to those excerpts which directly addressed the inclusion of men in AWM activities. This was followed by open coding, capturing distinct ideas related to the research question. I also open coded the observation data and the conversation with the current AWM president. After open coding, I began to look for connections among the codes, grouping them into themes. During this time, I continuously refined codes, adding new excerpts and eliminating others as well as refining the code definitions to better represent the data.

The results of the analysis revealed how the discourse of inclusion mediated the largely gender-inclusive structure of AWM. In the following sections, I present findings from the thematic analysis that detail the AWM's reasons for inclusion, the reflective ways they navigated inclusion while still serving the interests of women and other nondominant genders, and the possible consequences of their choice to include men in their events.

Results

The themes that arose as a result of the thematic analysis – *reasons for inclusion*, *benefits of inclusion*, *challenges of inclusion*, *navigating inclusion*, and *replication of domestic labor inequities* – were consistently shared among at least four of the five participants and often identified across multiple data sources (i.e., individual interviews and focus group interview). Overwhelmingly the statements about the inclusive practices of AWM were in support and often with a sense of advocacy. I first detail the reasons for gender inclusivity at AWM events.

Reasons for Inclusion

Early in my observations of the AWM Teas I asked Mariam (year 4) about her perspective on the decision to include men at the events. Her response was reflective of the group's prefiguring of gender dynamics in their department and in mathematics in general. She shared:

> In an ideal world, we don't want to have an AWM because everybody's equally welcome and represented in their department. And if we want to reach that ideal state, we can't… these male colleagues are also in your program, and they're facing almost the same things, and there is a lot to gain by talking to them also. So, it's good to include them in events, I feel like, because in the future we want that. (Mariam, year 4, Interview 1)

Mariam shares her view that women and men have experiences in common in the PhD program and proposes the possibility that there is support to be gained through connections with men. Furthermore, Mariam's description of an ideal world includes equal representation and inclusion of women. Other AWM members viewed the inclusion of men as a logistic matter.

Both Ida and Lucy, the present and founding president of AWM, viewed the inclusion of men as necessary for the group to be operational due to there being few women in the PhD program. Ida discussed the need to have men initially serve as principal members because at the time AWM restarted there were not enough women to fill all the roles of the Executive Council. Ida reflected "…when we started the group, we didn't have enough women to run the group, so we needed men to volunteer to help run the group and do a lot of this important stuff" (Ida, year 6, Interview 1).

Although the principal members of the Executive Council were all women at the time of the study, the need to have men participate persisted. Lucy, the current AWM president, explained how the continued gender imbalance still necessitated the inclusion of men:

> But I think for the social events, it's tricky. I think AWM wants to be known and active in our department, and because there are fewer women, I feel like sometimes if we make our events smaller, then maybe we're not as well known in the department. You need a sort of a critical mass of just people to meet your goals. (Lucy, AWM president, private conversation)

Lucy's explanation reveals the strategy of including men for the greater good of the group and to reach their goals of being recognized in the department. As evidenced by Ida and Lucy's explanations, the gender makeup of the graduate students necessitated the inclusion of men in the AWM group. Participants also discussed the benefits and of including men in AWM activities.

Benefits of Inclusion

In asking the participants directly about their reasoning for including men, they responded that events structured by inclusion benefited the department community, prevented AWM from being isolated, and benefited women. During the focus group Cindy compared her experience in her undergraduate studies to her experience in AWM at SWU. She said:

> I think for AWM here, I don't know how it is at other schools, but I know, for, like in undergrad, women in STEM…there were no men involved whatsoever. And here there's like a lot of men who actively attend our events, or volunteer to help with stuff. And so, I think that overall helps contribute to a better environment in the department. (Cindy, year 2, Focus Group)

Cindy expresses the possibility of isolation in creating a group that is segregated and connects the value of community to AWM's practice of inclusion, explaining that the inclusion of men is beneficial to the department as a whole. In addition to the benefits to the department, inclusion was explicitly connected to benefiting women. For example, Ida shared:

> It doesn't seem to matter so much if there are more men than women, which will happen if you have 60 people in an event in our math department because you don't have 60 women. I think it just matters that there are a lot of women to talk to and that there's a space where the women feel comfortable to talk to people and ask questions. And they're a focus. (Ida, year 6, Interview 1)

It is here that AWM's practice of inviting men into their events is distinct from a version of inclusion in which women are invited into a space in which they are marginalized. If the women being included in another, dominant group, the scenario Ida describes would structure participation in a way where women were being integrated into the dominant culture. However, because it was AWM that included the men, the social interaction is happening outside (to an extent) of a context in which men are the default. Through a prefigurative and figured worlds lens, gender inclusivity at AWM Teas facilitated the enactment of a mathematical context where women's presence was normalized.

Another reported benefit of inclusion was the potential for men to become allies to AWM as a group and women as individuals. Cindy stated "I feel like it kind of builds allies and like, oh, well if this guy comes to AWM Tea, he is probably supportive of women in math. (Cindy, year 2, Interview 1). Bringing men into the advocacy goals of the group demonstrates a prefigurative vision in which men are personally invested in the success of women. In addition to the reasons in favor of gender inclusivity in AWM, there was also evidence that the group was reflective of the potential challenges of including men in most of their activities.

Challenges of Inclusion

AWM members discussed differences between gender-inclusive and women-only events, including the times when inclusive events recreated a mathematical contexts in which women were marginalized. Cindy spoke about the differences in the AWM that were not inclusive:

> So, I feel like sometimes when it's just women or gender minority focused, maybe you would say things you wouldn't say when it's just anyone's welcome. And I think that's also valuable to hear and maybe feel like you're not as alone in some experiences or also I remember I got some good advice there. (Cindy, year 2, Interview 1)

Cindy details the experience of being to share more openly struggles related to graduate school at women and gender minority events. She also describes support and camaraderie as advantages of being more vulnerable at events without men. In addition to the possible missed opportunities for support in inclusive events, Ida recalled a time when inclusion backfired and lessons the group learned from this misstep. She recalled:

> I think it's important to always have women only events as well as mixed events. And in the beginning, we had a real wake up moment with this when our events started getting completely overrun with men…the men were just coming and talking about all their issues. It was coded, it was supposed to be a female space, but things were done in a way that was way more open to everyone…I think we had to be a lot more heavy handed. (Ida, year 6, Interview 1)

Ida describes early attempts to be inclusive by "coding" events as women focused. At times this implicit messaging was not enough to create an inclusive space that still centered women. She was candidly reflective that there are consequences to inclusion "I told you the reasons why we made the choices, and now I'm telling you the reasons why those choices, the costs of those choices." (Ida, year 6, Interview 1). Acknowledgement of the sacrifices, and sometimes consequences, of inclusion implies the value AWM sees in opening their events to men. Balancing the benefits and challenges of including men was discussed by the current AWM president as she discussed the group's plans to navigate these tensions.

Navigating Inclusion

In speaking with Lucy, the current president, about the inclusive nature of the AWM events she discussed a careful balance in their decisions to host events that included men:

> So, I think that this is something maybe in our officer meetings we don't necessarily explicitly talk about a lot, but I think we all think about it a lot because it's tricky because their math being a male dominated academic field…We do make a lot of friends in our classes and things like that, especially if we're doing a more social geared event that's like, oh, well we're very close friends with a lot of the men in our department. Whereas…the industry panel that's an actual conversation that's probably going to be brought up in our meeting… are we going to restrict that to underrepresented genders because it's more of a recruiting or talking about your experiences as a woman. (Lucy, AWM president, private conversation Jan 24, 2024)

Lucy describes wanting to include men in events since due to the social relationships shared among the men and women, echoing earlier sentiments that inclusion connects women more extensively to the department. In contrast, Lucy appears more reluctant to assume the inclusion of men in the industry panel event where the focus is specific to issues relevant to women and other nondominant genders.

Replication of Domestic Labor Inequities

As I attended weekly Teas, I observed what I viewed as an inequitable division of labor. Men regularly attended the weekly Teas and often outnumbered the women. While this didn't appear to have significant impacts socially, it became apparent that women were responsible for setting up and breaking down the events. This observation was verified in a conversation with the current AWM president. Lucy explained:

> If a historical or traditional idea of things are that women's jobs would be cleaning or cooking or things like that and then we have an event and if the majority of the attendees are men, but the women are the ones who are staying after and cleaning up and it's like… well who are we putting on this event for? (Lucy, private conversation, Jan 24, 2024)

Lucy is reflective of the implications of the labor division at the AWM events. Importantly, she asks the question "who are we putting on this event for?" This indicates that although the group assumes many reasons why inclusion is beneficial to their goal of supporting women, they also attend to the potential impacts of those decisions. So, while the discourse of inclusion allows the group to enact an idealized future of gender equity, there is still a tether to the reality of inequitable domestic labor among men and women.

Conclusion

AWM at SU subscribed to inclusion as an authoritative discourse that informed the structure of many of their activities. However, they navigated the complexities of inclusion by incorporating men into their events while also striving to maintain a space that supports women and gender minorities in mathematics. As such, AWM's approach to inclusion reflects both adherence to and negotiation of this discourse, as the group actively questioned who is served by inclusion but did not challenge inclusion as an assumed value to the group.

The framework of figured worlds and prefiguration highlights how AWM created alternative gender dynamics within the male-dominated space of their department. By including men in their events, AWM modeled a more integrated and aspirational environment where women were more visible and less isolated. This approach allowed for informal mentorship, allyship, and increased presence within the department, reinforcing the idea that gender inclusivity at AWM events benefits not just women but the entire academic community.

In tandem with the reported benefits of inclusion, members acknowledged the need for women-only events, recognizing that some conversations and experiences would be more accessible in a space free from dominant-group influence. Additionally, while inclusion was intended to foster an environment where women were more integrated, there were instances existing gendered power dynamics were reproduced. In this way the prefigurative enactment of gender equity in mathematics was contradicted by societal influences outside of the department.

Previous research indicates that women may significantly resist integrating their gender and mathematical identities due to the underlying bias associated with being a woman in mathematics (LaTona-Tequida, 2025, Solomon et al., 2011). This resistance reflects the need for a cultural shift in mathematics that positions women as more fully legitimate. In response, the gender inclusive practices of AWM at SU point to potential mechanisms to initiate this cultural shift. Participants expressed that spaces exclusive to women were somewhat isolating and potentially worked against their goals of being more integrated into their departmental community. The AWM activity, mediated by the discourse of inclusion, appeared successful in creating a departmental atmosphere that was beneficial to the participants.

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Expert Perspectives on the Set of Outcomes for a Non-Standard Presentation Counting Problem

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The objective of a counting problem is to determine the cardinality of a set of outcomes, but there is never a singular, unique way to conceive of those outcomes. I report on data of combinatorics experts solving a counting problem whose numerical solution is 2^6, a standard expression to see in combinatorics, but whose set of outcomes is not standard. I demonstrate four distinct models that the participants used to conceive of the outcomes, and how each of those led to different ways of explaining the solution. I argue that these differences among experts reflect the challenging nature of articulating what the set of outcomes for a counting problem is, and how to approach a solution using this set. These data may inform how counting could be presented to learners when adopting a wide view on the multiple possible sets of outcomes for any given problem.

Keywords: combinatorics, mathematicians, expert reasoning

Consider the Roads Problem: How many ways are there to traverse the diagram from point A to point B in Figure 1, if no line segment can be repeated? This counting problem is notable because the outcomes being counted—paths along a particular diagram—are not immediately similar to common archetypal problems. It is possible to create a bijection between the outcomes and length-6 binary strings, which means that there are 2^6 total outcomes, and the Roads Problem can be considered to involve arrangement with repetition. However, as I elaborate later, it is also possible to conceive of the outcomes as involving subsets, or simply as path diagrams independent of auxiliary representations. There is no singular set that can be described uniquely as the set of outcomes, and the model of outcome used can impact how a solution is found, and the form it takes. If a set of outcomes is the foundation of a counting problem, what does it mean for combinatorics if there are multiple contenders for what that set is?

Figure 1. Diagram for the Roads Problem.

This paper reports on data of nine combinatorics experts solving the Roads Problem during task-based interviews. I draw on these data because four distinct models of the set of outcomes emerged from the participants, which demonstrates the multiple faces of a set of outcomes even among experts. These results can provide additional context when working with K-16 students, particularly when conventional solutions might depend on a specific way of conceptualizing or encoding the outcomes.

Literature Review and Theoretical Framework

Student difficulties in combinatorics have been well-documented, including the challenge of distinguishing between problem types. Batanero and colleagues (1997) found that students can have difficulty distinguishing between problem types, and Lockwood (2014) reported that students sometimes use their 'gut' when the problem statement does not specifically say order

matters. Literature has increasingly applied an approach that values how students create and justify solutions based on the outcomes being counted. For example, systematically listing sets of outcomes (English 1991, 1993; Montenegro et al., 2022), or even thinking about how to do so (Halani, 2012) has been shown to correlate with successful solutions even at the undergraduate level (Lockwood & Gibson, 2016). Some of this work reports on how students conceive of outcomes, and how those conceptions impact the way they find or justify solutions (e.g., Lockwood et al, 2018; Wasserman & Galarza, 2019), showing that certain (sometimes unfamiliar or unintuitive) ways of encoding outcomes are more likely to lead to successful solutions than others.

Lockwood (2014) has advocated for a set-oriented perspective on counting, where attending to the sets of outcomes is a fundamental aspect of a solution. This perspective draws from her (2013) model of student reasoning, which examines the interactivity between *sets of outcomes, formulas/expressions*, and *counting processes.* The set of outcomes for a counting problem is the set of elements being counted, the cardinality of which is the solution to the problem. Formulas and expressions refer to the mathematical formulas and expressions (such as $\binom{n}{k}$ or 2^6) used in the course of the solution, and what is typically used to write the answer. I refer to mathematical expressions that solve a counting problem as *counting expressions.* Counting processes refer to internal or external processes a counter engages with as they solve a problem (such as listing). Wasserman (2019) argued that counting expressions can also represent a particular enumerative strategy or counting process. For example, although $\binom{n}{k}k = n\binom{n-1}{k-1}$ both count headed size k committees chosen from n people, and are thus numerically equal, they represent two distinct ways of counting those committees. The left-hand side first selects the members of a committee before choosing the head, and the right-hand size chooses the heading before selecting the remaining members. I adopt Lockwood's model for student reasoning in this paper, and I discuss how the duality of counting expression can be observed when different conceptions of a set of outcomes are used to solve a counting problem.

Four Models of the Set of Outcomes in the Roads Problem

To illustrate how the duality of counting expressions might arise when solving the Roads Problem, I present four different ways to model the outcomes. These models are my own formalizations of the approaches I observed from experts, and which I expand on in the Results section.

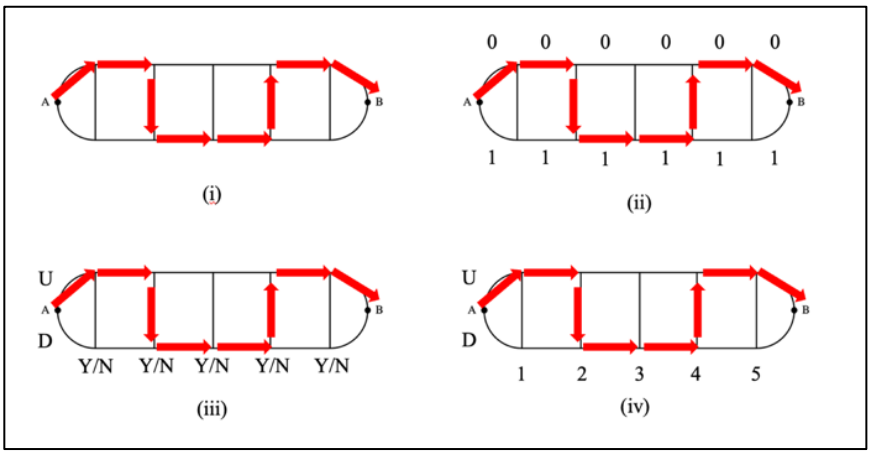


Figure 2. One path shown under four augmented diagrams.

Literal model. The literal model interprets outcomes as visual paths along the diagram. This model does not include an alternate way of encoding or otherwise representing an outcome. I refer to this model as literal because it most closely aligns with the outcomes as described in the problem statement.

Binary model. The binary model encodes paths as a six-character sequence of 0s and 1s, where each 0 or 1 refers to a path segment along the top or bottom of the diagram, respectively. This model does not directly encode vertical path segments, but rather those segments can be deduced from the encoded information. The path in Figure 2.ii represented in the binary model is 001100. This model creates a bijection between all possible paths and six-character binary sequences, and thus there are 2^6 possible outcomes.

Sequential model. The sequential model encodes paths as an initial decision to move up (U) or down (D) from point A, followed by a sequence of choices to cross (Y) or not cross (N) at each vertical path segment. The path in Figure 2.iii represented in the sequential model would be UNYNYN. There are two possibilities for the first decision (U or D), and two possibilities (Y or N) at each subsequent intersection, and so there are $2 \cdot 2^5$ possible outcomes. Note that I distinguish between $2 \cdot 2^5$ and 2^6 because the former illustrates how the first decision (U or D) is a distinct type of decision from the others.

Subset model. The subset model encodes paths as an initial decision to move up (U) or down (D) from point A, followed by a subset of $\{1, 2, 3, 4, 5\}$ to indicate which of the vertical segments occur in the route. The path in Figure 2.iv represented in the subset model would be U$\{2, 4\}$. There are two possibilities for the first decision (U or D), and 2^5 possible subsets of a 5-element set, so thus there are $2 \cdot 2^5$ possible outcomes. I have used 2^5 to represent the ways of counting subsets to reflect the reasoning of the experts reported on in this study, but another counter using the same model might have used a different expression to represent the subsets.

The binary, sequential, and subset models demonstrate distinct models of the outcomes, even if they are used to produce the same or similar mathematical expressions. It is not possible to exchange a representation in one model for a representation in another through a direct character exchange, and each model selects some but not all information about a path to create a representation. For example, despite both binary and sequential models using six-character sequences to encode routes, the outcome 001100 cannot be converted to UNYNUN by a simple exchange of 0s for N and 1s for Ys. Moreover, the binary model selects only information about horizontal segments, and the sequential model selects only information about the vertical segments. In terms of Wasserman's duality of mathematical expression, 2^6 and $2 \cdot 2^5$ represent distinct enumerative processes, and moreover the expression $2 \cdot 2^5$ can even take on more than one interpretation.

Methods

The data presented here were collected as part of a larger study that examined the role of representations in solutions to counting problems for both undergraduate students and combinatorics experts. Both populations were presented with a sequence of tasks meant to elicit the use of representations as part of the solution process. The task sequence for the expert population was comprised of eight problems, the first four of which were selected from the student sequence, and the second four of which were not given to students. I chose expert tasks from the student tasks because they were non-standard presentations of common problem types. Most experts participated in one 90-minute interview, and their work on the tasks comprised most of that interview. However, some experts participated in two shorter interviews due to

scheduling constraints, although they were given the same materials. Most of the experts only completed the tasks that were given to both populations, and only one expert completed all eight problems.

I focus here on expert solutions to the Roads Problem, which was the first task given to the experts. I present these data because prior analysis of student solutions on the problem revealed several approaches that reflected differences in how the outcomes were modeled, which seemed to impact how students approached a solution. While the comparison between the populations is not reported here, the expert solutions demonstrate similar impacts between model and solution approach. Moreover, several of the experts commented that the problem was not one they already knew how to solve, and none of them stated the answer prior to reasoning about the outcomes verbally. I thus argue that the work discussed here demonstrates how experts approach an unfamiliar problem that can be solved with their existing combinatorial tools, rather than how experts recall prior solutions.

The experts were recruited for the study because they were either combinatorialists or education researchers who had previously studied combinatorial reasoning. Six combinatorialists and three educators participated in the study. Their pseudonyms and areas of study are presented in Table 1. The research areas of the educators are only summarized to preserve anonymity. Seven of the participants were tenure-track, one was an instructor, and one was a postdoctoral scholar. Three participants had taught graduate-level combinatorics courses, and five had taught undergraduate-level combinatorics courses, or courses that included combinatorics.

Table 1. Participant demographic information.

Pseudonym	Expertise	
	Mathematician (M) or Educator (E)	Research Area
Austin	M	Algebraic and enumerative combinatorics
Craig	E	Student reasoning
David	E	Student reasoning
Erik	M	Algebraic and enumerative combinatorics
Leo	M	Graph theory and discrete geometry
Oscar	M	Logic, computability theory, and graph theory
Seamus	E	Student reasoning
Silvio	M	Group theory
Ted	M	Algebraic and geometric combinatorics

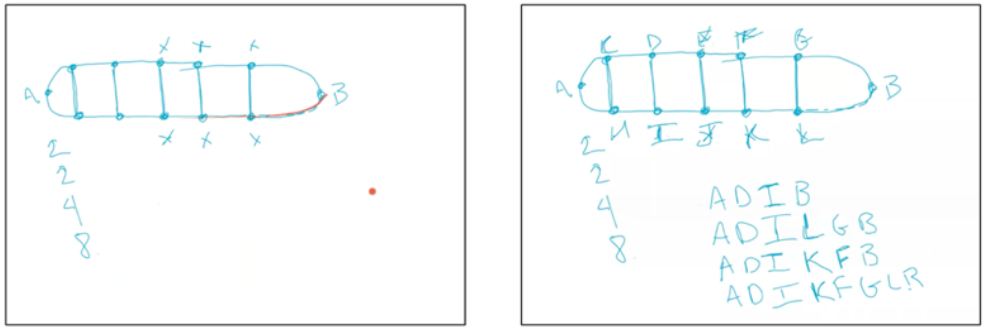
All interviews were conducted over Zoom, and both audio- and video-recordings of the interviews were collected. One participant solved tasks on a blackboard that was visible during the interview, but the remaining participants solved tasks on an electronic tablet whose screen was shared. I created enhanced transcripts of the interviews, where screenshots of the participants' work were used as references alongside the text of the interview. I performed thematic analysis (Braun & Clarke, 2022) on the transcripts to identify the different models of the set of outcomes, problem solving techniques, and how the participants leveraged their conceptualizations of the outcomes to identify a solution.

Results

I present the four models of outcomes I observed, and how these models impacted how the participants sought and justified their solutions. Present in many of these solutions, but not expounded on here, was reasoning about whether a path could move 'backward.' Some participants quickly reasoned that this could not occur, but for Austin this seemed to be the impetus that led to his model.

Literal Model

Craig was the only participant whose work primarily involved reasoning about the literal outcomes, and he spent approximately 37 minutes solving the problem. He began by tracing paths along the diagram with a stylus, which appeared as a red line on the screen that followed his stylus but was not permanent. He organized his tracing by categorizing the outcomes based on the first vertical segment to be part of the path. The first two paths he traced were those that went entirely on the top half or bottom half of the diagram, and the second two paths were those that went on the top or bottom half of the diagram until reaching the last vertical segment. Craig kept track of the number of paths he had traced mentally, and he would record the total number of paths in each category once he had traced all of them (see Figure 3.i). Although it seemed that Craig had developed an internalized way of keeping track of which paths he had or had not traced, he never verbalized it and the red line from his stylus did not appear on screen consistently enough to develop inferences. After tracing out the first four categories, he appeared to have difficulty keeping track of which paths he had already traced, and he labeled each vertex with a letter, as seen in Figure 3.ii. He then used the labels to record paths for a short time, but he commented "This also feels like it's a representational nightmare" before erasing them. It seemed that the primary use of the labeling was to keep track of which outcomes he had traced, and they did not seem to be directly relevant to reasoning about a mathematical expression.



(i) (ii)

Figure 3. Craig's sketching of the literal model, and his use of the model.

After observing that the first five categories included 2, 2, 4, 8, and 16 outcomes, Craig predicted that the last category would include 32 outcomes, and the total number of outcomes would be $2 + 2 + 4 + 8 + 16 + 32 = 64$. When I asked him to describe why there might be powers of two, he responded "Um, having to do with decisions […] each time you reach a point you have a decision. Um, the thing that I don't love about it is that I was trying to think about, uh, for a little bit why I got 2 and 2." By this, I infer that Craig connected the summand of 2^n with the last n vertical paths, but he could not justify why that pattern was inconsistent with the

number of paths that used zero vertical segments. I did not push Craig to further justify his solution. Craig appeared to have difficulty using the literal model, perhaps because they were distinct from archetypal outcomes for counting problems, and he had not developed techniques to identify structural patterns within the set.

Binary Model

Austin was the only participant to use the binary model of the outcomes, but he did so only at the end of his solution. The first part of his solution involved decreasing the number of vertical segments in the diagram, and then sketching each of the literal outcomes. By doing so, he observed that there were 2^{n+1} total paths for a diagram with n vertical segments, and he stated "So I'm envisioning a bijection between, um, paths and bit strings." This bijection is illustrated in Figure 4. Austin used this observation to claim that there were 2^6 paths around the original diagram because he knew that there are 2^6 six-character bit strings. This method uses prior knowledge about binary strings to create a solution, and Austin stated "it's unfamiliar […but…] it just absolutely gives up its secrets when you start doing small examples."

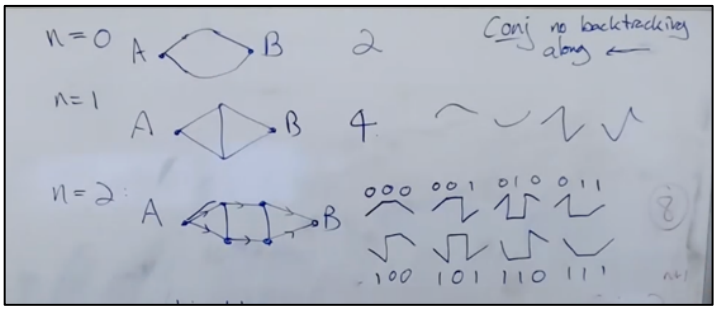


Figure 4. Austin's work using the binary model.

Sequential Model

David, Leo, Oscar, Seamus, and Silvio each used the sequential model. David characterized this method by "we could probably compute the, like, the total number of routes or paths in terms of the number of choices you have to make along the way," which draws an equivalence between a path and a process that would generate that path. Unlike the binary model, where each path was represented by an element of a well-understood set with a known cardinality, the sequential model counted the number of ways a process could be carried out by an agent that traversed the diagram. Similar to David, Oscar and Seamus used language like "if I go down I'd have to go to this point, and again I have two choices there," and "if we went on road two we'd have the same options but in… sort of reverse," indicating they imagined themselves or an agent traversing the diagram, and the path traced out by the agent depended on a sequence of decisions. To further distinguish this model from the binary model, the characterization of the path depended on which vertical components were taken, and each expert differentiated between the first decision (up or down from A) and the subsequent decisions, whereas the binary model depended on which horizontal components were taken and did not distinguish the first path segment from the others. Oscar summarized the method by stating "maybe the right way to think of it is… a route is 'do I take the bridge or not, yes or no?' And then also, but I, realizing that I have two choices initially." This statement seems to indicate that using a sequence of decisions was not merely a process used to count the literal model of outcomes, but instead a complementary way of conceiving of the outcomes themselves.

Subset Model

Erik and Ted used the subset model of the outcomes, which was like the sequential model but included a noticeable difference in terms of how the decisions related to time. Unlike the sequential model, where the experts seemed to think of a path as a sequence of decisions an agent would make while they traversed the diagram, the subset model seemed to make those decisions prior to an imagined traversal. For example, Ted stated "the idea I'm gonna do is say, like, which of these middle paths am I gonna cross?," which indicates selecting the paths prior to traversal, and Erik stated "So I think it really is just any collection of those five vertical roads tells me exactly what I'm going to do […] So, step one is do I go… Do I start up or down. Then, I pick any subset." Erik's statement indicates a distinction between the first decision and subsequent decisions, and both statements indicate choosing the paths prior to diagram traversal. This sentiment seems to be reflected in their final solutions, both of which counted subsets as a single object with cardinality of 2^5. By comparison, although 2^5 appeared in solutions using the sequential model, this expression seemed to indicate one unit repeated five times rather than a single unit. Ted's written work, shown in Figure 5, appears to reinforce that 2^5 stood for the single unit of subset rather than five distinct decisions.

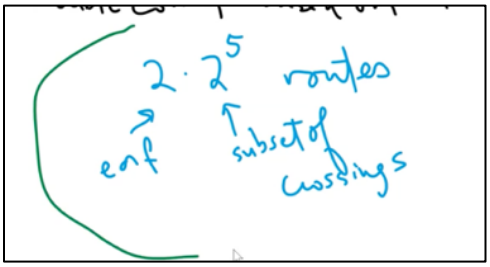


Figure 5. Ted's solution delimitating the first decision from the subset.

Discussion and Conclusion

The expert solutions discussed here shed some light on the nature of sets of outcomes in combinatorics problems. The literal outcomes as described in the statement of the Roads Problem—paths along a diagram—were used as the exclusive set of outcomes by only one participant. All other participants created alternative models of the paths (binary strings, sequences of decisions, and subsets) that seemed to inform how they found a solution and the form of the solution.

It did seem to be the case that the alternative models were strictly artifacts of counting processes the participants employed. The language used by the participants, such as when Oscar said "a route is 'do I take the bridge or not, yes or no?," shows that they were using the literal model to create a secondary model they could hold simultaneously. The second model also demonstrated beliefs about the outcomes that might provide insight on the relationship between outcomes and counters. For instance, the experts who used the sequential model used language that indicated the outcomes would be constructed as an agent traversed the diagram, while the experts who used the subset model used language that indicated the outcomes would be determined prior to an agent traversing the diagram. The binary, sequential, and subset model reflected different temporal relationships between the outcomes and the counter. Whereas the binary model did not include a temporal relationship, the sequential and subset models included language that referenced future or in-the-moment decisions, respectively. The models also showed a difference between counting static complete sets of outcomes (all binary strings), and processes that would generate a single outcome (sequential model).

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Exploring Aspects of Student Facility With Taylor Series in the Context of the Electric Dipole

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Using a grounded theory approach, we investigate students' facility using Taylor series expansions in the context of the electric dipole. Eight students participated in one-on-one teaching-learning interviews, during which audio and video data were recorded while they completed a written activity. We show that while physics students at this level are very familiar with the procedural aspects of Taylor series expansions, they may not recognize several common cues to use them to simplify functions. We find that sketching out the exact function and an approximation of it can help students learn to associate these common cues with Taylor series. In addition, we probe students' conceptions of the meaning of the expansion point and find that one difficulty in choosing an expansion point in a physics context is associating that 'point' with a spatial location. We conclude with recommendations for physics and calculus instructors.

Keywords: Taylor Series, physics, applications

Introduction

In their studies, physics students must often approximate complicated functions that describe physical systems. While crude approximations—such as setting small parameters to zero—may be sufficient in early coursework, they lose pedagogical value as students advance. A more versatile and effective method involves using a small ratio of parameters to construct a Taylor series expansion, which serves as an approximation with adjustable accuracy. This use case of Taylor series, which Jones et al. (2019) refer to as an "extension of a linear approximation," is an indispensable mathematical technique in a physicist's toolkit and plays a pivotal role in connecting mathematical rigor with physical intuition.

Usually students first encounter Taylor series towards the end of their introductory calculus sequence, although they use results of Taylor series in physics courses earlier. One early use is the 'small angle approximation', that $sin\,\theta \approx \theta$ for $\theta \approx 0$. However, applications like these are often treated as facts to be memorized, justified only with plausibility arguments. In calculus courses, they are expected to remember the general formula for the Taylor series expansion of a function *f(x)* expanded about some expansion point *a* $f(x) = f(a) + \frac{\partial f}{\partial x}\Big|_a (x-a) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_a (x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n f}{\partial x^n}\Big|_a (x-a)^n$ and be able to apply it to complicated functions that are already in terms of the expansion variable. However, in physics explicit prompting for Taylor series is rare: students are usually expected to pick up on contextual cues, such as one quantity being much smaller than another, or that an approximate expression is required (Wilcox et al., 2013). Physics courses expect students to understand when a Taylor series expansion is needed, to create their own small and dimensionless expansion variable, and only then to perform the expansion (Loverude & Li, 2013).

A common context for Taylor series expansions is to approximate the electric field at a distance *L* from an electric dipole, which consists of a pair of charged particles of equal magnitude and opposite sign, separated in space by a distance *s* (see Fig. 2). An expert would treat the variable *L* as a parameter that takes on values either much smaller or much larger than *s*, and would use the *parameter ratio* $\frac{s}{L}$ as the expansion variable for the Taylor series in order to

understand the field's behavior at relative extremes. The expansion point is the approximate value of the parameter ratio, and the zeroth-order term *f(a)* is often physically meaningful. Students may struggle with this even if they have mastered the computational techniques in a calculus course, as these contexts require sense-making with physical quantities.

In this paper we investigate problem-solving strategies of sophomore-level students when applying Taylor series to the electric dipole. We aim to identify specific student strengths and conceptual barriers—specifically those related to cue recognition and interpretation of the meaning of the expansion point—to better inform instructional scaffolding. Following a review of the literature on Taylor series in physics education, we outline our study design, present and discuss our findings, and conclude with instructional implications for both physics and calculus courses.

Our research questions are as follows:

1. Research Question 1 – Cuing with the Parameter Ratio: Do students in this population recognize that the request to find an approximate expression when the charge separation is much smaller than the observation distance—indicating a very small parameter ratio—serves as a cue to apply a Taylor series expansion, without explicit prompting? If not, what prompts allow students to come to the conclusion themselves that these phrases are cues to do so?
2. Research Question 2 – Conceptions of the Expansion Point: What are students' conceptions of the meaning of *a* (the expansion point) in a physics context?

Background

Knowing when and how to apply Taylor series in physics is often difficult, even for students who have attained procedural proficiency in their use from calculus courses (Smith et al., 2013; Wilcox et al., 2013; Loverude and Li, 2013). Some of this may be due to the increased number of steps required to use a Taylor series expansion in physics as compared to in calculus courses. One such step is determining the expansion point: Wilcox et al. (2013) and Smith et al. (2013) showed that students have difficulty when the expansion point is non-zero. Another step is to recognize the cues to use a Taylor series that experts recognize. Loverude (2025) observed that while students readily employ basic approximation methods when prompted, they may not yet intuitively select more sophisticated techniques that experts use to extract physical insights.

Common cues for students to use a Taylor expansion are outlined in the Wilcox et al. study. These cues are: explicitly asking for a Taylor series expansion, asking for an approximate expression for a complicated function, and the use of language or symbols implying that one parameter is much smaller than another. Their students were successful in picking up the cues, but their population was also from a course in which these cues would be taught. We sought to evaluate whether the second and third cues are sufficient for students—who had not had explicit instruction identifying these as indicators—to infer the appropriateness of a Taylor series expansion when given an intellectual need for such an expansion (Harel, 2008). We predicted, given the documented difficulties students have in transferring knowledge between courses (Loverude, 2017), that students would struggle to come to the conclusion that a Taylor series expansion is necessary. We wanted to explore their reasoning at this stage.

Methods

The student population sampled in this study consists of physics majors and minors at a large-enrollment R1 institution in the Pacific Northwest that graduates about 100 students annually. Approximately 15% of students self-identify as members of underrepresented minority

groups and about 27% identify as female. Math Methods I is a sophomore-level course, usually taught with minimal physics context and from the standard textbook (Boas, 2005). The course is typically taken after one year of introductory calculus and one year of introductory calculus-based physics. Similar to patterns across the U.S., the introductory physics sequence followed by most participants did not require students to solve problems using calculus (Loverude, 2025).

Eight students, all enrolled in Math Methods I, volunteered for this study in exchange for $200. All participants identified as female and used the self-selected pseudonyms Abby, Chad, Grace, Jake, Jenna, Priscilla, Summer, and Susan. While this sample was not representative of the course demographics, time constraints prevented further recruitment. Two participants, Jake and Summer, were non-traditional students who had transferred from community colleges; Jake had completed the same course in the prior quarter.

Participants engaged in one-on-one teaching-learning interviews (Nguyen & Rebello, 2011), completing a Taylor series-based activity with intermittent input from an interviewer. Previous studies on the use of Taylor series expansions in physics have asked students to complete the entire problem at once, with written responses analyzed afterwards (Loverude, 2025; Loverude & Li, 2013; Wilcox et al., 2013). Other studies have conducted teaching interviews in which students were explicitly prompted to use Taylor series (Smith et al., 2013). Teaching-learning interviews allow for deeper probing of student reasoning than analysis of written responses alone. Interviews were conducted in quiet rooms, lasted 1–2 hours, and were screen-recorded on an iPad. Audio was transcribed using AI software and manually corrected. Grace's audio was not recorded, so her data were only used when video and notes were sufficient.

Data processing followed the procedures outlined by Otero and Harlow (2009). After reviewing interview notes and recordings, the lead researcher identified sections of interest, usually where students struggled most. Content logs were written for about one-third of the activity questions; these logs helped us focus on three questions from the activity, reproduced at the end of this section. Student written data were integrated into the corrected transcripts.

Figure 1. Schematic of the data analysis process. We used grounded theory to generate codes which were grouped by theme (such as 'Expansion Point'), and report on patterns that emerged from within those themes (Attride-Stirling, 2001; Glaser & Strauss, 1967; O'Callaghan et al., 2024).

Our data were analyzed using thematic analysis informed by grounded theory (TAG) (Attride-Stirling, 2001; Creswell & Creswell, 2018; Glaser & Strauss, 1967; O'Callaghan et al., 2024). We used a grounded theory approach in generating a codebook from the raw data (Fig. 1, left). A primary focus of the analysis was on students' initial responses to the activity questions; codes for these were initially developed by the lead researcher during a review of the content logs and later validated through application. A second focus was on students' responses to interviewer prompts. Both the prompts and the corresponding responses were coded inductively. Both coding efforts contributed to the development of the final codebook. The lead researcher and two undergraduate research students applied the completed coding scheme on one interview before independently applying it to other interviews and reaching above 85% agreement.

We subsequently used our codebook to identify themes, which formed the basis of a thematic analysis (Fig. 1, center). Individual codes were grouped into broader themes (e.g., Expansion Point), and we describe several reasoning patterns that emerged as clusters of codes within each theme (Attride-Stirling, 2001; O'Callaghan et al., 2024). This analysis yielded the reasoning patterns which inform the discussion throughout this paper.

The activity used was adapted from the Tutorials in Mathematical Methods in Physics by Loverude, Thompson, and Christensen (Loverude et al., n.d.) to provide an intellectual need for a Taylor series expansion (Harel, 2008). The goal of the activity was to model the electric field produced by an electric dipole at an observation point P far away from the center of the dipole; that is, $L \gg s$ in the diagram.

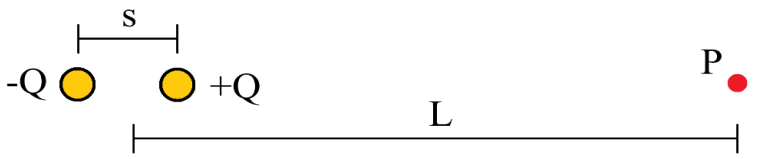

Figure 2. The figure given on the activity. The two circles ('+Q' and '-Q') are the equal magnitude, oppositely signed charges that constitute the dipole. L is distance from the dipole's midpoint to the observation point P and s is the distance between charges in the dipole. Students were to work in the regime where $L \gg s$.

In their study, Loverude and Li (2013) report that students have difficulty finding an appropriate expansion variable and getting their equations into a productive form for expanding in a Taylor series. Since we were unsure to what extent students would be cued to use a Taylor series—if at all—they were guided to write their exact expression in terms of the parameter ratio $u = \frac{s}{L}$ before expanding it as a Taylor series.

The relevant questions in the activity are listed below. Note that the 'expression in question 6' is the exact equation for the electric field as a function of u: $E = \frac{kQ}{L^2}\left[\frac{1}{\left(1-\frac{u}{2}\right)^2} - \frac{1}{\left(1+\frac{u}{2}\right)^2}\right]$.

10. Let's say instead that we want to know what the electric field is when u is very small, but not exactly zero. What could you do to your expression in question 6 to approximate the electric field for small values of u? (Answer: Expand it as a Taylor series and truncate.)
13. One of the most important Taylor series expansions to learn is the expansion for $(1 + x)^P$. Derive the Taylor series to three terms for this expression.
14. Look again at your expression in question 6. If we were to use the expansion that you just derived, what should p be equal to? What should x be equal to? *Asked verbally: What should a be equal to?* (Answers: p=*-2*, $x = \pm\frac{u}{2}$, and a=*0* since x is small.)

Findings and Discussion

RQ 1: Cuing with the Parameter Ratio

Findings. Our first research question is whether students would recognize that a very small parameter ratio and a request for an approximate expression are cues to use a Taylor series expansion, and if not, how they come to recognize this. Prior to Question 10, students were asked to describe the physical and mathematical results of setting the parameter s to zero and the

parameter L to infinity separately, and subsequently about the results of setting the quantity $u = \frac{s}{L}$ to zero.

To answer Question 10, Chad, Jake, Susan, and Summer initially suggested that (in Susan's words) they "do the limit as u approaches zero". Abby, previously informed by the interviewer that setting $u=0$ was not correct, settled on "a limit approaching one". Summer and Jenna suggested 'dropping' u, which appeared to mean setting it to zero rather than excluding the symbol. Priscilla only algebraically rearranged the symbols without altering the expression. After clarification that taking the limit as u approaches zero would yield the same result as setting $u=0$—which they had already done—Jake and Susan proposed plugging in a number close to zero (such as 0.001). No others had alternate ideas before discussing with the interviewer.

The interviewer then asked each student a variant of: "In introductory physics, you used the small angle approximation $\sin x \approx x$. Why is that valid?" Only Jake immediately related this to the Taylor series of sine. Four (Abby, Grace, Summer, and Susan) were asked to sketch graphs of $y = \sin x$ and $y=x$ overlaid and three of them eventually concluded that the approximation was due to a Taylor series. Among those who did not graph, none came to this conclusion. These results are shown in Table 1.

Table 1. Student reasoning in Question 10. Three of the four students who concluded that a Taylor series expansion was necessary drew the functions $y = \sin x$ and $y=x$ on the same axis. Three of the four who did not conclude such also did not draw any graphs.

	Concluded Taylor Series	
	Yes	No
Drawing	Abby, Grace, Susan	Summer
No Drawing	Jake	Chad, Jenna, Priscilla

Discussion. That five out of seven of our students initially suggested using a limit is not surprising, as prior questions likely primed them to think about limits. However, all of their initial suggestions would result in exact values for the electric field at various values of u, rather than in approximations valid over a range of values of u. These data agree with Loverude and Li (2013), who noted that students often interpreted $L \gg s$ as a reason to set $s=0$. Combined with the fact that their alternatives also give exact values rather than approximate expressions, this suggests that students at this level may struggle to conceptualize approximations of functions.

The graphing task proved useful in helping students connect mathematical tools to contextual use, though the interviewer's prompts also mattered. Before coming to the correct conclusion Abby and Susan were asked how to increase the domain of validity of the approximation, while Summer was asked how to improve its accuracy and did not conclude that the approximation was the result of a Taylor series. This suggests that prompting students to broaden the approximation's domain may be more effective, possibly because it avoids assumptions about the series' exactness; according to Martin et al. (2011), students commonly view truncated Taylor series expansions as being exact throughout the domain of validity.

These results show that the cues from Wilcox et al. (2013) may be valid if they are taught, but students who have not learned Taylor series in a physics context may not recognize them. While visualizations help, additional prompting about the nature of the graph may be necessary. This illustrates how effortful the transition from mathematical abstraction to physical application is for learners while also supporting the idea that students have emerging graphical-mathematical blended reasoning, which is present in experts (Zimmerman et al., 2023).

RQ 2: Conceptions of the Expansion Point

Findings. Before investigating students' conceptions of the expansion point, we first assessed their procedural facility with Taylor series. In Question 13 we asked students to find the Taylor series expansion of $(1+x)^p$ without giving them the general formula. Six of the seven immediately started taking derivatives to get the coefficients, while Jake attempted a summation expression. All of them ended up with the correct formula with minor errors. Most errors involved the expansion point a: three students dropped all powers of x-a, three omitted $-a$ from the powers of x (two—Jenna and Susan—did so intentionally, stating that $a=0$), and one had to be prompted to evaluate the derivatives at a.

In Question 14, students matched the result of their expansion to the exact expression of the electric field from earlier. Most incorrectly chose $p=2$ rather than the correct $p=-2$, which is consistent with Loverude and Li (2013). While most correctly or nearly correctly identified x, only Jake immediately concluded that $a=0$ without interviewer guidance.

When asked for the meaning of a, Jake paraphrased the interviewer's earlier explanation of a and concluded that $a=0$ since the expansion variable was close to zero. Priscilla offered that a is "something really close to the value of whatever x is" but needed prompting to identify its value. Chad also produced an acceptable definition of a without prior explanation by the interviewer but initially believed that 'point' referred to the spatial location point P, as did Abby. When told by the interviewer that $a=0$, Abby explained that zero makes sense because s and L are measured from the origin, another spatial location. Summer reported that a is "where we are basing the expansion", but then was unsure what it would represent, guessing the variable u, Coulomb's constant k, "the electric field", and $\frac{1}{2}$.

Table 2. Students' initial responses to Question 14. A question mark indicates that the student did not give a definite response without aid from the interviewer. Jenna and Susan had already chosen a=0 in Question 13, though with incorrect/no justification.

	p	x	a
Abby	2	±u/2	Point P
Chad	2	±u/2	Point P
Jake	2	±u/2	0
Priscilla	2	±u/2	?
Jenna	2	u	0*
Summer	2	Q or k/L^2	?
Susan	?	u	0*

Discussion. Students generally showed comfort deriving the Taylor series expansion for $(1+x)^p$. Errors in Question 13 mostly involved omitting or misplacing the expansion point a, suggesting that students are accustomed to dealing with expansions where $a=0$ (Martin, 2009; Habre, 2009) and have limited experience with making sense of a in physics contexts.

When applying the Taylor expansion to the physics context, identifying p and x became a straightforward extension of students' mathematics knowledge. Arriving at the conclusion that $a=0$ was more difficult. Though many students were able to offer workable definitions of a, this did not always translate into finding its value. Priscilla and Summer may also have resisted concluding that $a=0$ from a mental distinction between Taylor and Maclaurin series.

We did not expect to observe conflation of the expansion point with a spatial location. This issue likely arises from the diagrams used in physics problems: they are often spatial diagrams of the problem set-up and not graphs of the expression to be approximated, as are found in calculus.

While students demonstrated a good grasp of the procedural aspects of Taylor expansions, their difficulties identifying and interpreting *a* reveal that applying Taylor series in physics requires scaffolding extending beyond the procedural proficiency developed in calculus courses.

Conclusion

This study adds to evidence that while sophomore physics students are often procedurally fluent with Taylor series, they may not recognize when and how to apply them in physics contexts. Our findings focus on how students interpret common cues for approximation (Wilcox et al., 2013) and how they adapt the formula using physical parameters.

Consistent with Loverude (2025) and Loverude and Li (2013), no students identified the need for a Taylor expansion without prompting, often preferring to 'do the limit' instead. We found that graphing both exact and approximate expressions in the case of the small angle approximation helped students recognize that "find an approximate expression" is a cue to Taylor expand. We further conjecture that, for students who are not immediately helped by graphing, prompting them to consider how to extend the domain of validity of the approximation yields better results than prompting them to focus on increasing its accuracy within a fixed domain. We hypothesize that this is because the former avoids the conceptual challenge of reconciling the inexactness of a truncated Taylor series.

Choosing the expansion point was also challenging for students. Both Smith et al. (2013) and Wilcox et al. (2013) report on student difficulties with non-zero expansion points, but in neither study were students' conceptions of the meaning of the expansion point investigated. We show that even students who could self-generate reasonable definitions for *a* struggled to determine that *a=0* in this context. Some interpreted the expansion point as a spatial location, a confusion which is likely more prevalent in physics and engineering contexts than in mathematics because diagrams in these disciplines often depict spatial configurations rather than graphs of the function, so students cannot visually locate the expansion point.

Our evidence indicates that students need support in identifying when a Taylor series is an appropriate tool in physics contexts. To help them develop this sense, we recommend that math instructors use physical examples that are within students' zone of proximal development (Vygotsky, 1978) to illustrate the practical uses of Taylor series. For instance, the potential energy between two objects is $U = -\frac{GMm}{r}$, where *M* and *m* are the masses and *r* is the distance between the objects. The potential energy between the Earth (radius *R*) and an object at a height *h* above Earth's surface is $U = -GMm(R+h)^{-1}$. Using $x = \frac{h}{R}$ as the (very small) parameter ratio and expanding about *x=0*, this yields $U = -\frac{GMm}{R} + mgh$; the first-order term is the familiar expression for potential energy above the surface of the Earth. Examples like these give physical meaning to the parameter ratio and the expansion point while illustrating common cues to Taylor expand, like a small parameter ratio.

Lastly, we recommend that physics instructors be aware that even algorithmically proficient students may not understand or remember the mathematical properties of the tool they are using if those properties don't help them to perform the algorithm. In addition, math courses make definitional distinctions between similar concepts (like Maclaurin series, Taylor series, and Taylor polynomials) which students may retain in contexts where physics instructors are not.

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Exploring Sense of Belonging Among Black and Latina Students in College Algebra and Precalculus: Insights From a Mixed Methods Study at a Minority-Serving Institution

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In this sequential explanatory mixed methods study, I examined Black and Latina female students' sense of belonging in gateway mathematics at a diverse, open-access minority-serving institution. Using frameworks of intersectionality, sense of belonging, and authorizing student perspectives, I explored how students' identities shaped their experiences in college algebra and precalculus. I surveyed 1,136 students (pre-survey) and 639 students (post-survey) using an adapted Math Sense of Belonging Scale and analyzed data using ANOVA and ANCOVA. Results showed that mathematics affinity and expected grades were stronger predictors of belonging than race or gender. From the qualitative data, I identified six positive and six negative factors influencing belonging, including professors' mathematical microaffirmations and microaggressions, perception of caring faculty, peer collaboration, peer connections, mathematics self-efficacy, and perceptions of class diversity. These findings highlight the importance of caring faculty, collaborative learning, and peer support in fostering belonging in racially diverse mathematics classrooms.

Keywords: Sense of Belonging, Intersectionality, Gateway Mathematics Classes, Minority-Serving Institutions, Authorizing Student Perspectives

Despite increasing diversity in higher education, Black and Latina women remain underrepresented in STEM and face persistent barriers in introductory mathematics courses (Ong et al., 2016; Hatfield et al., 2022). College Algebra and Precalculus at open-access institutions often serve as gatekeepers rather than gateways to STEM pathways, characterized by low pass rates and unwelcoming environments (Seymour & Hunter, 2019). These students navigate racialized and gendered challenges due to their intersecting identities (Leyva et al., 2020; McGee & Bentley, 2017). Moreover, minoritized female students are less likely to report feeling a sense of belonging in STEM although classroom-level belonging is critical to persistence and success (Wilson et al., 2015; Rainey et al., 2018). However, most studies overlook the intersection of race and gender and rarely focus on diverse, open-access minority-serving institutions.

In this study, I examined how Black and Latina female students' identities influence their sense of belonging in gateway mathematics classrooms at a racially and ethnically diverse open access minority-serving institution. The research questions are: (1) How do Black and Latina female students' sense of belonging in the college algebra and precalculus classrooms differ compared to students in other racial and gender groups? (2) How does Black and Latina female students' sense of belonging in college algebra and precalculus change from the beginning to the end of the semester? (3) How do Black and Latina female students describe their college algebra and precalculus learning environment, experiences, participation, persistence, support systems and challenges as it relates to their sense of belonging?

Theoretical Perspectives

To frame my study, I drew on three interconnected theoretical perspectives: college students' sense of belonging, intersectionality and authorizing student perspectives. Strayhorn (2019) argued that belonging is a fundamental human need that must be satisfied before students can

achieve higher-order outcomes such as confidence and creativity. Belonging is context-dependent and closely related to mattering, which is a feeling that one is valued by others. It influences academic behaviors and is associated with engagement, persistence, and achievement. Absence of belonging can lead to anxiety, low self-esteem, isolation, and poor performance.

Moreover, I used Collins and Bilge's (2020) framework of intersectionality, which explores how overlapping systems of power, such as race, gender, and class, mutually shape individual experiences. For Black and Latina women in mathematics, intersectionality reveals how marginalization occurs at the intersection of race and gender, often referred to as the "double bind" (Delgado & Stefancic, 2012). Finally, I use Cook Sather's (2006) concept of authorizing student perspectives to center students' voices and lived experiences. This framework positions students as experts in their own educational journeys and calls for creating spaces where their insights are valued and used to inform meaningful change in classroom practice.

Literature Review

Much of the extant research on sense of belonging focuses on minoritized students at the campus level, particularly within four-year PWIs or research universities (Hurtado et al., 2015; Zumbrunn et al., 2014). Belonging is positively associated with academic achievement, persistence, and mental health (Strayhorn, 2019), while its absence contributes to STEM attrition, especially among minoritized and female students. Although campus belonging is important, the classroom plays a central role in shaping students' daily experiences and decisions to persist in STEM (Wilson et al., 2015; Riegel-Crumb, 2019).

Recent studies have begun to examine belonging at the classroom-level in mathematics education. Griffin (2023), McGrane and Rasmussen (2023), and Lahdenpera and Nieminen (2020) found that interactive learning environments, mentorship, and instructor engagement significantly enhanced belonging. Griffin's study showed the majority of female calculus students identified group work or interactive lectures as key contributors. At Hispanic-Serving Institutions, McGrane and Rasmussen found calculus support courses improved belonging, while Cawley and Wilson (2024) documented that microaggressions undermined it. Urbieta (2022) highlighted how socioeconomic challenges, such as housing instability, lowered belonging among low-income students.

Su (2015) introduced the concept of mathematical microaggressions to describe subtle, often unintentional ways instructors signal that certain students do not belong in mathematics. These include dismissive phrases such as, "It's obvious" or "You should have learned this before," as well as behaviors such as ignoring questions or rushing through the material. On the other hand, mathematical microaffirmations, which are intentional, affirming practices that validate students' mathematical thinking help foster positive mathematics identities and reinforce belonging. As Cawley and Wilson (2024) observed, students often remember how they felt in mathematics class very vividly.

Classroom belonging is tied to motivation, engagement, and confidence (Freeman et al., 2007; Wilson et al., 2015). Students report stronger belonging when instructors are perceived as caring and when active learning is encouraged (Kirby & Thomas, 2022). Despite its importance, research on minoritized women's mathematics classroom belonging remains limited, underscoring the need for further study in diverse institutions.

Methods

I used a sequential explanatory mixed methods design, integrating both quantitative and qualitative data to examine Black and Latina female students' sense of belonging in college

algebra and precalculus classrooms. The quantitative phase was conducted first to identify patterns in belonging, followed by a qualitative phase to interpret and expand upon those findings (Creswell, 2015). The study site was an ethnically diverse, open-access, four-year public college near a major Southeastern city. Of its 11,000 students, 32% identified as Black, 27% Latinx, 24% White, and 12% Asian. Over half were eligible for Pell Grants and 40% were first-generation. The institution holds designations as both an Asian American Pacific Islander Serving Institution (AAPISI) and a Hispanic Serving Institution (HSI).

Quantitative Phase

Participants were enrolled in College Algebra and Precalculus during Fall 2023, in which semesters are 15 weeks long. I visited all sections to administer pre-surveys (weeks 1-3) and post-surveys (weeks 11-14). Surveys were distributed via Qualtrics using QR codes, and only students who completed both were included in the final analysis. The sample included seven sections of College Algebra with support, 37 of College Algebra, and 14 of Precalculus. Courses ranged from 25-30 students. A total of 1,136 students completed the pre-survey (31.0% Black, 30.3% Latinx, 20.7% White, 13.4% Asian, and 4.5% Other; 51.7% female and 48.3% male), and 639 students completed the post-survey (30.4% Latinx, 29.1% Black, 21.1% White, 14.2% Asian, and 5.2% Other; 50.4% female and 49.6% male). The faculty included 44 full-time instructors (47% White, 29% Asian, 18% Black, 4% Hispanic) and five part-time instructors.

Students completed a demographic questionnaire and an adapted Math Sense of Belonging Scale (Good et al., 2012) revised to reflect classroom context. The scale included 30 Likert items (1=Strongly Disagree to 6=Strongly Agree) and demonstrated high reliability (α=0.94). Composite scores were calculated for pre-, post-, and change in belonging. I analyzed the data using SPSS, using an alpha level of 0.05. Descriptive statistics and histograms were used to examine distributions. To address Research Question 1, I conducted a one-way ANOVA on pre-belonging scores across race-gender groups. For post-belonging scores, I used a multi-factor ANCOVA controlling for pre-scores, with race-gender, faculty, mathematics affinity, and expected grade as factors. To address Research Question 2, I conducted a multi-factor ANOVA on belonging change scores using the same variables.

Qualitative Phase

In the qualitative phase, I explored students' experiences to interpret and deepen understanding of the quantitative findings. This approach aligns with the study's theoretical perspectives, intersectionality and authorizing student perspectives, which center students' voices and experiences. I collected two types of qualitative data: mathematics autobiographies and individual interviews. Thirteen participants (five Black, six Latina, and two Black/Latina women) who completed both pre- and post-surveys were purposefully selected to represent a range of belonging scores. Eleven of these participants also submitted mathematical autobiographies reflecting on K-12 mathematics class experiences, which informed the interview protocols. Semi-structured interviews (average 59 minutes) were conducted in person or via Zoom at the end of the Fall 2023 semester. Interview prompts included questions such as, "In what ways did you feel that you belonged in your math classroom?" and "Can you think of a specific example of a time in this class when you felt like you didn't belong?" All interviews were professionally transcribed and I verified the transcripts for accuracy. I analyzed the data using Braun and Clarke's (2022) six-phase reflexive thematic analysis to identify patterns and themes related to students' sense of belonging.

Findings

Quantitative Results

Research Question 1: How do Black and Latina female students' sense of belonging in college algebra and precalculus classrooms differ compared to other racial and gender groups?

Quantitative analyses revealed that Black and Latina female students' mean belonging scores were comparable to other race-gender groups. At the beginning of the semester, only one statistically significant difference emerged: Latina female students reported lower pre-belonging scores than Black male students (p=0.026, Mean Difference=-0.24). Black female students' scores did not significantly differ from any other group. By the end of the semester, when controlling for pre-belonging scores, race-gender was not a significant predictor of post-belonging (p=0.740), indicating no meaningful differences between Black and Latina female students and their peers. Post-belonging mean score for Black female students was 4.5 and for Latina students was 4.4, while the overall mean score was 4.4.

Research Question 2: How does Black and Latina female students' sense of belonging change from the beginning to the end of the semester?

There were no significant differences in change scores based on race and gender (p=0.541). However, other factors were significantly associated with belonging change: faculty (p=0.034), mathematics affinity (p<0.001), and expected course grades (p=0.012). These findings suggest that students' perceptions of their instructors, enjoyment of mathematics, and confidence in their academic performance, played a more substantial role in shaping belonging than race and gender.

Overall, students maintained relatively stable and high belonging scores throughout the semester, with minimal average change. While race and gender did not significantly influence post-belonging or belonging change scores, mathematics affinity and expected grades were significant predictors of students' sense of belonging. Faculty did not significantly impact post-belonging scores, but did influence changes in belonging over time, highlighting the importance of instructors. These quantitative findings provided a foundation for the qualitative phase of the study, and in the next section, I expand on these results.

Qualitative Findings

Through thematic analysis of interviews and mathematics autobiographies, I identified six key factors that promote a sense of belonging in diverse introductory mathematics classrooms:

1. Professors' mathematical microaffirmations – Small validating interactions that affirm students' mathematical thinking and effort.
2. Perceptions of professors as caring, supportive and helpful – Students felt more connected when instructors were approachable and invested in their success.
3. Encouraged peer collaboration – Structured group work fostered academic and emotional support.
4. Peer connections – Positive interactions and relationships with classmates contributed to a sense of community and inclusion.
5. Positive mathematics self-efficacy – Confidence in math ability both supported and was reinforced by belonging.
6. Perception of classroom diversity – Seeing diversity in race and gender helped students feel represented and included.

These factors often worked together to enhance belonging. For example, students who received mathematical microaffirmations were more likely to perceive their professors as caring. Mathematics self-efficacy was both a contributor to and outcome of belonging, forming a positive loop. A conceptual model (Figure 1) illustrates these interconnections.

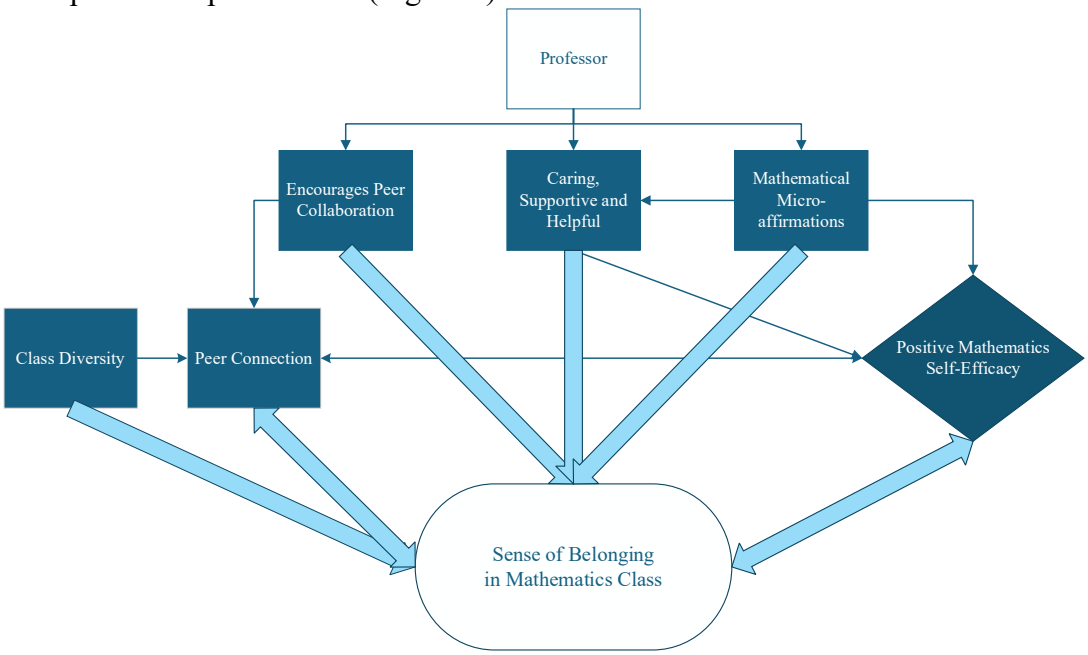

Figure 1. Six Factors that Promote Sense of Belonging in Mathematics Classrooms.

In contrast, students also described six factors that diminished their sense of belonging:

1. Professors' mathematical microaggressions – Subtle, dismissive comments that undermine students' mathematical confidence.
2. Perception of professors as uncaring or judgmental – Students avoided seeking help, fearing embarrassment or shame.
3. Limited peer collaboration – Lack of structured interaction hindered relationship-building.
4. Lack of peer connection – Isolation from classmates reduced feelings of inclusion.
5. Negative mathematics self-efficacy – Low confidence in mathematics ability weakened engagement with class activities and peers.
6. Past negative mathematics experiences – Previous struggles shaped current perceptions of belonging.

Students emphasized that when professors communicated, either explicitly or implicitly, that they were not valued, they became reluctant to be vulnerable about their mathematical challenges. This led to avoidance of interaction, increased frustration, and in some cases, disengagement. Similarly, lecture-only teaching approaches and lack of peer interaction reinforced feelings of isolation.

Notably, supportive professors and strong peer relationships served as protective factors, buffering students against negative experiences. Some students with low mathematics self-efficacy or prior negative mathematics experiences reported increased belonging when these support systems were present. As Britteny shared, "If you need help with something, [Dr. S.] is always there... [They were] easy to talk to so that made the course easier to get through because

when you're in a course for so long with a teacher, you have to get to know them, or it just has to be very easy to approach them. Like you bond over math."

Importantly, the qualitative findings reveal that students' sense of belonging is shaped more by relational dynamics, such as faculty support and peer interaction, than by the perceived difficulty of mathematics itself. Past negative experiences do not necessarily determine future belonging. When professors foster inclusive, affirming environments, students can develop a strong sense of belonging regardless of background or prior mathematics self-efficacy.

Discussion of the Integration of Quantitative and Qualitative Findings

The integration of quantitative and qualitative findings deepens our understanding of how a diverse institutional context influences Black and Latina female students' sense of belonging in mathematics classrooms. While the quantitative data showed no significant differences in belonging scores across race and gender groups, the qualitative findings revealed that diversity alone is not sufficient to foster strong belonging. As Hurtado and Guillermo-Wann (2013) suggested, representation matters, but it must be accompanied by relational and structural support. Students appreciated seeing peers who shared their racial or ethnic identity, but the ability to form supportive peer groups based on shared experiences played a more critical role in fostering belonging.

Racial and gender identity were not salient factors in shaping belonging for most participants in this diverse setting. Participants reported no experiences of racial or gender microaggressions, contrasting with studies at less diverse institutions. However, racial representation still mattered, and students valued being in classrooms where they felt seen and saw others who looked like them. The lack of significant differences in belonging scores across race and gender groups suggests that other factors, such as peer relationships and classroom dynamics, may be more influential in diverse classrooms.

Moreover, students' confidence in their mathematical abilities, shaped by prior educational experiences, influenced their engagement and sense of belonging. Students with negative K-12 mathematics experiences often started college with low mathematics self-efficacy, which impacted their classroom participation. However, when these students encountered supportive professors and collaborative environments, their sense of belonging improved. This suggests that mathematics self-efficacy may be more salient than racial or gender identity in shaping belonging, especially in diverse classrooms where students see others who look like them. It is important to note, that while race and gender did not directly affect belonging scores, students' past educational experiences are often shaped by systemic inequities, which are a direct influence of intersectional identity. This underscores the need to consider how historical and structural factors influence students' current sense of belonging.

Finally, while faculty was not a significant predictor of post-belonging scores, it was a significant factor in belonging change. Supportive faculty behaviors fostered belonging, while mathematical microaggressions and lack of support diminished it. Students did not explicitly connect these experiences to race or gender reinforcing the idea that relational dynamics were more salient in this context.

The integration of findings reveals that belonging is a complex, multifaceted construct shaped by students' prior experiences, classroom relationships, and institutional context. While diversity provides a necessary foundation, intentional pedagogical practices and relational support are essential for fostering belonging among minoritized students in mathematics classrooms.

Practical Implications for Gateway Mathematics Faculty

Creating a classroom where students feel they belong requires intentional, sustained effort throughout the semester. As Strayhorn (2019) emphasized, belonging must be actively nurtured and neglecting this need can contribute to students' academic and emotional struggles.

A strong link emerged between mathematics self-efficacy and students' sense of belonging. Faculty should adopt practices that build students' mathematical confidence, such as normalizing struggle, sharing personal educational experiences and journeys, and offering low-stakes assessments with revision opportunities. Creative assignments such as discussions, presentations and projects can also increase engagement and self-efficacy. Individualized support, especially early in the semester, is crucial for students with low-mathematics self-efficacy.

In addition, students shared that mathematical microaffirmations enhanced their sense of belonging, while mathematical microaggressions hurt it. Faculty development should include training on recognizing and using microaffirmations intentionally, while avoiding harmful language. Faculty peer observation groups can help instructors reflect on classroom communication patterns.

Moreover, clear and accessible office hours, both in person and virtual options, should be consistently communicated, especially to first-year students. Structured peer collaboration, with class norms co-constructed with students, and diverse groupings, can foster meaningful connections. Digital tools such as group chats may further support academic and emotional peer engagement.

Directions for Future Research

Future research should explore which faculty practices most effectively foster belonging by examining instructors whose students show significant changes in belonging over time. Classroom observations and interviews with faculty could reveal how instructor beliefs and behaviors shape belonging, especially through classroom norms. Additionally, studies should investigate the experiences of students who withdraw from or stop coming to mathematics classes, centering their perspectives to better understand attrition. Finally, expanding research to include students with disabilities, LGBTQ+ students, and varied learning environments (e.g., online, hybrid, elementary mathematics education and advanced courses) would offer a more comprehensive understanding of belonging in mathematics education).

Conclusion

This mixed methods study examined the sense of belonging among Black and Latina female students in gateway mathematics courses at a diverse, minority-serving institution. I identified that mathematics self-efficacy, faculty-student interactions, and peer relationships were more influential than race or gender in shaping students' sense of belonging in diverse mathematics classrooms. Faculty practices, such as using mathematical microaffirmations, demonstrating care, and fostering peer connections, played a critical role. Students also actively built peer relationships, even in classrooms lacking collaborative structures.

The study highlights the importance of centering student voices and identifying both positive and negative factors that shape belonging. Importantly, it shows that intentional faculty practices can buffer against negative mathematics experiences and foster inclusive learning environments. As Alexa shared about her belonging experiences, "You're not in it by yourself. You're going to have someone else to help you along the way… not just being there because you have to but being there because you feel like you're needed here." Mathematics classrooms can and should become spaces of belonging for all students.

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Undergraduate Students Metacognition While Constructing Mathematical Proofs

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This phenomenological case study examines evidence of undergraduate students' metacognitive reasoning while completing proof construction tasks. Two pairs of undergraduate mathematics majors participated in task-based video recorded interviews where they were presented with contextual scenarios and asked to prove or disprove a related mathematical statement using think-aloud protocols. Participants then took part in stimulated recall interviews where they were shown video clips from their task completion and asked to reflect on their thought processes. This contributed report explores each student's access of metacognitive knowledge and the subsequent actions they took during these proof tasks.

Keywords: Metacognition, Problem Solving, Proof

Mathematical proofs and their underlying logical structures are foundational to the development of mathematical knowledge (Hanna & Barbeau, 2008). The common vocabulary, deductive reasoning, and formatting used to develop and write these proofs are used to communicate new mathematical knowledge (Ernest et al., 2016). Undergraduate mathematics proof-based courses aim to imbue the necessary mathematical knowledge, writing and reasoning skills in students for them to be able to read, construct, and analyze mathematical proofs. The development of these skills is not confined to individual courses, but it is declared to be one of the overarching goals of an undergraduate mathematics program (Committee on the Undergraduate Programs in Mathematics [CUPM], 2015). Although research has explored the cognitive processes related specifically to the proof construction process (e.g., Hanna, 2020; Inglis & Alcock, 2012; Moore, 1994; Tall, 1998), considering that research recognizes metacognition's impact on student success with other mathematical tasks (e.g., Mamona-Downs, 2002), investigating metacognitive aspects of proof construction explicitly may inform how metacognition impacts proof construction and how to leverage this knowledge to improve teaching of proof construction.

For the purposes of this research, I utilize the definition of *metacognition* as "any knowledge or cognitive activity that takes as its' object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about, one's own thinking" (Lerman, 2020, p. 608). Metacognitive processes of *monitoring* and *control* describe the interaction between the *meta-level* (internal metacognitive knowledge and related models of the object-level) and the *object-level* (external to the meta-level, anything external to the individual and their metacognitive knowledge). *Monitoring* is the process where the "meta-level is informed by the object-level" (Nelson & Narens, 1994, p. 11) such that the information taken in during the monitoring process can change the models at the meta-level. *Control* is the process where the "meta-level modifies the object level" (Nelson & Narens, 1994, p. 11). Knowledge at the meta-level can cause action at the object-level, either through starting or ceasing an action. The choice not to take a certain action is part of the control process. In the case of an individual working on constructing a mathematical proof, the theorem statement, their written work, any other related cognitive processes, would all be at the object-level. The meta-level would encompass the individual's judgements about their proof construction process (e.g., how they perceive their own success) and any knowledge of strategies useful to this particular type of proof.

Metacognitive knowledge is partitioned into *declarative*, *procedural*, and *conditional* metacognitive knowledge (Schneider & Artelt, 2010; Shraw & Dennison, 1994). *Declarative* metacognitive knowledge encompasses knowledge a person has about themselves, their cognitive knowledge and abilities. *Procedural* knowledge describes individuals' knowledge of relevant strategies, heuristics, and other tools. (Schneider & Artelt, 2010). *Conditional* metacognitive knowledge is the knowledge of when and why specific cognitive tools and strategies are useful in a given context (Lerman, 2020).

Due to the focus on students' metacognitive processes rather than the validity or rigor of their resulting proof product, I utilize a broad definition of *proof construction* as the process of using a series of logical assertions deduced from a set of premises and previous assertions, and/or a set of instructions that when followed are representative of these deductions, where this collection of assertions and instructions results in an argument that "justifies a claim to oneself and others" (Blanton et al., 2003, p. 113). *Proof* is then defined to be this resulting collection of assertions.

The purpose of this phenomenological case study is to contribute to the field of metacognition research in mathematics education by examining the experiences of undergraduate mathematics students' metacognitive processes during proof construction. Specifically, this research aims to address the questions: What declarative, procedural, and conditional metacognitive knowledge do undergraduate mathematics students demonstrate or claim to have and use during construction of a mathematical proof? What metacognitive processes do undergraduate mathematics students engage in during construction of a mathematical proof? In particular, what monitoring and control processes do undergraduate mathematics students engage in during construction of mathematical proofs? And, in what ways do the cognitive and metacognitive processes of undergraduate students inform each other during construction of mathematical proofs?

Theoretical Perspective and Methodology

Entering this research from a social constructivism theoretical perspective, I considered individual's constructed conceptions, acknowledging that these conceptions are inextricable from the social contexts in which they were constructed (Patton, 2002). Vygotsky claimed the process of learning to include the appropriation and use of tools and signs (Grossman, 1999) that occurs during social interaction. Vygotsky not only considered social interaction with others but interaction within the self, intramental thinking (Ernest, 2010). This sort of internal interaction with the self is enmeshed with metacognition, the individuals reasoning about their own cognition. I specifically considered intramental and intermental thinking, (i.e., the internal and external respectively) (Ernest, 2010). Restructuring one's own conceptions through the process of metacognitive monitoring is analogous to the restructuring of conceptions that occurs with the appropriation of tools and signs inside of social settings. The metacognitive process of control therefore is enmeshed with the appropriation and use of tools and signs itself, as the metacognitive level impacts the individuals' external responses.

This study explores the phenomenon of individual's cognitive and metacognitive processes during the proof construction process. To examine this phenomenon, I conducted a qualitative study with four undergraduate mathematics majors as they completed paired proof-based tasks in an interview setting. Utilizing the participants' verbalizations during the tasks and their self-described experiences and reflections in follow-up interviews, I investigated indications of metacognitive processes for each student and the subsequent actions the students took in their proof construction process. The four undergraduate student participants were from proof-based undergraduate courses at a mid-size university in the United States. Prior to interviews, students

were paired based on similar mathematical backgrounds and experience working on mathematics collaboratively together.

Data Collection

The four participants in this study were mathematics majors at a mid-size university in the United States. They each had previously completed several proof-based courses and were currently enrolled in an abstract algebra course designed to specifically focus on proof writing. To address that individuals' conceptions are "formed through their interactions with each other as well as by their individual processes" (Ernest, 2010).

Each pair of participants were given two tasks, and were audio and video recorded and monitored remotely via Zoom© while Livescribe© pens recorded written data. Two to three days after a task, each individual completed a stimulated recall interview, watching selected video clips from their task completion, and asked to reflect on their experience and thinking during those clips. For each task, one pair was given a true claim (Version A), and one pair was given a false claim (Version B) and asked to prove or disprove the given claim. For the second task, each pair was given a different version of the task from the one they were given for the first task. Both tasks are given below

Task 1 Version A prompt. [Adapted from Gilbert and Larson (2012)] A gamekeeper gives player A a positive integer. The players then take turns changing the integer they are given to a smaller integer and then pass that smaller integer back to their opponent, player B. If the integer you received and are changing is even, the two legal moves are: subtracting 1 from the integer *or* dividing the integer by 2. If the integer you received and are changing is odd, the two legal moves are: subtracting 1 from the integer *or* subtracting 1 from the integer, then dividing this result by 2. The game ends when the integer reaches 0, and the player who makes the last move (changing an integer to zero) wins the game.

Prove or Disprove the Claim: Assume the starting integer is given to player A. Any starting integer that when factored is written as $2^k * l$ where k is an even integer and l is an odd integer, will guarantee that player A wins the game.

Task 2 Version A prompt. On a rectangular $n \times m$ grid, there are candies on each square of the grid. The bottom left corner square in the grid contains a bitter candy, all other candies are sweet. The two players take turns picking squares. When a player picks a square, they must eat every candy that is on the square they picked, or on any remaining squares of the grid above and to the right of the square they chose. The players take turns doing this until one of them eats the bitter candy (on the red square). Whichever player eats the bitter candy loses the game. Suppose there are two players, Player 1 and Player 2, where Player 1 makes the first move.

Prove or Disprove the Claim: For a game played on any $n \times m$ rectangular grid, where n and m are both positive integers, Player 1 will always have a winning strategy. (Excluding the trivial case of a 1×1 rectangle)

Data Analysis

Task interview data and stimulated recall interview data were transcribed. All utterances were coded, first deductively to distinguish verbalizations describing internal reasoning versus verbalizations describing something external. The Model of Sign Appropriation and Use (Ernest, 2010, p. 44) was referenced during this deductive coding to help make these distinctions, and code internal reasoning as intramental.

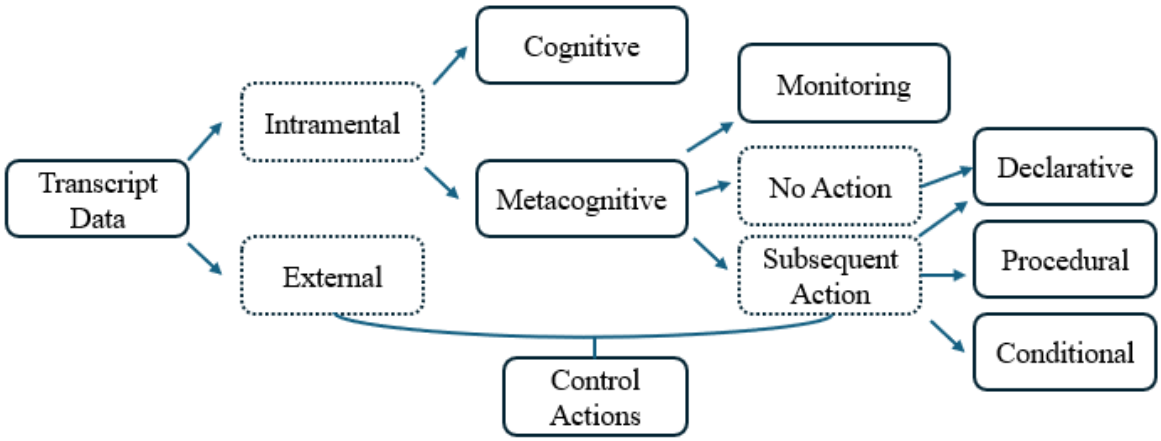

Figure 1. A Flow Chart of the Data Analysis Process

Once verbalizations regarding internal reasoning had been identified, the subject of these utterances was determined. If the subject was something external, that data was coded as cognitive. If the subject of the verbalization was other internal reasoning of the individual, the data was coded as metacognitive.

Data coded as metacognitive was then coded into three categories. Any metacognitive data that was immediately followed by the participant taking an action (or stated intentional inaction) was coded as having a subsequent action. These codes were paired with external coded actions to identify indicated metacognitive control. Metacognitive data where participants stated some intentional reflection about something external was coded as indicating potential metacognitive monitoring. Statements that were declarations of some metacognitive knowledge were coded as stated knowledge, and then deductively coded as declarative. All metacognitive data was deductively coded as indicating declarative, procedural, or conditional metacognitive knowledge. Any data that was coded as metacognitive control, was then connected with external actions and control actions were inductively coded into types of actions.

Consider this example quote from Harper, "I don't know if this general tree thing is ever going to end…Any pattern…has probably already showed up and we just didn't notice it. Let's go back to actual numbers." This excerpt is related to Harper's internal reasoning and therefore was coded as intramental. Then, because the subject of these statements is Harper's own thinking and reflecting on that reasoning and the success of her strategy, this was coded as metacognitive. Harpers initial statement was followed by an action coded as evaluating strategy, and her eventual action was coded as trying examples, as indicated directly in the quote. The relevant metacognitive knowledge was coded as procedural, related to the use of strategies and heuristics.

Results

The control actions, actions participants took immediately following the access of their metacognitive knowledge, included the following: Re-reading the prompt, Reading Aloud, Trying Examples, Conjecturing, Evaluating Argument, Evaluating Strategy, Intentional Inaction, Taking Notes, Math Drawing, Using a Tangible Object, Generalizing, Checking Work, Re-working examples, Organizing Proof, Writing Proof, and Re-reading Proof. Each of the participants engaged in most of these control actions at some point over the course of both tasks. Table 1 offers examples of data for a few of the control action codes, the quoted data, and descriptions of the example control actions.

Table 1. A table of a few demonstrated control actions participants took (Sparks, 2025).

Metacognitive Control Process Actions During Proof Construction		
Control Actions	Description	Example Transcript Data
Re-Working	Re-visiting examples for understanding (not for verification)	"wanting to go back and run through the logic again, remembering what I was trying to prove"
Evaluate Strategy	Actively considering the value or effectiveness of their current approach	"I don't know if this general tree thing is ever going to end…Any pattern…has probably already showed up and we just didn't notice it. Let's go back to actual numbers."
Organizing Proof	Determining what will be included in the proof or the order	"I don't know, the most efficient way to write this? Let's hit let's write out the tree and then. Yeah, so B can give two…"
Writing Proof	Deciding to move from "scratch work" to writing down a proof or part of a proof	"…I write chaos all over the place on a page just to give myself a sense of what's going on…then I…write it out using more formal language and a series of steps that you can follow…"

Each of the participants had their differences in what types of metacognitive knowledge or processes they demonstrated. Lola often accessed declarative metacognitive knowledge about herself and her proof skills. She also was frequently able to access conditional metacognitive knowledge regarding when and why certain strategy uses might be beneficial in the proof process. Lola also regularly demonstrated evidence of monitoring, reflecting on external outcomes and using that information to inform strategy use or control actions. Harper consistently accessed procedural metacognitive knowledge, using knowledge of strategies and heuristics to initiate control actions.

Riley primarily indicated procedural and conditional metacognitive knowledge and consistently initiated the control actions of evaluating strategies and re-working examples. Spencer indicated declarative metacognitive knowledge, referencing his own perception of his strengths, weaknesses, or affect related to the task completion.

Discussion

There were several noteworthy results that came up during this data analysis. Because mathematical proofs constitute their own genre of writing (Selden & Selden, 2013) one of the insights this research provides is the connection between the access of metacognitive knowledge and the subsequent control actions regarding writing specific components of the proof construction process. The initiation of formal proof writing and the organization of proof structures both were impacted by students' metacognitive knowledge. In particular, in the majority of instances of these proof writing related control actions, students accessed procedural

metacognitive knowledge prior to taking these actions. Participants also made declarative metacognitive knowledge statements that also preceded writing proof and organizing proof related actions, often related to knowing at what point in their proof process they feel has worked well for them to start these components of a task.

In addition to the writing specific aspects of proof construction, another significant result is the control action of re-working examples. Students often accessed metacognitive knowledge before choosing to re-do examples. In this case, not re-doing examples to check for error, but instead to identify reasoning and patterns useful for translating to a more general argument. This is an aspect of these participants' proving process that did does not necessarily make an appearance in other forms of problem solving. While understanding or explaining reasoning is a common aspect of problem solving, the need to structure a generalized argument is not necessarily an expectation or necessity of other forms of problem solving. This data indicates that students are accessing and utilizing metacognitive knowledge to inform the decision to rework examples for this purpose. As this research continues, the types of metacognitive knowledge accessed before, during, and after re-working examples should be explored further.

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Theoretical Reports

A Conceptual Framework of Emergent Utility of Advanced Mathematics for Inservice Teachers

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In this theoretical report, we present a conceptual framework that details inservice teachers' emergent utility perspectives towards advanced mathematics and their advanced mathematics coursework. We leverage a river journey metaphor to help illustrate aspects of our conceptual framework. Along the river are utility points that refer to opportunities for teachers to reflect on the usefulness and relevance of advanced mathematics content in relation to its impact on their teaching. Then, we provide examples of both preservice and inservice teachers from extant literature and an initial study conducted in a real analysis course to highlight the four main components of the conceptual framework. We end with implications for designers aligned with the instructional design theory of Realistic Mathematics Education for creating courses for secondary mathematics teachers.

Keywords: Emergent Utility, Inservice Teachers, Emergent Modeling, Teacher Education

Although inservice and preservice teachers have questioned the usefulness or relevance of upper division or advanced mathematics courses for their professional teaching (e.g., Cofer, 2015; Wasserman, 2017; Zazkis & Leikin, 2010), there is growing research that demonstrates how mathematics teacher educators may structure these courses to support teachers in seeing the utility of these courses for their own learning and teaching (e.g., Cruz et al., 2024; Goodchild et al., 2021; Qiu et al., 2025; Wasserman et al., 2023). For example, one common direction is the inclusion of direct content connections between advanced content and content relevant to secondary mathematics. A real analysis connection (Wasserman et al., 2019) could be the concept of continuity as seen in secondary school and a differential equations connection (Apkarian et al., 2023) could be the concept of function. Additionally, there are non-content connections that can be made such as relating experiences in mathematical practices between secondary and university mathematics (Cruz et al., 2024). Collectively, this body of research aims to address the double discontinuity first introduced by Klein (1932, 2016). Our work builds alongside this work, yet, what motivated us is slightly different. Inservice teachers take advanced mathematics courses to pursue graduate degrees, which means they bring experience and knowledge of secondary school mathematics and students to their own graduate learning. More specifically, drawing on Klein, inservice teachers may experience this double discontinuity anew.

The purpose of this theoretical contribution is to call attention to the differences between how inservice and preservice teachers view the relevance of advanced mathematics courses. We propose a conceptual framework that has potential applications for design aligned with Realistic Mathematics Education (RME; Freudenthal, 1973) as an instructional design theory and the concept of emergent modeling. This conceptual framework describes how inservice or preservice teachers (IPSTs) may progress towards seeing how their advanced coursework could have an impact on their teaching practice (i.e., the utility of their courses for their teaching). While both inservice and preservice teachers are important for the conceptual framework, and as such will be discussed, our contribution focuses on inservice teachers. At first, we consider IPSTs together as one entity, because both inservice and preservice teachers have preconceived notions about advanced mathematics. However, the experiences that inservice and preservice teachers bring to

their advanced mathematics courses may create fundamental differences in how they engage with the coursework. Thus, we distinguish between each population by discussing both existing literature regarding the support of preservice teachers in making connections and using our preliminary data from inservice teachers.

To illustrate our emergent utility conceptual framework, we draw on the river journey metaphor from Rasmussen & Keene (2019) on waypoints in one's mathematical learning. Rasmussen and Keene (2019) think of waypoints as opportunities for students to engage and connect various ways of reasoning with mathematical concepts (p. 2). These waypoints are content based. We, however, take a different perspective on waypoints. We refer to utility points as opportunities to reflect on the utility of content in relation to its impact on one's teaching. That impact may be different for inservice and preservice teachers. To describe our emergent utility conceptual framework in detail we first situate ourselves in our larger theoretical lens. We then describe the framework using the river journey metaphor and provide examples using an example from existing literature and from empirical data we have collected.

Theoretical Background

We adopt the Emergent Perspective as an interpretive framework to make sense of both students' conceptions of individuals' mathematics ideas and socially embedded classroom practices (Cobb & Yackel, 1996). By adopting the Emergent Perspective, we conceptualize learning as a coordination between students' cognitive development and engagement in mathematical activity. Inservice teachers enrolled in advanced mathematics coursework bring their beliefs about themselves as an instructor, their students, the mathematics they teach, and previously used pedagogical practices. Inservice teachers' previous pedagogical experiences must be considered as a lens through which they consider the relevance of advanced mathematics coursework. Though this framework describes inservice teachers' perspectives on the utility of advanced mathematics coursework, we conjecture it is still viable as a design framework for recommendations about designing courses for preservice teachers. Although preservice teachers may not have experience teaching in a classroom, their perspectives about advanced mathematics coursework might be informed by their beliefs and opinions about their future students (Scherer & Bertram, 2024) or their perception of their mathematical ability (Beisly et al., 2024).

To describe our conceptual framework, we use the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1973). In RME, mathematics is a human activity. Then, the design heuristics of RME support instructors in developing materials that guide students in engaging in mathematical activity. This initial activity can be connected, with help from the instructor as a brokering agent, to formal mathematical logical structures as they are realized by the extant community of mathematicians (Rasmussen et al., 2005). This transition from informal to formal reasoning is directly related to the RME design heuristic of emergent modeling. We leverage this heuristic in our proposed framework. Providing IPSTs with opportunities to develop more formal ways of reasoning may guide them in making instructional moves that will help connect their own students' informal ways of reasoning towards the more formal. From the lens of RME, designers create mathematical tasks intended to elicit students' informal activities. These initial ideas might include sketches, graphs, input-output tables, etc. Students' initial activity forms the foundation for generating ideas and can be folded into classroom discussion by the instructor to become a model of more formal mathematics. As students gain experience working in this experientially real setting, their attention may begin to shift away from the experientially real setting towards mathematical patterns and relationships. Once these mathematical relationships enter the classroom discourse, they can be used by the

instructor to connect to the more formal mathematical ideas that they have in mind. Once the students' model(s) become a way to reason about mathematical relationships, as opposed to a way to represent the applied problem, the student uses it as a model for more formal mathematics (Rasmussen et al., 2005; Zandieh & Rasmussen, 2010).

In the next section, we introduce the notions of an IPST's utility of and utility for, advanced mathematics content and later provide examples of the four emergent utility perspectives of our conceptual framework. It should be noted that we view teachers' emergent utility as inextricably linked to the emergent modeling heuristic. As inservice teachers develop new ways of reasoning about advanced mathematics content, it is likely that the ways they perceive the relevance of this content are informed by the lens of their own teaching experiences.

Emergent Utility Conceptual Framework

To help illustrate our framework, we rely on the river journey metaphor (Rasmussen & Keene, 2019). The river metaphor is intended to convey students' non-linear progression through mathematical ideas that may require more or less time based on their previous instructional experiences and current mathematical reasoning. The journey is non-linear in the sense that some students might return to prior waypoints, spend more time in certain areas of the river, or branch off to explore aspects of the river they find interesting. More specifically, we conceptualize the specific ways in which teachers reason about the utility they continue along the river journey as key utility points. We do not comment on the exact nature of the content waypoints, as they are intended to be left general for the sake of encompassing any advanced mathematics course in which IPSTs might enroll. We also want to mention that the amount of time one spends in the river investigating each utility point will vary for each individual and might be impacted by external factors (e.g., shift in the river's features representing a change in the course structure).

When they first enter the river, IPSTs may have preconceived notions about whether or not the course will be helpful for their teaching, as indicated by the starting point in Figure 1. This could be based on the IPST's perception of students' mathematical ability, or their general impression of the advanced mathematics content. At first, IPSTs might perceive the utility of the advanced mathematics coursework as developing their understanding of related course content, but ultimately view the course content as irrelevant for their teaching. Thus, they may choose not to spend as much time at the content waypoint or utility points because they do not view the concepts as relevant for their teaching. Such an IPST develops a utility of a model of perspective towards the advanced mathematics content, as pictured by the top, blue, dotted line in Figure 1.

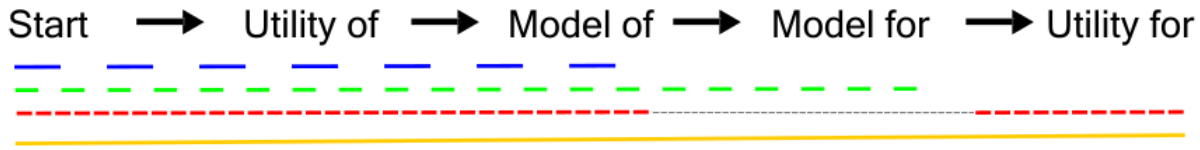

Figure 1. The Emergent Utility Conceptual Framework.

We view advanced mathematics courses as a river filled with opportunities, that some students may perceive as hurdles (e.g., rapids or rocks), to engage in iterative emergent modeling processes where one can spend more or less time building models for the relevant content waypoints and utility points. As IPSTs continue to build models of and for advanced mathematics content, they require opportunities to reflect on their learning in terms of its

relevance to teaching secondary mathematics, that is, time to spend at each utility point. Some IPSTs may come to develop models for a majority of the course content, but decide that the time they themselves spent exploring the river would take too long, or be too challenging for their own students. Such a teacher has developed a utility of a model for perspective, as pictured by the second, green, dotted line in Figure 1.

As the IPSTs begin to spend more time at each utility point, they begin to reason about the broader mathematical context and how the content they teach is situated within it. Here, inservice and preservice teachers likely have different knowledge around 'the content they teach' - an inservice teacher is returning to the river and their journey is now informed by previous experiences as a teacher. In other words, they themselves have guided students down a river. It may take more or less time for each individual teacher to reflect on the relevance of the course content to their teacher to the point they are willing to make shifts in their teaching. Eventually, with enough time at each waypoint, the IPST might come to envision changes in their pedagogical decisions about content they will, or have previously taught, that they now realize is directly related to the advanced mathematics content they are learning about. Such an IPST has begun to perceive the utility for advanced mathematics content to impact their teaching practice. In other words, IPSTs are developing an emergent model of/for their own teaching, as it is informed by their new models for advanced mathematics content. In this way, IPSTs are making connections between their formal ways of reasoning about course content and using these connections as a new pedagogical lens to make decisions about their own instruction. However, some IPSTs may not have enough time to spend traversing the river and discovering or reinventing all of the content waypoints but still believe that the course is relevant for their teaching. These IPSTs develop a model of the utility for advanced mathematics content, as pictured by the third, red, dotted line in Figure 1. Such IPSTs may struggle to provide explicit examples of which advanced mathematics content will shift their teaching in specific ways. Despite this, such IPSTs still indicate the impact of the course content on them as educators, which is why the red dotted line is broken by the smaller gray portion in Figure 1.

The IPSTs who develop a model for the utility for advanced mathematics have a new, or renewed, vision of how their formal ways of reasoning about advanced mathematics content will shift their teaching and can provide explicit examples. Such IPSTs have spent enough time at each content waypoint and utility point to reflect on the relationship between their formal ways of reasoning about the advanced mathematics content to determine ways in which this reasoning will shift their teaching practices. This is pictured below in Figure 1 by the unbroken, bottom yellow line. More specifically, an IPST who develops a model for the utility for advanced mathematics content is making new, or different, decisions about course content, pedagogical moves, and curricular innovations in their secondary mathematics classroom.

It should be noted that, for preservice teachers, this may be their first journey down a specific river. For inservice teachers, this may be their second, third, or fourth journey down a specific river. Revisiting the content waypoints additional times, visiting utility points with more teaching experiences, and coordinating this time down additional river journeys may impact the amount of time a specific teacher spends before developing an emergent utility perspective.

Examples of Emergent Utility Perspectives From Literature on Preservice Teachers

First, we consider the results of Liang et al. (2023). The three preservice teachers (William, Maria, and Sterling) were each interviewed twice about their transition to university math positioned as students (first interview) and future teaching practices, using inverse functions as the example topic (second interview). We begin with William because while he had a very

positive view about his experiences in advanced tertiary mathematics courses, he continually referred to secondary mathematics as "copycat" math. This was because, according to William, secondary students tend to mimic what the teacher does to solve math problems. This, from William's point of view, is not sufficient when it comes to the learning of mathematical proof at university. William's overall view of tertiary mathematics was that it was fun and interesting for him, but was reserved for individuals who developed a "shield" to survive a transition to proof course to get to the more interesting coursework. This belief about his potential students presupposes an approach to his secondary teaching (forcing students to "use their brains"), but he also appeared to view inverse functions solely as applicable for solving equations. A preservice teacher like William would have a utility of a model for (green line) advanced proof-based content with teaching the concept of inverse functions to his future secondary students. This is because while he has some ideas about how his tertiary mathematics experiences will influence his teaching; he did not appear to leverage his experiences in determining concrete ways to shift his future teacher practice.

Because of her experiences in secondary instructional settings, Maria conceptualized secondary classrooms as competitive (p. 116). She felt pressure from her teachers and instructional settings such as standardized exams, and her impression was that she had to practice problems repeatedly without a strong conceptual understanding in order to perform well (e.g., receive a high grade). However, Maria summarized her experience in tertiary math classrooms as exciting, and recommended to others to "dig deep". Though Maria did not identify herself as a "successful" tertiary mathematics student, she valued the notion of introducing secondary students to proofs that extend beyond the typical geometrically focused examples. A preservice teacher like Maria would have a model of the utility for (red line) logical proof techniques in secondary mathematics classrooms. This is because though Maria was open to the idea of including her experiences in tertiary math classrooms in her secondary teaching, she appeared to require time to "dig deeper" in her math studies and required more opportunities to reflect on ways in which more formal ways of reasoning might shift her future secondary teaching.

When asked to reflect on his experience transitioning from secondary to tertiary mathematics courses, Sterling focused on if and only if, or bidirectional, proofs because he initially struggled to make sense of them. Sterling then provided an example to illustrate how he would support his future secondary students in making sense of these types of statements (p. 119). This example, which involved creating if-then statements about the solutions to the equation x^2=2, illustrates an instance when reversing a true statement does not always produce a new true statement. By providing a non-example of a bidirectional statement to his imagined future students, Sterling was supporting them in making connections he himself struggled with in the transition from secondary to tertiary mathematics. The fact that Sterling was able to reframe an example of a secondary mathematics concept through the lens of his learning in tertiary mathematics contexts, coupled with claims that Sterling possessed a strong background in mathematics at the tertiary level, indicate that Sterling had developed a model for the utility for (yellow line) proof-based content (specifically bidirectional proofs) with teaching the concept of inverse functions.

Examples of Emergent Utility Perspectives From Inservice Teachers

We continue to illustrate our framework by presenting four examples from a study conducted with 10 inservice teachers enrolled in a graduate real analysis course. The following quotes are responses from reflections about the Fundamental Theorem of Calculus (FTC) and continuity and differentiability, with emphasis added.

When responding to a reflection about understanding the fundamental theorem of calculus as related to the concepts of rate of change and accumulation, Demi stated the following:

> *Demi*: Referring to the FTC in this manner would be interesting to a certain level of student. However, to a run of the mill calculus student who is just trying to master the skill of taking the integral I believe would overwhelm them. I would not show this until they were comfortable with the skill or with great explanation beforehand.

This demonstrates that Demi views the relevance of this real analysis content through their own teaching experiences and beliefs about "a certain level" of student. Demi contends that the formal aspects of the FTC would be more difficult than needed in their own instructional context. Demi views the utility of this advanced mathematics through her own teaching experience and appears to believe that because the content was challenging, it means that it would "overwhelm" their students. Such a teacher would possess a utility of a model of (blue line) perspective as her beliefs about her students appeared to mediate her willingness to engage with content waypoints and utility points throughout the course.

When responding to a reflection about comparing the formal, epsilon-delta definition of continuity versus a more informal definition, Eloise stated the following:

> *Eloise*: Honestly, when first completing the homework, I was unsure I really liked/approved of the analysis definition of continuity. However, after doing more problems and reading more about it, it has started to make more sense to me. I find it very interesting since I have never seen continuity defined this way but am starting to appreciate it because I enjoy having a deeper understanding of calculus and it's cool to continue to expand that understanding more still. I will say however that I do not think I will actually change the definition I use to teach continuity … the analysis definition is harder to understand especially for a first-time calculus student.

As Eloise self-describes that she understands the formal analysis definition yet indicates a lack of usefulness in their own teaching due to curricular and testing tensions. Throughout the course, Eloise demonstrated reasoning about continuity that was highly interconnected with other course content but ultimately decided that the reasoning they developed was "harder to understand" for their students and therefore they would not "change the definition" they teach. Whereas Demi's own reasoning about the reciprocal process of differentiation and integration mediated their perception of the level of difficulty of the content and its relevance for their students, Eloise made connections between the course content and their prior teaching practices but still decided to not make changes to the way they teach the concept of continuous functions to secondary students. This teacher possesses a utility of model for (green line) perspective.

When discussing the possibility of teaching derivatives Hannah thought about it "in a variety of ways so the concept can be fully understood." In this way, Hannah appreciates a conceptually focused approach to introducing students to the notion of differentiability. However, Hannah also believes that it would be "less confusing if you did it by practicing finding and taking derivatives." It appears that even though Hannah is open to the impact that advanced mathematics courses could have on conceptual understanding, the connections they made between "finding and taking derivatives" as a process and the relationship between this process and the notion of differentiability are seemingly disparate. Because of this disconnect, such a teacher who is open to changing their teaching practice based on their renewed understanding of the content, but the ways in which these instructional changes can occur is mediated by their own reasoning, would possess a model of the utility for (red line) differentiability as it relates to teaching derivatives to secondary students.

In the same reflection assignment as Eloise, Jocelyn responded:

> *Jocelyn*: Prior to taking this class, I know that I took a Real Analysis class in my undergraduate mathematics classes, but I fell back into the thought that a continuous function is just when you can draw the graph of the function without picking up your pencil. This is the way I have taught my students up until this school year, which is, I believe, on the level of their understanding. But seeing how this idea is used for differentiability makes me understand that this surface level of understanding does not prepare students for what they will see in calculus. I am able to see the connection between continuity and differentiability, which helps me understand both concepts on a deeper level.

It should be noted that Jocelyn was one of the few teachers with calculus teaching experience. A teacher like Jocelyn, who uses formal ways of reasoning to make shifts in their teaching practices has constructed a model for the utility for (yellow line) continuity as it relates to teaching AP Calculus content.

Conclusion

The nature of the transitions between the previously discussed emergent utility perspectives requires further attention and will likely change depending on the individual IPST. However, the purpose of this paper is to set a theoretical basis to inform researcher-designers about the nature of teachers' emergent utility towards advanced mathematics coursework. Inservice and preservice teachers engage with utility points differently. Likely, preservice teachers simultaneously engage with the advanced mathematics content for the first time and have not yet had opportunities to engage with students as a teacher. Inservice teachers have an additional challenge. Their teaching experiences informs the lens through which they view the utility of advanced mathematics coursework, and it could be the case they have a harder time making shifts in their teaching based on formal ways of reasoning about advanced mathematics content.

The ways in which an IPST navigates the river will be dependent on their previous experiences and beliefs. An inservice teacher will likely view the relevance of advanced mathematics content through actual teaching experience (e.g., Demi and Jocelyn) and therefore envision changes to previously implemented curricular materials. A preservice teacher (like William, Maria, and Sterling) can provide imagined examples of activities to include, or problems to address, but these are not considered in light of other issues that emerge from the actual practice of teaching, course creation, implementation, and classroom management. Preservice teachers are leveraging their own formal ways of reasoning and making decisions about whether it applies to imagined students, whereas inservice teachers are levering experiences with actual students and determining whether their own formal ways of reasoning are approachable to their instructional context. Because inservice teachers are likely engaging with the advanced mathematics content in a graduate level course, their new river journey can now be viewed through the lens of both previous instructional experiences as well as their prior teaching. This is what we meant earlier when we said that inservice teachers may be able to spend more time or make it further along the river in their journey. As such, designers must keep this in mind when determining whether certain activities are appropriate for preservice teachers, inservice teachers, or both.

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Conceptions of Derivatives in Calculus: Merging and Extending Earlier Frameworks

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Researchers studying derivative conceptions often use Zandieh's (2000) framework, which emphasizes graphical, symbolic, verbal, and physical representations within three process-object layers of ratio, limit, and function. While foundational and aligned with traditional calculus instruction, recent literature on derivative teaching reveals additional conceptions. I present the Conceptions of Derivative (CoD) framework, which expands Zandieh's work to offer a more comprehensive perspective on contextual and cognitive processes when thinking about derivatives. The framework provides researchers with a comprehensive lens, while offering instructors awareness and language for intentional targeting of derivative conceptions based on their students' conceptual needs.

Keywords: Derivatives, calculus, conceptions, representations

When considering various conceptions of derivative in calculus, many researchers rely on Zandieh's (2000) framework or its extended versions (e.g., Roundy et al., 2015). As a cognitive framework for learning derivatives, Zandieh's framework has also shaped the research on derivative teaching (e.g., Hitier & González-Martín, 2022) possibly due to its resonance with instructional and curricular approaches to derivatives (Jones & Watson, 2017).

However, in my research on eight experienced college calculus instructors' derivative tasks (Gerami, 2023, 2024), I encountered conceptions beyond those captured by Zandieh's (2000) framework. This motivated the development of an updated framework—*Conceptions of Derivative (CoD)*—to accommodate both my empirical findings and more recent theoretical perspectives on other approaches to thinking about derivatives. This report begins with reviewing Zandieh's (2000) framework and its extended versions, followed by a discussion of its limitations. I then present the extended CoD framework.

Zandieh's (2000) Framework and Its Extensions

In a foundational piece, Zandieh (2000) constructed a theoretical cognitive model for students' understanding of the concept of derivative. For Zandieh, an individual's understanding of derivative is defined "in terms of its overlap or lack of overlap with the mathematical community's notion of that concept" (Zandieh, 2000, p. 104). As shown in

Table 1, the framework is organized by four columns, called contexts/representations, and three rows, called process-object layers. Therefore, each cell in the framework refers to a conception of derivative within a representation or context, and a process-object layer.

Table 1. Zandieh's (2000) framework for the conceptions of derivative

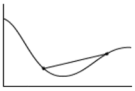
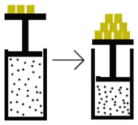
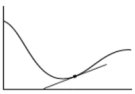
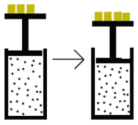
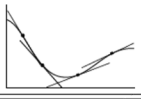
		Representations/Contexts				
		Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Difference Quotient)	Other
Process-Object Layer	Ratio					
	Limit					
	Function					

Defining representation using the notion of a person's different images of the same concept (i.e., concept image; Tall & Vinner, 1981; Vinner & Dreyfus, 1989), she captured four mathematical representations for the concept of derivative: graphical (slope), physical (velocity), verbal (rate of change), and symbolic (difference quotient). Zandieh (2000) also included an "Other" column within the representations/contexts to capture representations that may appear different but are structurally similar. In an example, she noted that rate of change of temperature is another physical manifestation of derivative, like velocity, stating that "Many other physical examples are possible, and there are variations possible in the graphical, verbal, and symbolic descriptions" (p. 105). Nonetheless, for her, these *other* representations within the process-object framework still describe the similar "underlying structure" of the same concept of derivative.

For the rows, Zandieh (2000) used the term "process-object" from Sfard's (1991, 1992) work to capture the duality of some mathematical concepts: processes acting on previously defined objects, and objects as a result of other processes acting on them. To refer to different process-objects within the concept of derivatives in her framework, Zandieh introduced the term "layers" to capture the hierarchical structure of the three process-object rows, as each layer is found by using that layer as a process over the previous layer as an object. The three layers in the framework are ratio, limit, and function. The first layer, ratio, is the object of the average rate of change between two points, which can be represented graphically as a secant slope, average velocity between two times, or symbolically as $\frac{f(x_0+h)-f(x_0)}{h}$ or $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. To find the next layer of derivative—the *limit* layer, we hold the ratio layer as an object and use the *limit* layer as a process on it to decrease h as the horizontal distance between the two points. This gives us the limit of the ratio layer as an object: $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, referred to as $f'(x_0)$ or the derivative of the function at a point. Finally, at the final layer and via the functioning process, we find the instantaneous rates of change at every possible point on the function. This results in the derivative as a function: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Facing challenges to capture all the ways that university students made use of their understanding of the derivative to solve derivative problems, Likwambe and Christiansen (2008) added the *instrumental* representation to capture the "only learning exhibited" by most of their university student interviewees (p. 24). To define this new representation, Likwambe and Christiansen relied on Skemp's (1978) and Lithner's (2003) notion of instrumental understanding as the ability to use a rule or procedure, while specifying it for finding derivatives using *derivative rules*. Likwambe and Christiansen further argued that the instrumental representation is its own category because it does not have any layers (i.e., it does not 'live' in any one specific layer of Zandieh's [2000] framework).

Later, Roundy and colleagues (2015) built on Likwambe and Christiansen's work to further extend Zandieh's (2000) framework by: 1) changing velocity—the physical representation's example—to *measurement* to allow for changes in quantities other than time; 2) adding the *numerical* representation, as calculated values for ratio of changes at the ratio layer and limit layer (for small Δx values), and sequence of numerical ratios of differences at the function layer; and 3) categorizing instrumental representation outside of the rest of the framework as a symbolic representation within the function layer, because using "rules for symbolic derivatives does not require a conceptual understanding of the derivative" (p. 923) and the derivative rules only exist at the function layer. Figure 1 shows this extended framework.

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2-y_1}{x_2-x_1}$ numerically	
Limit		"instantaneous ..."	$\lim_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \cdots$	... depends on x	tedious repetition

	Symbolic
	Instrumental Understanding
Function	*rules to "take a derivative"*

Figure 1. Roundy and colleagues' (2015) extended framework.

Limitations and Resolutions

While applying Zandieh's (2000) framework to analyze eight experienced college calculus instructors' derivative tasks (Gerami, 2023, 2024), several limitations emerged that motivated the development of an extended framework.

Limitation 1: Placing Non-Limit Approaches

Some instructors in my study conceptualized tangent lines through approaches fundamentally different from the traditional limit-based method. Rather than finding tangents by pushing two points together via limits, they used infinitesimal reasoning—zooming in on a function until it appears flat. This approach represents a distinct epistemological perspective rooted in infinitesimal and differential calculus (Ely, 2021; Keisler, 2011; Samuels, 2017), not merely a different graphical representation of limits.

Similarly, while Roundy et al. (2015) appropriately included instrumental understanding (derivative rules) in their framework, as many students use them to think about derivatives, they positioned it within the limit-based structure. However, students using derivative rules do not conceptualize the rules through process-object layers from ratio to function; it is my understanding as a calculus instructor and mathematics education researcher that they bypass this conceptual development entirely. This suggests instrumental approaches should be placed in a separate epistemological category. Moreover, while Roundy et al. added instrumental understanding within the function layer, I also knew that one could use derivative rules to also find derivative at a point as well as "average derivative" or average rate of change between two points.

These observations revealed three distinct *Conceptual Epistemologies* for understanding derivatives: limit-based, infinitesimal/differential-based, and instrumental/rule-based. Each represents a fundamentally different way of conceptualizing how they are derived.

Limitation 2: Conflating Context and Representation

I also had difficulty with the representation columns *Physical* and *Verbal*. It seemed to me that the physical representation conflates mathematical representation with contextual application. Physics problems are not a way of representing derivatives mathematically; they are contexts where derivatives can be applied. Similarly, while Roundy et al.'s (2015) change to *measurement* addressed some concerns, the distinction between their measurement and numerical columns remained unclear. Moreover, one could use derivatives on problems regarding change within other fields, such as physics, biology, chemistry, and economics.

To resolve this issue, I separated contextual applications into a distinct component called *Contextual Framing*—the context in which derivatives are used. This resolution would also recognize that any epistemological approach can be applied across various contexts.

Limitation 3: Verbal "Representation" as Conceptual Signals

Lastly, my research (Gerami, 2024) suggested that the verbal representations directed instructors toward other derivative conceptions. Rather than being representations themselves, these verbal descriptions serve as signals to other ways for conceptualizing derivatives, which can be different for different people. To address this limitation, I drew on semiotics (de Saussure, 1959) to distinguish between the *signified*, or the mathematical representation itself, and the *signifier*, or the verbal label often used to reference it. This separation clarifies how language can guide conceptual understanding while maintaining mathematical representation as the focus.

The Conceptions of Derivative (CoD) Framework

The Conceptions of Derivative (CoD) framework addresses the limitations discussed above by reorganizing how we conceptualize derivatives using four distinct but interconnected components that capture the multifaceted nature of derivatives (Figure 2).

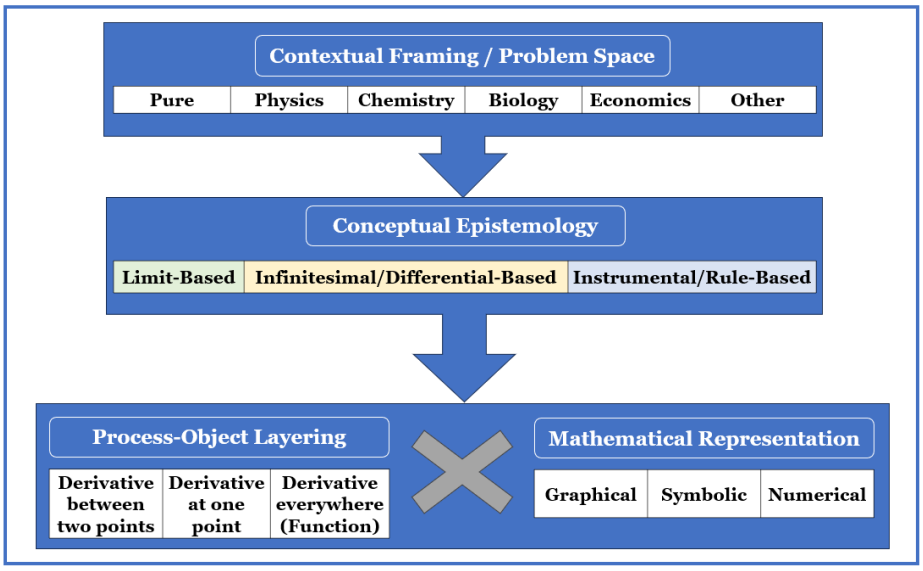


Figure 2. The four main components of the Conceptions of Derivative (CoD) framework

Contextual Framing is the context through which we frame or interpret derivative problems. Common contexts in calculus are pure mathematics, physics, chemistry, biology, and economics.

Conceptual Epistemology captures fundamentally different approaches to understanding what derivatives are and how they are derived. The framework identifies three distinct epistemologies:

limit-based (traditional approach using limits of difference quotients), infinitesimal/differential-based (using infinitesimal quantities or differential reasoning), and instrumental/rule-based (procedural approach using derivative rules).

Mathematical Representations encompass the three primary ways derivatives can be expressed mathematically: graphical (visual representations), symbolic (algebraic expressions), and numerical (calculated values). Following semiotic principles, each representation is understood through both its *signified* (the mathematical object) and *signifier* (the verbal label used to reference it).

Process-Object Layering maintains the hierarchical structure of derivatives from Zandieh's (2000) framework but adapts it to work across different epistemologies: 1) between two points (average derivative/rate of change), 2) at a point (instantaneous derivative), and 3) everywhere (derivative function). Unlike Zandieh's framework where layers are tied to limit processes, these layers can be conceptualized through any of the three epistemological approaches.

The comprehensive framework is shown in Figure 3. As the arrows show, the framework suggests that one encounters a derivative problem within a particular contextual framing. They then approach the derivatives via a specific conceptual epistemology. Within that epistemology, they can work across three layers using various mathematical representations, with verbal signifiers guiding their selection and interpretation of representations. I next explain the layers and representations within each conceptual epistemology.

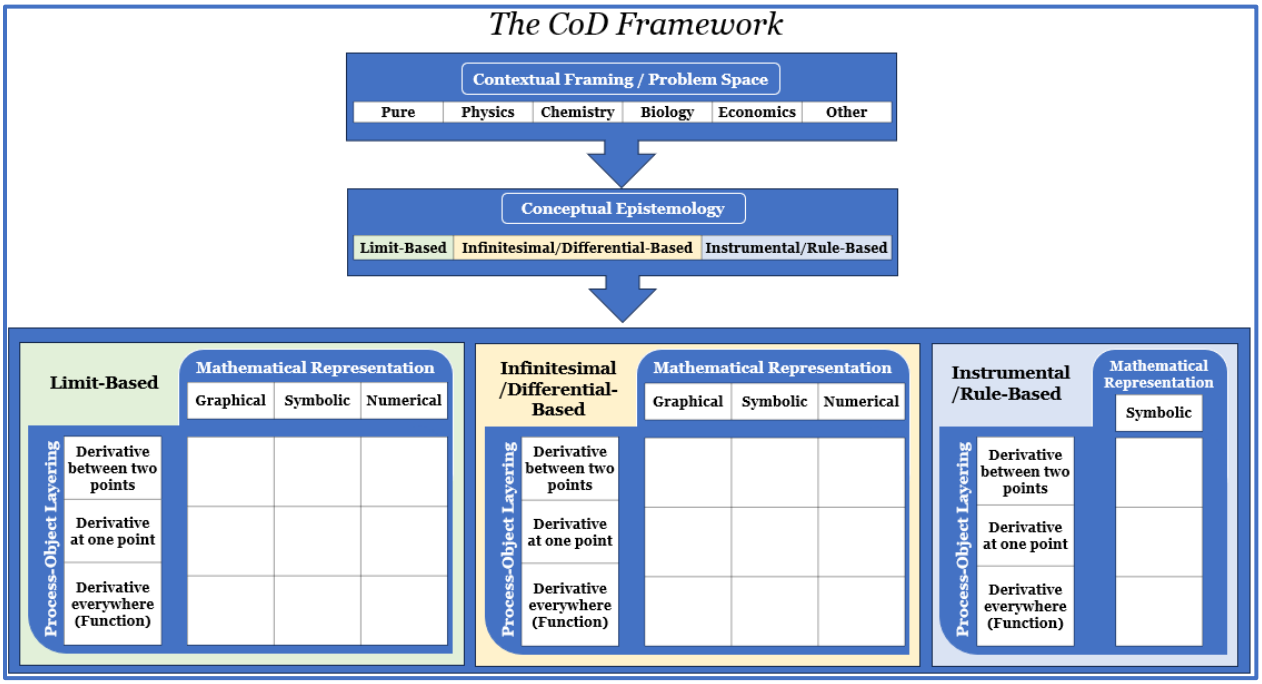


Figure 3. Conceptions of Derivative (CoD) framework

The limit-based conceptions in Figure 4 closely follow Zandieh's (2000) framework, where derivatives emerge through limiting processes. As I have reviewed these above, I do not repeat the details here. Signifiers like *secant/tangent slopes*, *difference quotient*, and *rate of change* are often used to direct attention to specific representations within each layer.

Limit-Based

Process-Object Layers	Representations: Graphical – Signified	Graphical – Signifier	Symbolic – Signified	Symbolic – Signifier	Numerical – Signified	Numerical – Signifier
Ratio		Slope of secant line	$\frac{f(x_0+h)-f(x_0)}{h}$, $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, or $\frac{f(a)-f(b)}{a-b}$	Difference quotient	Secant slope as a number	Slope value
Limit		Slope of Tangent line	$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, or $\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$	Limit of the difference quotient	Tangent slope as a number	Table of tangent slope values
Function		Graph of the derivative function	$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, or $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$	Derivative as a limit	Tangent slopes as numbers	A list of slopes for "all" points

Figure 4. The limit-based conceptions of CoD

Infinitesimal/Differential-Based

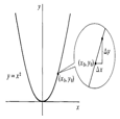
Process-Object Layers	Representations: Graphical – Signified	Graphical – Signifier	Symbolic Infinitesimal – Signified	Symbolic Infinitesimal – Signifier	Symbolic Differential – Signified	Symbolic Differential – Signifier	Numerical – Signified	Numerical – Signifier
Ratio		Slope of a line through two points	$\frac{\Delta y}{\Delta x}$	Slope on a line through two points or "real rate"	$\frac{\Delta y}{\Delta x}$	Slope on a line through two points	Slope as a number	Slope value
Zoom		Local slope	$st(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x})$ The real number infinitely close to the answer for every non-zero infinitesimal Δx	Infinitesimal rate	$\frac{dy}{dx}$ *at* x_0 *Or* $\frac{f(x_0+dx)-f(x_0)}{dx}$	A quotient or a differential equation $dy=[\blacksquare]dx$ *at* x_0	Local slope as a number	Local slope value
Function		All local slopes	$st(\frac{f(x+\Delta x)-f(x)}{\Delta x})$ for every x and every non-zero infinitesimal Δx	The derivative function	dy/dx *Or* $\frac{f(x+dx)-f(x)}{dx}$	The derivative f' or a quotient or a differential equation $dy=[\blacksquare]dx$	Local slopes as numbers	A list of local slopes for "all" points

Figure 5. The infinitesimal/differential-based conceptions of CoD

The process for the infinitesimal- or differential-based conceptions in Figure 5 is 'zooming in.' At the ratio layer, derivatives are real rates of change between two points ($\frac{\Delta y}{\Delta x}$). At the "Zoom" or derivative at a point layer, the rate becomes infinitesimal ($\frac{dy}{dx}$); graphically, this involves zooming in until the function appears linear and one can find the slope. While I cannot provide details here due to space constraints, the infinitesimal approach uses the standard portion

of hyperreal numbers, whereas the differential approach uses first-order approximations, so they differ symbolically (see Ely, 2021; Keisler, 2011). The function or everywhere layer extends zooming process across the function's domain. Signifiers like *infinitesimal change* and *differential* are often used to target these conceptions.

Instrumental- or rule-based conceptions in Figure 6 focus on procedural ways of deriving derivatives. Unlike other epistemologies, this approach can be used bottom-up. The everywhere or function layer is found when rules are used to generate derivative functions. The derivative at a point layer can be found by evaluating derivative by 'plugging in' a specific x-value into the derivative function. Although not commonly used, the derivative between two points layer can be found by finding the average of two derivatives from the at a point layer. The symbolic representation and signifiers like *taking derivative* or *using derivative rules* (e.g., power, quotient) are often used for this epistemology.

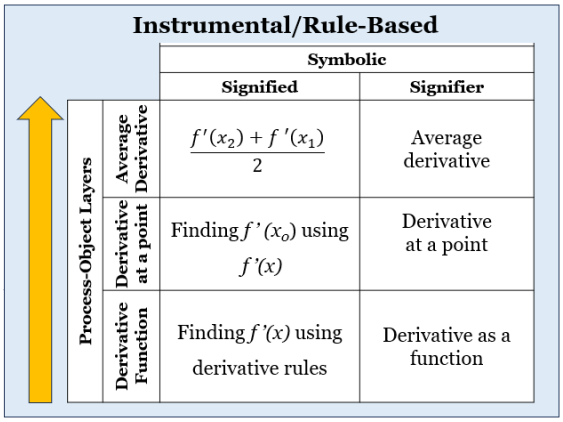


Figure 6. The instrumental/rule-based conceptions of CoD

Conclusion

The Conceptions of Derivative (CoD) framework addresses critical gaps in existing theoretical process-object models by recognizing that derivative understanding encompasses multiple epistemological approaches, not just limit-based reasoning. Rather than privileging a single epistemological perspective, the framework recognizes three distinct conceptual epistemologies—limit-based, infinitesimal/differential-based, and instrumental/rule-based—that can be used within various contextual framings. This offers researchers language to discuss derivative conceptions beyond those in traditional frameworks. Future research can use this framework to investigate how contextual framing influences epistemological choices, how instructors target various conceptions, and how students navigate between different conceptions. Different epistemologies appear to privilege different areas of study. Given that calculus courses are increasingly targeted for specific student audience like biology and business majors, the framework allows instructors and curriculum designers to choose which derivative understanding to emphasize and how.

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If (Critical) Love is an Answer, What is the (Research) Question?

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Research in math education calls upon "love" more and more, but rarely with careful theoretical development. Hence we offer a theory of "critical love" here, drawing upon pedagogy scholarship, activist writing, and queer theory. Critical love calls us beyond a superficiality of knowing in our minds to a deeper knowing in our beings and actions. We elucidate the theory by identifying its essential components, organized along four horizontal frames: change, power, connection, and practice. We conclude with implications of critical love as an orientation in mathematics classrooms, teacher professional development, and pedagogy research.

Keywords: love, critical, embodiment, transformation, queer

In this report we develop a theory of critical love, which we offer as a theoretical framework for our community's exploration of new ways of being as mathematics teachers, researchers, and learners. Critical love calls us beyond a superficiality of knowing in our minds to a deeper knowing in our beings and actions. Indeed, we affirm that "knowing differently reaches its deepest and most meaningful level when we are able to use this knowledge in service of living differently" (Ahluwalia, 2022, p. 313). We offer critical love as a conceptual resource for guiding transformation of knowledge into action. In this way, we endeavor partial work toward Green & Lopez's (2025) recent call for the development of "love as a grounding for STEM education" (p. 2) and Clingan's (2015) older call for us to "consider what might be possible if love were intentionally and specifically identified as a methodology, pedagogical practice, and value in… education" (p. 1). After a review of additional literature, we highlight sources of theoretical motivation – especially queer theory and the *Revolutionary Love Project* – before then moving into the 3 + 1 + 3 + 2 = 9 components of our conceptualization of critical love. We conclude with some implications for practice and potential uses of the theory.

Literature Review

Our suggestion of love as a potentially powerful force in pedagogy is not a new one (Freire, 1970; hooks, 1994; Anderson, 2002). However, in a mathematics culture which values logic and precision (Ernest, 2016), many folks shy away from the term for its perceived emotionality and imprecision. We acknowledge a key peril in the use of ambiguous and weighted terms like love: the potential harm of dual meanings. Johnson et al. (2018) highlight this risk of misnaming, saying that "when teachers embody fake love, it produces this bogus relationship… [that is] pretentious and masquerading" (p. 54). For example, in a study of STEM preservice teachers Green & Lopez (2025) note that "[the preservice teachers] rehearsed the dominant discourses in the United States which work to reproduce social injustice… while [also] giving strong commitments to social justice in terms of radical love" (p. 22). Rather, Johnson et al. (2018) call for a "practice of humanizing love… that is bounded in action" (p.48), a call echoed by social justice educator Brooks (2017) who states that "a critical theory of love… reunite[s] and rehumanize[s]" (p. 108). Tran and Pierre (2025) argue likewise, drawing upon hooks (2000) to state "when love is oriented purposefully, it becomes an anchor that empowers us to humanize each other" (p. 5). Thus, to love critically must be an act of humanization.

We therefore find resonance between the fact that a growing body of scholarship calls for (re)humanization of mathematics (Rosa & Orey, 2016; Ormseth et al., 2017; Gutiérrez, 2018; Simic-Muller, 2023; Kalinec-Craig, 2025; MacArthur et al., 2025), just as more and more math pedagogy literature calls upon love. In their development of a framework of ethnic studies for mathematics education, for example, Yeh et al. (2021) ground in an "orientation to [radical] love… drawn from activist and critical scholarship" (p. 73). However, they do not develop a deeper formulation of what they mean by "radical love" except to say that it is "interdependent, collective, healing, and nurturing" (p. 73). Likewise, in their "loving critique" of asset-based mathematics pedagogy, Paris & Alim (2014) avoid a theoretical framing of love and only remark that "love" refers to their "position of deep respect" (p. 1). Basile & Azevedo (2022), drawing upon Paris & Alim, also engage in a "loving (self-)critique" of their work as STEM educators, but offer only that their use of "love" reflects a "willingness and courage to deeply interrogate" (p. 1087) and their "patience and forgiveness" (p. 1095) in the process. My colleagues and I likewise engage in loving self-critique of our work in the ACT UP Math project on critical transformation in math education (Black et al., this issue). As such, even as mathematics educators call upon love, they do so without a robust theoretical development. Hence we offer a theory of "critical love" here, drawing upon and synthesizing the wisdom of pedagogy scholarship, activist writing, and queer theory.

Theoretical Framing

Queer theory provides a theoretical framing that: embraces fluidity of selves and identities (Morris, 2000); questions everything, especially things thought of as normal (Rands, 2019); embraces uncertainty (Shlasko, 2005); deconstructs binaries (Fifield &Letts, 2013); holds criticality of power relationships (Mayo & Rodriguez, 2019); and interrogates categorization (Brim & Ghaziani, 2016). For additional discussion, we refer the reader also to Chapters 2 and 3 of Hall's (2003) *Queer Theories*, and Chapter 1 of Browne and Nash's (2010) *Queer Methodologies*. We reference the preceding literature throughout our discussion of critical love. In particular, the *simultaneous holding of countless complexities* binds critical love together (see Table 1), aligned with queer theory's call to "engage with multiple [potentially contradictory] understandings" at once (Shlasko, 2005, p. 129).

We also draw inspiration from the *Revolutionary Love Project* (RLP), led by activist, documentary filmmaker, and faith leader Valerie Kaur. The RLP conceptualizes love as having three orientations: "love for others, love for opponents, and love for self", and teaches this practice of love "backed by research and infused with ancestral wisdom". While we draw upon the literature of the RLP – especially their *Educator's Guide* (2020), prepared by the project's Director of Education, Dr. Melissa Canlas – we organize critical love without reference to the recipient (e.g. self, others) of love. Instead, we focus on love as a force of individual and systemic transformation, which requires action and labor. Indeed, "love is a form of sweet labor" (Canlas, 2020, p. 17) and "love is an action, a participatory emotion" (hooks, 2000, p. 135).

A Theory of Critical Love

We elucidate our theory of critical love by identifying its essential components, organized in Table 1. We tabulate the elements of critical love to offer some conceptual structure and to facilitate the reader's remembering. However, queer theory calls us to hold caution around rigid categorization and hierarchy among the components. Throughout our discussion, therefore, we endeavor to explicate some of the ways that the components are inextricably interwoven.

Table 1. The essential components of critical love, tabulated in relation to one another.

	Individual	link	Systems
Change	Courage	*simultaneous holding of countless complexities*	Envisioning
Power	Humility		Criticality of Power
Connection	Curiosity		Interdependence
Practice	Embodiment Discernment		

Conceptual Core of Critical Love

First, critical love requires ***courage***. Courage empowers us to try, "fail", change, and try again, a cycle necessary for our individual and collective growth. Freire (1998) named the necessity of this cycle in teaching: "It is impossible to teach… without the courage to try a thousand times before giving in" (p. 3). Likewise, the RLP conceptualizes "transition" – or change – as a key practice of love, which they define as " the fiery process that is required to move from one reality into another" (2020, p. 118). Critical love calls us to willingly engage in this "fiery process", which places us in a state of vulnerability and thereby requires courage. As education researcher Ohito wonders, in conversation with colleagues Lyiscott et al. (2021): "Can we be courageous enough to *name* exactly who and what we fear? If we were to unstick from our fear and unfurl our tongues, then what would we feel free/d to say?" (p. 5). We invite the reader to consider what they fear – and why – as they consider the other components of critical love.

Humility forms another essential element of critical love. Again, we draw upon Freire (1985), who wrote "many things that today still appear to me valid… could be outgrown tomorrow, not just by me, but by others as well" (p. 11). Humility requires critical acknowledgement of where we are now, and loving imagination of where we might grow. One perspective might position humility as an antithesis to ego; aligned with queer theory's rejection of binaries, critical love takes a different stance. "The Hebrew word *anavah*… means 'to take up your God-given space in the world.'... [and] is translated into the English word *humility*... [to] take up space and just honor the full vision of who we are" (Lyiscott et al., 2023, p. 262). Critical love, through humility, calls us to continually grow in order to honor the "full vision" of ourselves. Not because we are "not enough", but rather because we each have a deep potential for fullness and the power to influence our world through our beings. Thinking of the classroom, Darder (2017) explicates that "knowledge production is lived out by the teacher through humility and openness… [and] this shift necessitates… the willingness of teachers to take responsibility for what they know" (p. 101). Thus, humility calls us to not only name what we do not know, but also take responsibility for what we do know, ever-aware of continual transformation at the fuzzy and amorphous interface of "known" and "unknown".

Along that interface, critical love calls us to ***curiosity and wonder***. The RLP encourages us to regard persons (including, at times, ourselves) with the mantra "you are a part of me I do not yet know" (Canlas, 2020, p. 25; compare to the Mayan concept of In Lak'ech as applied to math education in Gutiérrez, 2017). The phrase "I do not yet know" evokes a perpetual sense of curiosity and a readiness to construct new knowledge. "To wonder is to cultivate a sense of awe and openness to others' thoughts and experiences... wonder is the wellspring of love" (Canlas, 2020, p. 25). Through wonder we broaden our perspectives and learn from one another; in particular, we learn from our students just as they learn from us. We therefore offer critical love as a partial answer to Britzman's questions "How does one take joy in having ideas, in changing

one's mind, in encountering the work of learning?" and "What sort of relations [exist] between learning to love and loving learning?" (Britzman, 2000, p. 44). In a state of curiosity and wonder, we both learn to love and love to learn – an ontology aligned with queer theory's insatiable "interrogatory edge" (Rands, 2019). That said, we find an even deeper call in the seeing of others as "parts of us that we do not yet know": the practice pushes us to perpetually see the complexity of other persons and to feel that complexity reflected in ourselves. "Wonder is an orientation to humility [in recognition that] others are as complex and infinite to themselves as we are to ourselves" (Canlas, 2020, p. 25). We arrive therefore at critical love's conceptualization of the link between the self and systems outside of the self.

The ***simultaneous holding of countless complexities*** of personhoods presents a profound challenge. We again seek theoretical guidance in queer theory which "hopes to overwhelm our capacity to 'get it,' to bring us to a point where we… [have] neither knowledge nor resistance to knowledge… lacking answers, we are able to embrace questions, engage with multiple understandings, and imagine new possibilities" (Shlasko, 2005, p. 129). Curiosity and wonder help us "embrace questions" while courage and humility help us "imagine new possibilities". Here we elucidate critical love's holding of complexity, which requires that we can "engage with multiple [potentially contradictory] understandings" at once. In this way, critical love holds wariness around the idea of accessing an objective understanding and invites us to proactively recognize – and even celebrate – our subjectivities.

Recognition of our subjectives reminds us of the ever-presence of our history and position (Milner, 2007; Aguirre et al., 2017), and thereby leaves space for holding multiple understandings at once. Indeed, "queer theory attempts to understand the world (as do all theories) but makes this attempt while simultaneously recognizing the impossibility of ever developing a complete or unbiased explanation" (Shlasko, 2005, p. 125). Deeper still, critical love celebrates our subjectivities as reflective of our (individual and collective) capacity for transformation: "Queer thinking argues subjects and subjectivities are fluid, unstable, and perpetually becoming" (Browne & Nash, 2010, p. 1). In this way, critical love frees us from the chase of the unknowable and illusory capital 'T' 'Truth'. "The point of queering is not to fit the 'all' into one or more norms, but to mark, deconstruct, and pass beyond the regulatory impulse to fit being(s) into finite categories" (Fifield & Letts, 2014, p. 398). Rather, no one person or group of persons holds 'the' truth and critical love calls us to not just continually learn from one another, but also recognize (even celebrate) our dependence on one another for that learning. In an interview with Davis and Sarlin (2008), Michael Hardt describes love "as a social muscle" that has to involve "a kind of training… in cooperation with many". Likewise, Lanas and Zembylas (2015) write, drawing upon Nancy (2003), that "the transformational power of love arises precisely from the fact that it breaks the I" (p. 38).

We use ***interdependence*** to refer to this reciprocal dependence on one another for mutual knowledge production. For Darder (2017), in the classroom that reciprocity occurs through dialogue: "dialogue cultivates and nurtures students' curiosity and imagination toward a greater critical capacity to confront dialectically the content of their study and the task of constructing new knowledge" (p. 91). While critical love views dialogue as a manifestation of interdependence, there lies a deeper orientation at work. The RLP challenges us to "see no stranger" and critical love embraces the challenge as an invitation to depend on one another for our perpetual transformation. As Coles welcomes in conversation with colleagues Lyiscott et al. (2021), "I invite you to be transformed by my being, a being often falsely constructed as needing to be transformed by dominant power structures." (p. 3). Critical love calls us to transform one

another not just through dialogue, but also through relationships and curiosity – even in the face of ever-present pressure by the dominant systems of our context.

In this mutual transformation lies a deep sense of needing one another: I come to know myself and our world more deeply only in relation with others, whom I recognize as parts of myself I had not yet known. Implicit in this conceptualization of interdependence lies a perpetual expansion of each individual's meaning of "we". Critical love pushes back on worldviews built on "us" and "them", aligned once again with queer theory's rejection of binaries (Fifield & Letts, 2013). Traditional academic structures in mathematics, in contrast, overemphasize the transformation of students by instructors – which some authors describe as mathematical enculturation (Bishop, 1991; Perrenet and Taconis, 2008) – and thereby position students as a "them" separate from an "us" of instructors. Such a division limits the reciprocity of mutual transformation, and critical love views hierarchy as an obstacle here.

As such, critical love ***interrogates power relations***. Like Lanas and Zembylas (2015), we view love as "always connected to power" (p. 38) and as a powerful force for "the long-term process of transforming power in our institutions and everyday lives" (p. 40). Drawing upon prior conceptualizations of Nepantla – a liminal space at the in-between of two worlds (Anzaldúa, 2002; Gutiérrez, 2012) – and tension (Gutiérrez, 2015; Myers et al. 2023), Hagman et al. (2024) highlighted the tension between "power over" and "power with" in transformation efforts in pedagogy. Critical love likewise embraces this idea of Nepantla, and reaffirms the deep value of students as partners in recognition of students' "unique perspectives and contributions" (Green & Lopez, 2015, p. 19; also see Matthews et al., 2018; Cook-Sather et al., 2018; Bolick et al., 2024). At the same time, critical love holds wariness around the implicit binary that positions "power over" as a sort of opposite to "power with" or "shared power" (Brooks, 2017, p. 107). A binary framing might (subconsciously) celebrate "students as partners" as an "opposite" to more traditional "power over" models.

Instead, critical love asks: what could a non-binary conceptualization of power look like? Critical love calls for us to consider the varied forces of power in our lives – interpersonal and systemic – and not only seek out "power with" others, but also exercise deep trust and vulnerability by welcoming others to have certain "power over" us. As Darder (2017) writes, "teachers must recognize that there exist no human relationships, let alone educational practices, without the explicit or implicit existence of authority… [and] authority expresses itself in a multitude of ways" (p. 100). We must consider the systems and relations which give us power over others and give others power over us, and be critical of that power with a blend of courage, humility, and interdependence. Drawing upon Lewis's theorization of 'presence' (2022), Ahluwalia (2022) calls us to open up a "'third space' for more complicated, relational analyses of power" (p. 315). Critical love lives inside this "third space".

Critical love therefore requires that we ***envision*** alternative – or "third" – spaces of research, education, and relationship. Indeed, critical love knows us to be capable of, as Ohito envisions, "a future pluriverse expansive enough for myriads of humans to coexist in and cohabit peacefully" (Lyiscott et al., 2021, p. 5). Likewise, in their pedagogical guidance on culturally-relevant STEM education, Tran and Pierre (2025) describe love as "visionary" in its movement "toward justice", and that "love emphasizes the relationality of shared humanity, a type of love that is deeply personal, political, and collective" (p. 5). In pursuit of such a vision, the RLP grounds in the concept of reimagining: "to reimagine is to explore a vision of a relationship, community, and world where we all flourish" (2020, p. 80). We note not only a growing body of literature on (re)humanization of mathematics, but also a growing body of literature on

flourishing in and through mathematics (Su, 2017; Tunstall and Ferkany, 2017; Hill et al., 2021; Khan et al., 2021; Su, 2024). In cultivating a vision for the world in which we all flourish, we not only push back against injustice, but also ground in a vision of collective liberation. Such envisioning requires immense courage: "Dare to reimagine an outcome, a solution, a relationship, a future that does not leave anyone behind… imagine cultures, contexts, and institutions that run on love instead of fear" (Revolutionary Love Project, n.d., p. 2).

Foundational Elements of Critical Love

The prior seven components form the core of critical love. We turn to two foundational elements, which support the translation of more abstract notions into action. As the RLP reminds us, "theory can guide us but [it] must also be activated through action" (Canlas, 2020, p. 11).

The practice of critical love requires ***embodiment***. As a social justice educator, Brooks (2017) argues that "love must be embodied" (p. 111) for knowing "intellectually" is "insufficient to bring about justice" (p. 109). Here we arrive at a pivotal departure in our conceptualization of critical love from the "epistemic love" conceptualized by STEM educators Tran and Pierre (2025), which they describe as having four typologies: tangibility, emotionality, intellectuality, and audacity. We see in these an overemphasis on the role of the mind, for love also lives in the body and it is through the body that we find deeper transformation in our ways of being. As Ahluwalia (2022) notes, "the body knows" and it is through attunement with the body that we can "understand what is being stirred up in [us] as [we] try to do this work" (p. 311). We must practice awareness of the sensations that arise in our bodies, and exercise loving wonder of their meaning. Indeed, the RLP grounds in the metaphor of cyclical "breathing" and "pushing", for love is a perpetual "choice to labor and take action" (2020, p. 12). Without breath, we burnout; without push, we get stuck.

Critical love affirms that we are so much more than fragmentary minds elevated by elaborate flesh vessels. We are whole human beings, "not mathematical machines. We live, we breathe, we feel, we bleed." (Su, 2017). Likewise for the people all around us. In the classroom, for example, Darder (2017) affirms that the "perception of our students as embodied human beings" is fundamental to "a liberatory classroom practice", for "a student enters the classroom as a whole person and should be respected and treated as such" (p. 87). Embodiment calls us to ground in a perception of others as embodied human beings. Through such a grounding, we engage in a continual practice that teaches our bodies new ways of moving and being.

Finally, critical love calls us to ***discernment***. At the tension point of "simultaneously holding countless complexities" lies a fiercely tangled struggle of determining when and how to love in each moment. We note a possible orientation which resonates for us: the idea of "returning to our name(s)", employed by Lyiscott et al. when they first elucidate (2021) and later reflect (2023) on their editorial vision for *Equity and Excellence in Education*. Through the lens of "naming", discernment calls us to thoughtful consideration of what constitutes our names, and how we might return home to them "in loving accountability" (2023, p. 259). With courage, can we discern "who and what we fear" (2021, p. 5) and how fear pushes us from our names? With humility, can we discern our fullness of self and when we stray from it? With curiosity, can we discern "who should do the listening in a given moment" (Canlas, 2020, p. 69)? Can we discern with whom we might grow our interdependence? Can we discern which systems give us power over others and how to hold it, if we even should? Can we discern when we avoid others having power over us, and why? Can we listen to our bodies when they communicate, and discern the message?

Implications for Practice

As Johnson et al. (2018) wonder, couldn't "a pedagogy of love" serve "as an embodied practice that influences holistic teaching?" (p. 47). We affirm a resounding yes, and highlight some implications of critical love's *embodiment* on how we regard our students. We chose embodiment for this short paper as it deeply challenges our orientation to the teaching of mathematics, a discipline often positioned as disembodied and outside of human activity (de Freitas & Sinclair, 2013; Gutiérrez, 2017; Battey & Marshall, 2024; McNeill et al., this issue). We recognize the field of embodied cognition in mathematics education as a partial implication (cf. Edwards et al., 2014; Abrahamson, 2020; Khatin-Zadeh, 2022; Soto, 2024; also see MacArthur et al., 2025, p. 24-25 on "body and emotions" in the math classroom). However, we argue that critical love maintains an even deeper call. Darder (2017) cautions against "standards [in the classroom] of disembodied expectations removed from the organic responses of student bodies" where "teachers conduct themselves 'professionally' in ways that distance them from the possibility of expressing authentic human love" (Darder, 2017, p. 83). Critical love calls us to remember the body as an indivisible unit, not merely a vehicle for "cognition".

A simple practice might invite students to attune to their breath in the math classroom (cf Peper et al., 2016; Walkins, 2021). For another practice, I reflect on my first TA assignment in a calculus class of 200 students: for all exams, the instructor – in an act of critical love – would bring in a small "test box" of extra snacks, water, and personal hygiene supplies to support students in caring for their bodies. Indeed, some neuroscientific research has considered the impact of nutrition on brain function (Gómez-Pinilla, 2008), while numerous pedagogical studies have considered the impact of snacks (Kanarek & Swinney, 1990; Burrows et al., 2017) and hydration (Beezhold et al., 2018; Pawson et al., 2013) on university students' success.

Critical love also carries potential for teacher professional development. In our literature review, we highlighted Green & Lopez's (2025) recent study on training of STEM preservice teachers, in which they found that teachers "rehearsed dominant discourses" while also giving "strong commitments to social justice in terms of radical love" (p. 22). Rather than avoid the term "love" in training – and have it called upon anyway – we wonder about the incorporation of thoughtful dialogue on critical love as "an anchor that empowers us to humanize each other" (Tran & Pierre, 2025, p. 5). Hauk et al. (2014) draw upon Bennett's (2004) model of intercultural sensitivity to foster an "intercultural orientation" for post-secondary math instructors. We offer critical love as a potentially fruitful framing, for we find alignment between critical love and Bennett's "stages" of ethnorelativism (acceptance, adaptation, and integration). Moreover, we argue that the linearity and strong hierarchy of Bennett's model limit its application, while critical love actively avoids such structures via queer theory. Indeed, Lanas (2017) calls for "intercultural education in teacher education" that moves away from "seeing the relationships between teacher and learners as static" and is "connected more explicitly to a theoretical conceptualisation of love" (p. 1). As Green & Lopez (2025) put it, "further research could explore how love, serving as a foundation for STEM education, can prompt teachers to reevaluate… notions of individualism [and] competition" (p. 25), notions the prevailing culture of mathematics channels into broader STEM (Battey & Marshall, 2024).

Lastly, we offer critical love as an orientation in our research work. In our literature review, we highlighted various STEM and mathematics education research which call upon love without a deeper theoretical framing (Yeh et al., 2021; Paris & Alim, 2014; Basile & Azevedo, 2022; Black et al., this issue). With an orientation of critical love, we can expand our openness to new research perspectives and engage in generative loving (self-)critique.

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Two Approaches to Rates and Foundations for Derivatives

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We distinguish 2 approaches to coordinating quantitative meanings for rates, proportional relationships, and variation. The first approach—based on coordinating unit rates, multiple batches, and chunky variation—has dominated existing research on rates and rates of change. The second approach—based on coordinating measurement rates, variable parts, and smooth variation—has not been recognized as distinct from the first. We examine affordances of each approach for reasoning about 2 aspects of rates essential for instantaneous rates of change and derivatives: They are single measures that remain approximately constant as changes in 2 variables tend to 0. We argue that the second approach complements the first in important ways and conclude that it should be recognized as an important focus in future empirical research.

Keywords: Rates, Derivatives, Calculus

Past research has recognized that the origins of students' and teachers' challenges with derivatives can often be found in their reasoning about quantities in combination with multiplication and division, variables, rates, and functions (e.g., Boyce et al., 2021; Byerley, 2019; Larsen et al., 2017; Thompson & Harel, 2021). One common challenge is that students and teachers often focus on rates as pairs of amounts rather than as single measures (e.g., Byerley & Thompson, 2017; Frank & Thompson, 2021; Johnson, 2015b; Yu, 2021). In calculus, a view of rates or slopes as pairs of amounts is problematic for developing an understanding of instantaneous rate of change and the derivative of a function. In particular, students who view rates and slopes as pairs of amounts will not be able to coordinate average or instantaneous rates of change of a function with changes in the input variable (e.g., Carlson et al., 2002; Yu, 2024).

In this theoretical report, we distinguish two approaches to mature meanings for rates that we term *unit rate* and *measurement rate*. Although mathematics education research has rarely articulated the distinction between these two meanings, we argue that the distinction matters because each presents different affordances for supporting two aspects essential for understanding how local, linear approximations lead to derivatives. First, a (constant) rate, rate of change[1], or slope is a *single quantity* that measures a quality such as steepness or speed by coordinating changes in two quantities multiplicatively. Second, understanding that m is the instantaneous rate of change of a function at a point relies, in turn, on the idea that *rates can remain constant or approximately constant as* Δx *and* Δy *tend smoothly and continuously to zero*. Our contribution rests on critical examination of ways that past theoretical research on quantitative reasoning has treated multiplication and measurement.

Unit Rates and Measurement Rates

Our delineation of two approaches to rates begins with the distinction between multiplicands and multipliers (e.g., Greer, 1992). If one invokes multiplication when considering a situation where three children each have four cookies, the number of cookies one child has is the multiplicand and the number of children is the multiplier. Sharing or partitive division results

[1] We view rate of change as an application of rate and therefore we use the term rate to include rate of change in the remainder of the article.

from dividing the product by the multiplier to determine the multiplicand. An example would be: "If 12 cookies are distributed evenly among three children, how many cookies does each child get?" Measurement or quotitive division results from dividing the product by the multiplicand to determine the multiplier. An example would be: "If there are 12 cookies, how many children can each get four cookies?"

Given the close connections among division, ratios, and rates, that there are two distinct quantitative meanings for division suggests, in turn, that there should be two distinct quantitative meanings for rates. We introduce these two meanings using red and blue paint mixed in a 4-to-3 ratio. When reasoning with *unit rates*, we consider an initial batch containing 4 gallons of red paint and 3 gallons of blue paint. We imagine variation occurring through partitioning and iterating this initial batch, and we use division based on equal sharing to determine that 4/3 gallons of red paint correspond to each 1 gallon of blue paint. If n is a natural number, then for n gallons of blue paint there will $n \bullet 4/3$ gallons of red paint. More generally, if r and s are natural numbers, then for r/s gallons of blue paint there will be $r/s \bullet 4/3$ gallons of red paint. Here, n and r/s function as *multipliers* acting in parallel on two fixed multiplicands, 4/3 gallons of red paint and 1 gallon of blue paint. A similar argument explains a second unit rate, 3/4 blue gallons for every 1 gallon of red paint. Beckmann and Izsák (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have referred to such reasoning as the *multiple-batches* perspective on proportional relationships. The unit rate meaning and multiple-batches perspective have dominated past research on rates (e.g., Lobato & Ellis, 2010; Tallman et al., 2024; Thompson, 1994; Thompson & Thompson, 1994).

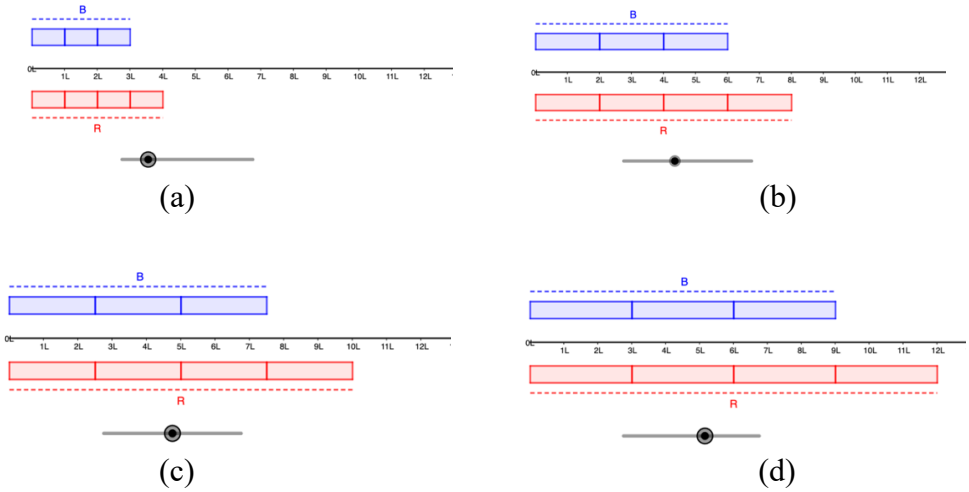

Figure 1. Covarying volumes of paint. All seven parts contain (a) 1 liter. (b) 2 liters. (c) 2.5 liters. (d) 3 liters.

In contrast, when reasoning with *measurement rates*, we organize the same mixture into 4 parts of red paint and 3 parts of blue paint. We imagine variation occurring through increasing or decreasing the volume of each part in such a way that any one part still has the same volume as all remaining six parts. Figure 1 shows screen shots from a dynamic illustration of covarying volumes in which each of seven equal-sized parts increases and decreases in parallel. In contrast to reasoning with unit rates, here we focus on measurement division and ask how much of the blue paint makes the same volume as the red paint. Because the number of parts of blue and red paint are fixed, and because all seven parts contain the same volume, it always takes 4/3 of the blue paint to make the same volume as the red paint, and it always takes 3/4 of the red paint to make the same volume as the blue paint. Notice that if B is any volume of blue paint, the

corresponding volume of red paint is 4/3 • B; and, if R is any volume of red paint, the corresponding volume of blue paint is ¾ • R. The measures, 4/3 and ¾, are the two measurement rates in this situation. In contrast with unit rates, with measurement rates variation is accomplished by varying a *multiplicand*, B or R, while the corresponding multiplier, 4/3 or ¾, remains fixed. Beckmann and Izsák (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have referred to such reasoning as the *variable-parts* perspective on proportional relationships. The measurement rate meaning and variable-parts perspective has not been recognized in past research on instantaneous rates of change and derivatives as a second, distinct approach.

Research on Rates Grounded in Quantitative Reasoning and Covariation

Research on rates has often been grounded in Thompson's (1994) account of quantitative reasoning. The two features of his account on which we focus are (a) characterizing numerical operations and quantitative operations as two distinct categories of mental operations and (b) characterizing rates as reflectively abstracted constant ratios, or multiplicative comparisons of two quantities varying in constant ratio (pp. 190–192).

Thompson and colleagues discuss multiplication as operating on quantities sometimes and on numbers other times. Thompson and Saldanha (2003) characterized "conceptualized multiplication" as one-to-many relationships: "Envisioning the result of having multiplied is to anticipate a multiplicity" (p. 103). This perspective is consistent with thinking of ratio comparisons where one amount is so many times as, or as much as, another amount. Thompson and Saldanha then discussed multiple proportions that lead to "numerical multiplication." To illustrate, they gave the following analysis for the area of a rectangle that has dimensions 12 inches and 2 centimeters. One side is 12 times as long as 1 inch, one side is 2 times as long as 1 centimeter, and the area is 24 times the area of a rectangle of dimensions 1 inch by 1 centimeter (p. 103). This analysis treats 2 • 12 as a numerical operation and foregrounds three separate one-to-many relationships where the measures associated with each factor play symmetric roles. Such symmetry contrasts with the conceptually distinct roles played by multipliers and multiplicands discussed above, making it harder to recognize the distinction between unit rates and measurement rates.

How one conceptualizes rates interacts not only with one's conception of multiplicative comparisons but also with one's conception of covariation between two quantities. Existing research has emphasized partitioning and iterating two quantities in parallel (e.g., Lobato & Ellis, 2010; Tallman et al., 2024; Thompson & Thompson, 1994). Thompson and Thompson (1994) provided a particularly succinct statement using speed as the example: "If you go m distance units in s time units at a constant speed, then at this speed you will go (a/b) • m distance units in (a/b) • s time units" (p. 283). This coordination can be envisioned as partitioning both m distance units and s time units into b equal-sized parts and taking a copies of the result. Here multiplication is operating on two quantities, distance and time, in parallel. Other researchers have also examined partitioning and iterating in parallel as a starting point for mature conceptions of rates—for instance, Lobato and colleagues discussed composed units (e.g., Lobato & Ellis, 2010), and Peck (2020) described a "unit rate strategy" as a main outcome in a learning trajectory for slope. Appeals to reflective abstraction have left in a black box how partitioning and iterating two quantities in *parallel* leads to perceiving a fixed multiplicative comparison *between* amounts of each (e.g., between distances and durations of time).

More recently, researchers have examined connections between chunky and smooth variation (e.g., Castillo-Garsow, 2012; Castillo-Garsow et al., 2013) and students' conceptions of ratios and rates (e.g., Johnson, 2015a; Tallman et al., 2024; Yu, 2023, 2024). Castillo-Garsow et al.

explained that a chunky conception of variation focuses on concatenated intervals and endpoints of those intervals. Such reasoning applied in parallel to two quantities is consistent with Thompson and Thompson's (1994) description of constant speed (quoted above) and with Lobato and Ellis' (2010) discussion of composed units. Castillo-Garsow et al. explained further that a smooth image of change focuses on continuous progression through the flow of time. Researchers have recognized that smooth covariational reasoning is critical for building calculus from quantitative and covariational reasoning (e.g., Thompson & Carlson, 2017, pp. 442–443). Still missing from these more recent studies are explanations for how operating on two quantities in *parallel* through partitioning and iterating leads to fix multiplicative relationships *between* the two quantities.

Measurement and Variation

In this section, we connect conceptions of quantity, measurement, variation, and the asymmetric roles of multiplicands and multipliers. Mathematics education research has maintained a distinction between conceiving of a quantity and measuring that quantity. As one example, Thompson (1993) explained: "A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (p. 165). Interestingly, work in mathematics has given a rigorous treatment of such measurement.

1901 Otto Hölder formulated seven axioms for quantity that did *not* include measurement (Michell & Ernst, 1996) but from which he derived measurement. To begin, Hölder's axioms allow one to describe whole and rational number multiples of a quantity: Given any specific quantity Q and any natural number n, Q can be iterated n times to form a quantity $n \bullet Q$ and can be partitioned into n equal-sized parts to form a quantity $(1/n) \bullet Q$ (Michell & Ernst, 1996, sections 3 and 7). It follows that for any positive rational number m/n, where m and n are natural numbers, there is a quantity $(m/n) \bullet Q$, defined as $m \bullet ((1/n) \bullet Q)$, which can be formed by partitioning Q and iterating the result. Note that m/n is a multiplier acting on a multiplicand, Q, and that this action creates chunks. Following Hölder (Michell & Ernst, 1996, sections 10, 11, and 15), one can use completeness of the real numbers to show that for any positive real number t there is a well-defined quantity $t \bullet Q$ whose measure with respect to Q is t. Here, one views the measure, t, as the multiplier that varies as it acts on Q, a fixed multiplicand. We refer to such reasoning as *variation of measure*. The reliance on chunks—in the case of rational multiples—and on completeness—when t varies through irrational numbers—implies that reasoning rigorously about smooth variation in combination with measured quantities is non-trivial.

In contrast, it is also possible to view variation where a quantity increases and decreases without attention to a measure. For example, if a volume of water is poured continuously into a vase, then there is a specific volume in the vase at any given time even if we do not know its measure with respect to any particular unit of volume. This conception of variation aligns with smooth variation. As explained within the paint example, when one applies this perspective to covariation of two quantities, variation occurs in the multiplicand while the multiplier remains fixed. This perspective also resembles that used by Newton: In his history of calculus, Edwards (1979) stated that Newton regarded the speed of a point moving along a straight line as intuitively apparent on physical grounds (p. 192) and took smooth, continuous variation of quantities as given by nature, viewing time as flowing continuously and other quantities as increasing continually in time (pp. 210, 271). We refer to such reasoning as *unmeasured variation*.

Rates, Variation of Quantities, and Derivatives

As stated in the introduction, a quantitative notion of instantaneous rate of change rests on two ideas: Rates or slopes are single quantities and rates of change can remain constant or approximately constant as Δx and Δy tend smoothly and continuously to zero. We agree with Thompson and Carlson (2017) that smooth continuous variation is foundational for building calculus from quantitative and covariational reasoning (pp. 442–443). Below we articulate two approaches to coordinating rates, proportional relationships, and quantitative variation.

Unit rates, multiple batches, and chunky variation

Here we summarize an approach to mature reasoning starting with unit rates. Much of the research on rates has relied proportional covariation in which two quantities are of different types, such as distance and time or height and volume. Such approaches rely on the multiple-batches perspective and on *variation of measure*, at least initially. This follows because, if D and T are two unlike specific quantities such as a distance and a duration of time, then one cannot measure the other directly. The only way to determine new pairs in the same linear relationship is by letting D and T each function as a measurement unit for its own type of quantity (e.g., D measures distances and T measures durations of time). Thus, to explain multiplication as an operation on quantities varying at the same rate, D and T must be iterated and partitioned in parallel, where variation occurs by varying the multiplier, as in varying r/s in $(r/s) \bullet D$ and $(r/s) \bullet T$. As explained above, such variation involves chunks and extension to irrational multiples relies on completeness of the real numbers.

Measurement Rates, Variable Parts, and Unmeasured Variation

Here we summarize an approach to mature reasoning starting with measurement rates. Measurement rates fit most naturally in situations where the two covarying quantities are of the same type—for instance, two lengths or two volumes. In such situations, reasoning about proportional covariation can rely on the variable-parts perspective, *unmeasured variation*, and multiplication as a quantitative operation *between* two quantities. The numbers of same-sized parts are not limited to whole numbers: They can also be positive rational or irrational numbers. For example, if the number of parts is 2.3, then there are 2 whole parts and 3 mini-parts, each 1/10 the size of the whole parts. Covariation occurs by changing the size of one part, and therefore the size of each part or portions of whole parts. Moreover, variation occurs by varying the multiplicand, as in varying B and $4/3 \bullet B$ in the paint example. The parts are free to vary smoothly and continuously.

Entry Points into Derivatives

In this section, we examine the affordances and constraints of unit rates and measurement rates for developing instantaneous rates of change and the derivative. Approaching instantaneous rates of change through unit rates and variation of measure has several advantages. First, considering time as an independent variable may appear self-evident and natural to students—for instance, when thinking about the relationship between speed, distance, and time. Second, a wide range of further applications involve variation of unlike quantities. Examples include the relationships among current, time, and charge; among density, volume, and mass; and among pressure, surface area, and force. Third, past research (e.g., Lobato et al., 2013; Lobato & Siebert, 2002; Tallman et al., 2024; Thompson, 1994) has reported that students of various ages can spontaneously generate sound reasoning in which they partition and iterate in parallel. These results suggest that unit rates and variation of measure provide an accessible entry point for

initial reasoning about covariation and proportions in a range of situations where derivatives can be applied. At the same time, our discussion to this point leads to the conclusion that the current practice of engaging students primarily with speed and further situations that use pairs of dissimilar quantities poses an impediment to developing instantaneous rates of change and the derivative because it is difficult to relate unlike quantities through multiplication directly.

Approaching instantaneous rates of change through measurement rates and unmeasured variation of the same type of quantity has a contrasting profile of advantages. Here, it makes sense to interpret the difference quotient $\Delta y/\Delta x$ as a measure that is the result of measurement division: How many or much of Δx it takes to make Δy. This approach makes direct use of the connection between division and measurement, presents rates of change as single numbers that relate Δy and Δx directly, and affords reasoning about smooth and continuous variation.

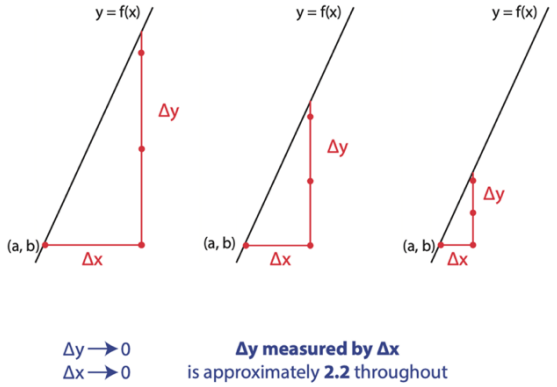

Figure 2. Difference quotient as measure: As Δx tends to zero, the measure $\Delta y/\Delta x$ remains close to 2.2.

To further illustrate a measurement rate interpretation of instantaneous rate of change in the non-linear case, we consider two points on the *x*- and *y*-axes that correspond to the *x*- and *y*-coordinates of a point moving smoothly and continuously in the coordinate plane. We use the variables *x* and *y* for the distances of these two points from the origin. We can regard the distances *x* and *y* as unmeasured covarying quantities and the curve that the object travels along as the graph of their relationship. In this situation, if Δx is small enough, the horizontal and vertical distances will change approximately uniformly relative to each other over the interval. Therefore, the measure of Δy in terms of Δx will remain approximately constant. Figure 2 is zoomed in on a small portion of a graph that shows snapshots of Δx (and Δy) decreasing toward zero from a point (*a*, *b*) on the graph. On this zoomed-in portion of the graph, the situation is similar to a linear situation, where $\Delta y/\Delta x$ would remain a constant measurement rate, regardless of the size of Δx (and Δy). Because of this approximate linearity, $\Delta y/\Delta x$ tends to a limit as Δx (and Δy) goes to zero, and this value is the instantaneous rate of change of the relationship at the given point, the derivative of *y* with respect to *x*. It indicates how *y*, the vertical distance of the moving object, is changing with respect to changes in *x*, the horizontal distance, at the given point. In Figure 2, assume that the derivative at the point (*a*, *b*) is 2.2. Then, at that point, the change in the vertical distance of the object is 2.2 times as much as the change in its horizontal distance. Therefore, the vertical speed of the point is 2.2 times its horizontal speed at that instant. When the point's horizontal distance changes by a sufficiently small amount from *a*, its vertical change in distance from *b* is approximately 2.2 times as much. So, for all changes Δx that are sufficiently small, when the horizontal distance of the object is $a + \Delta x$, the vertical distance of the object is approximately $b + 2.2 \bullet \Delta x$.

Discussion

Our delineation of two distinct approaches to coordinating quantitative meanings for rates, proportional relationships, and variation has implications for both research and instruction. With respect to the research, current quantitative approaches to rates have sought to combine reasoning with unit rates and multiple batches with smooth covariation. As one example, Tallman et al. (2024) reported a teaching experiment with one preservice teacher, Samantha, using a series of constant speed tasks involving a character named Clara. A main goal of the teaching experiment was for Samantha to develop an understanding of constant speed where she would anticipate that "any accumulated distance is some multiple k (where $k \in R+$) of the distance Clara travels in one unit of time, and that it therefore takes Clara that same multiple of one time unit to travel that distance" (p. 12). Samantha was able to solve the tasks appropriately by partitioning distances and times in parallel using chunky reasoning. At the same time, the authors argued that Samantha's reliance on chunky variation constrained her reasoning, and they hypothesized that smooth covariational reasoning is essential for constructing rates as reflectively abstracted constant ratios (p. 22). From our point of view, the constraints in Samantha's reasoning may have been an artifact of researchers presenting situations for which it was appropriate to partition and iterate two quantities in *parallel* with an expectation that such reasoning would support a different multiplicative relationship *between* those quantities. We understand this design to be based on a mix of the two approaches to rates and covariation that we have identified and explicated.

With respect to instruction, our analysis suggests that students might benefit from reasoning about rates with pairs of like quantities. We introduced measurement rates using volumes of paint. Further examples in the literature suggest students may be able to generate measurement rates reasoning spontaneously. As one example, Moore (2014) reported a teaching experiment during which Zac developed a preference for using the radius of a circle as a unit of measure and came to reason productively about multiplicative relationships between the length of the radius and arc lengths and sides of right triangles. Moore did not highlight distinct meanings for rates; but, from our point of view, Zac reasoned about measurement rates within a variable-parts perspective. As a second example, Yu (2023) reported a short intervention on linear motion designed "to move students from interpreting a speed of 1.7 m/s as 'for every additional second the person moves an additional 1.7m' into a meaning as 'for any amount of time that passes, x, the person moves 1.7x cm'" (p. 966). During the unit, college calculus students worked with a dynamic computer environment that allowed them to change horizontal and vertical lengths smoothly and to consider "how many times as large the change in the output is (the length of the vertical segment) with respect to the change in the input (length of the horizontal red segment)" (p. 972). Thus, from our perspective, Yu combined measurement rates with smooth covariation.

We hope to inspire further consideration of new possibilities for coordinating quantitative understandings of rates, proportional relationships, and covariation. Past research has focused on unit rates, multiple batches, and variation of measure, but measurement rates, variable parts, and unmeasured variation have characteristics that scholars have repeatedly described as important for calculus, including derivatives. Thus, we see further investigation into students' reasoning with measurement rates as an important direction for future research.

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Quantitative Meanings for Graphs: States, Transformations, and Covariation

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Research on students' and teachers' quantitative reasoning continues to underscore its importance for their learning and development. This importance requires that researchers continue to make strides in identifying salient and important ways of reasoning quantitatively. In this paper, we delineate four forms of quantitative reasoning to characterize both students' images of situations and their graphing meanings related to those images. Specifically, we differentiate between students conceiving quantities' changes via states reasoning, transformational reasoning, and gross or quantified covariational reasoning. We connect these forms of reasoning to their meanings for graphs when attempting to represent those quantities.

Keywords: Covariational Reasoning, Quantitative Reasoning, Graphing, Cognition

Quantitative reasoning involves conceiving a situation so that it entails measurable attributes (i.e., quantities) and relationships between them (i.e., quantitative relationships; (Smith III & Thompson, 2007; Thompson, 2011). *Covariational reasoning* involves conceiving the ways in which quantities vary in tandem (i.e., covary; Saldanha and Thompson (1998), Carlson et al. (2002), and Confrey and Smith (1995)). Together, quantitative and covariational reasoning (QCR) form a critical foundation for student development and teacher actions at all grade levels (e.g., Ellis, 2011; Johnson, 2015; Steffe & Olive, 2010; Tallman & Frank, 2020; Thompson, 1994). Over the past two decades, we have engaged in research to build models of students' and teachers' reasoning about experiential contexts in association with their QCR and graphing relationships in coordinate systems. In this report, we describe QCR forms that have emerged during this work: state, transformational, gross, and quantified. State and transformational reasoning entail thinking about quantities at distinct states. Gross and quantified reasoning additionally include images of covariation. We illustrate each form in the context of conceiving of a situation and constructing a graph to re-present quantitative relationships.

Covariation and Re-presentation

Carlson et al.'s (2002) and Thompson and Carlson's (2017) frameworks are two of the most used in the field, both because of the authors' prominence as QCR researchers and the generalized nature of the frameworks. We have drawn significantly on Carlson and Thompson's frameworks, with Carlson et al.'s (2002) attention to direction and amounts of change being a bedrock for our work in understanding students' graphing meanings (e.g., Liang & Moore, 2021; Moore, 2014; Paoletti et al., 2024). Thompson and Carlson's (2017) levels have aided us in describing student meanings for linear/non-linear relationships (Paoletti & Vishnubhotla, 2022).

A popular context-specific area for QCR research is students' coordinate-system and graphing meanings. Specific to our focus, coordinate systems provide a representational context for students to re-present quantitative relationships they have constructed in experiential contexts (e.g., taking a road trip or building a fenced pen). Lee et al. (2020) called this a *quantitative coordinate system*. Moore and Thompson (2015) termed such reasoning *emergent (graphical)*

shape thinking (*EGST*) if such actions involve covariation. In our use of covariation frameworks to describe participants' EGST and quantitative coordinate systems, we experienced the need to clarify or reframe aspects of them. These needs included incorporating a more intentional focus on magnitude reasoning, non-variational forms of reasoning that maintain the spirit of EGST, and the extent to which amounts of change are coordinated and compared.

Four Forms of Reasoning: States, Transformations, and Gross/Quantified Covariation

In Table 1, we delineate four forms of reasoning to discuss QCR and EGST: (1) state reasoning; (2) transformational reasoning; (3) gross (covariational) reasoning; and (4) quantified (covariational) reasoning. Each form is quantitative; each involves two quantities existing in a paired, possibly deterministic relationship. Each involves understanding that each quantity can take on a multitude of magnitudes, and that changing the magnitude of one quantity might involve changing the magnitude of the other.

Table 1. Four Forms of Reasoning about Two (or more) Quantities

Reasoning	Description
State	Quantities' paired magnitudes are only dependent on the state under consideration. Changing from one pair to another involves imagining a different instantiation of the situation.
Transformational	Quantities' paired magnitudes are only dependent on the state under consideration. Changing from one pair to another involves transforming between instantiations and reconstructing the magnitudes.
Gross (Covariational)	Quantities' paired magnitudes are dependent on the state under consideration. Also, one pair can be changed to another by imagining a quantity's magnitude increasing/decreasing with increases/decreases in the other quantity's magnitude.
Quantified (Covariational)	Quantities' paired magnitudes are dependent on the state under consideration. Also, one pair can be changed to another by constraining gross covariation by an invariant property of their simultaneous changes.

A person engaged in *state reasoning* conceives the relevant quantities as occurring in distinct instantiations called states. The quantities' magnitudes can change in the sense that there can be different magnitudes at distinct states. The magnitudes at a state exist independent from those at other states. In the context of imagining a road trip, any moment in the trip is a state. At every state, there exists some distance from the start and some distance from the destination. Because each moment exists independent of another, conceiving another state involves switching attention from one state to the next state *and then* constructing and determining anew the two distance magnitudes (Figure 1a; the red magnitude represents the distance from the start and the blue magnitude represents distance from the destination). The shift in states produces distinct distance magnitudes, which may or may not have different sizes from other states. For example, a person engaged in state reasoning might focus on the halfway point in their journey and identify that the two distances are equal. They might then shift their focus to a rest stop and, with their attention shifted, conceive how far the stop is from the start and end. An individual engaged in state reasoning understands that the path constrains possible states. They also might determine the two distances sum to a constant magnitude at every moment in the trip, conceiving this as an invariant property defining the quantities' deterministic relationship at any state.

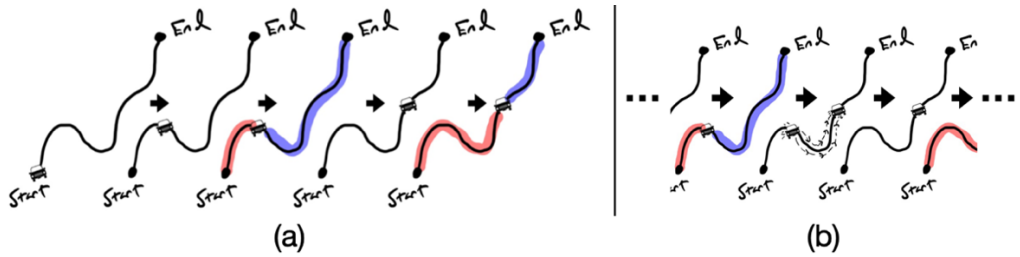

Figure 1. (a) State and (b) transformational reasoning about a road trip and distances.

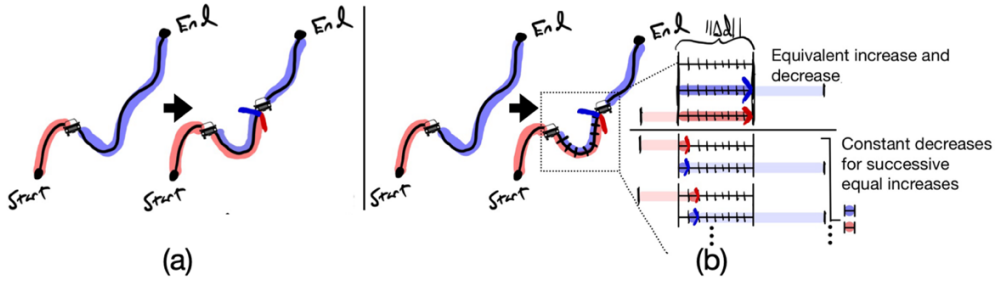

Figure 2. (a) Gross and (b) quantified reasoning about a road trip and two distances.

Transformational reasoning is equivalent to state reasoning, but the person imagines transforming the context from one state to produce another state. With a road trip, this involves imagining the car traveling along the road from one state to another, whether smoothly or in chunks (see Figure 1b, which is Figure 1a with the additional conception of the car traveling from one state to another). Like state reasoning, an image of the quantities' magnitudes must be constructed anew at the ending state due to the magnitudes not being sustained when conceiving the transformation. The states are no longer independent of each other, but the quantities' magnitudes constituting each state remain independent from those at other states. Both state and transformational reasoning allow for the quantities' magnitudes to change via being different sizes at different states. Each includes understanding the car's physical position can change, but neither includes images of the (co)variation of two quantities magnitudes.

A person engaged in *gross (covariational) reasoning* also understands quantities occur in distinct states and conceives states with respect to each other; any state is a *snapshot* occurring within the transformational image. Gross covariation additionally involves the quantities' magnitudes being sustained during a transformation. A person conceives one state of the quantities' magnitudes as dependent on the other state through a process of covariation; the ending state can be produced through a process of covariation emanating from the beginning state. In the case of the road trip, this might involve an individual reasoning that as they proceed from an early rest stop to some state later in the trip, their distance from the start increases and the distance they have remaining decreases (Figure 2a, indicated by the lengthening red magnitude and the shortening blue magnitude). They could conceive this as a loose process of simultaneous increase and decrease, or they could conceive the simultaneous increase and decrease in terms of the specific magnitude each quantity changes in total (e.g., explicitly identifying the magnitude increase in red and the magnitude decrease in blue). They might hold in mind that at any state during that covariation the two distances sum to a constant magnitude.

The aforementioned three forms of reasoning can entail holding in mind an invariant property defining the quantities' overall magnitudes at each state (e.g., they sum to a constant magnitude). A person engaged in *quantified (covariational) reasoning* also constructs a relationship in the *variations of each quantity* so that an invariant property *constrains their*

covariation when transforming from one state to another. Gross reasoning dealt with general intervals of increase and decrease and possibly considering specified total change. Quantified reasoning involves more precisely comparing the quantities' variations. In the case of the road trip, this could involve a person reasoning that as they change states, *any increase in the distance* from the start necessitates a decrease of *equal magnitude* in remaining distance (Figure 2b). Or a person might reason that as the distance from the start increases by *successive and equal magnitudes*, the distance remaining decreases by *a constant magnitude* (Figure 2b). In the former, the person quantified the gross covariation by comparing and generalizing one distance's variation relative to the other (i.e., each quantity changes by the same magnitude). In the latter, the person quantified the gross covariation by comparing one quantity's variation across fixed variations in the other (e.g., one quantity's magnitude changes constantly for constant magnitude changes in the other). In each, the quantities' covariation, whether chunky or smooth, was refined through constructing and quantifying amounts of change. Reiterating a point made above, quantified reasoning need not entail reasoning about specified values; in the road trip situation an individual can coordinate the two distance magnitudes without determining specific values in a unit (Figure 2b). As with the previous three forms of reasoning, the person might also understand that the total distances sum to a constant magnitude at each state. They might also consider how the additional invariant property of covariation maintains the invariant sum property.

Four Forms of Reasoning: EGST

To illustrate these forms of reasoning in an additional context and in relation to EGST, we use a fencing task adapted by authors of this paper to generate models of students' thinking: *Imagine constructing a rectangular playpen out of 20 feet of fencing. Construct a graph representing all possible width and length combinations for the playpen.* The task prompts re-presenting the relationship between the dimensions as they occur in the playpen context.

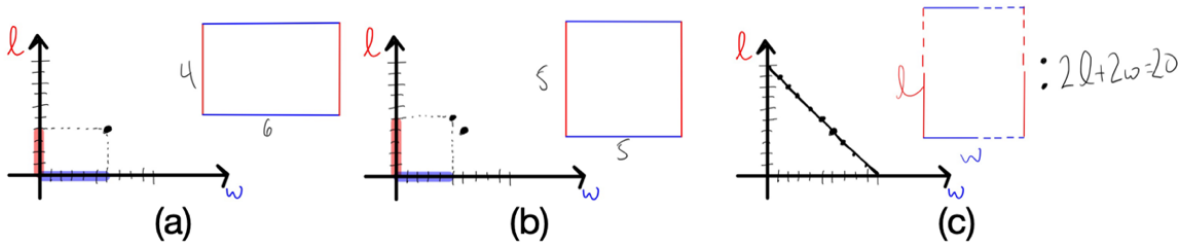

Figure 3. (a-b) Re-presenting paired magnitude states and (c) a sketched graph.

State reasoning involves understanding that each paired dimension is specific to a distinct, constructed playpen. The width and length magnitudes can change because of constructing a different playpen, with each state of paired dimensions being strictly dependent on the playpen one imagines constructing. The student might also understand the sum of playpen sides is 20 (e.g., $2w + 2l = 20$) at each state, and they might use this to determine dimensions at a state (e.g., choose $l = 5$ and determine w). Because the paired quantities are understood as dependent on a specific playpen, a student reasoning in this way produces a collection of points to capture each distinct playpen. They understand that each point captures a distinct playpen's paired magnitudes through the uniting of the axes-magnitudes re-presenting those quantities' magnitudes or values (Figure 3a-b). Having produced numerous points, they might also sketch a graph that captures a perceived visual pattern so that the graph captures every possible playpen (Figure 3c).

In the case of transformational reasoning, the person imagines transforming the pen from one state to the other; they have constructed a collection of playpens so that a playpen can be

produced by modifying another. The potential transformations a student might imagine are numerous, creative, and unpredictable. One involves imagining squeezing the pen in one direction while stretching it in another. An explicit image of length and width magnitudes are not sustained during such a transformation. Re-presenting such a relationship graphically is equivalent to that of state reasoning, except the person might imagine how the point may move for each transformation of the pen. A drawn graph conveys the appropriate path for that movement. Stopping the movement along a drawn graph creates a state in which they can determine the width and length pairs for that playpen. An image of the quantities' magnitudes and their covariation is not sustained during the movement.

Moore and Thompson (2015) introduced EGST with an eye toward graphs as emerging through quantitative schemes constrained by an invariant property in the quantities' covariation. EGST is not immediately relevant to state or transformational reasoning due to an absence of images of (co)variation. We perceive both forms to be compatible with Moore and Thompson's more general purpose of articulating EGST as a quantitative meaning (Moore, 2021; Moore et al., 2022; Moore et al., 2024; Thompson, 2011, 2012, 2013). The collection of points emerges from an individual's reasoning about paired quantities' magnitudes, their re-presentation, and (potentially) an invariant property constraining the pairs (e.g., constant perimeter). We thus characterize such actions as *non-variational quantitative meanings* for graphing.

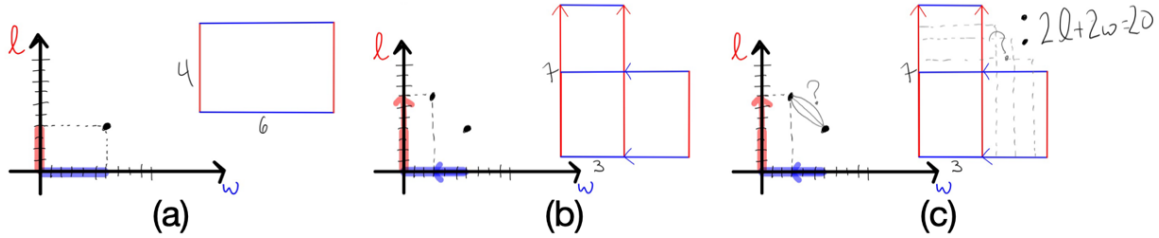

Figure 4. Re-presenting (a-b) paired magnitude states, (b-c) gross covariation, and (c) sketched graphs satisfying that gross covariation.

Gross (covariational) reasoning entails not only understanding that each pair of values corresponds to a constructed pen, but also that one pen can be conceived as a transformation from another pen through variations in the pen's length and width. Like above, a student reasoning in this way produces a collection of points to capture each state and they understand each point is a re-presentation of the state's paired quantities' magnitudes (Figure 4a-b). Differing from state or transformational reasoning, they also anticipate transforming from one point to the other via the quantities' covariation. For instance, they could reason about intervals of increase and decrease for the quantities that achieve the desired total variation amounts (Figure 4b-c). They could imagine these increases and decreases smoothly, and they might anticipate throughout the covariation that the quantities' total magnitudes at each state maintain the invariant property of a 20-foot perimeter. Because the quantities' covariation does not entail a quantified invariant property beyond (potentially) their resulting states maintaining the overall perimeter, the individual sketches a graph that results in that overall covariation. In doing so, they may or may not anticipate that the graph's emergence can occur in multiple ways to re-present that amount of variation between states (e.g., three possible graphs are shown in Figure 4c). Furthermore, the individual may or may not anticipate that, for whichever graph is accurate, the pen will transform in whatever way produces that graph (see dotted lines in Figure 4c representing an anticipated pen transformation). As with the previous forms of reasoning, having produced numerous points, a student might anticipate the sketch of a graph that captures a perceived pattern, the gross covariation, and a point for every state.

Both gross and quantified (covariational) reasoning involve determining invariant properties of covariation that constrain the transformation of the pens. With gross reasoning, the quantities' covariation within the two states is only constrained by achieving a total amount of variation. With quantified reasoning, an invariant property between the quantities' covariation is constructed so that one state can be transformed to another state in a way that additionally satisfies some property further quantifying the directional covariation. A student reasoning in this way might first produce one point re-presenting the pen's dimensions at some chosen state (Figure 4a from above). Then, they might use gross covariation and a conceived invariant property of that covariation to capture additional points or sketch a graph via re-presenting that property. An example of this is reasoning that the width and height magnitudes must covary such that any increase or decrease in width requires an equivalent decrease or increase in length, respectively (Figure 5a). For example, a width increase of 1 requires a height decrease of 1. The person's graph would then emerge as a result of coordinating the quantities' covariation under the constraints of that invariant property (Figure 5b). As with the road trip situation, a person could also reason that for successive, equal decreases in height, the width increases by a constant amount. A graphing action reflecting this might be the student denoting equal decreases in the height-axis magnitude and pairing that with equal increases in the width-axis magnitude. In either case, the individual might also consider how the invariant property in the quantities' covariation results in states that satisfy the invariant property of a constant perimeter.

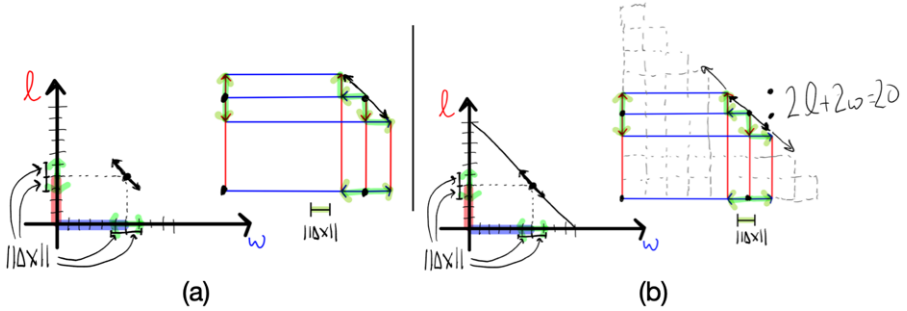

Figure 5. Re-presenting (a) quantified covariation and (b) a sketched graph constrained by that covariation.

All forms enable sketching a graph that captures states precisely. Using gross or quantified reasoning, an individual can construct a graph via EGST by first constructing points to capture snapshots and then sketching a graph that captures those states and invariant properties of covariation. Gross reasoning does not entail relative comparisons of variations, whereas quantified entails comparing variations between quantities (e.g., both quantities increase by the same amount) or within quantities (e.g., for equal increases in one quantity, the other quantity increases by decreasing amounts). With quantified reasoning, if the invariant property is quantified in terms of generalized properties (e.g., for a unit-magnitude change in quantity A, quantity B increases by 10% of its prior magnitude), then an individual can construct a graph via EGST strictly using one state and generating *all* other states using that invariant property.

A Brief Return to Informing Frameworks

We distinguish certain forms of non-variational reasoning from forms of QCR, and there are numerous connections to the frameworks within which we experienced the need to clarify these forms (Carlson et al., 2002; Thompson & Carlson, 2017). One need was introducing forms of reasoning that foreground magnitude reasoning. Whereas prior frameworks do not make magnitude reasoning explicit and often refer to or use numerical value, we describe our forms of reasoning with respect to magnitudes to be consistent with Thompson's (2014) emphasis on

magnitude-based QR. As a second need, we clarify distinctions between Carlson et al.'s (2002) directional and amounts of change reasoning. Gross reasoning can entail loose intervals of increase and decrease as well as identifying specified amounts of increase or decrease. The former is consistent with directional reasoning and the latter is consistent with amounts of change reasoning. Quantified reasoning is also consistent with amounts of change reasoning, but it makes explicit the coordinated comparison of amounts of change. To illustrate, gross reasoning includes understanding that as quantity A increases from 1 to 2 to 3, quantity B increases from 2 to 5 to 9. Quantified reasoning adds operations to compare these increases and conceive quantity B increasing by *increasing amounts* as quantity A increases by *successive constant amounts*. Thus, gross and quantified reasoning underscore important differences in how students can construct and reason about amounts of change. As a third motivating need for our forms of reasoning, Thompson and Carlson's (2017) framework foregrounds distinctions between images of variation, and specifically between chunky and smooth images (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). Our forms differentiate between reasoning that entails thinking of quantities as merely taking on different amounts (i.e., state and transformational reasoning) and that which entails explicit images of quantities' covariation (i.e., gross and quantified reasoning). Differences between smooth and chunky images are critical and relevant to each form of reasoning we identify, and we thus perceive our forms to complement those distinctions.

Closing

A question remains as to how an individual might conceive a drawn graph that connects the points they have produced through state or transformational reasoning. With both forms of reasoning, a student might conceive the drawn graph as capturing the infinite number of states that exist and maintain an invariant property among those states (e.g., constant fence amount and rectangular dimensions). In the case that a student has previously constructed and abstracted EGST meanings, the production of a graph via state reasoning can provide the source material for their constructing the quantities' covariation. For example, a phenomenon could be so physically and quantitatively complex that it is difficult for an individual to construct a covariational relationship within the phenomenon. They may instead determine paired values, construct an invariant property between the paired values or produce a substantial collection of them, and then sketch a graph. With the graph at hand, they could then engage in EGST to infer the covariational relationship between the quantities in the phenomenon. If compelled, they could then return to the phenomenon to judge the viability and further explore the relationship.

Having already constructed and abstracted EGST meanings, a student reasoning as above anticipates that in sketching their graph they are producing a trace constrained by covariational properties. What of a student who has not constructed and abstracted EGST meanings, but produces a drawn, continuous graph using state or transformational reasoning? Based on our work and the work of our colleagues (Byerley & Thompson, 2017; Moore, 2021; Moore et al., 2022; Moore, Silverman, et al., 2019; Moore, Stevens, et al., 2019; Thompson, 2013, 2016; Thompson et al., 2017), we are confident that without an explicit and persistent subsequent focus on quantities and their covariation, students are more likely to develop meanings rooted in indexical or declarative associations with mathematical facts (e.g., a line rising up-and-to-the-right means positive slope) than meanings stemming from QCR that can accommodate novel and unconventional situations. Without an explicit and persistent focus on quantities and their covariation, students are placed in a situation where they can associate the shape of a graph with capturing distinct paired values, invariant properties among those total values, and declarative facts, but the graph's emergence as an organic phenomenon between states is left unaddressed.

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Toward a Framework of Caring Relations in Undergraduate Mathematics Education

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Introductory college mathematics courses remain significant barriers to persistence in STEM, especially for students from historically marginalized backgrounds. While recent reforms have emphasized pedagogical changes, relatively little attention has been paid to the social and emotional dynamics of these classrooms. Drawing on work from K-12 education, we argue that relational dimensions of teaching, such as care and cultural responsiveness, are essential for supporting equitable student success in undergraduate mathematics. This paper introduces a conceptual framework for understanding caring relations in college math instruction, using insights from research on warm demander pedagogy and sociocultural theories of learning. The framework centers on the instructor's dual role in challenging students intellectually while supporting them relationally. We offer this as a tool to guide research and professional development, aiming to expand how we define and study effective instruction in ways that humanize and affirm all students.

Keywords: Caring Relation, Mathematically Caring Relationship, Culturally Relevant Care

Despite many reform efforts, introductory college mathematics, especially developmental courses (e.g., intermediate algebra) and gateway courses (e.g., calculus), remain critical barriers for many students. These courses are characterized by high failure and withdrawal rates, and many students leave them with less confidence and a weakened sense of mathematics identity (Benken et al., 2015; Valentine, Konstantopoulos, & Goldrick-Rab, 2017). Developmental mathematics sequences, in particular, often derail students' academic trajectories, especially in community colleges, and disproportionately impact students from racially and socioeconomically minoritized backgrounds (Larnell, 2016; Mishra et al., 2025; Bailey, Jeong, & Cho, 2010). Research has also shown that subsequent courses for STEM majors (e.g., precalculus, calculus) continue to serve as barriers to progressing in their chosen field. Large national studies have shown that disproportionate numbers of students, especially women and underrepresented and minoritized students, leave STEM majors after struggling in calculus even with other indicators of adequate preparation (Bressoud et al., 2015; Ellis, Fosdick, & Rasmussen, 2016; Seymour & Hewitt, 1997).

Mathematics plays a unique role as a gatekeeper to STEM fields, which makes these patterns all the more concerning. Students who hope to pursue health sciences, engineering, or other quantitative fields are often stopped short, not because they lack interest or ability. They are hindered by confusing placement systems (Ngo, 2019; Park et al., 2021), an emphasis on procedural skills over conceptual understanding, or early classroom experiences that leave indelible marks on their mathematics identities. These hurdles disproportionately affect students who are marginalized in STEM, especially those who did not have access to high school calculus or who get misrouted by current systems (Sadler & Sonnert, 2018; Leyva et al., 2021a). These barriers are rarely just academic; they are relational and emotional too. Students who have long struggled in math often need to put in far more cognitive and emotional energy just to engage in the kinds of learning behaviors that support success.

Students' ability to engage in meaningful learning is tightly linked to their sense of belonging, prior experience, self-efficacy, and the ways in which they are invited into or pushed

out of mathematical communities. For students who have not historically felt affirmed in mathematics classrooms, engagement becomes even more difficult (Leyva et al., 2021a), and research has shown that providing caring and supportive relationships with students has had great impacts on their achievement (Battey et al., 2016). Without attending to how students feel, identify, and interact in these spaces, we may not be able to adequately support all students' successful mathematics learning (Boaler, 2022; Leyva et al., 2021b).

Affective and Relational Dimensions of Instruction

While reforms in undergraduate mathematics have focused heavily on pedagogy and cognitive engagement, recent work has begun to explore the affective and relational dynamics that shape student experience and success. A small but growing body of research suggests that instructor behaviors and classroom culture significantly influence students' sense of belonging, identity, and persistence, particularly for students from historically marginalized groups. Some studies have ventured to understanding how faculty relate to their students. In their study of a Real Analysis class, Bennett and colleagues (2025) examined how instructors viewed students and analyzed demographic disparities in course outcomes. They also included survey and interview data from students, who reported that their classroom experience was strongly shaped by how the instructor responded to their ideas and contributions. Selbach-Allen and Delaney (2025) presented a case study of a successful community college instructor who exemplified the traits of a warm demander (discussed in more detail later in this paper), a relational teaching model more commonly seen in K–12 education. They described how the instructor intentionally built personal relationships with students and engaged them in conversations beyond course content, demonstrating that he understood them as full people with goals and challenges beyond mathematics.

Students' experiences have also been analyzed in an attempt to understand what supports have been beneficial to them. Leyva and colleagues (2021a; 2021b) detailed racialized and gendered experiences of students in college math classrooms along with students' perceptions that these classes were intended to weed them out from STEM trajectories. Other work has shown that STEM mathematics students have indicated that mathematics professors and peers can impact sense of belonging to the classroom due to mathematical, racial, or gendered microaggressions (Cawley et al., 2023; Cawley & Wilson, 2024). Park (2025) explored female students' sense of belonging in college math classrooms, finding that small instructor actions, such as microaffirmations or microaggressions, had measurable impacts on students' perceptions of inclusion and emotional safety. In an analysis of student math autobiographies, Cawley and colleagues (2025), found that many upper-division students reported consistently positive math experiences, and those with more affirming experiences were more likely to reference instructors in their narratives, suggesting that supportive instructor-student relationships may play a defining role in how students perceive their mathematical journey. Stone-Johnstone (2025) studied a co-requisite support course for intermediate algebra and found that students described the class as warm, welcoming, and comfortable. They highlighted the instructor's responsiveness and demeanor as key to why they felt safe seeking help—often preferring her office hours to those of the primary course instructor.

While studies on belonging, identity, and student experience are increasingly visible, the field still lacks a cohesive framing of what instructors could and should be doing to improve students' social and emotional experiences in undergraduate mathematics classrooms. This gap between widespread attention to pedagogy and relative silence around relational dynamics points to a need for further development of theoretical tools that can help us name, study, and support

this work. The culture of a mathematics classroom is critically important—especially for students who are most vulnerable due to their demographic background or previous experiences with the subject. These students need environments where they feel supported, and instructors who can successfully encourage them, push them to work hard, set high expectations, and help them navigate the inevitable barriers that arise in the postsecondary learning process.

While at first glance an undergraduate mathematics instructor's job is simply to impart mathematical knowledge to students, the instructor also impacts the way a student engages with and sees themselves as part of the mathematics community. Also known as mathematics identity (Martin, 2012), the "dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts" (p. 57) can impact a student's learning and engagement with mathematics. Research has identified that students' mathematics learning and identity has often been impacted by their previous experiences (Cawley et al., 2025), and that in many cases this is influenced by their instructors' beliefs, supports, and interactions (Andersson & Wagner, 2017). Mathematics trauma has also been identified to impact students resulting in a negative change in their disposition towards the subject, impacting their mathematical learning (Gunter & Gunter, 2023). General discourses around mathematics (e.g., "I'm not a math person", "mathematics is objective") can also impact a student's mathematics identity (Anderson, 2007). Instructors need to be aware of these possible experiences that are specific to mathematics and consider the power of their interactions with their students.

We argue that learning is a socio-cultural endeavor (Vygotsky, 1978), and that an instructor plays a large role in the way a student engages in their mathematics learning. That is, teaching mathematics content is not the only role that an instructor has—the instructor also nurtures and supports a student's mathematical development and sense of belonging. Evidence has shown that students need to be in an emotionally engaged state to be open to the processes in which we meaningfully learn (Immordino-Yang & Damasio, 2007). A student must feel enough trust in the learning environment to feel vulnerable enough to engage in deep learning (Delpit, 2013; Schonert-Reichl, 2017).

Notion of Care in Mathematics Education

We believe that undergraduate math instructors care about the work they do. We believe they care about their students and imparting mathematical skills to them. However, studies have shown that having high expectations about mathematical understanding alone is not sufficient to support students' mathematical learning and development (Lee et al., 1999). The notion of *care* has been discussed in relation to teaching and learning, and specifically in K-12 instructional spaces (Wentzel, 1997). Care is seen as a fundamental basis of humanity (Heidegger, 1927) and given the nature of teaching, Watson (2021) argues that teaching should be seen "as a caring profession…not only for nurturing intellectual subject-based development but also for the wellbeing of students" (p. 79). Care can be seen as both a virtue ("He is a caring person") or a relation (a relationship between two people; Noddings, 1999). Noddings (2002) contributed a theory of care in which both the teacher and the student participate, requiring reciprocity from both the teacher and the student. In her theory, a *caring relation* is one in which a teacher demonstrates care for a student(s), but one in which a student must show evidence that the care was received. The evidence by a student could be explicit (a smile or "thank you") or implicit (the student finishes a problem successfully or is eager to share a thought in their group; Noddings, 1999). This cyclical communication helps the instructor understand that care has been received, thus maintaining this relation. In this same way, it provides feedback to the instructor if

specific caring actions are not being received. Structuring *caring* as relational provides identities for both the cared-for and the carer.

Hackenburg (2010) extends Noddings' notion of a caring relation and defines a *mathematically caring relationship* "as a quality of interaction between a student and a teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning" (p. 237). Hackenburg (2005) argues that mathematics teachers may act as carers, but that they start "to act as mathematical carers when they work to harmonize themselves with and open new possibilities for students' mathematical thinking, while maintaining focus on students' feelings" (p. 47). This indicates that a mathematics instructor must have pedagogical content knowledge (Ball, Thames, & Phelps, 2008) to know and understand how to help students cognitively arrive at the learning outcome while also being aware of students' feelings. Finding that balance within each student (and a group of students) contributes to this caring relation in a mathematical way.

While Noddings and Hackenburg developed the ideas of care that can be envisioned within mathematics education, they do not attend to care in a critical way. In 1975, Kleinfeld attempted to address the issue of classroom instruction of Native Alaskan students who were placed with non-native teachers. In many cases, this pairing destroyed students' cultural identities in the name of "education". Kleinfeld did see examples, however, of teachers who were successful in building relationships with their students that utilized and respected cultural traditions, leading to successful learning outcomes. Based on her findings, Kleinfeld defined the notion of *warm demander*, which was a teacher who showed both personal warmth and also "active demandingess". That is, a warm demander shows care both in showing concern and attention to students in culturally relevant ways while also holding high academic expectations, demonstrating a belief that students can achieve such goals (Bondy et al., 2013).

In a different vein, Maloney and Matthews (2020) highlight two types of care in their work: transactional care and empathetic care. Transactional care is received by a student from an instructor who is a cheerleader, yet the instructor is not active with the student and waits for initiative from the student. Instructors who engage in transactional care may see students in a deficit mindset or as unmotivated if they do not initiate with the instructor. Instructors that exhibit transactional care demonstrate a care that is inert and a "one size fits all" care for all students, demonstrating that students are "cared about" but not necessarily cared for. The authors separate this from empathetic care, which is an active and constantly shifting care, in which the instructor sees students as complex beings and celebrates mathematical strengths while scaffolding mathematical weaknesses. Instructors who show empathetic care leave students feeling cared *for*, not just cared *about*. Maloney and Matthews found that students can sense the difference between these two types of care exhibited by their instructors, and that students who felt most connected to their math classroom were those with teachers showing empathetic care.

Considering the notion of warm demander and the value of empathetic care, we contend that a caring relation is also attuned to the identities of the student as well as the teacher, understanding that both identities can shape/influence that caring relation. In most considerations of care, "inadequate attention is given to issues of power" (Thompson, 1998), and therefore it is extremely important for an instructor to understand and develop awareness of their identity and inherent power as they demonstrate care for their students. *Culturally relevant care* is "characterized by a strong sense of community, rigorous demands, an integration of different cultures, and a general affirmation of one's humanity" (Watson, Sealy-Ruiz, & Jackson, 2016, p. 981). It extends beyond traditional care presented by Noddings (2002) in that instructors are "expected to recognize the inequities faced by students based on race [or other marginalizations]

and work to challenge them" (Watson et al., 2016, p. 985). Culturally relevant care requires an asset-based, justice-oriented perspective for students traditionally marginalized in the schooling system. Culturally relevant care "acknowledges the goal of justice to safeguard the rights of all students—to be respected and to be treated fairly" in which instructors need to have an understanding "of systemic, political, economic, and social structures that disproportionately appropriate opportunity according to race and a commitment to challenge and alter them" (Parsons, 2005, p. 26).

However, through the lens of culturally relevant care, what is seen as "caring" teacher behaviors can be easily misinterpreted as people show and interpret emotion, interest, respect, and body language differently in different cultures (Averill, 2012) and specific "caring" practices can be experienced differently by different students (Thompson, 1998). Cawley & Wilson (2023) describe mathematical microaffirmations as the subtle actions that can support students by affirming they belong in mathematics yet caution that some microaffirmations for students may be received as microaggressions due to these differences.

Because of this, we argue that the cyclical relationship in a caring relation is extremely important to consider. This feedback loop can help instructors to understand how their demonstration of care is being received and make subtle changes to ensure that students are receiving care in the way that supports them best. This means that an instructor needs to understand that caring relations with individual students will be uniquely different. It also implies that an instructor needs to be acutely aware of their identities (e.g., race, gender, language, educational background, etc.) and the way these impact the power dynamic within a caring relation. Being in touch with one's own identity can help bring perspective on the privileges and marginalizations that may exist within a caring relation.

Toward a Conceptual Framework of Care

This paper proposes a conceptual framework of a *caring relation* within the postsecondary mathematics classroom that attends to the theoretical perspectives above. Figure 1 shows a conceptual framework of care in an undergraduate mathematics classroom. A caring relation between an instructor and a student is cyclical and requires interactions from the instructor (the carer) and the acknowledgment of the caring actions by the student (the cared for), whether explicit or implicit. An instructor's interactions should be influenced by:

1) high mathematical expectations for the student which is informed by the instructor's pedagogical content knowledge (PCK) about the mathematics content being learned and for future mathematics content yet to be learned
2) reflection on their personal, cultural, and mathematical identities, bringing awareness to their privileges and marginalizations within the caring relation, acknowledging power dynamics as instructor
3) culturally relevant care in which the instructor learns about students' cultural and lived experiences and incorporates that into the classroom while also being apprised of current systemic, political, and social structures that may impact their students

In this conceptual framework, we see the instructor's caring interactions as malleable based on the three attributes above and that these attributes can be developed and refined both over time and on a student-by-student basis. While instructors cannot control how students respond or receive interactions, we believe that instructors can develop their interactions based on these attributes to shift and better attune care towards their student.

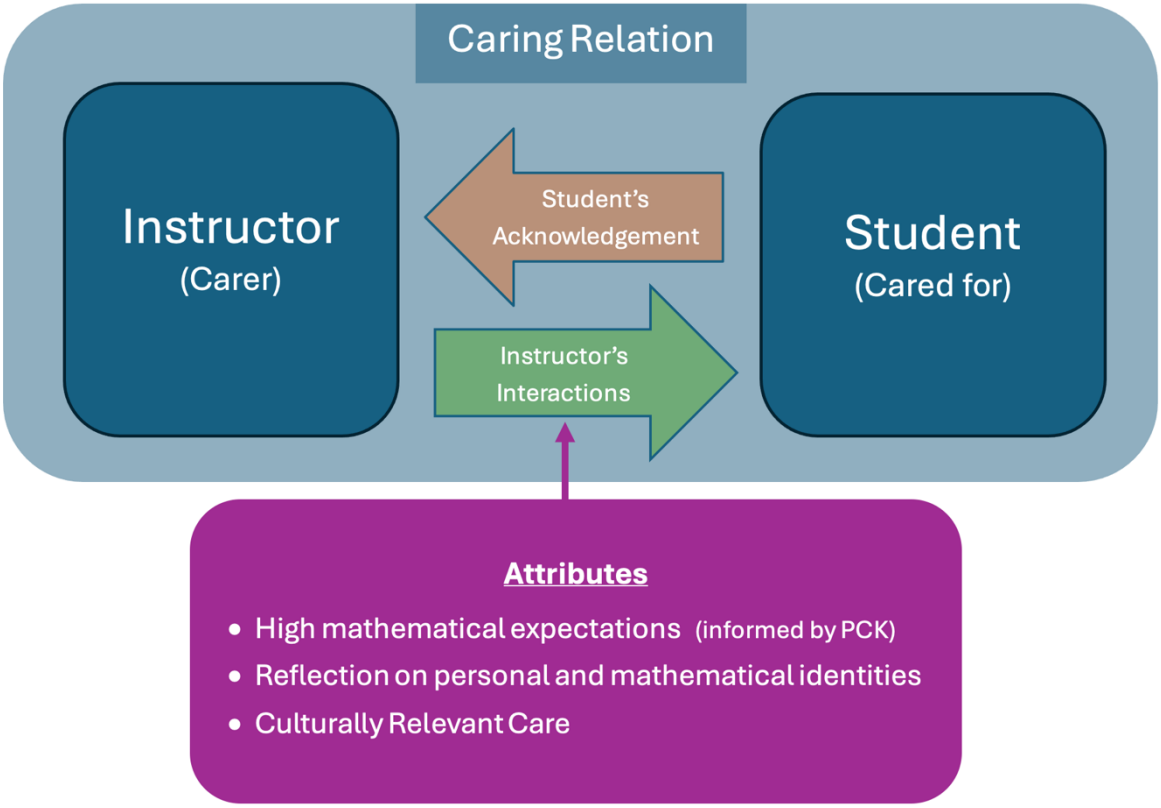

Figure 1. Conceptual Framework of Care in an Undergraduate Mathematics Classroom

So Why Does this Matter for College Mathematics?

We claim that colleges and universities are a unique environment in which to consider this work. College instructors often receive little or no pedagogical training when completing their graduate degrees (Robinson & Hope, 2013) and therefore may not have the skills and tools needed to engage with students. Providing a framework like this with attributes that can be considered in the effort to build a caring relation with students helps instructors to consider the different aspects needed to support a student. This is a unique setting also in that college students are generally adults who may hold different identities/obligations than the instructors. Instructors therefore are expected to enact care with fellow adults, which is seen as different than enacting care with children. College and university students usually hold more responsibility, life experience, and obligations than students in K-12 making it especially important to show sensitivity and culturally relevant care.

We posit that effectively communicating care and building caring relations with students is an important aspect of effective mathematics instruction at the undergraduate level. In a college setting students are transitioning to and expected to be independent learners. Their level of meta-skills to plan and attend to their own learning becomes increasingly important. These meta-skills are also often tied to affective factors that impact students' comfort with engaging with material or inclination to avoidance or procrastination. Higher education is also a choice – students make the choice to pursue college and so it is possible for students to drop out or stop out. Mathematics is well known to be a major reason why students either drop out, stop out, or change majors from STEM to non-STEM (Chen, 2013) and students sometimes struggle with the change from K-12 mathematics expectations to college-level expectations (Hernandez-Martinez,

2016). Students often carry mathematical experiences that impacted their sense of belonging in college mathematics, shaping their perspectives in mathematics and their ability to do mathematics (Cawley & Wilson, 2024). Therefore, mathematics instructors need to make concerted efforts to demonstrate care towards students.

We also have found in our experiences with professional development with community college and university mathematics faculty that the sense of care and concern shared in the aspect of instruction is most usually placed/emphasized on the *mathematical content*. This is often pushed by the urgent need to meet course outcomes and to prepare students with prerequisite material for the subsequent courses. Therefore, the attributes of care mainly focus on the mathematics content, and many times this can be done without pointed concern for the student or setting mathematical expectations that are both reasonably in a student's grasp. Our framework posits that it is simply not sufficient to care that a student "gets" the material (research has shown that students do not always carry forward prerequisite knowledge; Engelbrecht, Harding, & Du Preez, 2007) – students need to be seen and supported as who they are, and this requires more attention and intention. Some mathematics instructors teach with the hope that their students will see beauty in mathematics just like they do. However, it may take a long time for some students to warm up to mathematics as a field and feel confident and happy as a doer of mathematics. We believe that if a student has a caring professor who attends to these attributes in their interactions with students, students will be able to work on their dispositions towards mathematics. In fact, students tend to feel an obligation towards a teacher who acts as a warm demander even if they may not show interest in mathematics (Delpit, 2013). We see this as an intermediary step to bringing students into developing a better relationship with mathematics and shaping their mathematics identity to be one of confidence and positive disposition.

Conclusion

In this paper, we have argued that caring relations are an essential yet under-theorized component of effective undergraduate mathematics instruction. By synthesizing prior research on care, identity, and relational teaching models such as warm demander pedagogy, we proposed a conceptual framework that centers the instructor's role in building supportive and responsive environments for students. This framework underscores that care is not simply a personality trait or a passive orientation, but rather an active, culturally attuned, and pedagogically informed set of practices. We argue that such caring relations are especially important for students from historically marginalized backgrounds, for whom traditional mathematics instruction has often been alienating or exclusionary. Instructors who wish to support students' mathematical development must attend to their students' cognitive, emotional, and identity needs, not as separate concerns, but as deeply intertwined aspects of learning.

This framework offers a starting point for further research and dialogue. Empirical studies can explore how instructors enact care in varied postsecondary settings, how students perceive and respond to these actions, and what institutional structures best support this work. We also invite researchers and practitioners to consider how care interacts with issues of power, positionality, and discipline-specific expectations in college mathematics. By better understanding and enacting caring relations, we believe instructors can help more students persist, thrive, and reimagine their place in the mathematical community.

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MATH-AI: An Integrated Conceptual Framework for Student-AI (LLM) Interactions

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With the rising popularity of generative artificial intelligence, particularly large language models (LLMs), it is imperative to understand how students are leveraging these rapidly evolving tools in their learning. Although researchers have begun to address issues of academic integrity, acceptance, and adoption, few conceptual frameworks exist for analyzing how students interact with LLMs in the context of mathematics problem solving. This paper introduces the MATH-AI framework, a synthesis of theories of metacognition, student agency, trust in automation, and human processing and regulation to examine how AI mediates math problem solving and potentially reconfigures the learning process. A case study of two students is provided to demonstrate the framework's application and illustrate distinct patterns of trust, processing, and self-regulation during AI-mediated mathematics problem solving. These cases highlight the potential risks and opportunities associated with LLM use and provide a foundation for future empirical studies and development of subject-specific AI guidelines.

Keywords: Undergraduate Mathematics, LLM, Artificial Intelligence, ChatGPT, Learning

Despite rapid adoption and integration of generative artificial intelligence (GenAI), especially large language models (LLMs), within higher education, little is known about how these tools shape cognitive processes during math problem solving. Influenced by institutional promotion and social normalization (Cialdini et al., 1991), many have come to view adoption of LLMs (e.g., ChatGPT, Google Gemini) as inevitable. Nonetheless, some worry about how prolonged use may interfere with skill development and learning (Nguyen et al., 2025).

While studies have addressed human-AI interactions within the university context, they often focus on issues of acceptance and adoption (Ivanov et al., 2024; Xu et al., 2025), academic integrity (Almisad & Aleidan, 2025), and perceptions among students and faculty (Tsiani et al., 2025). These studies provide useful insights about how individuals make decisions about how to use GenAI but do not address how such tools reconfigure student thinking and reasoning.

This paper proposes a framework for analyzing student-AI interactions with the aim of revealing conditions that support cognitive engagement and processes essential for learning math. While models of AI-human interaction typically center factors that influence adoption, perceived utility, or ethics, this framework highlights foundational academic practices such as studying, reasoning, and ultimately learning. We draw on frameworks from cognitive science (i.e., cognitive processing), educational psychology (i.e., self-efficacy, self-regulated learning), and human-computer dynamics (i.e., trust) to develop an interdisciplinary model for examining student-AI interactions during mathematics problem solving. By applying an interdisciplinary, empirically guided approach, this model serves as a conceptual tool for examining student-AI interactions and as a foundation for subject-specific, evidence-based AI use guidelines.

Theoretical Framework

Student Beliefs & Technology Adoption

Students who believe in their ability to learn often work harder, persist longer, and perform better academically (Bandura, 1997). At the same time, students hold different beliefs about their

abilities according to domain (e.g., math abilities vs. writing abilities). On a smaller scale, they gauge task-specific *self-efficacy*: whether they believe they can perform specific actions related to a task (Bandura, 1986; Pajares & Miller, 1995). Students with low *math* self-efficacy (Li et al., 2021) often exhibit higher levels of math anxiety, negatively impacting performance (Beilock & Maloney, 2015). In contrast, feelings of competence and autonomy have been linked to cognitive activation (e.g., attention, engagement, and thinking; Lu et al., 2022). Researchers have noted the important role teachers play in supporting students' task- and domain-specific self-efficacy by providing targeted feedback and guidance (Lu et al., 2022; Van der Beek et al., 2017). Open questions remain whether students prompt LLMs like ChatGPT in ways that result in such support, and whether they act on generated feedback during math problem solving.

Apart from beliefs about their own abilities, students' beliefs about chatbots' abilities may also influence use. Research on attitudes towards technology adoption suggests that positive attitudes often predict adoption and use (Ajzen, 1991). In the case of ChatGPT, students who hold positive attitudes extract more benefits by asking more detailed questions or experimenting with prompting strategies, leading to more productive uses than those who remain skeptical (Zhuang, 2025). However, students' trust in generated output can also enable overreliance. Theories of trust in automation describe over trust as a form of miscalibration whereby users rely on an agent to perform tasks beyond its capabilities. On the other hand, distrust often leads to underuse or disuse (Lee & See, 2004). Trust, therefore, guides reliance; a student who over trusts an LLM may override their own thinking/reasoning, uncritically adopting inaccurate output while students who avoid the tool may miss out on opportunities for support.

Cognitive Processing, Offloading, & Self-Regulation

Students' perceptions and beliefs manifest in how they engage both with concepts and tools during problem solving. Prompting LLMs may reduce memory demands by providing immediate information. Decreasing the working memory requirements of a task can reduce cognitive load (Paas et al., 2003), but the benefits of cognitive offloading (reducing cognitive demand by externalizing some part of a task) are contingent on whether offloaded demands are required to realize the learning goal (Risko & Gilbert, 2016). Offloading *mastered* procedures (e.g., arithmetic, symbol manipulation) may free cognitive capacity for conceptual reasoning, whereas offloading reasoning processes central to the learning goal may threaten learning.

To investigate when offloading supports or undermines learning, it is necessary to consider the types of cognitive processing students engage in during AI interactions. Craik and Lockhart (1972) differentiate shallow from deep processing where shallow processing consists of surface-level encoding (e.g., tending to appearance or sound), typically resulting only in short-term retention. In contrast, deep processing involves creating associations, facilitating long-term retention (McLeod, 2025). Between shallow and deep processing, the ongoing and sometimes incomplete process of self-explaining promotes integration of new information (Chi et al., 1994).

In addition to cognitive processing, students' capacity for self-regulation further shapes how they engage with LLMs. Self-regulated learning involves setting goals, monitoring one's own understanding, and adapting strategies in response to feedback (Zimmerman, 2000). However, unlike human instructors, LLMs do not provide formative feedback on students' progress toward conceptual understanding unless they are prompted to do so. Students must also monitor their progress while tending to the accuracy of output. Strong self-regulation skills can reduce uncritical acceptance of LLM output, while the lack of such skills may leave students vulnerable to over trusting tools and/or disengaging from problem solving altogether.

Practicing problem solving is a key aspect of studying mathematics. Studying, a key form of self-regulated learning, usually occurs without direct expert/teacher guidance, requiring students to rely on their own strategies or collaborate with peers. It often involves gathering and synthesizing information and usually results in the production of observable indicators of cognitive processing such as notes, annotations, or summaries (Winne & Hadwin, 1998). LLMs introduce a new dynamic; these tools can generate similar outputs, potentially reconfiguring how students approach studying. To understand this shift, it is important to examine the self-regulation strategies students employ when working with LLMs.

Potential Outcomes of AI Reliance During Math Problem Solving

Beliefs, patterns of use, and self-regulation determine the outcomes of students' interactions with LLMs. Depending on the quality of cognitive engagement, AI can support learning or pose significant risks including metacognitive miscalibration, changes in overall self-efficacy, and entrenchment of robust misconceptions. Recent reports (OpenAI, 2025) announce that ChatGPT has shown its strongest performance on academic tasks to date. As LLMs improve and grow in popularity, students may begin to treat them much like the internet, as general-purpose stores of knowledge. Studies have shown that internet searching can create the illusion of knowledge as individuals mistake access to information for a reflection of personal understanding, even in the absence of understanding (Fisher et al., 2015). These experiences may be further exacerbated by *metacognitive miscalibrations* where students overestimate their own knowledge (Lichtenstein & Fischhoff, 1977), and by new AI-specific challenges where students overestimate tool accuracy.

AI systems are designed to promote engagement, not challenge users (unless prompted to do so) and are more likely to reinforce assumptions embedded within prompts. Thus, an LLM is more likely to enable robust misconceptions if students interpret output as confirmation of understanding. Robust misconceptions are especially resistant to change and may become further entrenched rather than corrected in the absence of opposition (Smith et al., 1993).

Persistent reliance on LLMs may also influence math self-efficacy. If miscalibrations lead to overconfidence, lack of knowledge may be made apparent through formal assessments and academic performance. Furthermore, shallow forms of processing may become habitual through repetition. These risks can negatively affect students' learning outcomes and even their agency as they come to depend on LLM tools in ways that limit their abilities to regulate and sustain their own learning (Krook, 2025). Taken together, these constructs highlight the complex ways in which LLMs may interfere with cognitive processes and subsequent learning outcomes, providing a foundation to theorize student-AI interactions as part of the math learning process.

Conceptual Framework

As GenAI becomes increasingly present in educational contexts, we must clarify the cognitive and social processes students engage with when working with AI and identify frameworks that help math educators integrate these tools without undermining learning. Guided by these goals, the present framework emerged as a response to observations of students interacting with AI in ways that reflected less-than-optimal use. This framework is empirically informed by recurring patterns observed in interview data with undergraduates solving algebra problems with ChatGPT. Students in this study often operated in a space between having no strategy and having some relevant prior knowledge that *could* be activated. In these moments, ChatGPT enabled them to pursue solution pathways and eventually produce answers. However, interactions rarely resulted in advancements in conceptual understanding. Many students accepted output without fully integrating it with their own mathematical thinking. These

empirical insights illuminated the need for constructs that highlight levels of cognitive processing during AI interaction. They also underscored the need for interdisciplinary knowledge that captured students' self-perceptions, metacognitive monitoring, and sociocultural perspectives on tool use to describe the complex ways in which students navigate AI assistance/availability and their own cognitive faculties.

To capture the complex interplay between students' beliefs and perceptions, cognitive processes, and how these manifest in LLM use patterns, we propose the MATH-AI framework which encompasses five key dimensions: *metacognition*, *agency*, *trust*, *human processing/regulation*, and *AI mediation*. This framework integrates established constructs to address emerging challenges in mathematics education and offers a structure for educators and researchers to analyze use patterns and link them to potential benefits and risks. The five dimensions of MATH-AI are conceptualized as follows:

1. ***Metacognition*** refers to students' abilities to monitor and evaluate their own knowledge and performance. It captures how students assess their understanding during AI interaction, whether monitoring supports or undermines cognitive engagement.
2. ***Agency*** refers to students' sense of ownership/control over their learning. It is shaped by beliefs about their ability to learn and do math (self-efficacy) as well as their willingness and motivation to engage in the learning process. It captures students' orientations toward using AI (as a scaffold or a crutch). This is a dispositional construct that highlights whether students position themselves as active rather than passive in their engagement. Research has established connections between self-efficacy and motivation, yet little is known about how these are influenced by AI interaction. Early findings suggest a tendency of students to depend on AI, diminishing their agency (Darvishi et al., 2024).
3. ***Trust*** encompasses students' judgements about output reliability. Particularly, whether trust is appropriately calibrated to specific tasks or if it enables overreliance.
4. ***Human Processing & Regulation*** captures the cognitive/metacognitive moves students make during problem solving including the *depth* of cognitive engagement and the *regulatory* actions taken to monitor, evaluate, and adapt strategies (e.g., interrogating/verifying/seeking explanations). While agency reflects willingness and stance toward learning, processing and regulation describe enacted behaviors and strategies resulting from that stance. This construct draws attention to how students position AI in practice, as either a collaborator/scaffold or a shortcut to bypass deeper processing.
5. ***AI Mediation*** operates through all prior dimensions of the framework, highlighting the ways AI tools intervene in learning processes. For example, by offering scaffolds, feedback, or alternative representations or by encouraging reliance that diminishes agency and depth of processing.

These five dimensions are operationalized across three phases: input, interaction, and output. Inputs include students' preconceptions, beliefs and internalized views. Interactions encompass how students engage with ChatGPT, including prompting and verbalized processing. Outputs capture potential consequences of sustained patterns of use, including habitual cognitive processes, impacts on agency, and the risks/opportunities for learning. Each of the phases also incorporates a feedback mechanism consisting of students' reflections during and after AI use.

Students' self-efficacy (*input*) largely contributes to persistence, efforts to seek understanding, and the types of follow up questions they formulate. When working with AI, students with higher domain- and task-specific efficacy may be less likely to cede important

problem-solving actions to an LLM. Similarly, perceptions of tool efficacy influence attitudes, trust, and choices about whether and how to use such tools during math problem solving.

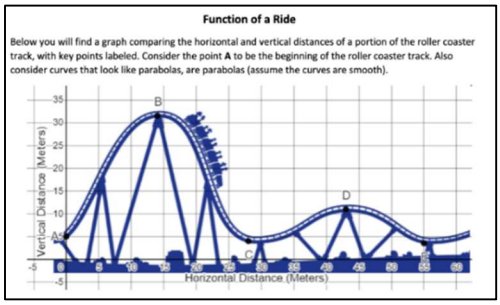
During AI *interaction*, students make decisions (perhaps subconsciously) about the depth of cognitive processing they engage in and their level of reliance. Importantly, a student may exhibit high levels of *productive* reliance. For example, using an LLM to generate practice problems, critique their reasoning, or explanations can promote self-regulation by helping students monitor their knowledge and discuss misconceptions. In contrast, reliance that offloads reasoning without verification may stall learning.

Sustained patterns of use may develop into habits that either support or threaten learning (*output* phase). Outcomes depend on prompting strategies and maintaining effort during problem solving. Over time, these outcomes can have lasting effects on students' broader sense of self-efficacy, metacognitive abilities, and ultimately student agency.

MATH-AI Framework Application

To illustrate application, we draw from a subset of data collected as part of a larger project investigating if/how students engage in productive struggle when working with ChatGPT. This subset includes undergraduates who completed an algebra task, Figure 1 during think-aloud interviews. Algebra was selected because it is a subject most undergraduates share prior knowledge/course experience in, unlike the other tasks involving statistics and calculus. During interviews, researchers encouraged students to share their thinking.

We present two contrasting cases not meant to represent generalizable findings but meant as illustrative applications of the framework and as a starting point for the development of subject-specific guidelines. Analyses are organized according to process order (input → interactions → outputs) to show how student beliefs may influence the ways they interact with ChatGPT and how these interactions can lead to specific outcomes.



(a)

1. What is the domain and range of the function?
2. Find the intervals where the function is decreasing and increasing. How did you know?
3. Where would you scream? Describe your ride as you travel the roller coaster. Include in your description your trip from point to point, whether you are moving up or down, and discuss what is happening to your speed.

(b)

Figure 1. Image Provided with the Roller Coaster Task (a) and the first 3 questions included in the task (b).

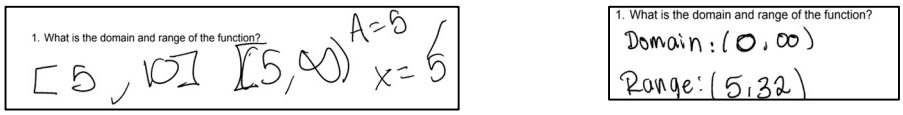
Case 1: High Trust, Shallow Processing

Despite expressing reservations, Participant 1 (P1), a food and nutrition major, described high behavioral reliance across coursework and a positive attitude toward ChatGPT for math problem solving: "I started using it because I saw how easy it was. You just put in the question …my friends were like, 'you don't use ChatGPT? Why?' I was just like, I don't know, I just don't like the cheating of it." P1's experiences are of particular interest because they show how LLM tools may be perceived as detrimental while simultaneously being promoted through social and institutional norms: "It made me lazy, I feel dumb. I don't feel like I'm learning…I don't think it's fair that [the institution] is pushing it on students…how do you expect students not to cheat if you're giving us ChatGPT?" P1's problem solving involved reading tasks, assessing their own knowledge, and prompting ChatGPT whenever they perceived a gap.

Inputs. P1 exhibited low self-efficacy both in general, as evidenced above, and across tasks: "I don't even know how to solve this…I haven't heard domain and range since high school so I'm asking ChatGPT to explain." Their positive attitude towards ChatGPT for math problem solving enabled a high level of trust, evidenced by unquestioned acceptance of generated outputs. When asked to explain their solution, they replied: "[ChatGPT] is giving me an example, the domain is -2 to 4 and it's saying that it goes upwards…so in this case it would be x = 5 because that's where it starts and then it ends at x = 4, in their case that's where it would end, and in my case I would say…x = 10 so I believe my domain would be that."

Interactions. Subsequently, P1 processed output only at a shallow level: "I'm just using their example to connect it to mine as much as I can." P1 did not monitor their own understanding or knowledge acquisition and ultimately did not identify and fix errors: "I honestly think it did a good job at [explaining domain and range] …I still don't fully understand…but it did help me understand just to solve the question." This reflection suggests that P1 did not actively participate in solving and instead ceded the process to ChatGPT. Poor performance coupled with high confidence in generated output resulted in miscalibration as P1 expressed high confidence in the solutions they provided, shown in Figure 2a, despite their inaccuracy.

Potential Outcomes. Sustained use of LLMs which follow this interaction pattern can lead to substantial detriment as users may repeatedly practice incorrect procedures that further entrench misconceptions and unproductive habits. In this case, shallow processing resulted in incorrect solutions and misconceptions without P1's awareness. More concerning than the incorrect answers, however, was the lack of evidence of developing conceptual understanding on topics the participant clearly expressed confusion about before interaction. P1 continued to struggle with domain and range after interacting with ChatGPT, despite having encountered these concepts in high school. This pattern illustrates how high trust combined with shallow processing can lead to miscalibration, potentially impeding the development of mathematical competence while also eroding students' broader sense of agency in their learning.



(a) (b)

Figure 2. Written Answer to Question 1 Provided by Participant 1 (a) and Participant 2 (b).

Case 2: High Trust, Deep Processing

Participant 2 (P2), a business management major, reflected on how ChatGPT has changed their approach to mathematics: "I remember, if I needed to do a math problem, looking it up on the internet and finding places where it didn't cost money…it would just be hard. I would always have to figure it out on my own." They also described how sustained AI reliance seemed to undermine their learning: "I'm not doing very well in the [math] class…at the beginning of the year, when I didn't understand stuff…I would just put it into ChatGPT…but now I'm trying a lot harder to do it on my own, so that I understand it for the final."

Inputs. P2 described low self-efficacy and high reliance. They also expressed efforts to shift away from dependence and toward building understanding. P2 described a high level of trust in ChatGPT, noting how easy it was to complete assignments by prompting the tool.

Interactions. P2 prompted ChatGPT for conceptual help, reflecting: "I always have to remember domain is this way [traces a horizontal plane] and range is this way [traces a vertical plane]," before prompting: "how to find domain and range." However, ChatGPT's lengthy,

decontextualized output caused frustration: "That makes no sense…I don't know how that helps me find the domain." Rather than abandoning the tool, P2 re-prompted with, "when looking at a graph with porabalas [sic]." P2 reflected on output: "this made me think about how to find it with numbers, so I said by actually looking at a graph" and exhibited deep processing, making connections both to real-world contexts and prior math knowledge:

> This [the right of the graph] has to connect with this [left end of the graph] eventually, right? It's not just going to be cut off…this has to come back to point A, which makes me feel like the domain is all real numbers…but a rollercoaster isn't infinite…obviously this is the end, so I feel like it has to stop here.

Potential Outcomes. P2 transitioned from frequent prompting to relying on their own thinking and integrated ChatGPT's examples with their own ideas: "If it's like the example, then the range is negative infinity to 32…but then there's the bottom of it [points to the bottom of the graph] …so I think it has to stop there." Although P2 struggled to interpret output, additional prompting enabled progress toward making sense of domain and range in the situated context of a rollercoaster. P2's written answer is shown in Figure 2b. Unlike P1, who relied on ChatGPT and shallow processing, P2 engaged critically with output. This pattern of use can support learning as students question their understanding and make connections to both prior knowledge and real-world contexts.

Discussion

The MATH-AI framework focuses on how meta/cognition, agency, trust, human processing, and AI interaction intersect to reconfigure the ways students approach common mathematics processes such as problem solving and studying. While this study was limited to two cases, it provides a foundation for empirical analyses to further extend and refine the framework, explore its applicability/comparisons across diverse student populations, and inform the design of interventions to guide students towards productive uses of LLMs for learning math. For example, a self-regulation framework informed by these analyses might prompt students to ask: Can I explain this concept without assistance? Can I solve a similar problem on my own?

To best support mathematics self-efficacy, self-regulation and cognitive processing, educators must intentionally promote productive uses of LLMs. Regardless of instructor views, AI is here to stay. The critical question now is how its mediation will shape students' dispositions and strategies for learning mathematics. As social-cognitive theories remind us, self-efficacy develops through mastery and regulation, while models of normative behavior suggest that students' beliefs about these tools may ultimately be overshadowed by widespread availability and adoption. At the same time, theories on cognition highlight the potential for LLM scaffolding and the risk of offloading essential cognitive processes. Together, these perspectives emphasize that the integration of AI in math education can lead to starkly different outcomes: either supporting learning or entrenching patterns of dependence that reduce students' opportunities for deep thinking and cognitive development. By applying the MATH-AI framework to research and teaching practice, educators can gain a deeper understanding of how to support mathematical thinking and learning in an AI-powered world.

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Mathematical Modeling Knowledge and Affect for Teaching: A Holistic Framework for the Competencies Required to Teach Mathematical Modeling

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In this theoretical paper, we build on prior frameworks on teacher knowledge and affect to propose a holistic framework for the competencies required for teaching modeling—a framework that integrates teacher competencies from both the cognitive and affective domains. We illustrate elements of the framework through the case of one secondary teacher's work from a modeling-focused professional development program. Finally, we describe the implications of our framework for researchers and educational stakeholders.

Keywords: mathematical modeling, teacher knowledge, affect, modeling competencies

Mathematical modeling (hereafter referred to as modeling) is the process of using mathematics to describe and make predictions about real-world situations. As modeling has increased in popularity across K-16 education, the modeling education field has developed numerous programs to help educators effectively teach modeling. For example, over the last two decades, research programs have designed modeling-focused professional development programs for teachers (e.g., Jung & Brady, 2016), investigated the competencies required to teach modeling (e.g., Borromeo Ferri, 2018; Greefrath et al., 2022; Wess et al., 2021), and introduced mathematical modeling in teacher-education programs (e.g., Anhalt et al., 2018). However, this research has predominantly focused on the knowledge required for teachers to teach modeling from a cognitive point of view and often emphasizes a pedagogical perspective (over a subject matter perspective). Yet, to develop a framework that provides a "complete picture" of the competencies required to teach modeling, there is a need to include both cognitive and affective perspectives (Thompson, 1992).

Chamberlin (2019) stated that modeling, cognition, and affect are intertwined with one another, and he proposed a theoretical framework that connects these three constructs. According to his framework, there is an interrelationship between cognition, affect, and modeling where each element influences one another and are in a continuous positive feedback loop with one another. That is, according to Chamberlin's (2019) framework, to successfully model, one needs to engage in successful cognitive and affective processes. Similarly, we contend that to *teach* modeling successfully, a teacher should acquire both cognitive competencies (i.e., knowledge for teaching) and positive affect (i.e., affective competencies required for teaching). In this theoretical paper, we propose a framework for the competencies needed for teaching modeling that combines both knowledge (cognition) and affective perspectives for *teaching* modeling. In light of this goal, we first provide the relevant background that supports the development of the framework. Next, we provide illustrative vignettes of the framework components using data from a secondary teacher, Olivia (pseudonym). Finally, we highlight the implications of our proposed framework for the modeling teacher education community.

Background

The Knowledge Domain

Several frameworks have long shaped research on teacher knowledge and competencies for teaching mathematics. For example, Shulman (1986) categorized teaching mathematics content knowledge into subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Building on Shulman's (1986) work, Ball et al. (2008) presented a Mathematical Knowledge for Teaching (MKT) framework by unpacking SMK and PCK into more fine-grained domains. These include common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), horizon content knowledge (HCK), and knowledge of content and teaching (KCT). CCK refers to the general mathematical knowledge, not specialized or unique to teachers. SCK pertains to technical knowledge unique to teaching (e.g., looking for patterns in students' errors). KCS represents teachers' understanding of how students think and learn and the ability to anticipate what students find confusing or engaging. Teachers' awareness of how mathematical topics connect across the curriculum and discipline is associated with their HCK. Finally, KCT combines mathematical knowledge with teaching practices like sequencing content and identifying suitable instructional methods (Ball et al., pp. 399–403).

Although Ball et al.'s (2008) framework offers important foundations for mathematical teaching, few frameworks have described the additional demands required for teaching modeling. For example, Borromeo Ferri and Blum (2010) proposed a framework for the competencies required for teaching modeling. Their framework proposed that teachers of modeling should acquire skills along four dimensions: (i) theoretical (i.e., knowledge about modeling cycles, and modeling), (ii) task (i.e., knowledge about modeling tasks), (iii) instructional (i.e., knowledge about implementing modeling tasks), and (iv) diagnostic. Building on this framework, Wess et al. (2021) presented modeling-specific PCK by outlining four facets of knowledge: (i) knowledge interventions (i.e., ability to apply adaptive strategies and to assess their effect on learners' solution process), (ii) knowledge about modeling processes (i.e., competencies of diagnosing students' modeling phases), (iii) knowledge about modeling tasks (i.e., the skills in understanding modeling task characteristics and analyzing and developing them), and (iv) knowledge about modeling aims and perspectives (i.e., knowledge of modeling cycles and their suitability for various purposes in classrooms).

The Affect Domain

Although the interconnection between affect and cognition remains debated, affect is generally considered as consisting of the "non-cognitive aspects of human thought" (Hannula, 2012, p. 137). Research on mathematical affect takes various perspectives (Chamberlin, 2019), but the most prominent of these perspectives is McLeod's (1992) three-dimensional framework for affect. The framework comprises emotions, attitudes, and beliefs, with beliefs representing the most cognitive component and being less amenable to change (see Table 1 for the definitions). Di Martino and Zan (2011) proposed that attitudes function as a bridge between emotions and beliefs. Consistent with this view, DiMartino and Zan (2011) proposed a framework for attitude where emotional disposition, vision of mathematics, and perceived competence were included. In Di Martino's and Zan's (2011) framework, emotional disposition is defined as one's likes/dislikes towards mathematics, vision of mathematics is conceptualized as one's beliefs and values regarding mathematics, and finally perceived competence is conceptualized as "perception of being/not being able to succeed in mathematics (Di Martino & Zan, 2011, p.476).

Table 1. Definitions of emotions, attitudes, and beliefs, directly taken from Phillip (2007, p. 250).

Emotions	Feelings or states of consciousness, distinguished from cognition. Can be positive or negative.
Attitudes	Manners of acting, feeling, or thinking that shows one's disposition or opinion. Can be positive or negative.
Beliefs	Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensively, and are harder to change than attitudes.

While students' mathematical affect has been studied through varied conceptualizations, research on teachers' mathematical affect is still emerging. Phillip (2007) notes that while beliefs are considered part of affect in research, they are treated as a distinct construct in studies on teacher affect. Research on teacher affect has documented that teachers' mathematical attitudes inform their mathematical beliefs and in turn instructional practices (Charalambous, 2015; Wilkins, 2008). At the same time, research has also documented that teachers' attitude towards mathematics plays a role in students' attitude towards mathematics (Mensah et al., 2013), and ultimately students' mathematical performance (Ma & Kishor, 1997). Thus, teachers' affective competence is an integral part of students' mathematical (modeling) learning. The takeaway is therefore two-fold: (1) teachers' positive affect correlates with positive teaching and learning outcomes, and (2) frameworks for modeling (and mathematical) knowledge required for teaching should consider both knowledge and affective domains to provide a holistic understanding of the competencies needed to teach modeling.

Conceptual Framework for the Competencies Required to Teach Modeling

We integrated theories (Prediger et al., 2008) of teacher knowledge (Ball et al., 2008; Borromeo Ferri & Blum, 2010; Wess et al., 2021) and affect (Di Martino & Zan, 2011; McLeod, 1992) to propose a framework for the competencies required for teaching modeling (Figure 1). The framework proposed in this paper consists of two major *domains*: knowledge and affect. At the core of this framework is the knowledge perspective. Surrounding the knowledge domain lies the affective domain (in green), which is influenced by the knowledge domain. The positions of the affect and knowledge domains were inspired by Mandler's (1989) work, highlighting that one's actions (based on their knowledge) result in affective responses.

The knowledge domain is further divided into *dimensions* (i.e., subject matter knowledge for modeling (SMK-M) and pedagogical content knowledge for modeling (PCK-M)) and *components* (theoretical knowledge, curricular knowledge, instructional knowledge, and assessment knowledge). This framework combines modeling knowledge for teaching with modeling specific perspectives of teacher expertise to teach modeling effectively. We adapted Ball et al.'s (2008) categorization of mathematical knowledge for teaching to modeling. SMK-M refers to the understanding of the modeling process and the mathematical content that teachers must possess to guide students through the modeling process. PCK-M refers to the knowledge required to effectively implement modeling in the classroom and diagnose students' modeling. Within SMK-M, we situate *theoretical knowledge*—knowledge about modeling cycles, modeling tasks and their suitability for various purposes. Both horizon modeling knowledge (HMK) and mathematical common content knowledge (CCK) impact teachers' knowledge of mathematics that then influences how they engage with and make sense of modeling, whether it involves applying basic mathematical fluency (CCK) or understanding broader mathematical connections across topics and levels (HMK).

We positioned *curricular knowledge* as the blending (Fauconnier & Turner, 2003) of SMK-M and PCK-M, as it draws on both concepts. It represents a bridge where teachers' knowledge of modeling intersects with their ability to transform that knowledge into meaningful learning experiences through modeling tasks. This overlap justifies its position in the framework. Knowledge about modeling task characteristics encompasses the theoretical perspectives underlying modeling task design. Similarly, *knowledge about content and curriculum* (KCC) and *specialized content knowledge for modeling* (SCK-M) are also situated within curricular knowledge as both require an understanding of mathematics and the ability to situate that in curricular structures. Teachers must know how modeling fits into the broader mathematical curriculum (SCK-M) and align with the content standards (KCC).

Instructional knowledge belongs in PCK-M because it is about transforming content knowledge about modeling into effective teaching practices. *Knowledge about intervention* requires pedagogical decisions on how to guide and support students during modeling tasks. On the other hand, *knowledge about modeling and teaching* focuses on teaching modeling in ways that foster mathematical understanding. Both components go beyond just knowing modeling to how to teach it, both as a vehicle and as content. Finally, the *assessment knowledge* in modeling refers to the teachers' knowledge of modeling, alongside their ability to anticipate and interpret the ways students are likely to approach modeling tasks. This knowledge enables teachers to better diagnose students' understanding of the modeling process as they go beyond grading answers to truly assess students' modeling competencies and learning of modeling.

To develop a framework for the affective competencies required for teaching modeling, we drew on McLeod's (1992) framework for affect. Specifically, we asked what are some qualities that are required for teachers to have a positive disposition towards teaching modeling (Jacobson & Kilpatrick, 2015)? We identified the following: (1) constructive emotional dispositions towards modeling, (2) constructive orientation (i.e., positive attitudes) towards teaching modeling, (3) positive perceived competence in modeling, and (4) optimistic vision (i.e., positive beliefs) towards modeling. Constructive emotional dispositions towards modeling refer to positive emotions and attitudes that teachers develop to meaningfully engage in the teaching of modeling. Constructive orientation towards teaching modeling refers to attitudes that promote modeling in the classroom environment. Positive perceived competence in modeling refers to teachers' confidence in tackling modeling tasks. Finally, an optimistic vision towards modeling is having a positive outlook on the broader impacts of modeling on the individual, community, and societal level.

Illustrative Vignettes from Olivia

To describe components of our framework, we drew from one secondary teacher's (Olivia) participation in a modeling focused professional development (PD) program. The overarching goal of the PD program was to develop teachers' proficiency in teaching modeling at the secondary level. A total of 24 secondary teachers (grades 6-12) participated in the PD program. As part of the PD program, Olivia engaged in seven modeling tasks (which consisted of solving and reporting the solution through formal and informal reports), submitted seven task reflections regarding her experiences working on the tasks, designed three modeling tasks, implemented one modeling task in her own classrooms (grade 9 and 10), and submitted a final report reflecting on her experience designing and implementing the task. These sources serve as illustrative vignettes for describing the functionality of our framework. Given space constraints, we present a selection of competencies and will elaborate on others in our talk.

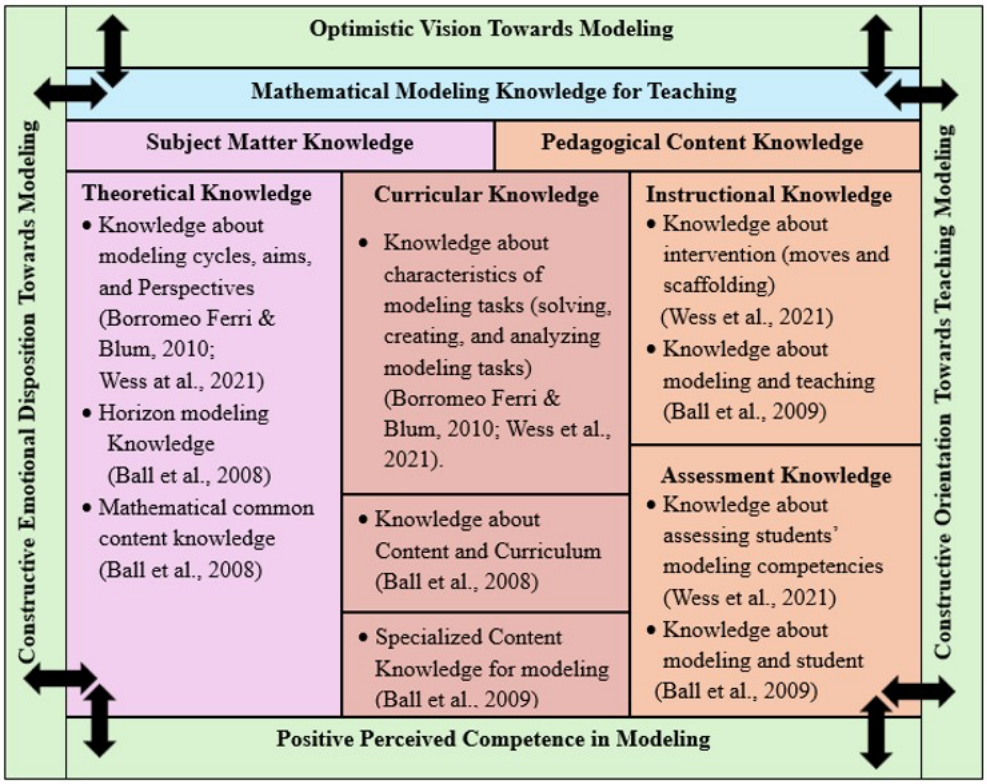

Figure 1. Competencies required for teaching modeling

Knowledge about Characteristics of Modeling Task

Olivia developed the *Eliminating Bottles* task (Table 2), which exhibited robust integration of both modeling task characteristics and socio-ecological awareness. The task exhibited the following task principles: (1) alignment with content standards, (2) learner-centered, (3) model development, (4) model documentation, and (5) reality principle (Aguirre et al., 2019; Galbraith, 2006; Jung et al., 2023; Lesh et al., 2000). First, the task was connected to the state's grades 9-10 content standards, such as surface area and volumes, ensuring that while students address broader social issues of plastic waste prevention, they would also meet formal curriculum standards. The task is learner-centered as the students were given the agency in making assumptions, creating a model, and interpreting the results in meaningful ways. Olivia's understanding of task characteristics also influenced her expectations for the task. For example, Olivia explicitly stated and collected students' poster reports that included the problem statement, assumptions, model formulation, validity, and improvements, addressing the model documentation principle. Finally, the task exhibited the reality principle through drawing on real-world issues of plastic waste that support students' socio-ecological awareness—a critical component of modeling tasks.

Table 2. Olivia's Eliminating Bottles task prompt

The Natural Resources Defense Council claims that 20 billion plastic bottles are thrown away each year. To reduce plastic waste, many public spaces are installing bottle filling stations to encourage the use of reusable water bottles. Develop a procedure to calculate the amount of plastic the [institution name] filling station has eliminated based on the reported number in the photo. Relate this measurement to something you are familiar with by describing how much space the plastic would cover (e.g., number of football fields, gym courts, etc.,).

Knowledge about Intervention

Olivia's knowledge about intervention was evident in her modeling task implementation report, where she documented her strategies to support students' understanding of the modeling process. For example, Olivia cited using several intervention strategies, including probing students' thinking when she felt that the students were stuck, incorporating contingent scaffolding or breaking tasks into smaller chunks for diverse learners, monitoring student work, facilitating small group interactions through the use of pocket questions (ready-to-use questions), creating groups based on her (Olivia's) knowledge of students, providing manipulatives for collecting data, and offering extension activities for those who finished early. These instructional approaches demonstrate Olivia's ability to design targeted interventions for implementing modeling tasks in her classroom.

Knowledge about Assessing Students' Modeling Competencies

Olivia assessed students' modeling competencies using an assessment rubric that evaluated specific stages of modeling cycles. The rubric reflected the modeling stages that Olivia prioritized and that she expected students to understand. These include making sense of the problem, assumptions and model creation, data collection, validating, and communicating the process clearly. She also added a quantifying description (Figure 2) to each modeling stage to communicate her expectations and provided fair feedback on students' work to support students' model development. Prior to students engaging in the task, Olivia shared the rubric and reviewed both the rubric and project expectations with the entire class, ensuring that the assessment process was open and transparent (Assessment Standards for School Mathematics, 1995). In addition, Olivia also diagnosed students' blockages informally as she walked around the class and listened to students' conversations, which supported Olivia's contingent interventions.

Modeling Task Rubric

	Excelling	**Progressing**	**Emerging**
Make Sense of Problem	Problem statement and conclusion are both made in context. Objective of the solution is identified. **30**	At least one, either the problem statement or conclusion, is made in context. The objective of the solution is not clearly identified. **20**	Problem statement and conclusion are identified but lack context. **10**
Assumptions and Model Creation	All assumptions listed are justified or based on background knowledge. All assumptions made are utilized in the model. **50**	One or more assumptions made lack justification. OR Not all assumptions made were necessary for the model. **40**	At least one assumption is listed but lacks justification. OR Model lacks evidence of use of assumptions. **30**

Figure 2. Excerpt of Modeling Task Rubric Developed by Olivia

Constructive Orientations Towards Teaching Modeling

Olivia demonstrated multiple constructive orientations towards teaching modeling in her task reflections and implementation reflections. For example, in her implementation reflection, she expressed that she wanted to challenge herself in implementing more modeling tasks in her classroom due to the overwhelming engagement she received from her students. She also expressed that implementing modeling tasks can be interdisciplinary. This idea is evident in her task reflection, where she reflected that if she were to implement modeling tasks with historical contexts, "[she] likes the idea that [she] could work with the history department and create a lesson that ties in with [the history department] and can integrate history into [her] classroom and mathematics into theirs." In other constructive orientations, Olivia provided guardrails for

students who may struggle with open-ended questions and designed modeling tasks that draw on students' personal interests, mathematical experiences, and local community-based knowledge to support mathematically rich discourse.

Positive Perceived Competence in Modeling

Through her engagement with the modeling tasks in the first part of the PD program, Olivia developed confidence in her modeling skills. For example, in her modeling task reflections, Oliva made statements such as "I think in the future I will be less afraid to create multiple models knowing that I will eventually adjust them to make them fit the task better," "I am gaining more confidence in my ability to attack these modeling tasks. I'm becoming more eager to attempt them, and I am realizing that I can be very creative in how I approach problems," and "I am impressed with my ability to clearly explain my processes and procedures in context and effectively discuss the applications my model has to the real world." These statements demonstrate her positive perception of her ability to model, which in turn has an impact on how she may teach modeling.

Discussion

The framework presented in this paper advances the holistic understanding of competencies required for teaching modeling by positioning cognition and affect as inseparable domains (Chamberlin, 2019). On the cognitive side, teachers require both SMK-M and PCK-M for teaching. On the affective side, teachers require positive emotions and attitudes, together with confidence in tackling and promoting modeling with an optimistic vision of its broader impact on learners and society. We argue that competencies along both domains must co-exist for teaching modeling successfully. This is because teachers with strong pedagogical content knowledge may avoid implementing modeling tasks if they lack self-efficacy in their own modeling competencies or in addressing students' difficulties during modeling. Conversely, teachers with positive affective orientation but with limited cognitive resources for modeling knowledge (Hammer, 2000) may struggle to design and implement modeling tasks effectively.

The integration of cognition and affect in our framework has several implications for teacher education in modeling. First, the design of PD programs can foster teachers' positive attitudes towards modeling and teaching modeling by incorporating tasks with various educational, cultural, and societal contexts that are equally challenging but also feasible at the same time. This PD environment will provide opportunities for teachers to (i) experience struggling in tasks in a productive manner and to (ii) appreciate the utility of modeling in describing and predicting real-world issues. PD programs can also consider incorporating book studies and journal reflections, where teachers are continuously encouraged to reflect on their learning both as a modeler and as an educator of modeling. Second, teachers' growth in modeling competencies for teaching should be assessed through both knowledge and affect domains. Evaluators can leverage tools such as longitudinal cross-sectional analysis of focus group interviews (Tuthill et al., 2020), and pre-post surveys of modeling specific attitudes (e.g., Asempapa & Brooks, 2022), to track changes in teachers' affective orientation towards teaching and learning of modeling. Finally, researchers may use the framework both to inform the design of modeling-specific PD programs and data collection, and to undertake empirical validation through its application to data. This work contributes to the fields of teacher education and modeling by challenging the prevailing emphasis on PCK for modeling. We argue that attention must also be given to *what* teachers need to know as *learners of modeling* and *how* teachers should *feel* towards the teaching and learning of modeling for its successful implementation in K-16 classrooms.

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Interrogating Proof Norms: "It's kind of like telling me, 'Oh, act like a robot.'"

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We draw on poststructuralism to interrogate assumptions of what counts as "normal" in proof. Guided by post-qualitative inquiry, we share moments from an unstructured interview with an undergraduate student, Sherine, alongside research literature to question two dominant proof norms: depersonalization and the absence of the problem-solving process. Our interrogations reveal how proof norms potentially reinforce messages that exclude relationality, storytelling, and emotional connection from the proof process, as well as newcomers to proof who benefit from seeing the nonlinear process of problem-solving. We argue for broadening what gets recognized as legitimate proof, where proofs can be both formal and personal, polished and messy, logical and human.

Keywords: Poststructuralism, Proof, Norms, Values

We invite you to read two proofs for the divisibility property: for all integers m and n, if $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ (Figure 1). You, like others we have shared these with, might notice that the proof on the left feels like a "normal" proof[1] whereas the proof on the right feels unusual[2]. You might catch glimpses of my (the proof author - Saúl's) personality and uncertainties. I hope you feel a sense of connection between us- me and you (the reader).

By assumption, there exist integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 a$. For any integers m and n we have $mb + nc$ $= m(k_1 a)$ $+ n(k_2 a)$, by assumption, $= (mk_1 + nk_2)a$. Thus, $mb + nc$ is divisible by a.	My goal is to show that a divides $mb + nc$. To me, this means that I can write $mb + nc = ka$ where k is an integer. I'm really not sure how I will get there but I'll start with what I know. Since I know that a divides b, then I know that there must be an integer r so that I can write $b = ra$. Since I also know that a divides c, then I also know that there must be an integer s so that I can write $c = sa$. Hmm. It might sound a bit repetitive, but I like to keep track of where I have been to help me with where I'm going. Next, I will consider m and n to be integers. Since $b = ra$ and $c = sa$, I can substitute them in $mb + nc$ and write $mb + nc = m(ra) + n(sa) = mra + nsa$ Haha. N. S. A. Acronyms make me laugh for some reason. Anyway, I can see that I can factor out a from $mra + nsa$ so I can also write $mb + nc = (mr + ns)a$. It's not looking how I want yet but maybe I need to review my goal. My goal is to write $mb + nc = ka$ with k being an integer and that is not what I see. Wait! m, r, n, and s are all integers so $mr + ns$ is also an integer! I'm going to define k to be $mr + ns$ and say $k \coloneqq mr + ns$. Nice, nice, nice! I think I'm getting to where I want to be! Since $mr + ns$ is an integer, then k is also an integer because it is equal to $mr + ns$. Now I can write $mb + nc = ka$! I've reached my goal! This means that a divides $mb + nc$! #That'sShowbiz!

Figure 1. A "normal" proof, and a personalized proof

Entering this project, we (Saúl and Kristen) began feeling uncomfortable labeling the proof on the left "normal". Our experiences working with students and reading mathematics education literature suggested that what is typically considered "normal" in proof may not resonate with all learners, and that some students might find value in proofs that depart from these expectations.

[1] You can find this proof in Houston's (2009) textbook: *How to think like a mathematician: A companion to undergraduate mathematics*.
[2] I (Saúl), did in fact breach proof norms (Dawkins & Weber, 2017)!

For instance, Brown (2018) shared an example of a student who personalized a justification for the role of axioms in proof using culturally familiar language like "ese" and "holmes." Vroom and Alzaga Elizondo (2024) found that students appreciated existence proofs that included the author's problem-solving process, suggesting that such transparency offered a more relatable view of proving. Weber and Melhuish's (2022) observations further underscored our concerns: "The way to be 'good' at mathematics is, at least in part, to emulate the language, argumentation, and culture of mathematicians" and "deference to the discipline of mathematics (as conceived largely by a select few Western, white men) is likely to reinforce existing power differentials inherent in the system" (p. 307). In what ways, then, does this "normal" proof reinforce those dynamics? In this work, we draw on poststructuralism to interrogate taken-for-granted assumptions about what counts as a "normal" proof and whose ways of knowing it centers.

Poststructuralism: An Interrogation Lens

Rather than treating proof as universal and neutral, we understand it as embedded in particular institutional and cultural norms that govern who and what counts as mathematically legitimate. From a poststructuralist perspective, "truth" in mathematics is not fixed or discovered in isolation from people; instead, it is discursively created through language, practices, and social expectations for mathematical activity (Foucault, 1980).

Identifying mathematical proof discourses is no small task; discourses are so deeply embedded in practice that they often go unnamed. We appreciate Dawkins and Weber's (2017) work, which surfaced some taken-for-granted expectations through an extensive literature review where they identified different *norms*, or accepted expectations for practice, and *values,* or shared goals or orientations, in the context of proof. One such value, which we will continue to interrogate, is *independence from the author* – or that "mathematics knowledge and justification should be independent of (non-mathematical) contexts, including time and author" (p. 129). Two norms that uphold this value are the *depersonalization* norm – "mathematical proof is written without reference to author or reader's agency" (p. 129) and the *absence of problem-solving* norm – "a proof is an autonomous object, not a description of a problem-solving process" (p. 130). This means that a "normal" proof, for instance, avoids the pronoun "I", and authors' personas are concealed along with how they came to understand the argument.

Gee's (1989) framework on *primary* and *secondary discourses* helped us conceptualize how students might experience these norms. Primary discourses are developed in everyday settings through interactions with family, friends, and cultural communities, while secondary discourses are those required to engage in institutional spaces. Mathematical proof discourses (e.g., the norms and values of proof) are secondary discourses. Participating in proof may feel natural or comfortable for students whose primary discourses align with these academic expectations. But it might feel unnatural for others whose primary discourses do not align as much. This discomfort reflects a mismatch between proof discourse and the discursive practices students bring with them. This is *not* a deficit in the students, but a narrowness in the dominant norms.

Researchers studying the teaching and learning of undergraduate proof have largely engaged with the "predict" or "understand" paradigms of inquiry (Stinson & Walshaw, 2017), which are supportive to study the effects of teaching interventions or try to explain phenomena that are part of the teaching and learning of proof. For instance, such paradigms have been useful in investigating how students understand proof (e.g., Guajardo & Lew, 2024; Vroom & Alzaga Elizondo, 2024; Zazkis & Rouhani, 2024). We argue that there is more to learn from a "deconstruction" paradigm, such as the use of poststructuralism, which encourages scholars to continuously examine what is taken for granted and reveal what is unsaid when people settle into

norms (Lather, 2006; Stinson & Walshaw, 2017). While mathematics education researchers have drawn on poststructuralist theory in other contexts (e.g., Dubbs & Herbel-Eisenmann, 2021; Johnson et al., 2022; Stinson, 2008, 2013), this lens has not been widely applied to the study of proof in undergraduate mathematics. We believe that poststructuralism offers a powerful way to interrogate what is deemed as "normal" proofs, providing motivation to disrupt dominant discourses.

Our Positionalities

I (Saúl) was born and raised on the Texas border, surrounded by people who looked a lot like me. I've gone through different iterations of how I identify myself, not really knowing what to settle with (Hispanic, White, Latino?). As I write this, I identify as Latino/e. Despite taking mathematics courses at a Hispanic-Serving Institution, I struggled to see myself as a doer of mathematics until my enculturation into mathematics. While I enjoy research, I don't feel that I can *do* research. Like mathematics, it has always felt presented as a matter of fact. A long-term goal for my writing is to explicitly show the vulnerability of my work, along with how I, with the support of those around me, can do research. As Kristen and I attempt to make mathematics more inclusive, I also hope to do this for research.

I (Kristen) am a white woman who grew up in a deeply gendered space in the Appalachian Mountains. Much of my research has focused on students' *access* to mathematical norms (e.g., Vroom, 2022; Vroom & Ellis, 2024), but through ongoing collaborations, including with Saúl, I've started to think more critically; How can I support "changing the game" rather than just "playing the game" (Gutierrez, 2009, p. 6)? This shift, and especially our interrogation lens, is uncomfortable for me as someone who avoids rocking the boat. At the same time, it's important for me to help open up academic spaces, especially mathematics, to make room for more ways of being.

How We Interrogated the Norms

I (Saúl) engaged in an unstructured interview with an undergraduate student, Sherine. Sherine described herself as an Arab/Middle Eastern cisgender female person and used she/her pronouns. At the time of the interview (Fall 2024), she had recently completed an online Introduction to Proofs course over the summer. From the outset, I shared my purpose for the interview: to interrogate dominant norms in mathematical proof by learning from Sherine's experiences. In the first part of the interview, Sherine read a section from Dawkins and Weber (2017) titled "Value 2: mathematical knowledge and justification should be independent of (non-mathematical) contexts, including time and author." This approach was inspired by Stinson (2008, 2013), as we aimed to "[ascribe] more agency to the individuals in recreating or shifting meanings of the discourse" (Gutiérrez, 2013, p. 44). The second part involved her reading the full proofs shown in Figure 1. In both parts, I opened with the same prompt: *What do you think, wonder, and feel as you read or now that you have finished reading?* The interview was intentionally unstructured; we did not prepare follow-up questions in advance so I (Saúl) could be fully present with Sherine and let the data emerge organically rather than be shaped by prior categories. The interview lasted approximately 1.5 hours and was audio- and video-recorded.

Our interrogation process was non-linear and messy. We read and re-read, talked with Sherine, shared ideas with other researchers, re-listened to our conversations, and repeatedly revised through discussion and writing. This messy process was guided by post-qualitative inquiry (St. Pierre, 2024) and writing as a method of inquiry (Richardson & St. Pierre, 2005; St. Pierre, 2015). These invited us to release rigid structures in data collection and analysis,

embracing the uncertainty and creativity that arise when we allow something new to take shape (St. Pierre, 2015). We reemphasize that our goal was to interrogate the norms themselves, a theoretical endeavor, not to describe the entirety of our conversation with Sherine. Her conversation helped us, Saúl and Kristen, feel and notice our reactions to these norms. To do so, we often returned to the recording, at first to become familiar with the conversation, and then to notice connections and moments that pulled at us. We then paid attention to what seemed troubling about the norms. Alongside this, we regularly met to discuss, read, write, and reflect, allowing our engagement with the literature to shape our understanding and what we wrote. Our writing is personal, shaped by who we are, our interpretations, and our affective responses to the discourses.

We also want to emphasize that we view Sherine, as well as ourselves, as both shaping and reproducing mathematical proof discourses. In fact, Sherine indicated that she has reproduced some of the norms we are interrogating due to her teacher inexplicably reproducing them, even saying, "It never occurred to me if it makes sense." In our writing, we highlight moments where Sherine questions dominant discourses and also critique moments where she appears to reproduce them. We intend for you to read them *not* as a critique of *her*, but as a critique of the *discourse*. We also acknowledge that all of us act in ways that are sometimes inconsistent; we do not interpret her seemingly conflicting comments as evidence of confusion or deficiency. Rather, we use a "yes, and" approach to discourse: contradiction is not failure; it is part of being human within systems that are full of tension.

Interrogating Proof Norms

In what follows, we interrogate the two norms upholding the independence from the author value. We do so by drawing on the conversation with Sherine using research insights to argue how these norms can be exclusionary. We invite you to join our interrogation; ask yourself, how is this dominant way of doing proof messed up, and for whom?

Interrogating the Depersonalization Norm

Let us start with a quote cited by Dawkins and Weber (2017) in their discussion of this norm, where Davis and Hersh (1981) satirically describe "the ideal mathematician" as:

> [following] an unbreakable convention: to conceal any sign that the author or the reader is a human being. It gives the impression that, from the stated definitions, the desired result follows infallibly by a purely mechanical procedure. (p. 36)

Sherine questioned this "unbreakable convention," explaining:

> We know that they are human, so why would we conceal that fact? It's kind of like telling me, 'Oh, act like a robot.' [...] And it should be more like human, because you want people to be like, 'Whoa, oh my god, a person did that,' not think it's some robot. It is a huge achievement if it was really good, you know? [...] Humans are amazing, let them use their brains. So, like the idea that we're robots…I'm sorry, but we're better than robots, and robots aren't necessarily perfect.

Here, we see that the depersonalization norm sends troubling messages to students like Sherine to act in an unnatural way, "act like a robot," to suppress their humanness and ingenuity. Sherine expressed that proofs should be "more like human" to showcase the creativity of the person who authored the proof. We agree. Proofs should be allowed to reflect the identity of the author so that students can experience proof as a personal and creative practice.

When discussing an instance of a "more human" proof through the use of "I", Sherine explained, "Some people feel like they need to avoid using 'I,' so they would use the royal 'we'.

[…] ‘We’ is perceived better than ‘I’.” While Sherine acknowledged she has adopted this convention in her proof writing, she also explained, “if the proof is sound and looks good, then who cares if they used ‘I’?”. While scientific writing aims to view its subject matter as autonomous, authors are expected to avoid passive voice in writing. Burton and Morgan (2000) offer insight into why “we” might be perceived as better explaining that the royal “we” avoids passive voice while meeting the goal of making the subject matter independent of the author:

> The unwritten assumption here is that, in most cases, the author’s persona is not relevant. This assumption is consistent with the commonsensical belief that the subject matter is neutral and that the use of such impersonal language reflects this neutrality. (p. 437)

We want to highlight two points. First, the subject matter is *not* neutral (Bishop, 1990) and perpetuating that “truth” through the proof text reinforces that damaging narrative. Secondly, as Weber and Melhuish (2022) pointed out, avoiding personality to be more objective is not universal. This discourse is rooted in Western and masculine institutional traditions (Bishop, 1990; Hottinger, 2016; Leyva et al., 2021) and promotes generalization and colorblind practices (McNeill, 2025; McNeill et al., 2022).

While Sherine expressed a desire for proofs to be more transparent about the human author, she also gravitated towards the “normal” proof on the left in Figure 1, explaining,

> It’s cleaner. It does not have like extra words. Like, if I’m reading a proof, let’s be real, all mathematicians are kind of lazy. Like, nobody wants to read that whole script, right? We just want: proof, know what’s going on, and that’s it.[...] I wouldn’t say [the personalized components of the proof] adds anything. I don’t like it. It's just kind of there for no reason. Like, it’s not really helping me with understanding. Like, it’s not helping me with the proof and it’s not helping me with understanding. So, like, it’s not really useful.

What do *you* think? Does the personalized part add something? We believe it does. It adds a relational dimension between me (Saúl, the author) and you, the reader. I am being transparent about my persona, my humanness, and in doing so, I invite you to connect with me, to see yourself in the text, rather than encountering something sterile and robotic. And yet, the idea that my persona is irrelevant in proof writing is reproduced in the norms of mathematics; “good” provers (mathematicians) view relational elements as extraneous and undesirable: “Let’s be real, all mathematicians are kind of lazy. Like, nobody wants to read that whole script, right? We just want: proof, know what’s going on, and that’s it.”

The depersonalization norm conflicts with the epistemologies of students, particularly non-masculine and non-Western students, whose primary discourses value relationality and emotional connection (Bishop, 1990; Keith, 1988). Storytelling and other narrative-based ways of communicating have long played a central role in how many non-dominant communities teach, learn, and construct identity (Genz, 2011; Lauer & Aswani, 2009). These practices are not only valid forms of expressing knowledge, they are also more intimate. They stand in contrast to the depersonalized proof norm that prioritizes disembodied logic, objectivity, and detachment.

Our point is that the depersonalization norm implicitly sends messages that non-Western, non-masculine people cannot count as practicing mathematical proof unless they strip them of their history and culture. Proof can be exclusionary for those whose primary discourses involve personal and emotional connection, since it may invalidate how they engage with mathematical proof. How else do you see the depersonalization norm as messed up, and for whom?

Interrogating the Absence of the Problem-Solving Process Norm

Like the depersonalization norm, we see that the absence of the problem-solving process norm hides the humanity of the proof-author, and additionally, the proof-reader. Here, what gets hidden is not the *who* of the author, but the *how* they may have thought through the problem, struggled, and found a way through. When Sherine reflected on Dawkins and Weber's (2017) subsection about the absence of the problem-solving process, Sherine explained:

> I feel like describing the [problem-solving] process might be only for people who just started dealing with proofs instead of people who know how proof works.

Sherine's comment suggests that process, or the nonlinear, human work of figuring something out, is seen as something for newcomers. Then, after reading the personalized proof, she shared:

> It [the proof with the problem-solving process] makes sense, it's an explanation, but that's not what a proof *is*. Proofs do not explain stuff. Proofs show you why is something… Like proofs *prove* something.

Here, Sherine reproduces the view that proofs should not include the problem-solving process. To her, a defining characteristic of proof is that it only shows the validity of the claim, rather than it also explains why it's valid. Sherine did not dismiss the value of process. She saw it as helpful, just not something that belonged in a proof. She explained, "You should write the proof alone, and then maybe if you want to describe the process, that would be something not included in the proof" - a kind of footnote for beginners. We see that putting the process in the margins outside the "real" proof is narrowing for a couple of reasons. First, as with the depersonalization norm, this norm erases a relational dimension of mathematical work. Proofs are not simply abstract objects; they are human-made texts written for human readers. When the process is hidden, there is little sense that someone was here, thinking through this, maybe even thinking of you and your sensemaking as they wrote.

Second, the absence of the problem-solving process undermines support for newcomers and reinforces exclusionary norms about who belongs in mathematical spaces. Without an open discussion of how proofs are thought out and written, a curtain is maintained that separates those presumed to have an innate mathematical ability (people whose primary discourses are closely related to the secondary discourse for mathematical proof) and those who are not presumed to have it (people whose primary discourses are not closely related to this secondary discourse). Maintaining a norm that omits the process reinforces a kind of gatekeeping. Leyva et al. (2021) shared that instructional practices such as indicating to students that if they are not understanding course material soon enough, they should consider dropping the course, have been perceived by students as messages that they are not good enough to participate in any mathematical practice including those pursuing degrees such as Business since mathematics courses can be requisites for those majors. When proof-writing excludes process, a similar message may be conveyed: that unless you already understand how to read these polished arguments, you don't belong. This is particularly harmful for students whose primary discourses are not closely related to the secondary discourse necessary for mathematical proof.

In this way, the absence of the problem-solving process norm not only withholds intellectual support; it communicates a broader ideological stance about who is presumed to be a "real" mathematician and who must continually prove their right to participate. How else do you see the absence of the problem-solving process norm as messed up, and for whom?

An Argument to Interrogate Proof Norms

In this paper, we've argued that the depersonalization and absence of the problem-solving process norms are not neutral or purely mathematical. These norms are socially constructed and,

as we've shown, can function in exclusionary ways. By defining what counts as proof in such narrow terms, these norms obscure both the humanity of the author and reader. Secondary discourses are easier to acquire by those whose primary discourses are already similar to the secondary discourses (Gee, 1989). Therefore, these norms also privilege those whose prior experiences, discourses, and identities align with dominant mathematical values, values historically shaped by Western, masculine traditions, while invalidating relational, human, and process-oriented ways of engaging with proof.

Drawing on poststructuralism, we troubled what's been taken for granted. As Lather (1991), drawing on Caputo (1987), writes, the goal of this kind of deconstruction isn't to settle things once and for all, but "to keep things in process, to disrupt, to keep the system in play, to set up procedures to continuously demystify the realities we create, to fight the tendency for our categories to congeal" (p. 13). Through this lens, we questioned the binary between objective and personal proofs, pushing back on the idea that including the author's personality, thinking, or struggle somehow devalues a proof.

To be clear, we're not arguing that depersonalized proofs have no place. There are contexts where those norms may serve useful purposes (we suspect mathematics research journals). But that type of proof is only one representation, among many, that we should value and allow students to encounter. While we believe the implicit messaging alone justifies expanding these norms, there is also evidence that mathematicians themselves often adopt personalized styles and voice (Contreras et al., 2025). At the same time, some instructors may feel particularly compelled to emphasize precision and formality to their students - "sometimes to the point of being more pedantic than they might be with experienced mathematicians" (Rupnow & Randazzo, 2023, p. 305). We encourage broadening what gets recognized as legitimate proof, and inviting students into forms that are more relational, expressive, and reflective of the thinking behind the final product.

As an anecdote, when we wrote personalized proofs as we prepared for this project (and, frankly, this personalized conference paper), something opened up for us. It felt freeing. It felt vulnerable. It felt different from what math and academia usually allow. We think this is a promising direction for future research: What happens when students are invited to personalize proofs? How does it feel to write in ways that reflect your own thinking and voice? And just as importantly, how does it feel to read this kind of proof, or this kind of academic paper?

More broadly, we see this work as part of a growing movement in our field to reexamine and resist "normalities," such as those of you who are questioning "normal" curricula and teaching practices (e.g., Leyva et al., 2022; Osibodu, 2024). We invite you to join us in exploring poststructuralism, not only as a way to critique, but also to imagine differently. For some, deconstruction may offer a motivating approach that safeguards against dogmatism and reminds us that even cherished mathematical (educational) categories are not innocent. As Lather (1991) puts it, deconstruction continually demystifies the realities we create. It keeps the system in play. Let's keep the system in play, where proofs can be both formal and personal, polished and messy, logical and human. Where there's room for more kinds of knowing, more ways of belonging, and more stories of how we figured things out.

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Defining a "Differentials-within-limits" Framework for First-year Calculus

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Work in first-year calculus has clearly shown the power that "differentials" afford for first-year calculus learning, but it also speaks of the need to build ideas of limits. While some researchers have loosely described informal differentials and limits within the same paradigm, their actual conceptual definitions and relationships have not always been explicitly constructed. Continued progress in this area could benefit from theoretical consideration of: (a) providing clearer intuitive definitions for differentials inside a limits-based approach to calculus, and (b) defining more exactly the relationships between differentials and limits. We provide here a theoretical framework for building a simultaneous differentials-and-limits approach to first-year calculus.

Keywords: calculus, differentials, limits, differentials-within-limits

There is a persistent theoretical issue facing the teaching and learning of first-year calculus. On the one hand, the research literature in recent years has painted a very clear picture of the need for first-year calculus to be heavily grounded in quantities and quantitative reasoning (Jones & Ely, 2023; Kertil & Gülbağcı Dede, 2022; Oehrtman & Simmons, 2023; Thompson & Dreyfus, 2016; Yu, 2024). These efforts are often centered on the need for *differentials* (e.g., *dx, dt*) to have quantitative meaning, which can then also interact with other quantities to, in turn, construct meanings for derivatives, *dy/dx*, and integrals, $\int_a^b f(x)dx$ (Hu & Rebello, 2013; López-Gay et al., 2015; Roundy, Dray, et al., 2015). Further, most STEM fields utilize free-standing differentials to capture quantitative relationships at small scales, such as $dI = r^2 dm$ or $dS = dQ/T$. Consequently, these paradigms have proven powerful for enabling students to develop deeper understandings of calculus and to become proficient users of calculus beyond the mathematics classroom (Amos & Heckler, 2018; Pina & Loverude, 2019; Roundy et al., 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012; Weber et al., 2012).

On the other hand, there has been a historical tension in the mathematics community between differentials and limits (Katz, 2009). The historical outcome was to discard differentials in favor of limits, leaving typical modern calculus based on limits exclusively, without differentials. It has taken considerable effort by researchers to redeem differentials to a "legitimate" status within teaching (Dray & Manogue, 2010; Ely, 2017; Martínez-Torregrosa et al., 2006; Rogers, 2005). In an interesting pendulum swing, however, some of these differentials-based reform efforts have now gone so far as to do away with limits, using (essentially) the hyperreal number system in their place. Because much of modern mathematics is based on the real numbers and on limits, this could create baggage for students in underdeveloping limits per se and in using a number system not generally accepted in the broader mathematical community.

What should calculus education do? Reasoning with differentials as quantitative objects is essential, but the limit is a necessary concept for the general mathematical community. The obvious answer seems to be to create instructional paradigms that coherently leverage *both* limits and differentials, yielding the benefits of each. In fact, some calculus education researchers have more-or-less done this, using approaches that in essence contain both differentials and limits (Jones, 2015; Oehrtman & Simmons, 2023; Weber et al., 2012). The problem is that some of this work has been done without careful attention to exactly *what* a differential is within a limit

structure, and what its *properties* and its *relationship* to a limit are. We admit that our own prior work falls into this category, and we are guilty of this same laxity. Additionally, other paradigms that have tried to define differentials more explicitly (Thompson et al., 2013; Thompson & Dreyfus, 2016) tend to not cohere very well with the growing body of work on differentials in math and STEM education. Similarly, paradigms that view differentials only as "linear approximation" tend to miss out on imbuing expressions like dy/dx with direct quantitative meaning – a major benefit of differentials. We find that continued progress could benefit from building clearer theoretical frameworks on a differentials-and-limits approach that is also built on and compatible with the body of work in this area. Thus, in this *theoretical* paper, we contribute by supplying one possible set of definitions and relationships. To be clear, our framework is focused on *intuitive, conceptual* definitions for use in first-year calculus, rather than on *formal mathematical* definitions used for proof in the mathematics community. We anticipate that one could disagree with our specific definitions and properties, and we welcome discourse in this area, and we welcome others providing alternate definitions or improving upon ours.

Background

While we aim to add theoretical clarity to the simultaneous use of differentials and limits, we most certainly want our framework to stay true to the important work that has been done on differentials so far. The major theme in the literature is that a differential is an informal construct, representing a "tiny bit" or an "infinitesimal bit" of a quantity (Amos & Heckler, 2018; Hu & Rebello, 2013; Jones & Ely, 2023; López-Gay et al., 2015; Von Korff & Rebello, 2012). In STEM education, dt might refer to a tiny bit of time or dA might refer to a tiny bit of area. Many STEM educators intentionally leave differentials as an intuitive idea, without providing any formal mathematical definition. For example, Von Korff and Rebello (2014) state that they use "d" to mean something "vanishingly small, without giving a mathematical definition of what it means to be… vanishingly small" (p. 698). The main reason for doing so is that a formal definitions from within non-standard analysis can be as problematic for first-year calculus as the formal ε-δ definition of the limit has been. Rather, it is the informal notion that affords so much reasoning power (Jones & Ely, 2023). Simply put, differentials for first-year calculus work best when kept *intuitive* and *informal*, and we adopt this stance for our framework.

Based on the power of differentials, a natural question is: Why not use only differentials and abandon limits altogether? Some have actually taken this approach, using what is essentially a less formal version of non-standard analysis and its hyperreal number system (Dray & Manogue, 2010; Ely, 2021; Keisler, 2011). In this camp, a differential is a literal "infinitesimal": a non-zero number smaller than all real numbers. While this approach certainly has solid mathematical footing thanks to the work of Abraham Robinson's group in the 1960s (Robinson, 1961), the issue for us with this approach is that non-standard analysis is not generally utilized in the mathematical community and most modern mathematics is founded on the real numbers instead. We are concerned with developing a number system in our students' minds that can, and probably will, clash with subsequent mathematics study. It also makes using differentials a much harder "sell" to the broader mathematical community, impeding progress toward adopting differentials in instruction. Thus, our framework does *not* use a hyperreal version of differentials. Rather, out intent is for differentials to be a construct that can readily cohere with limits.

Next, we note that the research literature on differentials describes two distinct possible conceptualizations of differentials. Von Korff and Rebello (2014) described that some differentials have a "change" nature, such as dt representing a tiny change in time. On the other hand, they also described that some differentials have an "amount" nature, such as ds

representing a tiny length on a curve. This distinction is important, because both types are used across STEM fields. Integrals such as $\int_R \rho\, dV$ and $\int_R r^2 dm$ make use of differentials that can be thought of as a tiny bit of volume (dV) and a tiny bit of mass (dm).

Finally, as quantitative objects, differentials can interact with other differentials or quantities to produce the key constructs of derivatives and integrals. A derivative can be seen as the ratio of tiny changes represented by two differentials, *dy* and *dx*, to produce *dy/dx* (Roundy, Dray, et al., 2015). An integral can be viewed as summing the products between a differential quantity and a "macroscopic" quantity, $f \cdot dx$, to produce $\int_a^b f(x)dx$ (Jones & Ely, 2023) – though other relationships beyond products are also possible (Ely, 2017; Simmons & Oehrtman, 2017).

Situating Our Theoretical Contribution

As mentioned, several researchers have already discussed approaches to calculus that essentially use both limits and differentials in the same paradigm (Jones, 2015; Oehrtman & Simmons, 2023; Weber et al., 2012). In these approaches, limits are examined in their own right, and derivatives and integrals are still defined as limits: $\lim_{\Delta x \to 0} \Delta y/\Delta x$ and $\lim_{\Delta x \to 0} \sum f(x_i)\Delta x$. But once the differential forms are provided, *dy/dx* and $\int_a^b f(x)dx$, the differentials are allowed to have meaning as tiny changes or amounts of a quantity. These approaches have proven powerful, yet the issue is that the co-existing nature of differentials and limits has not been carefully and explicitly articulated. On the other hand, when attempts have been made at articulating their interrelations (Thompson et al., 2013; Thompson & Dreyfus, 2016), the definitions have not been very compatible with the growing body of STEM research on differentials described in the previous section. Additionally, another attempt we have seen at describing the relationship was given as $dx = \lim_{\Delta x \to 0} \Delta x$, but such a definition has a major flaw. The value of that limit is unambiguously zero, meaning that all differentials would be size zero. Such thinking is akin to the problematic "collapse metaphor" described by (Oehrtman, 2009), wherein all differentials collapse to nothing and essentially disappear from the quantitative relationships altogether.

We wish to produce a framework for differentials and limits that (a) provides theoretical clarity, while (b) still aligning with and strengthening the important work done in this area in recent years. This theoretical paper contributes by describing one such possible theoretical framework to cohere differentials with limits. To be clear, we reiterate that our framework is meant to propose *intuitive, conceptual* definitions for first-year calculus education, and not *formal mathematical* definitions intended for proof.

The Basis of the Framework: Processes and Objects in the Limit Concept

Our theoretical framework for differentials and limits is based on Sfard's (1991, 1992) idea of processes and objects. Of course, many researchers have discussed processes and objects, including Arnon et al. (2014), Gray and Tall (1994), and others, but for our current purposes we only need the general idea of these processes and objects, as found in Sfard's work.

A *process* is defined as a dynamic action that can be imagined as happening. For example, one potential *process* for a limit, $\lim_{x \to a} f(x)$, is that of *x* sweeping through numbers as it moves closer and closer to *a* (Cornu, 1991; Fernández-Plaza et al., 2013; Oehrtman, 2009) (see Table 1). Yet, a different *process* that can be conceptualized for a limit is graphically zooming over and over toward a section in the neighborhood of $x = a$ (Ely & Ellis, 2018; Ely & Samuels, 2019) (see Table 1). Notice that these two intuitive processes are both, essentially, "never-ending" (Davis & Vinner, 1986), because one cannot literally enact *x* passing through infinite real

numbers on its way to a, nor can one zoom in a true infinite amount to a single point within the real numbers (see Table 1).

Next, a process can result in some kind of outcome that can be thought of as an entity in its own right. For limits, as x gets closer and closer to a, the function $f(x)$ might hone in on a single numeric value, L (or remain steady at L). This outcome can be conceptualized as its own entity, and when it is, it becomes an *object* existing in the person's mind (Sfard, 1991) (see Table 1). The entire symbolic expression $\lim_{x\to a} f(x)$ can now be seen as referring to this object, "L" (Gray & Tall, 1994). These objects can then be used in further processes, creating even more objects.

Table 1. Pre-existing definitions from the literature regarding limit processes and objects.

Idea	Definition
Limit Process, Approach	One type of limit process is dynamically *approaching*. For example, in $\lim_{x\to a} f(x)$ we can hold an image of x getting continuously closer to a.
Limit Process, Zoom	A distinct limit process is dynamically *zooming*. In $\lim_{x\to a} f(x)$, we can imagine of zooming forever into a portion of a graph around $x = a$.
Limit Process as Never-ending	There is never a moment at which the limit process "completes." While it results in an object, the process itself never attains a completed state.
Limit Object	The limit object is the outcome that results from the limit process. It is defined as the value we assign to the limit, as in $\lim_{x\to a} f(x) = L$.

Defining Differentials *Within* Limits

With these conceptual ideas for limits in place, we can now *conceptually* define a differential as a secondary object that also is produced from *within* the limit process. In other words, we do not simply see differentials *and* limits together, but we see differentials coming into existence *within* the limit process. Table 2 contains our definitions, which are intentionally kept informal for first-year calculus. They are not meant to be treated as formal mathematical definitions.

Table 2. Definitions of differentials as secondary objects from within the limit process..

Idea	Definition
Differential, based on approaching	Consider the approaching imagery for the limit process in $\lim_{x\to a} f(x) = L$. Once someone perceives that the input variable x has become incredibly close to its target a (even if never reaching it), they can "pause" the limit process at that moment and consider the remaining difference between x and a. In other words, the differential is the *object* produced by "pausing" the approaching process once x is extremely close to a, and by considering the remaining discrepancy.
Differential, based on zooming	Consider the zooming imagery for the limit process instead. As the zooming continues to tiny scales, one can again "pause" the limit process at some moment and consider the very tiny "amount" of a quantity remaining in the zooming window. Just as before, the differential is the *object* produced by "pausing" the zooming process once it has reached very small scales, and by considering the tiny bit in the zooming window.
Output Differential	One can also create a differential for the *output* variable. In $\lim_{x\to a} f(x) = L$, as x gets close to a, or as one zooms into a graph near $x = a$, one can also imagine a differential exiting for the output variable, *df* or *dy*.

Properties of Differentials in This Paradigm

Having provided our baseline conceptual definitions, we now provide specific properties that informal differentials have in our framework (Table 3).

Table 3. Properties of differentials in our framework.

Idea	Definition
Subjectivity of Closeness	What is "close enough" in the limit process for it to be paused, and for a differential to come into existence, is subjective both on context and the reasoner's perspective. In a scientific context, a differential often comes into existence once reaching the threshold of measurement tools, or when further measurement yields a "linear regime" (Roundy, Weber, et al., 2015). For math functions, a differential might exist once the error is in such remote decimal places that they contribute very little to the variable's value.
Lack of a Specific Size	As a direct consequence of the previous property, differentials do not have a single, specific size nor value. That is, the differential object is *not* defined by a numeric value, as opposed to the limit object. Rather, the differential object is defined by the conceptual idea of a tiny remaining change/amount.
Non-hyperreal	While a differential does not have an inherent size, it still resides in the realm of very small real numbers, and is *not* a non-standard hyperreal.
Contextual Assignment of Values	While differentials have no inherent size, for pragmatic reasons one can still assign a numeric value to a differential in a given situation, when context dictates, by "pausing" the limit process at a point deemed close enough or small enough. Upon pausing the process, one can imagine using that current state to assign a temporary value to a differential. This is actually a common and important practice in many STEM fields, which use measurements at small scales to produce contextual values for differentials (Roundy, Weber, et al., 2015). How small is small enough is context-dependent; in one situation a time interval of 0.01 seconds might be a "differential" amount of time based on that context, where in another situation a time interval of 0.01 seconds might be too big to be considered a "differential" amount of time.
Zero Value for Output Only	For $\lim_{x\to a} f(x) = L$, if $f(x)$ attains and remains at L at some point in the limit process, the differential *df* (or *dy*) is zero, a legitimate value for the *output* variable differential only. The *input* variable differential (*dx*) cannot be zero
Inheriting Quantitative Properties	Differentials automatically inherit all the quantitative properties of the quantity represented by that variable. For example, if *F* represents a force measured in Newtons, then the differential *dF* is automatically also a force, presumably also measured in Newtons (Hu & Rebello, 2013).

How is This Framework Different From Previous Work?

One could ask: Since this framework is also based on informal, conceptual ideas, does it add anything to what already exists in the literature? The main difference we see is that previous work tends to define differentials as "tiny bits" without giving much more theoretical clarity on how such tiny bits connects coherently to limits. Our framework clearly states one possibility where differentials come into existence as objects and what their relationship to limits is. By articulating them explicitly, we are better positioned to construct instructional activities to simultaneously build limits and differentials in the same calculus classroom.

Differentials-within-limits in Action: Differentials as "Models"

Our framework used the notions of limit processes and objects to provide a conceptual definition for differentials, and to lay out several conceptual properties of differentials. The next big question for our framework is what implications these definitions and properties have for first-year calculus teaching. These implications can be encapsulated through the idea of a mathematical *model*. In brief, a model is a mathematical description of an object or system, even if the model is not a perfect representation of the object or system being studied (for more on modeling, see Blum et al., 2007; Cai et al., 2014). We consider a "differential" to be precisely this – a *conceptual model* of what happens at small scales. To illustrate, imagine a string that produces a pitch when plucked, where a shorter length (*L*) corresponds to a higher pitch (*f*). Then consider the limit, $\lim_{L\to 0+} f$. In the limit process, we can think of the string becoming increasingly short and imagine "pausing" the process once we've reached some incredibly small length. This tiny remaining length is what we have defined to be a differential, *dL*, and it would correspond to a very high pitch. While we can never truly make the string be of length zero, because there would no longer be a string nor a pitch, we can model the phenomenon of the limit process by considering what happens as the string length becomes differential-sized. The *dL* is a *model* of an ever-shrinking string length, paused at a specific moment, and its relation to the output variable (*f*) can be examined within the differential model to generate expectations of what the limit value might be (i.e., ∞ or a finite value). The limit process can then be resumed and paused again, and the model *dL* can be reexamined.

Importantly, differentials are commonly used in this way across the sciences. Differentials such as *dm*, *dI*, *dF*, and *dU* are models of systems dealing with small masses or small energies, which can be given values through small-scale measurements (Roundy, Weber, et al., 2015).

Derivatives and Integrals as "Best Models" Within This Framework

Because derivatives and integrals are also defined as limits and use differential notation, our framework, including the idea of models, also extends to these concepts, though in a specific way. Derivatives and integrals are not only models, but are what we term *best models*.

To explain, in our definitions and properties for differentials, *dy/dx* would be a literal ratio of two tiny changes or amounts. But, there is a key theoretical issue to resolve. Take, for example, $y = x^2$. The derivative evaluated at $x = 3$ should be $dy/dx = \lim_{x\to 3}(x^2 - 3^2)/(x - 3) = 6$. The issue is that no matter where the limit process is paused, there is *no* specific point at which a literal ratio of tiny changes gives exactly "6." Even if the limit is paused when *x* is a distance of 0.0001 of 3, the tiny changes at that moment give 0.00060001/0.0001 = 6.0001, which is not exactly 6. How do we resolve this problem in our framework? First, differentials in our framework have no specific inherent size, and assigning a specific value like this would be done only for pragmatic, contextual reasons. Thus, we would not necessarily claim that *dy/dx* = 0.00060001/0.0001, unless the temporary assignment of those values is done for some practical reason, such as in experimental measurement. Even so, it is still a good model for what the derivative ultimately would be. Yet, there remains an uncomfortable mathematical issue in how we can state that *dy/dx* = 6, rather than *dy/dx* ≈ 6, when we define a differential through pausing the limit process. This is where the idea of a *best model* comes in.

The derivative is fundamentally meant to answer the question: "What is the *single* numeric value that best represents the idea of an instantaneous rate of change at a given point?" In other words, we want a *best model* for the tricky "instantaneous rate" question. The answer certainly isn't 6.0001, because if we un-pause the limit process and re-pause it later, we would have a

new, different discrepancy between x and a, and a different momentary ratio, say 6.0000001. Thus, none of these numbers functions as a single *best model* for the instantaneous rate, because unpausing the limit process will always refine our ratio past it. Since the differentials dy/dx are meant to *conceptually model* any arbitrary pause in the limit process, there is only one single number that encapsulates the whole set of ratios we could get by pausing at different moments – the value "6." That is, the limit object (the other object produced in the limit process) serves as this best model. Here is where the lack of inherent size for differentials is a benefit in our framework. We can say $dy/dx = 6$, because there are no specific values that dy and dx each have, and 6 represents the single *best model* we can construct for ratios of very tiny changes (i.e., an instantaneous rate). This also allows us to make statements such as $dy = 6dx$, because this statement is a *best model* for what to predict for an arbitrarily small change in x. If one imagined some miniscule change in x (dx) and tried to predict how much the output variable changed, then without specifying any particular size or value for dx, the single *best model* is that dy would be 6 times as large as dx (Yu, 2020).

Integrals function in a similar way. The definite integral fundamentally addresses the question: "How much of a quantity do we have, when variation in one of the quantity's constituents prevents using a simple relationship?" Take the example of water flowing through a pipe. If it had a constant rate, then given some elapse in time, we could calculate the amount that flowed through the pipe as $A = r \cdot t$. This is what Oehrtman and Simmons (2023) call a *basic model*. But, suppose the rate is not constant. We might then imagine the limit process of zooming into smaller and smaller time intervals, until the rate appears essentially constant. If we pause at that point, we create a differential time interval, dt, and a differential amount of water in that time interval, $dA = r \cdot dt$. Oehrtman and Simmons (2023) call this a *local model*. This local model is precisely the *best model* for how much water we have in that differential time segment. Of course, zooming can never reach a point where the variation in the rate is totally eliminated, but since differentials have no specific size, they are a *conceptual representation* of the best model for constructing a tiny bit of A in each piece. No other model functions better for capturing the amount of A. This is why we assign an equals sign for an integral value, $\int_a^b f(x)dx = I$, rather than $\int_a^b f(x)dx \approx I$, because no other number is a better model for what we should get for the total amount of the quantity over the given interval.

Conclusion

To recap, we had noticed that there was a need in calculus education research to construct an explicit framework for using informal differentials within a limits-based approach that coheres with the large body of work that is being done in differentials, integrals, and derivatives. In other words, we did not want a framework that would run counter to this important work. Using the literature as our launching point, we laid out both conceptual definitions and conceptual properties of differentials *within* the broader limit architecture. We then elaborated on the implications of this framework for instruction, based on the idea of *models* (for differentials, generally) and of *best model* (for derivatives and integrals, specifically). Of course, ours is one possible framework and we welcome discourse in this area. We believe that others' ideas could contribute to theoretically refining our framework, or creating alternate theoretical frameworks. Regardless, we believe that continued progress in calculus education research will benefit from having these explicit frameworks to work with. They can lead to instruction that better articulates for students what a differential is and how it fits within limits, so that students are able to learn *both* limits *and* differentials within a single course, leveraging the benefits of each.

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An Argument for Teaching Logic With Euler Diagrams Instead of Truth Tables

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This theoretical report integrates insights from various lines of research to posit an argument that early logic instruction for proof-based mathematics should use set-based conditions for the truth of conditional statements rather than the widely used truth table definition. We claim the truth table definition poses unnecessary barriers for students, in part because it does not adequately express mathematical practice. In contrast, the set-based meaning seems highly productive and may better support students in reasoning about conditionals as objects, which our experiments suggest is a highly propitious development for learners.

Keywords: logic, conditionals, truth tables, Euler diagrams

It is standard practice in early proof courses to teach mathematical logic so that students can learn precise conditions for declaring mathematical statements true or false. This occurs especially often in transition to proof courses and discrete mathematics courses (David & Zazkis, 2020; Schumacher, Epp, & Solow, 2015). The most common approach to teaching the logic of statements uses truth tables (Alcock & Sa, 2025), which define the truth of compound statements – such as disjunctions, conjunctions, and conditionals – as a function of the truth of their component propositions. For instance, the conditional "if p, then q" is only false under the condition that p is true and q is false. Based on our joint experiences on two multi-year projects researching the teaching and learning of logic for mathematical proving, we have become convinced that this approach should be replaced with set-based definitions and Euler diagrams. In this theoretical paper, we set forth our arguments for this recommendation drawing on relevant literature from psychology, mathematics education, and our own research experiences. While many of these arguments have appeared in our prior work, we have not put them together in a unified argument meant to prompt a shift in teaching, as we intend for this argument to do.

To build this argument, we pursue a few lines of inquiry that challenge the preference for truth tables to help students understand mathematical language, specifically mathematical conditionals. The first principle we challenge is whether the truth table expresses a common-sense meaning of "if… then" statements from everyday language or how people reason about such statements. We share evidence from psychological research that the truth table definition of conditional truth (often called the "material conditional") fails as a descriptive account of everyday usage. Although this does not undermine the utility of this definition for mathematical usage, it points to the fact that it is a novelty being imposed upon student language use. Second, we challenge whether the material conditional adequately expresses what conditional claims mean in mathematics. We share some mathematics education research that challenges it as a descriptive model for how mathematicians reason. Further, we identify some counter-intuitive "facts" that arise when the truth table definition of "if p, then q" is taken to correspond to the mathematical notion that "p implies q."

The third component of our argument focuses on the adequacy of set-based truth conditions and Euler Diagrams to support student reasoning about conditionals in mathematics.

Specifically, a universally quantified conditional "For all $x \in S$, if $P(x)$, then $Q(x)$" is true whenever the truth set of P is a subset of the truth set of Q, which we shall refer to as the subset meaning (see also Dawkins, 2017; Dawkins, Roh, & Eckman, 2023). We offer three main points in favor of teaching with the subset meaning and Euler diagrams. First, we review some psychological research that suggests that Euler diagrams better facilitate novice learning, suggesting that there may be greater ease of use than has been found for truth tables (Hawthorne & Rasmussen, 2015). Second, our research suggests that some parts of the proof-based curriculum require that students learn to reason about conditionals as objects, meaning they conceptualize implication as a fixed relationship between two predicates. This kind of understanding is better facilitated by Euler diagrams than by truth tables. Third, we argue that set-based truth is a better descriptive account of how conditionals operate in mathematical language.

Point 1: The Truth Table Definition Fails as Descriptive Model for Everyday Conditionals

As Toulmin (1958) argued, scholars use the term *logic* to refer to at least four different kinds of endeavors. Two of them are descriptive, and two of them are prescriptive. In its descriptive mode, *logic* refers to either how individual people reason (Toulmin's *psychology*) or to how groups of people reason in some idealized or normative sense (Toulmin's *sociology*; note this second meaning blends in a prescriptive component). In its prescriptive mode, *logic* refers either to a means for building desirable arguments (Toulmin's *technology*) or to a formal system akin to mathematizing language (Toulmin's *science*). The descriptive sense of logic is at play when someone asks, "What is the logic in that?" while mathematical logic tends to focus on prescriptions that render language rule-based and consistent. Although the association of logic with processes of reasoning has been deeply criticized by some logicians (e.g., Frege and Husserl; see Schroyens, 2010, for helpful discussion), the potential association has led to two extensive lines of inquiry. Philosophers and logicians have pursued the question "can we build formal systems of logic that incorporate elements of everyday argumentation and language use?" This has led to novel logics such as fuzzy logic (in which claims can have truth values other than *true* or *false*) or modal logics that incorporate different senses in which a claim is true (e.g., *possibly true* in addition to *necessarily true*). Psychologists have pursued the converse question, "can we identify which formal system best describes how people interpret and reason about conditional claims?" It is this line of research that most directly informs our current argument.

One of the dominant means by which psychologists have studied how people interpret conditional claims has been the Wason (1966) card task. We will describe the task in its generalized form. Participants are asked to test a rule of the form "if p, then q" regarding four cards displayed. Each card has one side relevant to p and another side relevant to q. The four displayed cards show cases corresponding to p, not p, q, and not q. Participants must indicate which cards need to be turned to check whether the rule is followed/violated. It is far beyond the scope of this paper to review this vast literature (using variants of the Wason task and related study designs), but we shall recount a few key elements from the findings.

First, though the Wason card task seems a simple operationalization of the material conditional definition for conditional claims, the standard rate at which untrained undergraduates provide the normative answer is about 10% (Evans & Over, 2004). According to the material conditional definition that a conditional is false whenever p is true and q is not, the normative solution to the task is to turn the p and not q cards, since these are the two that might yield a counterexample to the rule. The most common answer provided by most undergraduates to many forms of the task are to turn the p and q cards, meaning they select an irrelevant card and omit a

relevant one. Second, performance on the task varies greatly depending upon the context for the rule. If the rule corresponds to a known social rule such as "if someone is drinking, then they must be at least 21 years old," then performance on the task greatly improves. This runs contrary to the underlying notion that logical reasoning pertains to the form of an argument, not to its content. These findings drive a wedge between the prescriptions of formal logic and empirical findings about human reasoning about statements in conditional form. For these and similar reasons, Schroyens (2010) claims that "Classical propositional calculus is vastly inadequate as a theory about how people reason. Whatever merits one (e.g., a logician) wants to attribute to the propositional calculus, one merit it cannot possess is being a descriptively adequate theory of human reasoning" (p. 71).

What role should a descriptive account of everyday reasoning hold for the teaching and learning of logic in mathematics? We see two main implications. First, instruction should not assume as intuitive any logical notions that do not have empirical support as being widely held apart from mathematics instruction. Second, instruction should privilege ways of teaching the logic of mathematical statements that do not needlessly introduce contrasts with everyday language use, all other things being equal. The descriptive account that has better empirical support relies heavily on probabilistic notions. According to the suppositional account, one's affirmation of "if p, then q" in the everyday is most closely linked with the conditional probability of q given p is true (see Evans & Over, 2004; Oaksford & Chater, 2020). This account helps explain why some human reasoning in experiments aligns to the *defective truth-table* for conditionals, which only includes the cases where p is true (Evans et al., 2007; Wason, 1966), as the meaning of "if p, then q" entails assuming p true. Cases in which p is false are taken as merely irrelevant to the conditional. Indeed, most anyone who has tried to teach the logic of conditionals using the standard truth table have found that convincing students that the conditional is true when p is false constitutes a steep uphill battle. The suppositional account justifies students in their resistance to this aspect of the material conditional.

We find these findings from non-mathematical settings insightful, and they help contextualize the rest of the paper. However, we should note that everyday meanings for conditionals need not restrain mathematical usage. There are fundamental distinctions. As Oaksford and Chater (2010) note, proponents of the suppositional account "argue that logical methods need to be supplemented with probabilistic notions not because classic logic is incorrect, but because they contend that outside mathematics, it rarely, if ever, applies to real human inferences and arguments" (p. 21). By contrast, mathematical inferences and arguments certainly align to logic in some manner (though it is worth considering the nature of this link carefully). Furthermore, we are concerned with student learning of technical meanings, rather than untrained usage of language (Alcock, 2025a). Next, we shall examine how the truth-table definition fails to express how mathematicians reason about conditionals, even in mathematics.

Point 2: The Truth Table Does Not Capture the Mathematical Meaning for Conditionals

As we noted in the introduction, the standard way of presenting the truth conditions for mathematical conditionals is the full truth-table. However, there are multiple reasons to question whether the material conditional truly aligns to mathematical usage. We will provide a few different strands of support for this claim.

Do Mathematicians Reason About Conditionals According to the Material Conditional?

First, if we assume mathematicians as paragons of logical reasoning, we may wonder how mathematicians respond to standard psychological tasks such as the Wason Card task. Inglis and

Simpson (2004) administered this task (with abstract content) to mathematicians and found that their answers differed from what is commonly observed among mathematics or history undergraduates, but still did not give the normative answer (conforming to the material conditional) more than half of the time. Mathematicians were more likely to avoid irrelevant cards (each irrelevant card was selected 19% of the time), but still only considered the not q card only 67% of the time.

In a more qualitative investigation of the process of reasoning, Inglis (2006) found that the best way to explain how mathematicians reasoned about mathematical conjectures (in conditional form) was the suppositional account. Specifically, he claims they assumed p true and then considered the probability that q was also. The distinction is that in the everyday, people will affirm a conditional even in the presence of counterexamples (instances of p and not q) while in mathematics, the observed reasoners sought to affirm that q was necessarily true without exception, ultimately by producing a proof. Excepting the affordance of various degrees of certainty in the truth of q, Inglis' account of the mathematicians' reasoning is compatible with the defective truth table account, which Hoyles and Kuchemann (2002) endorsed as adequate for usage in the mathematics classroom.

Attridge and Inglis (2013) provide a final piece of evidence supporting the link between mathematical conditionals and assuming p is true (consistent with the suppositional account or the defective truth table). In this study, the authors presented abstract reasoning tasks to students in mathematical and non-mathematical courses of study near the beginning and near the end of one year of secondary school studies. The goal was to compare the influence of mathematical study on abstract inference tasks with non-mathematics students as a comparison group. A year of studying mathematics (including no lessons on logic) decreased the rate of affirming the (invalid) converse inference, but also slightly depressed the rate of affirming the (valid) modus tollens inference from not q to not p. The authors explain this pattern as aligning to the defective truth table pattern of conditional inference.

Conditional Truth Versus Valid Inferences

Weber and Alcock (2005) asked mathematicians whether they affirmed the statement "if 7 is prime, then 1007 is prime," which everyone in the experiment took to be true according to the material conditional definition (1007 is non-obviously composite). Those authors found that mathematicians did not affirm this claim because 1007's primality does not follow from 7's primality (e.g., the inference "if x is prime, then $x + 1000$ is prime" is not valid). The authors concluded that the criterion by which these mathematicians evaluated the conditional was stronger than the truth table definition. Mathematicians required that q be a warranted inference from p. Along similar lines, Vargas (2020) notes, "This is the case for conditional statements which are not seen as formulas with truth values in themselves, but more as "licenses for inference" in logic programs" (p. 58). Janke and Wambach (2013, quoted in Vallejo-Vargas, 2023) add,

> a mathematical proof does not establish *facts*, but *implications*. A proof proves 'if-then-statements'. That means a mathematical theorem is not about a fact *B*, but about an implication 'If *A* then *B*'. Thus, the *absolute certainty* of mathematics does not reside in the fact, but in the *inferences*. (p. 471)

So, for teaching logic we should desire an account of conditionals that facilitates the important link between conditional statements and implications (Alcock & Sa, 2025).

It is precisely this intended link between conditional statements and implications that renders the material conditional an inadequate account of conditionals in mathematics. To see this

disconnect more vividly, consider that according to the material conditional, $(p \to q)\ or\ (q \to p)$ is a tautology. However, we would never want to affirm that for any claims p and q, one must imply the other. We raise this challenge because it allows us to endorse what we see as a more productive alternative to the material conditional. Most mathematical conditionals of interest are not conditionals involving propositions with a fixed truth-value, but rather universally quantified claims between predicates: "$\forall x \in S, P(x) \to Q(x)$." If we introduce this modification, the resulting expression $(\forall x \in S, P(x) \to Q(x))\ or\ (\forall x \in S, Q(x) \to P(x))$ is no longer a tautology. We will argue in the next section that an approach to conditional truth that builds upon universal quantification is more productive and instructionally adequate, namely the subset meaning represented using Euler diagrams.

Point 3: A Set-Based Definition for Conditionals is More Adequate for Teaching Logic

If we are to argue against the truth table definition of the truth of mathematical conditionals, we must offer a viable alternative. In our instructional design projects (NSF DUE #1954768, #1954613, and #2141626), we have explored such a viable alternative in the subset definition of conditionals. In this view, the conditional "$\forall x \in S, if\ P(x), then\ Q(x)$" is true whenever the truth set of P is a subset of the truth of Q. We encourage students to explore such relationships between truth sets using Euler diagrams. In this section, we provide three strands of support for the preferability of this approach.

Euler Diagrams Are Accessible for Novice Learners

One affordance of universally quantifying conditionals is that it brings them closer to the kinds of statements classically used in Aristotle's system of syllogisms. That system featured statements of the forms "All A are B," "Some B are C," and "No C are A." The development of this system preceded the development of modern logic by about 2000 years, which may speak to its relative intuitiveness, though this evidence is clearly only suggestive. Nevertheless, this allows us to draw upon research investigating how students reason about syllogisms in addition to the Wason Card tasks and inference tasks featured in the studies reviewed above.

Sato, Mineshima, and colleagues have conducted a series of experiments exploring how students solve syllogism tasks with relatively brief training (Mineshima et al., 2012, 2014; Sato & Mineshima, 2015; Sato et al., 2010, 2011). In many cases, they compare student success on syllogism tasks after one of three instructional treatments: brief training in Euler diagrams; brief training in Venn diagrams; and training only in sentential, non-diagrammatic approaches. Venn diagrams differ from Euler diagrams in that they always show the closed curves overlapping and use shading and *x*'s to indicate which regions are empty or occupied. Euler diagrams by contrast will show closed curves non-overlapping either to show one set is disjoint from another or is a subset of another (when one curve surrounds another). A brief summary of these studies' findings is that students perform better on syllogism tasks when using diagrams, and Euler diagrams support more accurate responses than Venn diagrams.

These authors claim that these findings demonstrate that such diagrams are highly accessible to novice learners. "The manipulation of diagrams in the inferential process is triggered without effort, if the spatial relations holding on external diagrams are governed by *natural* constraints" (p. 29, Mineshima et al., 2012). Although we do not echo their emphasis on the lack of effort involved for learners, we value their evidence that Euler diagrams are particularly helpful. "Our hypothesis is that Euler diagrams are *self-guiding* in the sense that the construction of diagrammatic proofs could be automatically triggered even for subjects without explicit prior knowledge of inferential strategies or rules, whereas Venn diagrams are not" (p. 7, Sato et al.,

2010). Their argument for this contrast, by which they explain their empirical findings, relates to the contrast between the underlying accounts of logic. Whereas the standard textbook treatment (based on truth tables) analyzes statements using quantification over individuals,

> in the field of natural language semantics, by contrast, quantifiers in natural language, such as *all*, *some* and *no*, are analyzed as denoting *relations* between sets… Thus, a sentence of the form *All A [are] B* is analyzed as $\boldsymbol{A} \subset \boldsymbol{B}$, rather than as the first-order representation $\forall x(Ax \rightarrow Bx)$. Similarly, *No A [are] B* can be analyzed as expressing $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$. (p. 2183, Sato et al., 2011)

They argue that Venn diagrams foster analysis by region of the diagram, while Euler diagrams foreground relations between the sets as units.

We argued in a previous section that findings about student thinking outside of mathematics inform logic instruction by helping us not place unnecessary barriers in students' ways. These studies of student performance on syllogism tasks suggest that Euler diagrams are particularly facilitating, likely because they foreground conditionals as relations between sets. The fundamental object relevant to truth-table analysis is the specific example object or a non-quantified statement about specific objects. Set-based truth conditions, especially as represented by Euler diagrams, support analyzing the truth of mathematical claims with the truth sets as objects. This means the truth of a conditional hinges on the relationship between two sets rather than a potentially infinite range of individual objects. Both views are ultimately useful in their way, but we contend that the view relating sets may be more facilitating for early logic learning. Insights from our studies in proof-based mathematics courses further support this claim.

Euler Diagrams May Support Students in Reasoning About Implications as Objects

In our investigations of student learning in introduction to proof courses, we have found that it is important for students to learn to reason about implications as objects, rather than actions or pseudo-objects (Norton et al., 2025). Building upon Sato et al.'s (2011) discussion, we can identify the distinction between noting a subset relationship between the sets $\boldsymbol{A}$ and $\boldsymbol{B}$ versus reasoning about the subset relationship as an object. As noted in Norton et al. (2025), this becomes particularly relevant when trying to act on a particular implication via operations such as composing implications ($p \rightarrow q$ and $q \rightarrow r$ yield $p \rightarrow r$), negating implications (which is distinct from negating either or both components), or transforming implications such as by inversion or conversion and reasoning about the properties of the transformed implication. We claim that set-based meanings for conditionals, especially when explored using Euler diagrams, may help foster understanding of implications as objects. While we acknowledge that this claim needs nuance in terms of how and how frequently this claim bears out in student learning, we also see evidence that truth tables are particularly unfit for fostering this kind of understanding.

Hawthorne and Rasmussen (2015) conducted one of the primary studies of how students reason about truth tables in learning logic in mathematics. One contrast they found was that the instructor conceived of the truth table as a unified definition of the truth of a statement while students often displayed a fragmentary understanding of the truth table. They understood each line as assigning truth values from the parts to the whole but did not reason about the truth table in a unified manner. We find it unsurprising that patterns of truth values do not support students in conceiving of a conditional statement as a fixed relationship between p and q, much less that it would support them to operate upon that object. Actions like composition, negation, and transformation can be carried out with truth tables by producing ever larger lists of T's and F's, but they do not provide a unified image of the meaning of the conditional to be acted upon.

By contrast, if students associate the conditional "if p, then q" with a subset relationship on their truth sets $\boldsymbol{A} \subset \boldsymbol{B}$, then we see easily how students could compose ($\boldsymbol{A} \subset \boldsymbol{B}$ and $\boldsymbol{B} \subset \boldsymbol{C}$ yields $\boldsymbol{A} \subset \boldsymbol{C}$), negate ($\boldsymbol{A} \not\subset \boldsymbol{B}$), and reason about transformations of this relationship (given $\boldsymbol{A} \subset \boldsymbol{B}$, what is known about $\boldsymbol{B} \subset \boldsymbol{A}$ or $\boldsymbol{A^c} \subset \boldsymbol{B^c}$?). We have explored some of these affordances in our teaching experiments (Dawkins et al., 2023; Hub & Dawkins, 2018). These experiments suggest these are viable learning pathways for at least some students, though further work is in order.

Another endorsement of the set-based meaning over the truth table definition comes from the literature we reviewed on the suppositional account and the defective truth table. As we noted before, students are most often troubled by the standard truth table because it claims conditionals are true when p is false. This whole question becomes a non-issue when analyzing conditionals using the set-based definition. Students are aware that to decide whether $\boldsymbol{A} \subset \boldsymbol{B}$ only objects in $\boldsymbol{A}$ are relevant and that objects outside of $\boldsymbol{A}$ certainly do not falsify the relation.

Subset Relations Capture Some Aspects of Mathematician Reasoning

In line with the potential compatibility between the subset meaning of conditionals and the suppositional account/ defective truth table, we think this could help explain some of the psychological findings about mathematician reasoning on the Wason Card task and abstract inference tasks. In particular, mathematical training particularly supports reasoners in avoiding fallacious inferences such as $q \to p$ and $\neg p \to \neg q$, but is less successful in promoting attention to the *modus tollens* inference $\neg q \to \neg p$. This seems reasonable given a strong association of a conditional with a subset relation, as subset relations are asymmetric (suppressing $q \to p$) and instances outside the subset are irrelevant (suppressing $\neg p \to \neg q$). It stands to reason that this same de-emphasis on cases where p is false would similarly suppress the *modus tollens* inference, as for a true conditional the truth sets of $\neg q$ and p are disjoint.

We will reference one final bit of evidence for the compatibility of mathematical practice and the subset condition. Alcock (2025b) investigated what diagrams mathematicians preferred for portraying conditional relationships. She found almost uniform preference for Euler diagrams over Venn diagrams. This means they chose representations that emphasize the truth conditions of a conditional, as Venn diagrams portray the region in which p is true and q is false, which may then be empty (the conditional being true) or non-empty (the conditional being false). While further work is needed to understand the nature of this preference, this supports our claim that the subset relation captures some aspects of how mathematicians reason about conditional statements in mathematics.

Conclusion

In this theoretical paper, we have drawn on a range of scholarship from different fields to build a case against the use of truth tables to teach logic for proof-based mathematics, which at present holds dominance in textbooks. We claim this approach puts unnecessary barriers between student reasoning from the everyday and mathematical practice, in part because the truth table definition does not adequately capture mathematical usage of conditionals. We offer the subset meaning as a viable alternative that avoids these challenges. Some studies we reviewed suggests that reasoning about conditionals in terms of set relationships is quite accessible, especially when explored using Euler diagrams. We do not mean to imply that there are not major challenges along that learning pathway, but we are convinced that this alternative approach has ample evidence to support a move away from the current status quo.

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How and Why Mathematics Faculty Use Evidence-Based Instructional Practices: A Revised Model of Instructional Decision-Making in Undergraduate Mathematics Education

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Evidence-based instructional practices (EBIPs) are widely recognized as effective, yet their use in undergraduate mathematics remains sparse. While prior work has explained why faculty adopt EBIPs, less attention has been given to what happens after that decision – namely, how instructors implement these practices, why they use some approaches over others, and what shapes their persistence. In this theoretical report, I review literature on individual and contextual factors influencing EBIP use and identify limitations in existing models of instructional decision-making. I then propose a revised model that integrates Lattuca and Pollard's (2016) framework with Schoenfeld's (2010) resources-orientations-goals (ROG) theory. The model emphasizes the context-dependent nature of mathematics faculty's EBIP use and centers implementation as the site where persistence is shaped. Using data from one case study, I illustrate how the model can trace the full cycle of EBIP engagement from adoption, to enactment, to the experiences that influence continued use.

Keywords: Evidence-Based Instructional Practices, Active Learning, Undergraduate Mathematics Education, Faculty Decision-Making, Instructional Persistence

In undergraduate mathematics, traditional lecturing remains dominant across national and linguistic boundaries (Artemeva & Fox, 2011; Johnson et al., 2018; Stains et al., 2018), despite decades of research supporting evidence-based instructional practices (EBIPs), such as active learning. EBIPs promote student-centered instruction, foster higher-order thinking, and often involve collaboration, with evidence showing improved exam performance and reduced failure rates even in large classes (Freeman et al., 2014; Theobald et al., 2020). However, EBIPs encompasses a wide range of strategies, from minimal techniques like incorporating clicker questions to fully restructured formats such as inquiry-based mathematics education (Laursen & Rasmussen, 2019). This variation highlights that EBIP adoption is not a simple binary and that mathematics faculty may implement these practices in diverse ways (Johnson et al., 2018; Knowles et al., 2023).

Existing models of faculty instructional decision-making (e.g., Lattuca & Pollard, 2016; Sturtevant & Wheeler, 2019) explain why instructors might choose EBIPs but say little about what happens after that decision, such as how and why instructors implement these practices differently, and how these experiences shape their persistence using them. In this theoretical report, I highlight the need for research from this perspective and propose a substantial revision to an existing model of faculty decision making – one that better accounts for the iterative and context-dependent nature of EBIP engagement in undergraduate mathematics. This work aims to inform change initiatives that seek to expand EBIP use in undergraduate mathematics education.

Literature Review

This literature review has two parts. I first discuss the current body of literature on why mathematics faculty use EBIPs in their instruction. I then discuss limitations in what is known about why faculty use the specific EBIPs they do and how their implementation experiences are

connected with their persistence using these practices, underscoring the need for models that account for this.

The Use of EBIPs in Undergraduate Mathematics Education

The uptake of EBIPs among mathematics faculty appears to be sparse (Johnson et al., 2018), reflecting a broader trend in collegiate STEM education (Froyd et al., 2013; Henderson et al., 2012; Larsen et al., 2019; Stains et al., 2018). Consequently, much of the research on why faculty use EBIPs comes from STEM education more broadly. This body of research has identified two main categories of factors that influence faculty decisions to engage in EBIPs: individual factors and contextual factors.

Contextual factors influencing EBIP use. Contextual factors refer to external, institutional, and departmental influences (Sturtevant & Wheeler, 2019). These include physical resources such as class size and classroom space (e.g., Apkarian et al., 2021; Henderson et al., 2012); teaching evaluations and student buy-in (e.g., Deslauriers et al., 2019; Shadle et al., 2017); time constraints (e.g., Johnson et al., 2018, 2019; Turpen et al., 2016); and the level of support and incentives provided by institutions and departments (e.g., Lund & Stains, 2015; Sturtevant & Wheeler, 2019).

Large classes and fixed seating can limit the student collaboration many EBIPs require, while flexible learning spaces are associated with greater uptake (Foote et al., 2014; Knaub et al., 2016). Faculty concerns about student resistance, often tied to increased cognitive effort or unfamiliarity, may discourage adoption, especially when negative teaching evaluations are feared (Apkarian et al., 2021; Deslauriers et al., 2019; Owen et al., 2020). Time constraints are also a common barrier: faculty cite insufficient time to prepare EBIPs, learn them effectively, or cover content, with concerns in mathematics often linked to preparing students for later courses (Johnson et al., 2018, 2019; Woods & Weber, 2020). Without adequate institutional or departmental support, these challenges can be exacerbated, particularly in environments that prioritize research over teaching; conversely, departments with student-centered norms tend to foster EBIP adoption through professional development, mentoring, and ongoing teaching-focused dialogue (Andrews & Lemon, 2015; Lund & Stains, 2015; Shadle et al., 2017).

Altogether, this literature suggests that multiple contextual factors, such as physical space, student buy-in, time constraints, and institutional culture, shape faculty's use of EBIPs. These factors can discourage adoption by creating logistical or motivational barriers, particularly when institutions do not actively support or incentivize teaching innovation.

Individual factors influencing EBIP use. Unlike contextual factors, which depend on specific settings (e.g., a specific institution or class), individual factors refer to personal influences such as orientations (e.g., beliefs, values, dispositions) and goals (Schoenfeld, 2010), identity, and prior experiences (Sturtevant & Wheeler, 2019). These factors shape instructional practice and can, in some cases, outweigh contextual influences in determining whether faculty adopt EBIPs (Andrews & Lemons, 2015; Johnson et al., 2019). The most salient of these factors are faculty's prior experiences, particularly those shaping their satisfaction with EBIPs, and their orientations and goals.

Experiences and satisfaction. Faculty experiences – as both students and teachers – play a critical role in shaping instructional choices. In mathematics, most instructors experienced lecture-dominated classrooms as students, and many replicate these methods, particularly if they were personally successful in them (Lortie, 1975; Woods & Weber, 2020). Conversely, dissatisfaction with lecturing – such as perceiving it as less engaging or less effective – can prompt faculty to explore EBIPs (Andrews & Lemons, 2015; Johnson et al., 2019). Positive

experiences implementing EBIPs tend to encourage continued use (Andrews & Lemons, 2015; Sturtevant & Wheeler, 2019), whereas negative experiences – such as challenges in implementation, lack of preparation time, or unexpected student resistance – can lead to discontinuance (Hayward et al., 2016; Sturtevant & Wheeler, 2019). These findings suggest that satisfaction, whether with lecturing or with EBIPs themselves, is a pivotal motivator for instructional change.

Orientations and goals. Faculty orientations and goals – such as beliefs about student potential, views on where learning should occur, and perceptions of professional responsibilities – strongly influence teaching choices (Johnson et al., 2018, 2019; Shultz, 2022; Vishnubhotla et al., 2022). Instructors who believe all students can learn mathematics and value in-class engagement are more likely to adopt EBIPs, while those with lower expectations for student potential or a belief that learning occurs primarily outside of class (e.g., while studying) are more likely to lecture extensively (Johnson et al., 2019; Woods & Weber, 2020). Additionally, instructors who prioritize covering content with precision often view lecturing as the most efficient way to meet their goals (Woods & Weber, 2020; Shultz, 2022), whereas those who emphasize student sense making may view EBIPs as better aligned with their objectives (Dawkins et al., 2019; Johnson et al., 2013). Beliefs about student engagement also matter: some faculty see frequent questioning during lectures as sufficient active learning (Woods & Weber, 2020), in contrast to how active learning is usually conceptualized in the literature through higher-order thinking involving discussion, problem-solving, and collaboration (Chi et al., 1989, 1994; Hattie & Timperley, 2007; Banes et al., 2018, 2020; Callahan et al., 2021).

In short, faculty adoption and persistence with EBIPs is shaped by a combination of experiences, satisfaction, and deeply held orientations and goals. These individual factors can motivate faculty to innovate or reinforce their preference for traditional lecturing, and interact with contextual factors that either support or constrain their choice to use EBIPs.

The Need to Conceptualize How and Why EBIPs are Used by Mathematics Faculty

Active learning encompasses a wide range of instructional strategies that vary in intensity and impact. Some strategies, such as partial notes, clicker questions, think-pair-share, or small group problem solving, can be flexibly integrated into traditional lectures (Alcock, 2018; Braun et al., 2017; Dawkins & Weber, 2023), while others require substantial restructuring of the course, such as flipped classrooms (Milman, 2012), inquiry-based mathematics education (Laursen & Rasmussen, 2019), or the Extreme Apprenticeship method (Rämö et al., 2015). Research has shown that mathematics faculty use a variety of such EBIPs (e.g., Johnson et al., 2018). However, much of the literature examining these strategies in mathematics education is interventionist in nature, often situated within professional development programs or targeted course reforms (e.g., Hayward et al., 2016; Johnson et al., 2013; Laursen & Rasmussen, 2019). While these studies have been valuable for demonstrating the benefits of active learning, some research suggests that such intervention contexts (e.g., professional development) may influence the learning gains observed (Andrews et al., 2011) and the kinds of instructor experiences that shape persistence with EBIPs (Andrews & Lemons, 2015; Sturtevant & Wheeler, 2019).

Given this variation and the potential influence of intervention contexts, this highlights the need for theoretical perspectives that can account for both how mathematics faculty adopt and implement EBIPs in a range of settings, including the different ways they enact active learning in their classrooms. While many EBIPs have distinct names and program structures, they often share core practices such as facilitating student collaboration, eliciting student thinking, and structuring opportunities for sensemaking (Freeman et al., 2014; Theobald et al., 2020). Hence,

conceptualizing how mathematicians use these practices – including how contextual and individual factors influence which practices they choose and how their experiences shape their persistence – can complement intervention-based research and provide insight into the mechanisms that make active learning effective and sustainable. In total, this underscores the value of theory that can account for both how and why mathematics faculty use EBIPs.

A Model of Faculty's Use of EBIPs in Mathematics Education

In this theoretical report, I propose a model of faculty's use of EBIPs to investigate how and why mathematics faculty use these practices in undergraduate mathematics. I build on Lattuca and Pollard's (2016) model of faculty decision-making, as modified by Sturtevant and Wheeler (2019), and integrate Schoenfeld's (2010) resources–orientations–goals (ROG) theory. The ROG framework complements the decision-making model's structure and terminology while offering a lens for understanding how faculty decisions manifest in classroom practice and influence their experiences using EBIPs. I also incorporate refinements from my literature review.

Conceptualization of EBIPs

Implementing EBIPs with fidelity requires skills and knowledge that take time and effort to develop (Andrews-Larson et al., 2019; Archie et al., 2022; Fukawa-Connelly, 2016; Hayward et al., 2016; Johnson & Larsen, 2012). Some scholars therefore conceptualize EBIP use along a continuum rather than as a strict binary (Braun et al., 2017). Rather than urging all mathematics instructors to immediately abandon lecture, others emphasize ways to adapt lecture to be more student-centered and equitable (Alcock, 2018; Dawkins & Weber, 2023; Rasmussen & Johnson, 2023). Moreover, evidence of EBIP effectiveness (e.g., Freeman et al., 2014) often reflects a range of implementation levels, including courses with minimal active learning (Rasmussen & Johnson, 2023). I therefore define EBIPs broadly as instructional practices that actively engage students in learning mathematics, spanning both flexible and transformative strategies, from partial integration of active learning techniques to fully inquiry-based approaches.

An Existing Model of Faculty Decision-Making

Lattuca and Pollard's (2016) model of curricular change, as adapted by Sturtevant and Wheeler (2019) to focus on EBIPs, describes how extra-institutional, external, and internal influences shape faculty motivation to change instructional practices. Extra-institutional influences situate decisions within broader socio-cultural contexts, such as the COVID-19 pandemic or national calls to improve STEM education (Conference Board of the Mathematical Sciences, 2016; Saxe & Braddy, 2015). External influences include institutional and departmental culture, professional norms, and disciplinary orientations, while internal influences include beliefs, knowledge, professional identity, and prior experiences. These categories align closely with the individual and contextual factors described in my literature review.

The model posits that influences interact to shape motivation, which in turn drives adoption or rejection of EBIPs. Motivation may stem from perceived utility, personal interest, value alignment, or perceived costs (Lattuca & Pollard, 2016). Sturtevant and Wheeler (2019) extended the model to include dissatisfaction with current practice as a key motivator for EBIP adoption, and dissatisfaction with EBIPs as a driver of their discontinuance, aligning with literature on EBIP persistence (Froyd, 2013; Henderson et al., 2012). Overall, this model conceptualizes EBIP use as a motivated decision, which is constrained or facilitated by salient influences, and the decision to quit or persist with these practices is influenced by one's dissatisfaction using them.

Revisions to the Model

In this theoretical report, I propose revisions to Lattuca and Pollard's model. First, since many individual and contextual factors from my literature review align with the influences in the model, I adjusted it to use this terminology. Specifically, I expanded internal influences to include all individual factors and external influences to include all contextual factors, renaming them individual factors and contextual factors, respectively. I also embedded the model within the socio-cultural context to capture extra-institutional influences.

Next, I incorporated Schoenfeld's (2010) ROG theory into the model. While Lattuca and Pollard's model offers a framework for understanding faculty's decisions to engage in EBIPs, Schoenfeld's ROG framework provides a finer-grained lens for understanding how those decisions manifest in the classroom. Resources include both physical and knowledge-based assets, with knowledge being central. Orientations refer to the beliefs, values, and preferences that guide instructional choices. Goals are the intended outcomes, from long-term aims such as fostering mathematical engagement to immediate objectives like covering a topic. In familiar situations, decision-making is often automatic, but in unfamiliar ones, instructors' choices align with their orientations and goals, narrowing which approaches they consider

To integrate ROGs, I added an implementation phase to the model to address a limitation Sturtevant and Wheeler's (2019) adaptation; namely, the pathway from the decision to engage in EBIPs to the outcome of persistence or discontinuation is underdeveloped, leaving unexamined the experiences that shape instructors' satisfaction or dissatisfaction. In practice, these outcomes emerge through the implementation of EBIPs, where instructors encounter a range of both challenges and positive experiences. Including an "Implementation of EBIPs" phase makes visible how individual and contextual factors continue to influence instructional choices after the decision to use EBIPs is made. As others have argued, implementation experiences can influence whether faculty continue using EBIPs (e.g., Andrews & Lemons, 2015), highlighting the need to understand not just decisions to adopt, but the lived enactment of these practices in context. Hence, this addition enables researchers to examine not only whether mathematics faculty choose to use EBIPs, but also how they come to use them in specific ways, shedding light on both the substantial variation seen in EBIP implementation and the role of implementation experiences in shaping persistence.

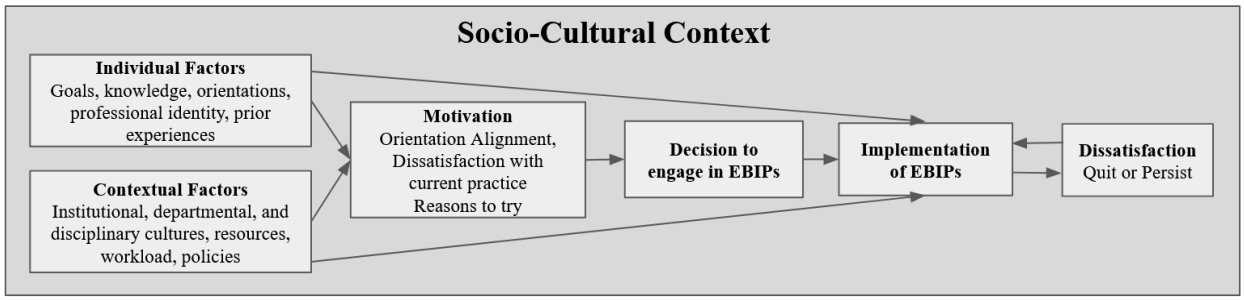

Figure 1. Visual Representation of the Revised Model. This revised framework models the influences that shape faculty engagement with EBIPs.

This model (Figure 1) provides a comprehensive cycle of EBIP use, not only looking at initial adoption but ongoing experiences. It distinguishes between individual and contextual factors, with contextual factors shaping individual ones. Individual factors include goals, knowledge, orientations, identity, and experiences. Contextual factors encompass institutional and departmental culture, resources (e.g., mentors, institutional support, classroom dynamics),

policies, and workloads. These factors, situated within a broader socio-cultural context, influence faculty motivation to engage in EBIPs and ultimately their decision to do so. Experiences implementing EBIPs, which are also shaped by individual and contextual factors, then influence whether faculty persist with or discontinue these practices continually (as indicated by bidirectional arrows in Figure 1).

An Example of the Model in Action

I draw from data from my dissertation to illustrate this cycle of EBIP use. Noah (pseudonym) is a tenured associate professor at a private research university in the western United States. He has several years of active-learning experience (including six years of flipped calculus) and, in Fall 2024, taught Discrete Mathematics using group worksheets and interactive lectures.

Motivation to Use EBIPs: The Decision to Engage in EBIPs

Noah's path to using EBIPs illustrates the influence of individual factors like goals, orientations, and prior experiences and contextual factors like departmental support. As a teaching assistant in an honors calculus course, he watched students "invent all these things for themselves," expanding his sense of what types of teaching were possible. Departmental culture later enabled him to use EBIPs, as he felt "more free to design courses" once in a department supportive of "teaching in different ways." Noah's prior experiences as a student and teacher also generated dissatisfaction with lecturing. While he "learned fine" from lecture plus homework, he recognized many of his students were "still learning…how to tell whether what they've done is correct," so they needed in-class opportunities for feedback, highlighting the need to bring active learning into his classrooms.

Implementing EBIPs: The Choice to Use Lecture and Group Work

Noah's resources, orientations, and goals explain how he implemented EBIPs: several days of interactive lectures followed by a class session devoted to small group problem solving. In discrete mathematics, his goal was for students to "learn how to read and write proofs." Firstly, Noah believed students needed initial exposure to content (orientation): "learning is also how the material is presented… they have to be exposed before they can sort of do [math]." While he preferred active learning, Noah described lecturing out of necessity, having abandoned a plan to flip his class due to limited time to prepare videos (time constraint): "I ended up lecturing content in class, and that's almost not as effective really." Still, he used active learning because "It's kind of basically kind of like getting everybody to come to office hours and work on homework…If I'm there in class when they're working on it, I can see what they're doing…" He described using group work specifically because "the process of explaining this to someone else" helps one identify "what you are stuck on," making group work better aligned with his goals and orientations than other options, such as individual problem solving in class. Hence, Noah was drawn to lectures followed by group work sessions since this let him cover content, while also providing opportunities for students to work together and receive feedback from him, while being less time intensive than flipping his class.

Dissatisfaction with EBIPs: The Choice to Persist

Lastly, an episode during the semester highlighted how Noah persisted past dissatisfaction stemming from student resistance to group work. When early group work fell flat during an initial observation, with limited interaction between students, he saw the problem not as a flaw in group work itself but as a mismatch between his orientation toward collaboration and his current

group work structure. In response, he moved from using unassigned groups to assigned pairs at shared whiteboards with a single marker to ensure collaboration. The next observation showed improved participation, including cross-interactions between pairs; Noah was "pretty happy with it" and reported using the structure again. This highlights Noah's adaptation as a persistence strategy: dissatisfaction during implementation triggered revision, not quitting, mediated by contextual factors and individual factors. In ROG terms, he mobilized resources (whiteboards), acted from orientations (collaboration) toward goals (improved proof writing).

Overall, Noah's case illustrates the revised model in action, highlighting how different factors influence one's initial motivation, EBIP implementation, and persistence. Moreover, this illustrates the potential affordances of research in undergraduate mathematics that conceptualizes the uptake and continued use of EBIPs in this way. Without this modification, the model would have had limited explanatory power beyond Noah's initial decision to engage with EBIPs. The modifications made it possible to analyze not only why Noah chose to use EBIPs, but also why he favors interactive lectures with small-group problem-solving (something the original model does not address), and how he persists despite dissatisfaction (with dissatisfaction accounted for in the original model but left vague). In particular, the model highlights how individual and contextual factors shaped both his use of EBIPs and his classroom experiences. These factors, in turn, gave rise to specific affordances and challenges, and revealed how Noah responded, both in the EBIPs he chose to adopt and in the ways he adapted them in order to persist.

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Theorizing Anticolonial Praxis in Undergraduate Mathematics Education

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Mathematics is frequently presented as a universal and objective discipline. Yet these claims of universality are culturally framed by whiteness and limit the possibilities for relational and ethics-driven approaches to mathematics that align with Indigenous knowledge systems. This theoretical report synthesizes insights from Indigenous perspectives, anticolonial theory, and critical mathematics education to critically examine the colonial underpinnings of mathematics. Leveraging these insights, we present a three-dimensional framework for anticolonial praxis addressing axiology, epistemology, and ontology, and seed an illustrative curricular possibility. We conclude with a call to action, urging instructors and researchers to negotiate the complexities of actualizing anticolonial praxis in institutions governed by colonial traditions.

Keywords: settler colonialism, indigeneity, curriculum, epistemology, ontology

Mathematics lags behind other disciplines in its readiness to radically reimagine the subject as just and liberatory (Gutiérrez, 2017a). At the undergraduate level, the lack of clarity about how mathematics may be reformed to advance equity can stifle broader institutional efforts. Consider Hispanic-Serving Institutions (HSIs) as an example. The rise of HSI designations has led to a wave of university efforts to decolonize curricula to provide course experiences that embrace Latin* ways of thinking, knowing, and being (e.g., Garcia, 2018). While equity-oriented research in mathematics education at HSIs has deepened our understanding of classroom social dynamics (e.g., Leyva et al., 2024), less attention has been given to how mathematical content itself is framed; HSI instructors continue to teach mathematics curricula that they believe to be culture-free (Shultz et al., 2024). In this way, the undertheorization of anticolonial approaches to mathematics may stand in the way of HSIs' progress towards truly serving Latin* students.

This theoretical report synthesizes anticolonial theory, critical mathematics education, and Indigenous perspectives to conceptualize anticolonial curricula and praxis in undergraduate mathematics. We open by challenging the myth that mathematics is universal, examining how its codification has been shaped by global power structures that reproduce white, imperial values (Bishop, 1990). Afterwards, we review relevant scholarship, emphasizing how more robust decolonial work in K–12 mathematics education and other postsecondary disciplines signals a pressing need for similar engagement in research in undergraduate mathematics education (RUME). Drawing on Indigenous and decolonial theories, we distinguish anticolonial praxis from common framings of decolonization and offer our conceptualization in the undergraduate mathematics context. We end with a call to action, urging mathematics educators to grapple with complexities of enacting anticolonial praxis within institutions rooted in colonial legacies.

The Colonial History of Mathematics

Mathematics is often viewed as neutral and objective (Bishop, 1990; Shahjahan et al., 2022). However, closer examination reveals cultural bias in narratives about mathematics. For example, mathematics curricula often highlight figures like Pythagoras in discussions of trigonometry,

while overlooking non-white contributors—such as the African architects of the Great Pyramids, whose achievements predate Pythagoras by millennia (McNeill & Gresalfi, in press; Raju, 2017). Such differences in recognition signify cultural power relations governing who controls the narrative of mathematics and, relatedly, which forms of mathematical knowledge are constructed as legitimate or worthy of preservation (Osibodu, 2024). These exclusions are not incidental; they function as mechanisms that uphold colonial and racialized hierarchies within the discipline.

The mathematics taught in Western education—which is governed by axiomatic logic and idealized notions of objectivity (Gutiérrez, 2017b)—is made possible only when a society has sufficient surplus to enable intellectual pursuits that serve no utilitarian or survival-based purpose (Zaslavsky, 1999). Exploitive labor structures and access to trade (Coates, 2011; Diamond, 1999) contributed to the development of such privileged forms of mathematics in Europe while mathematics in other areas of the globe remained deeply contextual and tied to social needs (Zaslavsky, 1999). These socially contextualized forms of mathematics were positioned as lesser by Europeans (Hottinger, 2016). European mathematics, along with the narrative that valuable mathematics is abstract, became globally dominant through settler colonialism (Bishop, 1990).

While the ontological distance between mathematics and social realities—a characteristic of European mathematics (Bishop, 1990)—is often cited as evidence that mathematics is neutral, it may actually enable mathematics to serve oppressive functions. Decontextualization allows mathematics to be engaged without ethical considerations and can operationalize imperialist values of commodification and control (Bishop, 1990; Ernest, 2018). Indeed, mathematics has played an important part in racial dehumanization (e.g., the three-fifths compromise; Ratliff, 2023), genocide (e.g., Israeli military strategy in the occupation of Palestine; the pseudo-science of the Holocaust; Alshwaikh, 2024; Ernest, 2018), capitalistic ecocide (Fúnez-Flores et al., 2024), and authoritarian governance (e.g., homeland security; Grewal, 2017).

Anticolonial Praxis as an Emerging Site of Inquiry in Undergraduate Mathematics

While decolonial and anticolonial scholarship has gained momentum across many educational fields, engagement with these conversations in RUME is nascent. For example, in a systematic review of 126 articles related to decolonization in higher education curriculum and pedagogy, Shahjahan and colleagues (2022) identified only one with a disciplinary focus in mathematics. Similarly, in reviewing engagements with decolonial theory in mathematics education Fúnez-Flores and colleagues (2024) identified only a single study from Bhattacharya and colleagues (2024) addressing postsecondary contexts. The study's project-based curriculum critiques dominant power structures and centers students' lived experiences, which the authors argue exemplify a decolonized curriculum. This curriculum engages Indigenous epistemology and ontology, which often proposes an environmental focus and relational ways of thinking (e.g., Kulago et al., 2021, Garcia-Olp et al., 2019). While this curriculum represents valuable progress, it does not match the depth of Indigenous engagement found in K-12 mathematics education.

This limited consideration of Indigeneity also appears in scholarship presented at the RUME conference—a venue that often reflects emerging priorities in the field. Within the last five years of RUME proceedings, we identified one theoretical report that engaged an Indigenous research paradigm and three other reports drawing on adjacent frameworks. Among these, it is notable that attention to instructional settings remains limited. For example, Hagman and colleagues (2024) use the Nahuatl term "Nepantla" to describe and theorize the tensions and instability inherent in equity-oriented departmental change efforts. The use of Nepantla to theorize such tensions in mathematics education research was first introduced by Gutiérrez (2008), drawing from Anzaldúa's (1999) original conceptualization of Nepantla as a liminal space of rupture and

transformation, where new forms of consciousness and resistance can emerge. Nepantla, as a space, disrupts Western dichotomies by embracing fluidity, multiplicity, and the coexistence of seemingly contradictory ways of being. Although Hagman and colleagues theorize with this Indigenous concept, its connection to instructional practice is remote and it is unclear whether the study's participants engaged this way of knowing. Similarly, Luecke's (2022) engagement with decolonial principles, though contextualized by a need to connect Western mathematics to Dakota/Lakota space, language, and culture at Tribal Colleges, focuses primarily on advancing an Indigenous research paradigm—a paradigm of "co-connecting knowledge" that is relational, reciprocal, and that decenters humans as the only source of knowledge. Although these inquiries importantly integrate Indigenous ways of being and knowing into scholarly conversations in RUME, they offer limited guidance to disrupt colonial paradigms in classrooms.

The ontological tensions Luecke (2022) identifies emerge within teaching inquiries in RUME, albeit in limited form. For example, critically examining the Western nature of mathematical representations as static and universal, Abreu (2022) proposes adopting mathematical drawing practices that are indeterminate and embodied. Similarly challenging the supposed universality that underpins undergraduate mathematics content, scholars also propose approaches that contextualize mathematical ideas or acknowledge the real-life limitations of mathematical abstraction. For example, although not published in RUME proceedings, Raju (2012) recognizes the physical impracticality of infinitesimals by advocating for grounding calculus curricula in Indigenous Indian approaches that achieve the major results of calculus without the use of limits. Abreu and Dickson (2024) propose instruction that invites stories that demonstrate isomorphisms and interrogates the topic's colonial underpinnings through questions such as, "Where does the persistent preoccupation for 'sameness in forms/structures' in dominant mathematics come from?" (p. 1315). Although these inquiries importantly integrate Indigenous ways of being and knowing into scholarly conversations within RUME, they offer incomplete models to guide anticolonial praxis.

The marginal theorization of anticolonial praxis in RUME exposes the field's complicity with colonial mathematical and instructional traditions (Gutiérrez, 2022; Raju, 2012), demanding reflection on what kinds of mathematical knowledge we validate and what educational possibilities we strive to create. Towards disrupting such complicity, we theorize anticolonial praxis in what follows.

Theorizing Anticolonial Praxis in Undergraduate Mathematics Education

Defining an Anticolonial Orientation

Efforts to critically address the legacy of settler colonialism have largely been framed as decolonization. However, the term decolonization has been co-opted in ways that retain limited connection to indigeneity and colonial violence, which can undermine its radical potential. Tuck and Yang (2012) caution against using decolonization as a metaphor, arguing that decolonization is a specific and material process grounded in the repatriation of Indigenous land and life. While we recognize the importance of preserving this meaning, we also recognize that higher education has been shaped by settler colonialism in ways that extend beyond land (Woldegiorgis, 2021). For example, while Latin* students at HSIs may lack Indigenous ties to the land on which the institution was built, these institutions are still structured by the colonial forces that may have dispossessed their ancestors in Latin America; dismantling those structures remains essential to actualizing justice (Garcia, 2018). Patel (2015) distinguishes between decolonial and anticolonial work noting that anticolonial more accurately reflects engagements that seek to understand and

counter colonial logics while acknowledging that such efforts, particularly within institutions of higher education, do not constitute decolonization. Adopting this perspective, we aim to theorize anticolonial praxis in undergraduate mathematics education as an orientation to curriculum and instruction that confront the epistemological and ontological hegemony that was mobilized through settler colonialism. We adopt an anticolonial perspective not to dilute the stakes of decolonial struggle, but to clarify the limits of our institutional locations and the transformative possibilities within them. We remain committed to disrupting the settler (mathematical) grammars that structure the very terms of legitimacy in education (Calderon, 2014).

A Framework for Anticolonial Praxis

Because mathematics is often perceived as socially neutral, many have not deeply considered how it could diverge from whiteness and colonial paradigms; this constrains radical reimagination of mathematics education as a site of liberation (Martin, 2019). Through a synthesis of decolonial theory, Indigenous epistemologies, and our own lived experiences with alternative pedagogies, we reimagine pedagogies of relationality, holism, and justice. Building on Gutierrez' (2022) theorization of Indigenous futurity, we propose a framework for anticolonial praxis consisting of three dimensions: axiology, epistemology, and ontology. Afterwards, we seed an illustrative curricular possibility to support the practical realization of this framework in undergraduate mathematics teaching.

Anticolonial axiology: centering political and spiritual desire. Axiology relates to ethics and value-driven action. Within anticolonial theory, axiology demands that we interrogate what is valued, who gets to define value, and how values are enacted in mathematics classrooms. We attend to political and spiritual dimensions of mathematics, restoring ethics in curriculum design.

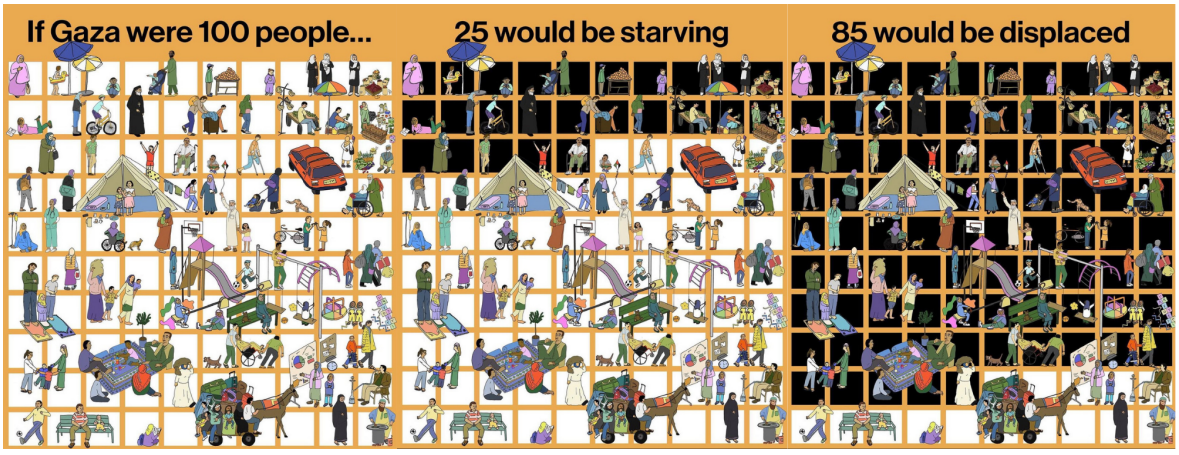

Figure 1. Chalabi's data visualization of genocide and displacement in Gaza

As Martinez et al. (2021) write, "an axiological departure of mathematics focuses on the shared experience lived while creating relationships with and across mathematics." When we are able to position the spiritual and mathematics as coexisting, "we can envision mathematics as alive, working with us as we engage in rich experiences. …[and] the experience of learning is no longer an individual task; it becomes a collective experience of knowledge construction" (p. 76). This view stands in opposition to dominant axiologies in mathematics education, where the value of mathematics is rooted in individual gain and the production of "rational and productive citizens" that would contribute to a nation's progress globally (Nikolakaki, 2016, p. 281). Such

framings are inextricably linked to capitalist ideologies, placing value on mathematics education as a tool for economic competitiveness rather than as a site for relational or spiritual engagement.

Embodying anticolonial axiology, Chalabi's (2024) data visualization (Figure 1) resists sanitized, apolitical norms, illuminating the discipline's potential to "live in harmony with values that protect life and enhance understanding" (Fasheh, 2012, p. 103). Visualizing the toll of the ongoing genocide and displacement in Gaza, Chalabi (2024) reveals embodied realities behind the numbers, rendering visible the lives and relationships erased by dominant data practices. In rehumanizing the data, Chalabi challenges capitalist and colonial tendency to quantify suffering for the purposes of policy, profit, or erasure, exemplifying anticolonial witnessing and solidarity.

Anticolonial praxis demands we reshape our views of what mathematics is for and integrate ethics as part of mathematical reasoning. Grounded in political and spiritual desire, anticolonial axiology values the potential for mathematics to foster connection and sustain collective life.

Anticolonial epistemology: honoring partiality and plurality in knowledge production. Epistemology relates to ways of thinking and constructing knowledge. Gutierrez (2017b) fosters epistemic plurality in her conceptualization of *mathematx*—a vision of mathematics where "multiple knowledges are valued and sought" (p.16) and such partialities are honored for their contributions to the whole. Mathematx calls on us to be epistemically disobedient—to refuse to recognize dominant forms of knowing as superior and to actively seek alternatives. This perspective intertwines with Anzaldúa's (1999) concept of Borderlands, which theorizes the space between categories, languages, cultures, lands, etc., as liminal, fluid, and inhabitable. The Borderlands are a third space, "greater than the sum of its severed parts," that allows for movement across and beyond imposed boundaries (p. 80). This perspective prompts mathematics educators to reimagine the field not as a bounded discipline, but as a site of possibility shaped by the cultural, historical, and experiential knowledge of those who engage with it.

Theorizing alongside Anzaldúa (e.g., 1999) and Gutiérrez (e.g., 2017b), Yeh and colleagues (2021) similarly call on mathematics educators to engage in acts of ontological and epistemological disobedience. They urge the field to interrogate long-standing borders that define what mathematics is, who it is for, and where it can be done. As an example in secondary mathematics, Yeh et al. (2021) assert that despite contemporary reforms emphasizing problem solving and reasoning "school mathematics is still typically defined as a discrete set of topics taught in a linear progression that ultimately leads to calculus" (p. 200). They ask, "If mathematics educators were no longer forced to guide students along a narrow path to calculus, what else could it mean?" (p. 200). They call for disobedience in the field, suggesting that as computers increasingly handle routine computations, mathematics education should focus on critical mathematical literacies—engaging with data and inquiry in ways that foster learners' community-based reasoning. This shift away from abstract mastery and towards the cultivation of relationally situated and multiple ways of knowing reflects epistemic plurality—one that moves fluidly across, within, and between boundaries.

Anticolonial ontology: mathematics as an interconnected way of being. Ontology relates to the nature of being. Kulago et al. (2021) emphasize how the dualism perpetuated by colonial knowledge systems severs connections between mind, body, spirit, and nature. In contrast, Indigenous epistemologies position knowledge as relational and dynamic—always in connection with others, land, ancestors, and the more-than-human world.

Anticolonial ontology resists the construction of the self as a disembodied knower, which Wynter (2003) names as a form of dehumanizing violence. Specifically, Wynter challenges the overrepresentation of "Man"—a specific, Western, bourgeois conceptualization of humans as

rational and detached from relational or spiritual ways of being. Through coloniality, this figure became the dominant standard through which humanity is defined, creating a *coloniality of being* that erases alternative ontologies. This framing aligns with dominant framing of mathematics as neutral, objective, and disembodied, which excludes ontologies rooted in relationality.

In contrast, an anticolonial ontology invites us to reimagine mathematics as a space of relational being, affirming our interdependence and entanglement with the world through land, community, and story. Rather than treating mathematics as a detached system of logic or symbols, this orientation considers mathematics as entangled with emotional and physical experience, lived histories, and collective futures. Martinez et al. (2021), for instance, describe a way of knowing and being that emerges through spirit, culture, and community. When learners engage mathematics in this way, they are not merely solving problems or mastering procedures. They are entering into a relationship with mathematics that is attentive to memory, land, family, and story. Through this lens, mathematics is part of becoming. This orientation does not seek to insert culture into mathematics as a contextual add-on. It requires a more fundamental shift in how mathematics is conceptualized. Instead of imagining mathematics as something to be applied to external realities, anticolonial ontology asks us to consider how mathematics already lives within those realities. Mathematics is world-making, shaped through embodied participation and connection.

A Curricular Seed: Integrating Light and Lifeworlds

I (the first author) choose to speak to you directly to ground our curricular seed in ontological intimacy, relationality, and ethics, thus embodying the principles of our theorization. Afterwards, I re-assume scholarly norms to analyze its constituent parts.

I credit Jake Morrow for this example—the farm manager at a school with a work program where I once taught. Jake came to me once with a calculus problem. He shared that for some crops, time to harvest depends not on the number of days that pass, but the cumulative amount of sunlight they receive. Locally, daylight hours varied significantly over the year. He wanted a mathematical model that could provide the date of the harvest given the date of planting.

I did not have the wherewithal to teach with this problem at the time, but now informed by Indigenous theories, here is how I envision it as anticolonial praxis. To begin, students would question what forms of mathematics would be needed; the answer is uncertain. It is unclear to me whether this problem might involve integrating a function whose inputs were calendar dates and outputs were the duration of daylight, or whether a more complex model is necessary. For example, could the higher light intensity near the summer solstice, when the sun is more directly overhead, influence growth in ways not explained by daylight hours alone? How do you account for days being discrete when a function for daylight over the year would likely be sinusoidal? These questions draw on our knowledge and experiences of the natural world as a guide.

We would have to try a variety of models. Given the local nature of the problem, we would draw on Jake's experience, or others who had a long history working with the land, to assess the utility of our models. In doing this, we would likely learn how harvests have been changing with the climate, and no longer follow their historical patterns. This would be an additional mathematical complexity to work into our model—one that might cause us to grapple with our fears and responsibilities related to an uncertain planetary future.

While this problem's original context was a school farm 1,000 miles away, it is also tied to injustices I witness locally and globally. I remembered this problem while creating a garden for my family. We are tired of eating food that supports corporate profits while degrading our health and we wish for less of our income to support forced labor (Bureau of International Labor

Affairs, 2025). However, we struggle to afford high-quality, ethically-sourced food. Of course, these are systemic challenges that affect many people more severely than they affect us, and inequities in land access make it difficult for some to grow food. I was pleased to learn our city has a number of community gardens that expand opportunities to work with the land, offering a potential site for service-learning if I were to teach around this issue. I am also reminded that our university, as a land-grant institution, was founded to develop agricultural knowledge for the region—knowledge that supplanted what was lost through the genocide and forced removal of Indigenous peoples (American Indian and Indigenous Studies, [masked] University, n.d.). I think of the educative possibilities of bringing this institutional history into a mathematics class, as we together consider what sophisticated forms of mathematics might have grown from that Indigenous knowledge. Perhaps this problem, and others, would have been solved long ago.

Situating the problem in our anticolonial praxis framework. Our curricular seed embodies our anticolonial praxis framework. Pertaining to axiology, our example holds meaning and embodies values of ecological justice and relationality. Students' mathematical engagement does not simply satisfy the requirements of schooling. Rather, it frames mathematics as intimately tied to the foods students grow and eat, food access inequities in their local community, and broader climate concerns. Epistemologically, the task of modeling has no clear "right answer." This invites multiplicity in interpretations and approaches, using lived experience and interconnected knowledge to explore models that are verified by their situated utility. Ontologically, mathematics becomes part of our constant quest to redefine who we are as people, and how we contribute to the world. Our experiences, embodiment, and intuition are inseparable from the mathematical endeavor, the outcomes of which have bearing on our ways of life.

Call to Action

The curricular possibility seeded here, and the larger anticolonial framework it represents, are not easily implemented in undergraduate mathematics classrooms as they currently stand. We teach and research within institutions that remain active sites of settler colonialism and are held accountable for reproducing colonial forms of knowledge therein. However, justice (in undergraduate mathematics education) cannot be achieved through incremental changes that do not fundamentally threaten these white paradigms (Martin, 2019; McNeill & Gresalfi, in press).

Our curricular possibility requires time investment in the relationships between ourselves, community members, and the ecosystem. Such reflective time is often limited when instructors are expected to cover the full range of computational and theoretical skills required for the next mathematics course; we anticipate instructors may feel concerned that their students would be "behind" in their mathematics learning as a result. We validate this concern. However, the first author is also reminded of their mathematics dissertation on symmetry groups of topological surfaces and of the limited hope they place in that work's capacity to yield spiritual, social, or planetary benefits. Indeed, it is difficult to imagine this would constitute the highest level of mathematics learning through Indigenous perspectives. When we consider how anticolonial praxis may impact students' preparation, we must ask what we hope to prepare students for. Are we preparing students to engage mathematics in supposedly ethics-free ways for advancement in a white supremacist, capitalist system, or to solve pressing social and global problems?

Informed by Gutiérrez' (2015) call for mathematics educators to nest in nepantla, we do not attempt to reconcile these conflicting instructional imperatives. An anticolonial vision cannot be squared with the status quo of postsecondary institutions. Still, in every lesson taught and every inquiry pursued lies the possibility of reimagining mathematics otherwise. We invite educators to embrace this possibility and carve pathways toward an ethical future (Gutiérrez, 2022).

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Untangling Active Learning: A Framework of Dimensions, Strategies, and Outcomes

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Active learning has become a widely cited cornerstone of undergraduate education reform; yet the construct remains conceptually ambiguous, leading to challenges in research synthesis, instructional implementation, and policy alignment. This paper synthesizes literature across mathematics and STEM education to trace this ambiguity to the conflation of three distinct but interconnected facets: dimensions of engagement, strategies for implementation, and outcomes of learning. To address this gap, I propose a theoretical framework that disentangles and defines these facets, providing analytic clarity for both researchers and practitioners. By situating this framework within ontological, epistemological, and axiological perspectives, the paper illuminates how differing research traditions conceptualize, investigate, and value active learning. This reframing enables systemic inquiry into how strategies activate specific engagement dimensions to produce desired outcomes in diverse contexts, informing instructional design. The framework thus offers researchers, practitioners, and policymakers a roadmap for advancing both scholarship and meaningful reform in undergraduate mathematics education.

Keywords: active learning, undergraduate mathematics education, student engagement, instructional strategies, learning outcomes

Calls for widespread adoption of active learning in undergraduate STEM education are ubiquitous (e.g., Conference Board of the Mathematical Sciences [CBMS], 2016; President's Council of Advisors on Science and Technology [PCAST], 2012). Meta-analyses and large-scale studies consistently report a positive impact of this pedagogical approach on student achievement and persistence (e.g., Freeman et al., 2014; Graham et al., 2013). Active learning has been credited with cementing the shift in focus to learner-centered environments, deepening learning "beyond surface-level knowledge acquisition", reducing achievement gaps for underrepresented students, and building student resilience (Duran et al., 2022; Kerrigan & Kwaig, 2024; PCAST, 2012; Theobald et al., 2020).

However, despite the theoretical arguments and mounting empirical evidence in favor of the construct, the adoption of active learning practices in undergraduate STEM has not kept pace (Børte et al., 2020; Nguyen et al., 2021). A significant barrier lies in the conceptual ambiguity surrounding the term itself: researchers and practitioners invoke "active learning" in divergent, sometimes conflicting ways—referencing engagement, strategies, outcomes, or a combination of them interchangeably. This definitional drift complicates efforts to synthesize the research and hinders its translation into effective, context-sensitive teaching practices. This ambiguity also weakens the influence of research on educational policy and large-scale reform initiatives.

The purpose of this paper is to bring conceptual clarity to the active learning construct by proposing and explicating a theoretical framework that distinguishes among three interrelated facets: dimensions of engagement, instructional strategies, and learning outcomes. This reframing of the construct seeks to clarify what characterizes active learning, how it can be effectively implemented, and what outcomes it produces. The framework paves a way for researchers, practitioners, and policymakers to interpret the active learning literature and make intentional decisions aligned with their goals.

Conceptual Ambiguity in Active Learning

Although "active learning" has become one of the most frequently invoked terms in undergraduate mathematics education reform, its meaning remains elusive.

Examining the Ambiguity

Often, active learning is thought of as the "antithesis of lecture" (Driessen et al., 2020). This broad conceptualization has created significant challenges in practice:

Multiple, sometimes conflicting interpretations. Bonwell and Eison's (1991) early definition of active learning as "activities involving students in doing things and thinking about what they are doing" highlights student engagement and reflective participation. Expanding on the idea of shifting students away from passively listening, Freeman et al. (2014) emphasize higher-order thinking and group work in their widely cited consensus definition of the construct. Laursen and Rasmussen (2019) stress the "inherently social nature" of active learning and emphasize equity in instructional design, "a characteristic that is notably absent in many scholarly definitions" (Williams et al., 2022). The use of broad descriptors—like "involving" and "doing"—leads to multiple, conflicting interpretations (Braun et al., 2017): Does active learning require group work? Are both conceptual and procedural tasks considered active learning? At what point does a traditional lecture, interspersed with some activities, qualify as active learning?

Variation in operationalization. Even when studies share a definition of active learning, they often operationalize it in very different ways. Some focus on a single technique (e.g., think-pair-share) while others examine whole-course transformations. Freeman et al.'s (2014) seminal meta-analysis on active learning included a wide range of courses, from lectures with "only 10-15% of the time devoted to clicker questions" to entirely "lecture-free 'studio' environments". These significant variations in intensity of practice limit comparability of research and risk rendering the construct too broad to be analytically useful (Lombardi et al., 2021).

Overlapping constructs. Active learning's porous boundaries with related approaches—such as inquiry-based, cooperative, and problem-based learning—have diluted the precision of the construct (Prince, 2004; Lombardi et al., 2021). Many studies treat these overlapping categories as interchangeable, making it difficult to isolate their individual impact. As a result, stakeholders often speak past one another while implicitly drawing on different conceptions of what *counts* as active learning.

Erasure of Context. Outcomes of active learning are not uniform and are moderated by context—class size, student identities, instructor beliefs and preparation, instructional choices, and institutional culture. However, active learning is often promoted as a "one-size-fits-all" solution for enhancing student learning and success. Ignoring the contexts in which it operates leads to a disconnect between conceptual understanding and practical application of the construct, widening the research-practice gap in mathematics education.

Consequences of Ambiguity

This conceptual murkiness has significant consequences. For researchers, the conflation of the characteristics, enactment, and outcomes of active learning complicates cumulative knowledge-building. Results are often aggregated over a wide range of interventions, which fails to provide a clear understanding of which strategies produce the desired outcomes under what conditions and why. For practitioners, this lack of clarity fosters misconceptions about active learning and hesitancy to adopt it. Thus, addressing this conceptual ambiguity and definitional drift is essential for advancing empirical research, supporting evidence-based teaching reforms, and enabling clearer dialogue within the mathematics education community.

A Framework for Active Learning

To address this ambiguity, this paper proposes a conceptual framework that unravels active learning into three interconnected but distinct facets: (1) dimensions of engagement, (2) strategies for implementation, and (3) outcomes of learning. These elements frequently overlap in the literature and are often used interchangeably, yet each plays a unique role in shaping how active learning is conceptualized, studied, and practiced.

Dimensions of Engagement

The first facet concerns the dimensions of engagement that characterize *how* students participate in active learning environments. Rather than treating engagement as a single construct, this framework recognizes multiple pathways through which learners can be actively involved in mathematical meaning-making. Synthesizing across scholarship (Edwards, 2015; Laursen & Rasmussen, 2019; Lombardi et al., 2021; Vale & Barbosa, 2023), five dimensions are identified. Table 1 outlines these dimensions with descriptions and illustrative examples.

Table 1. Dimensions of Engagement in Active Learning

Engagement	Description	Examples
Cognitive	Students' deep thinking about mathematical ideas	Higher-order thinking, purposeful meaning-making, self-regulation (planning, goal setting, adjusting approaches), metacognition
Social	Students' interaction with peers and instructors	Collaborative problem solving, co-construction of knowledge, mathematical communication
Behavioral	Students' observable participation and efforts	Use of movement and tools, task-focused persistence, resilience despite challenges
Emotional	Students' affective connection to learning	Curiosity, excitement, enjoyment, interest, sense of belonging
Agentic	Students' contributions to shaping the learning process	Initiating discussions, forming study groups, offering feedback to shape course design

Dimensions of engagement remain a relatively hidden paradigm of active learning research. This framework helps bring them to the forefront since they are essential to defining and understanding active learning. This multidimensional engagement model acknowledges that not every active learning technique will activate all dimensions uniformly, enabling researchers and practitioners to examine which aspects of engagement are activated by instructional choices.

Strategies for Implementation

The second facet centers on instructional strategies, the concrete practices and pedagogical approaches that operationalize active learning in classrooms. These strategies are the most visible aspect of the construct and range in complexity from brief, targeted activities (e.g., think-pair-share) to more comprehensive reforms (flipped classroom, using inquiry-based learning curriculum). Synthesizing the literature (Bonwell & Eison, 1991; Driessen et al., 2020; Freeman et al., 2014; Lombardi et al., 2021; Nguyen et al., 2021; Nardo et al., 2022; Martella & Schneider, 2024; Prince, 2004), the documented strategies can be condensed into three categories. Table 2 outlines these categories with descriptions and illustrative examples.

Table 2. Strategies for Implementation of Active Learning

Strategies	Description	Examples
Class Activities	Structured opportunities for student interaction, collaboration, and reflection	Think-pair-share, gallery walks, jigsaw, clicker questions
Assignments	Opportunities for scaffolding learning and extending engagement outside class	Pre-class work, formative checks for understanding, projects
Course Planning	Alignment of goals, materials, activities, and feedback processes for sustained participation	Backward course design, surveys, incorporating student input into curriculum and course design

By emphasizing the spectrum of strategies, this facet resists the tendency to flatten "active learning" as a single technique. For one, these strategies vary significantly in terms of intensity and design—for instance, a course that inserts occasional activities into the lecture bears little resemblance to a flipped classroom where all class time is dedicated to collaborative work—yet they are all labeled as active learning. Moreover, different strategies activate different forms of engagement. For example, clicker questions may primarily elicit cognitive and behavioral engagement, while group work may additionally support social, emotional, and agentic engagement. Thus, strategies need to be understood as means or *mechanisms* for activating specific dimensions of engagement, which in turn shape the learning results.

Learning Outcomes

The third facet addresses the learning outcomes associated with active learning. Outcomes are the most measurable aspect of this construct and encompass not only disciplinary knowledge but also developmental gains. This framework distills the outcomes reported in the literature (Bonwell & Eison, 1991; Freeman et al., 2014; Lugosi & Uribe, 2020; Theobald et al., 2020) into seven categories. Table 3 presents an overview of these, along with examples of impacts.

Table 3. Learning Outcomes of Active Learning

Outcomes	Examples of Impact
Cognitive	Increasing conceptual understanding, retention, problem-solving, metacognition
Academic	Improving exam performance, advancing toward credentials/ standards.
Relational	Enhancing collaboration, communication skills, interpersonal connections
Behavioral	Demonstrating greater task-focused effort, resilience, persistence
Affective	Experiencing greater motivation, confidence, sense of belonging
Agentic	Increasing ownership of and autonomy in learning, knowledge co-construction
Equity	Narrowing achievement gaps, providing equitable student support, fostering empathy and justice in learning spaces

This framework intentionally distinguishes academic from cognitive outcomes, recognizing that they may exist independently of each other (see Table 3). It also uses the term *relational* to emphasize the intimate, interpersonal connections often discussed in the literature under *social* outcomes. Importantly, outcomes mirror but are not identical to the engagement dimensions. For example, social engagement in class may foster collaboration (relational outcomes) while simultaneously influencing understanding (cognitive), belonging (affective), and co-construction

of knowledge (agentic) outcomes. By separating *process* (dimensions) and *results* (outcomes), this framework enables clearer theorizing about how strategies produce learning gains by activating specific dimensions of engagement.

Interrelations among the Facets

The dimensions of engagement characterize active learning (what it is), the instructional strategies highlight how active learning is enacted (how we come to know and observe it), and the outcomes capture why active learning matters (what values and purposes it advances). While conceptually distinguishable, these facets are dynamically interconnected: strategies are designed to activate specific dimensions of engagement, which in turn, produce context-situated learning outcomes. By making these relationships explicit, this multilateral model helps researchers and practitioners to navigate the complex landscape of active learning, enabling targeted research inquiry and better alignment of instructional choices with goals.

Note the connections of these facets, as described, to the ontology (what it is), epistemology (how we come to understand it), and axiology (what values shape its use) of active learning. In the next section, we will examine the philosophical underpinnings of the construct, both to strengthen the framework and to situate active learning within the broader traditions of mathematics education research.

Philosophical Grounding of the Framework

The ambiguity surrounding active learning is not merely a pedagogical challenge—it is deeply philosophical. Researchers (and practitioners) enter the conversation with different assumptions about what active learning is (ontology), how we can know it (epistemology), and why it matters (axiology). Making these philosophical perspectives explicit deepens theoretical rigor and enhances practical relevance.

Ontology: What Counts as Active Learning?

Ontology concerns the nature of the phenomenon. The lack of clarity surrounding the term "active learning", as described earlier, is fundamentally an ontological problem: Is active learning a set of instructional strategies, a mode of engagement, or a class of outcomes? Current literature often collapses these categories, producing competing ontologies. The framework addresses this tension by delineating three ontological layers—dimensions (process), strategies (practice), and outcomes (consequence)—and invites scholars to specify which aspect(s) their work investigates, thereby reducing ambiguity and supporting cumulative knowledge-building.

Epistemology: How Do We Know Active Learning?

Epistemology, here, examines the nature and origin of knowledge about active learning. Some studies on active learning employ positivist lenses, emphasizing empirical evidence and generalizability. These often examine outcomes of active learning (e.g., measuring exam achievement). On the other hand, studies focusing on dimensions and strategies (e.g., examining instructor beliefs, sense of belonging) often reflect interpretivist or social-constructivist paradigms. Distinguishing between dimensions, strategies, and outcomes helps researchers to align methodological choices with research questions.

Axiology: Why Does Active Learning Matter?

Axiology foregrounds the values and priorities that shape research and practice. Advocacy for active learning often rests on tacit or explicit value commitments: promoting equity, fostering

inclusion, nurturing belonging and agency, or advancing deeper understanding. By ensuring that values are centrally situated in how active learning is understood and discussed, this framework helps researchers and practitioners to align their values with their practice, inviting the field to more systematically investigate not just *what works* but also *what matters and for whom*.

Interplay of Ontology, Epistemology, and Axiology

Recognizing diverse ontological, epistemological, and axiological orientations clarifies why even a seemingly intuitive term like active learning can foster confusion. The proposed framework clarifies these orientations by making these philosophical underpinnings explicit: the dimensions of engagement that characterize active learning address the question, "What is active learning made of?" (ontology). Strategies enact active learning and shape the ways researchers and practitioners study it, answering the question, "How do we know active learning?" (epistemology). Finally, outcomes of learning reflect the values and intended purpose, answering the question, "Why does active learning matter?" (axiology). Rather than erasing differences among researchers, the framework offers a structure for explicitly naming them, enabling more transparent and productive inquiry and dialogue.

Implications

The framework presented here is not intended as an end to the ambiguities surrounding active learning but as an invitation—an invitation for researchers, practitioners, and policymakers to engage intentionally with the complexity of the construct. It provides a blueprint for engaging with existing literature and highlights where further research is urgently needed. The implications that follow articulate how this shift in perspective can strengthen both the research and practice of active learning, helping to realize its full promise.

Implications for Research

The framework calls for greater transparency in scholarship and highlights several areas where active learning research could be strengthened.

Attending to context. Research needs to move beyond dichotomous comparisons (e.g., "lecture vs. active learning") and investigate which strategies, activating which dimensions, in which context, produce which outcomes.

Clarifying ontologies. Researchers need to specify the aspect(s) of active learning they are studying. Is the focus on dimensions of engagement (what characterizes active learning), instructional strategies (how active learning is enacted), learning outcomes (why active learning matters), or a combination of them? This facilitates more productive synthesis and meta-analysis.

Aligning epistemologies with research goals. Selecting methodological approaches explicitly suited to the facet of active learning under study (e.g., dimensions may be better studied using qualitative approaches) and clarifying epistemological underpinnings will help researchers strengthen the validity of their work.

Integrating axiological commitments. By making explicit the values that motivate a study—such as equity, agency, performance, belonging—the framework encourages researchers to articulate their axiological commitments and critically examine whether the selected strategies and evaluated outcomes align with those commitments.

Methodological opportunities. This framework invites research that maps how strategies activate specific dimensions of engagement, how those dimensions shape outcomes, and the role of context in the process. Addressing this complex task invites methodological innovations that can generate roadmaps for both practitioners and researchers.

Implications for Practice

The framework also offers guidance for faculty, administrators, and institutions to implement active learning.

Context-sensitive implementation. Too often, research treats context as external to the learning model. This framework cautions against "one-size-fits-all" recommendations and highlights the need to adapt strategies to local conditions. For example, collaborative group work may foster social and agentic engagement in small classes. However, it requires adaptation in large lectures, such as the addition of teaching and learning assistants.

Roadmap for backward design. Practitioners often face the question, "How can I bring active learning practices into my classroom?" By distinguishing between strategies, dimensions, and outcomes, the framework provides a roadmap to backward design their choice. Instructors first identify the desired outcomes, then consider which dimensions of engagement are most likely to support those outcomes, and finally, select strategies that intentionally activate those dimensions given their specific context. For example, an instructor teaching a first-year calculus course at a public university may want to foster persistence and collaboration among students who often feel isolated in large lecture settings. To support these outcomes, the instructor would focus on social and behavioral engagement and might adopt strategies such as group problem-solving tasks facilitated by teaching assistants. By contrast, an instructor in a small seminar-style differential equations course might pursue the same outcomes through student-led discussions or peer-instruction activities. In both cases, reflecting on which engagement dimensions a strategy activates helps instructors design more balanced active learning experiences.

Fostering equity and belonging. By centering dimensions such as emotional and agentic engagement, as well as equity-based, relational outcomes, instructors can intentionally design learning experiences that not only boost achievement but also reduce opportunity gaps and support all students in the classroom, especially those historically marginalized.

Increased support for active learning. Clearer evidence-based research claims can strengthen institutional support (Børte et al., 2020), leading to better resource allocation and more targeted professional development opportunities.

Conclusion

Active learning has become both a rallying cry for reform in undergraduate mathematics education and a construct clouded by definitional ambiguity. As this paper has demonstrated, the conflation of engagement, strategies, and outcomes undermines research synthesis, yields generic policy recommendations, and leaves instructors uncertain about what active learning entails in their specific classrooms. By disentangling dimensions of engagement, instructional strategies, and learning outcomes, the proposed framework cuts through this persistent ambiguity by providing a robust lens for analyzing, comparing, and intentionally designing active learning experiences. For researchers, it provides sharper tools to frame and critique studies without flattening diverse practices to a single "active learning" variable. For practitioners, it serves as a reflective roadmap to design, select, and adapt strategies that align with their intended outcomes, context, and values—instead of reaching for one-size-fits-all solutions.

Most importantly, this clarity paves the way for real, equity-driven impact. When the links between strategies, engagement, and outcomes are made explicit, mathematics educators can design interventions that not only raise achievement but also build belonging, agency, and justice. Adopting this framework is therefore more than a conceptual exercise—it is a call to move beyond slogans and toward deliberate, value-based transformation in undergraduate mathematics education.

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A Case for Applying Unit Transformation Graphs to Model Students' Solutions to Transformation Geometry Problems

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In this theoretical report, we argue that unit transformation graphs are an appropriate and useful tool for modeling students' solutions to transformation geometry problems. Developed by mathematics education researchers working from a Piagetian theoretical perspective, unit transformation graphs explicate the mental actions that students use in solving problems, along with the objects (or units) on which they are acting. This theoretical report contributes to an understanding of how unit transformation graphs can be extended, beyond their current use, to model student reasoning in a new mathematical domain. We illustrate the potential efficacy of unit transformations in modeling students' solutions to transformation geometry problems with two researcher-generated solutions to a transformation geometry problem, and we call for future empirical research to model students' solutions to transformation problems with UTGs.

Keywords: Cognition, Learning Theory, Transformation Geometry, Second-Order Models

Transformation geometry—the study of geometry focusing on transformations (such as rigid motions and dilations) and properties that remain invariant under those transformations—is an important component of mathematics education at all levels. For nearly a century, mathematicians and mathematics educators have recommended that transformations play a greater part in geometry curricula (Betz, 1933; King, 2021; Usiskin & Coxford, 1972). The *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 2000) recommend that instructional programs from Grades PK-12 should "enable all students to…apply transformations and use symmetry to analyze mathematical situations" (p. 41). In the Common Core State Standards for Mathematics (CCSSM; Common Core State Standards Initiative, 2010), transformation geometry plays a broad role, elaborated below. At the postsecondary level, transformation geometry together with group theory can be used to classify Euclidean and non-Euclidean geometries—an idea introduced by Felix Klein in his 1872 address to the University of Erlangen, setting forth the influential Erlangen Program. Klein claimed, "The most important notion which is necessary in the following considerations is that of a group of spatial transformations" (cited in Herbst et al., 2017, p. 19).

Within the CCSSM, in Grade 8, students use hands-on activities to verify properties of rigid motions, describe the effects of rigid motions, and understand that two shapes are congruent if (and only if) one can be transformed into the other using a sequence of rigid motions. In high school geometry, students formalize notions of congruence and similarity in terms of transformations, leading to triangle congruence and similarity criteria (Wu, 2013). Transformations are also used in geometric proofs, such as proving that two shapes are congruent by describing a sequence of rigid motions that will map one onto the other. Consequently, middle and high school teachers need to be prepared to conceptualize and explain geometric transformations, including their use in proofs of similarity and congruence; evidence suggests this may pose a challenge (Hegg et al., 2018; Yanik & Flores, 2009).

Given the importance of transformations in students' geometry education, mathematics educators need to better understand how students conceptualize and reason about geometric transformations. From a constructivist perspective, understanding student reasoning in this way

involves engaging students in problem situations and, from observations of students' responses, posit answers to the question: "What mental operations must be carried out to see the presented situation in the way one is seeing it?" (von Glasersfeld, 1995, p. 78). Answering this question requires decentering from one's own mathematical knowledge and attuning to the mathematical knowledge of the student. In this sense, the researcher builds a *second-order model*, a model that can explain the researcher's observations of the student's mathematical reasoning (Steffe & Thompson, 2000).

Taking seriously the Piagetian position that students' mathematics is comprised of mental operations (i.e., mental actions), Norton et al. (2023) introduce *unit transformation graphs* (UTGs) as a tool for forefronting students' mental actions and illustrating their coordination in problem solving. UTGs are directed vertex-edge graphs, where vertices represent material being acted on (units) and edges represent mental operations (transformations applied to units). Although UTGs were originally used to model students' numerical reasoning, particularly about fraction multiplication (Norton et al., 2023), we hypothesize that UTGs can be extended beyond numerical contexts, modeling students' coordinations of mental actions generally.

In this theoretical article, we extend the scope of UTGs to the realm of transformation geometry. In doing so, we address the research question: *How, theoretically, can UTGs be used to build models of students' solutions to transformation geometry problems?*

Literature Review

We have research evidence that indicates students across Grades K-16 often experience challenges in making sense of and reasoning about geometric transformations. For example, middle school students, prior to instruction on transformations, may apply "holistic" strategies, rather than measurement-based analytic strategies, to solve transformation problems, with the latter being more likely to lead to correct solutions (Boulter & Kirby, 1994). Similarly, some undergraduate students may hold picture-based meanings of transformations without measurement-based or transformation-based meanings; students who hold either of the latter two meanings are more likely to successfully determine congruence and non-congruence of given shapes (Geotas et al., 2021).

Research has found specific transformations to present challenges for students. Dixon et al. (2022) found that, even after instruction (with traditional curricular materials), many middle school students struggle to solve problems involving rotations. Among 444 middle school students in their study, students answered an average of 59% of questions correctly, with only 46% of students correctly determining whether a given pre-image and image show a rotation and, if so, correctly locating the center of rotation. Hollebrands (2007) found that high school students may struggle to anticipate the results of transformations prior to carrying them out. Yao (2020) found similar results among preservice secondary teachers; moreover, they struggled to determine whether a composition of transformations can be performed as a single transformation. Frazee (2018) found that high school students may struggle to carry out reflections, especially over a slanted line, and to correctly locate a slanted line of reflection for a given pre-image and image, and they may struggle to carry out rotations and locate the center of rotation for a given pre-image and image. However, for both transformations, Frazee found significant conceptual progress after research-based instruction focusing on properties of transformations and developing students' mental processes for conceptualizing them. Research also indicates that preservice elementary school teachers, prior to research-based instruction, may struggle to identify examples from non-examples of reflections, rotations, and translations (Clayton, 2020) and to identify the correct center of rotation for a given pre-image and image

(Edwards & Zazkis, 1993). However, after research-based instruction (similar to that used by Frazee, 2018), Clayton (2020) found preservice teachers to make significant conceptual progress.

Secondary and postsecondary students may struggle to conceptualize transformations as not just motions, but as mappings (i.e., functions) from the plane onto the plane (Hollebrands, 2003; Yanik, 2011; Yanik & Flores, 2009). Hollebrands (2003) found that students' conceptions of domain were critical in understanding geometric transformations as functions. Yanik and Flores (2009) contributed a detailed account of one preservice teacher's trajectory toward developing a mapping-based meaning of translations, with vectors playing an especially important role. Although geometric transformations are formally defined as one-to-one and onto mappings on the plane (Martin, 1982), there are differing opinions about the nature, purpose, and productivity of thinking about transformations as motions versus as mappings (Edwards, 2003).

The challenges that K-16 students experience regarding geometric transformations underscore the importance of mathematics educators better understanding their conceptions and investigating ways in which they can be supported. We posit that to achieve such an understanding, mathematics educators need to better understand students' mental actions for carrying out, and anticipating the results of, reflections, rotations, and translations. This position is supported by research by Frazee (2018) and Clayton (2020), reviewed above, and by Johnson-Gentile et al. (1994), who found significant conceptual gains among middle school students who engaged in computer-based activities using Logo Geometry. They explained, "The need in Logo environments for more complete, precise, and abstract explication [of action commands given to the computer] may account for students' creation of conceptually richer concepts for motions" (p. 137). Although researchers have provided fine-grained analysis of student knowing and learning within transformation geometry contexts, including approaches using APOS theory (e.g., Hollebrands, 2003) and the Pirie-Kieren model (e.g., Yao, 2020), we argue that UTGs offer a uniquely fine-grained lens on students' mental actions that can provide important insight into students' conceptualizations.

Theoretical Framework

We frame our work in terms of Piaget's (1970) genetic epistemology; within this perspective, mathematics is constructed as a coordination of mental actions. A *mental action* is a transformation (a change in the state of some perceived/conceived entity) that may be carried out physically or entirely in one's mind. Within the realm of number, such mental actions include iterating, partitioning, and disembedding; these mental actions are applied to *numerical units*, which are entities conceptualized as 1 (Ulrich, 2015). For example, a student might iterate a unit of 1 to form a composite unit (e.g., 10), or partition a unit of 1 to form smaller equal-size parts (e.g., 1/2). They might disembed one of these parts (removing 1/2 from the whole), treating that part as a unit itself while keeping in mind its relationship to the original whole.

UTGs require researchers to identify the mental actions that students bring to problem situations and to model how they are coordinated. Consider the following example.

> *Cake Task*. Anita is having a birthday party. Her birthday cake is cut into 8 slices for Anita and her 7 friends. After the cake is cut, another friend arrives late to the party. Anita wants to share her slice of cake equally with the newcomer. What portion of the whole cake does Anita get? (adapted from Norton et al., 2023)

To solve this problem, a student might first take the whole cake as a unit. They could then *partition* the cake into eight equal-size pieces, then *disembed* one of those pieces to give to Anita. Then, when the newcomer arrives, this piece is further *partitioned* into two equal-size pieces, and one of them is *disembedded*. To relate the size of this piece to the whole cake, the

student might *iterate* it 16 times. Through this coordination of mental actions—partitioning, disembedding, partitioning again, disembedding again, and iterating—the student can conclude that Anita received 1/16 of the whole cake. This coordination of mental actions is illustrated by the UTG in Figure 1, where rectangular nodes represent units and arrows represent the student's mental actions on those units.

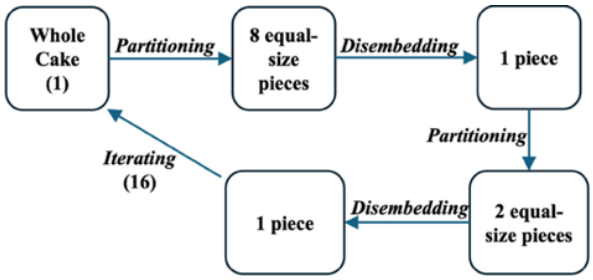

Figure 1. UTG illustrating a coordination of mental actions to solve the Cake Task

Spatial and geometric situations may involve a wide variety of units and mental actions performed on them. Following Antonides et al. (2025), we define a *spatial unit* as a perceived/conceived entity, such as a shape, that is spatially operable—that is, the entity is isolated as an object of attention and acted upon using spatial mental actions. Based on prior research on students' geometric reasoning, we generated a non-exhaustive list of mental actions that we hypothesized may be relevant in students' solutions to transformation geometry problems (see Table 1). We used prior research literature, where available, to suggest definitions of each mental action. We theorize definitions of mental actions not yet represented in the research literature. For example, a student might *unitize* a geometric shape and take it as material for spatial operation (Wheatley & Reynolds, 1996); they might *spatially disembed* part or parts of a geometric shape in problem solving (e.g., isolating one side of a square without losing track of the fact that the side is part of the square) (Barrett & Clements, 2003); or they might *spatially locate* an object in space (e.g., visualizing the location of a cube within a 3-D array, or a point in space corresponding to another point) (Battista, 2004). Understanding precisely how students temporally order mental actions on spatial units can reveal critical insights into their mathematical conceptualizations and reasoning (Antonides & Battista, 2022).

Table 1. Non-exhaustive list of mental actions hypothesized to be relevant in solving transformation geometry problems

Mental Action	Definition
Unitizing	Isolating perceived/conceived material as a distinguishable object through focused and unfocused attention (von Glasersfeld, 1981)
Spatial Disembedding	Isolating (unitizing) part of a spatial object, treating it as a separate spatial entity while maintaining its relationship to its original spatial whole (Barrett & Clements, 2003; Sarama et al., 2003)
Spatial Locating	Identifying the location of a point or other spatial object or region (Antonides et al., 2024; Battista, 2004)
Extending	Imagining a continuation of a spatial object such as a line segment

Directing	Establishing a spatially oriented relationship between two points
Reflecting	Flipping a spatial object, often (formally) over a line conceptualized as the line of reflection (or line of symmetry) (Clements et al., 2001; Edwards, 1991)
Rotating	Turning a spatial object in a particular direction (clockwise or counterclockwise), often (formally) in relation to a fixed point conceptualized as the center of rotation (Clements et al., 2001; Edwards, 1991)
Translating	Sliding a spatial object in a particular direction, often (formally) in relation to or described by a vector (Clements et al., 2001; Edwards, 1991)

Examples of UTGs in Transformation Geometry

To illustrate and support our theoretical argument, we provide two potential solutions (provided by the first two authors) to the K-Activity, described below, including mental actions used in these solutions (indicated by italics). We then illustrate how these two solutions can be modeled using UTGs.

K-Activity. There are two big K's, a black one of the top left, and a blue one on the bottom right [see Figure 2a]. Prove or disprove: The black K can be transformed to the blue K using a translation, a rotation, and a reflection, at most once each.

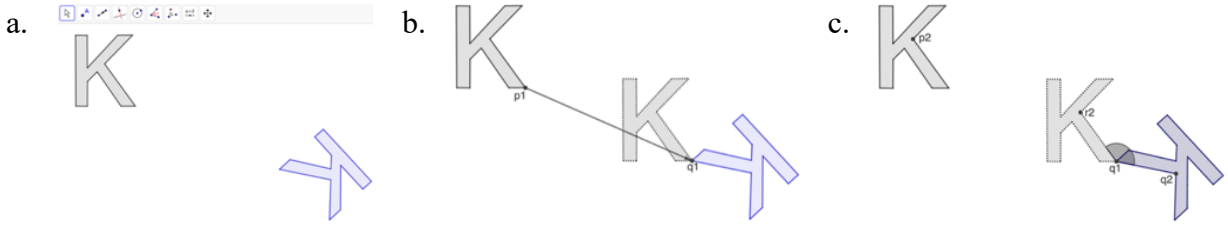

Figure 2. (a) Initial K-activity setup, and potential solution 1, involving (b) translation then (c) rotation

Potential Solution 1. In solving this task, a person might begin by *unitizing* the black K as an object of attention (a spatial unit) and isolating (*spatially disembedding*) a salient point p_1. By attending to the corresponding point q_1 on the blue K, the person *unitizes* the blue K as a spatial unit, then *spatially locates* and *disembeds* the corresponding point, establishing a relational anchor between the two K-shapes. Using these two disembedded points, they mentally construct a vector, $\overrightarrow{p_1q_1}$, by mentally imagining *directing* p_1 to q_1, and with this they mentally *translate* the black K toward the blue K, producing a new spatial unit (the translated K; see Figure 2b). Then, person may *spatially disembed* another point, p_2, on the black K, then *spatially locate* and *disembed* its translation, r_2, on the translated K, as well as the corresponding point, q_2, on the blue K (Figure 2c). The person may then *connect* points q_1 and r_2, forming a ray, and *connect* points q_1 and q_2, forming another ray; with these, they can imagine *turning* the former ray clockwise toward the latter, conceptualizing the angle $\angle r_1q_1q_2$. Then, with this angle, they can *rotate* the translated K, using q_1 as the center of rotation, in a clockwise direction. Through these coordinated mental actions, the person leverages fixed points and constructed angles as invariants to structure the sequence of transformations that map the black K to the blue K.

This person's solution can be modeled using the UTG shown in Figure 3. Rectangular nodes represent spatial units—material isolated as an object of attention and taken as input for spatial

mental actions (represented by arrows). The mental act of unitization is not represented by arrows, but rather by the creation of a rectangular node. Dark gray nodes represent the initial figurative material provided in the problem situation (taken by the problem solver as spatial units). Light gray nodes represent spatial units transformed in the process of accomplishing the translation of the black K. White nodes represent spatial units transformed in the process of accomplishing the rotation of the translated black K. The yellow node represents the final state of the black K.

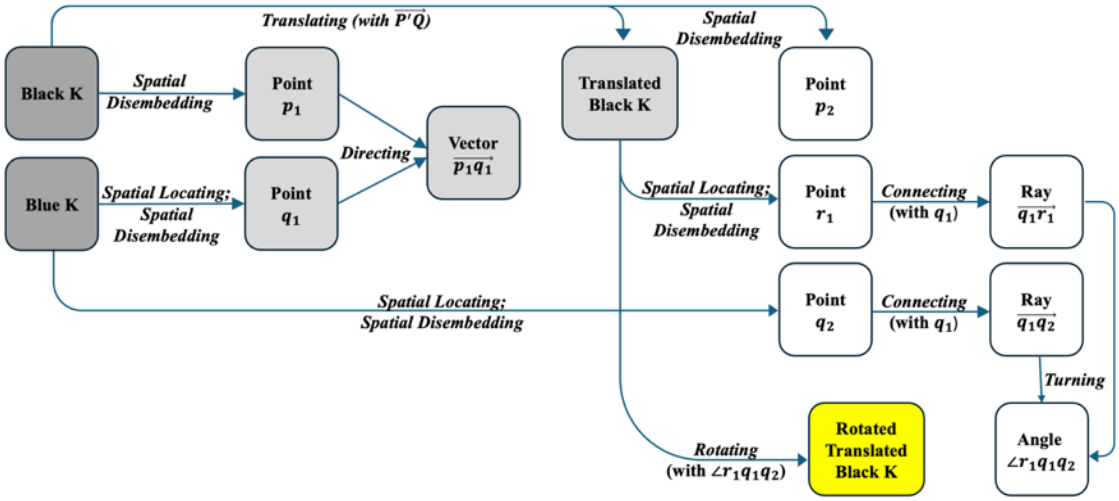

Figure 3. UTG illustrating the coordination of mental actions in Potential Solution 1

Potential Solution 2. As an alternative to the approach in Potential Solution 1, a person could first isolate the blue K as a spatial unit, via *unitization*, and from this *spatially disembed* its long flat side, then *extend* this side to form a line, L_1. They could repeat this process with the black K, *spatially disembedding* its long flat side and *extending* it to form line L_2. In doing so, the person can *spatially locate* a point, z, that they can use as the center to *rotate* the black K clockwise until its long flat side is visually aligned with L_1 (see Figure 4b). The rotated black K is *unitized* as a spatial unit that they will then translate. To carry out and describe this translation, the person could first *spatially disembed* a point p on the rotated black K, then *spatially located* and *disembed* the corresponding point, q, on the blue K. With these, they can conceptualize vector $\overrightarrow{pq}$ by mentally imagining *directing* p in the direction of q. Using this vector, they then *translate* the rotated black K (see Figure 4c), completing the transformation.

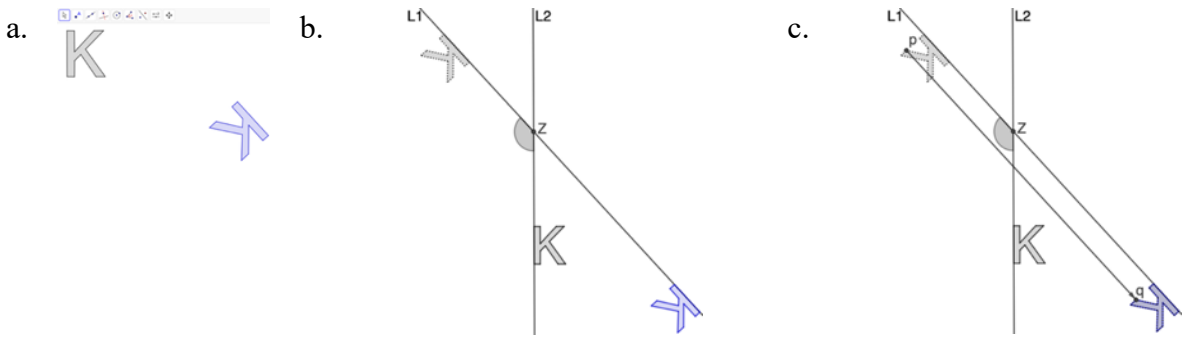

Figure 4. (a) Initial K-activity setup, and potential solution 2, involving (b) rotation then (c) translation

This person's solution can be modeled using the UTG shown in Figure 5. As with the previous UTG, rectangular nodes represent spatial units, and mental acts on those units are represented by arrows. Dark gray nodes represent the blue K and black K (the first two spatial units acted upon in the problem). Light gray nodes represent spatial units transformed in the process of rotating the black K. White nodes represent spatial units transformed in the process of translating the rotated black K. The yellow node represents the final state of the black K.

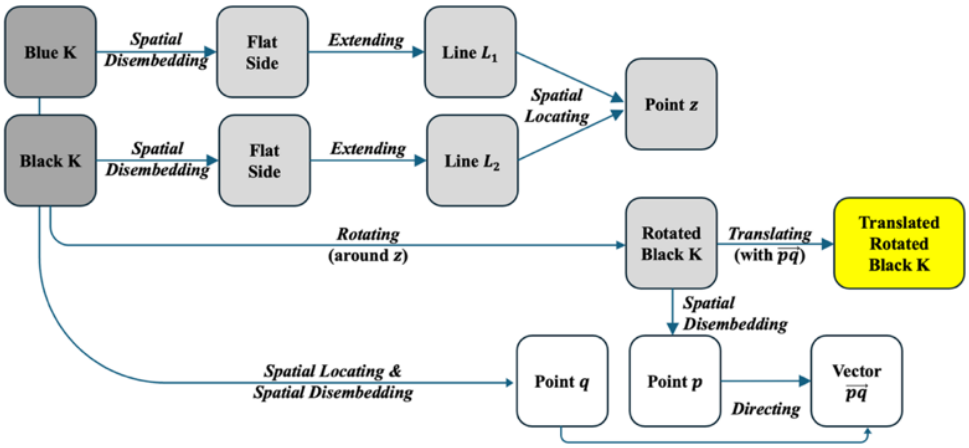

Figure 5. UTG illustrating the coordination of mental actions in Potential Solution 2

Concluding Remarks

UTGs contribute a more explicit lens on the units that students attend to and the mental actions that they coordinate on those units when solving transformation problems. Much prior work has documented the kinds of challenges students face, such as conceiving transformations as motions rather than functions (Hollebrand, 2003; Yanik, 2011), relying on picture-based rather than analytic strategies (Boulter & Kirby, 1994; Geotas et al., 2021), or struggling to anticipate composite transformations (Yao, 2020). UTGs offer a way to model not only the outcomes of these problem-solving attempts but also the evolving sequences of units and mental actions that students bring into coordination.

Additionally, our examples of the K-activity illustrate how different individuals may coordinate similar actions in different sequences: one beginning with translation then rotation, the other reversing that order. By modeling these variations, UTGs make explicit not only the actions themselves, such as unitizing, disembedding, locating, translating, and rotating, but also the ways these actions are sequenced and coordinated across units. This explicit attention to the interplay of units and mental actions offers insight into the diverse strategies students employ when reasoning about geometric transformations.

Finally, we suggest that UTGs can further illustrate the constructive ways that students coordinate units and mental actions, even when their solutions do or do not align with normative or textbook-like solutions. By making visible the variations in how students sequence mental actions, UTGs can reveal the resources that students bring to transformation problems, which in turn help researchers model student reasoning. With this in mind, we call for future research to engage students in solving transformation geometry problems and model their solutions with UTGs. These models can provide opportunities for researchers and teachers to support students in developing more productive strategies for solving transformation geometry problems while also highlighting how students' current reasoning can be a foundation for more productive ways of thinking in geometry contexts.

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Conceptual Foundations for Connecting School and Abstract Algebra: A Historical Analysis

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This paper uses a conceptual analysis of histories of algebra to identify a kind of reasoning, combinatorial reasoning, that was essential to the solution of polynomial equations. Historically, mathematicians' increased reliance on combinatorial reasoning in the solution of polynomial equations co-occurred with the rise of symbolic algebra. Algebraic symbols were often the context in, and on, which they engaged in combinatorial reasoning. Combinatorial reasoning, however, is often in the background of proofs or arguments—present but not framed as an essential kind of reasoning that could be developed to understand the proofs or arguments. The purpose of foregrounding this reasoning in this paper is to argue that it can serve as a conceptual foundation for designing coherent algebra curricula across grades 9-16—curricula that allow students and teachers to see deeper connections between school and abstract algebra.

Keywords: School Algebra, Abstract Algebra, Combinatorial Reasoning, History of Mathematics

Historically, mathematicians' work on the algebraic solution of the general polynomial equation[1] played an important role in the genesis of abstract algebra (Kleiner, 1986; Nový, 1973; Pesic, 2003; Stedall, 2012). However, one is hard pressed to see how the solution of polynomial equations, as presented in secondary curricula, is connected to ideas in abstract algebra, as presented in many modern-day textbooks. Secondary algebra curricula emphasize a range of techniques, often disconnected from one another, for finding the roots of polynomial equations (e.g., factoring, quadratic formula, rational root theorem, complex conjugate theorem) (Burch et al., 2021; Sangwin & Jones, 2017; Thompson, et al., 2012). Post-secondary abstract algebra curricula often begin with an abstract treatment of groups, making limited connections to the solution of polynomial equations in early parts of texts (e.g., Fraleigh, 2003; Herstein, 1997). This curricular structure can lead to a disconnect, particularly for future secondary teachers, between what they will teach to their future high school students, and what they are learning in their collegiate level abstract algebra courses (Wasserman, 2018).

For this reason, mathematics educators have recommended examining secondary curricula from an advanced perspective (Àlvarez & White, 2018; e.g., Lee & Heid, 2018; Serbin, et al., 2024), and using the history of algebra (Suoinmen, 2018; e.g., Barnett, 2017; Edwards, 1984; Blum-Smith & Coskey, 2017) to develop more coherent algebra curricula and experiences across grades 9-16. The purpose of this paper is to build on these recommendations. We do so by presenting a conceptual analysis (Thompson, 2008) that highlights how combinatorial reasoning with and on algebraic symbolism played a crucial role in advances in the algebraic solution of the general polynomial equation (Derbyshire, 2006; Stedall, 2012; Tignol, 1988/2016). Our examples start with Viète's formulas, and end with one of LaGrange's contributions to the solution of the quartic. These examples precede Ruffini's and Abel's work on the impossibility of the solution of the general polynomial equation of degree five (Ayoub, 1980; Pesic, 2003),

[1] Throughout, I focus on algebraic methods for solving polynomial equations and differentiate it from numerical methods (Rider Hamburg, 1976). Numerical methods are various approximation methods that give a numerical value for a root of a polynomial. Algebraic methods are ones that structurally relate the roots, factors, and coefficients of a polynomial to determine if the roots of the general polynomial equation can be expressed in terms of known coefficients.

and Galois's identification of general conditions for the solvability of polynomial equations (Edwards, 2012; Tignol, 1988/2016). We focus on this historical time-period (16th-18th century) because it laid the groundwork for the transition from viewing algebra as the solution of polynomial equations to viewing algebra as the study of structure. By identifying combinatorial reasoning as a foundational kind of reasoning during this period of time, we aim to make apparent a kind of reasoning that if emphasized in curriculum and instruction would allow students and teachers to experience a greater sense of connection between, and coherence across, secondary and post-secondary algebra experiences.

Background Literature: Context for Our Analysis

We use Kaput's (2008/2017) framework for algebraic reasoning as an organizational tool for discussing the algebra literature, which we use to motivate the reason for our analysis. Within Kaput's framework, secondary algebra research has tended to focus most heavily on strand 2, functions (Kieran, 2007; Warren et al., 2016), where mathematics education researchers have identified co-variational reasoning as a foundational kind of reasoning for this strand (e.g., Thompson & Carlson, 2017; Confrey & Smith, 1996). Less attention has been given to strand 1 at the secondary level (Kieran, 2007; Warren et al., 2016), a strand that includes the algebraic solution of polynomial equations. The studies that do attend to strand 1 often focus on students' development of structure sense as they engage in syntactically guided transformations on conventional symbol systems (core aspect B, Kaput's framework) (e.g., Hoch & Dreyfus, 2005).

Core Aspects and Strands
Two Core Aspects
(A) Algebra as systematically symbolizing generalizations of regularities and constraints. (B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbolic systems.
Core Aspects A & B are Embodied in Three Strands
1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning. 2. Algebra as the study of functions, relations, and joint variation. 3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.

Figure 1. Kaput's (2008, p. 11) framework for algebraic reasoning

One common focus of this work has been to investigate the role computer algebra systems (CAS) can have in supporting students to develop structure sense, particularly in relation to expanding and factoring algebraic expressions and solving polynomial equations (e.g., Saldanha & Kieran, 2006). Early work in this area emphasized that a skilled teacher's use of CAS could support students to develop a better understanding of equivalence (Nicaud, et al., 2004; Kieran & Saldanha, 2005), and of the relationship between a polynomial and its factors (Artigue, 2002), including relationships that might otherwise be too complex for secondary students to handle outside of CAS (LaGrange, 2003). As CAS have become more widely used by teachers, and the tools more powerful, researchers have tended to present fewer positive results (Artigue, 2010). More recent findings, for example, include that CAS can support students to develop non-reversible equation solving techniques (Jankvist, et al., 2019), that CAS tasks in textbooks tend to involve low procedural complexity (Davis & Fonger, 2015), and tend not to foster opportunities for students to engage in proofs (Jankvist & Misfeldt, 2019).

At the post-secondary level, there have been recent studies with practicing or future secondary teachers that have focused on understanding relationships between a polynomial and its factors (e.g., Burch, 2023; Tillema & Burch, 2022). One focal point of these studies has been

to support secondary teachers to use concepts they are learning in abstract algebra to unify ideas related to factoring that they will teach at the secondary level (Serbin, et al., 2024). Often these studies, like the studies at the secondary level, have been framed around secondary teachers' development of structure sense (e.g., Lee & Heid, 2018; Novotná & Hoch, 2008). While many studies related to the algebraic solution of polynomials have been framed around structure sense, there has been less attention to foundational forms of reasoning that produce it (Linchevski & Linveh, 1999; Novotná & Hoch, 2008) as compared to the much more extensive literature on functions and co-variation. The primary purpose of this paper, then, is to explicitly name an important form of reasoning, combinatorial reasoning, from which students could develop their structure sense related to the solution of polynomial equations.

Conceptual Framework

Thompson (2008, p. 1-31) "propose[s] a method by which mathematics educators can make tacit meanings explicit and thereby address problems of instruction and curricula in a new light." The method he proposes is conceptual analysis. Conceptual analyses involve specifying the mental operations that a person must carry out to understand a situation in the way they are understanding it (Glasersfeld, 1995, p. 78). This definition of conceptual analysis makes historical sources of mathematics ripe for conceptual analyses for at least two reasons. First, historical sources often capture mathematical understandings as they emerged, developed, and changed over time. It is rare, for example, the way an understanding is initially expressed is the way that it is used or understood at later points in time. This evolutionary process allows one to analyze differences in understanding, along with what features of an understanding maybe advantageous to aim for in current curricula. Second, often features of an understanding are implicit or taken as shared at the time an understanding is expressed (i.e., in a particular historical time-period). These implicit understandings may support the development of new understandings where the new understandings may ultimately be what gets encoded in curricula, while the implicit understandings fall away. For example, many of the concepts related to group theory (e.g., a group, a normal subgroup, a conjugacy class, factor group, etc.) were initially intimately tied to the solution of polynomial equations (Cajori, 1912; Barnett, 2017; Dehn, 1930). However, over time this connection has become much more hidden in curricula (e.g., Fraleigh, 2003). The benefit, then, of examining historical sources is that one can make implicit understandings explicit, especially, if implicit understanding or reasoning that was important to the genesis of later understandings is lost.

Elsewhere, I have given accounts of the mental operations that are involved in combinatorial reasoning, and which produce combinatorial structure, along with the consequences this has for differences in secondary students' or teachers' understandings (e.g., Tillema, 2018; Tillema, 2020; Tillema et al., 2024; Tillema et al., 2025). I summarize that work here by saying that these operations (along with schemes, coordination among schemes, and abstraction) produce "psychological slots," and a structure on these slots. From this perspective, some of the main differences among reasoners has to do with the psychological slot structure that they use to interpret situations. For example, a reasoner might interpret a statement like, $(a+b)^2$, using a "psychological slot structure" that is represented as $(__+__)(__+__)$. This is a 2-slot structure where each slot contains two slots—the larger slots are structurally connected by multiplication and the smaller slots by addition. The structure implies a recursive embedding of slots where the additive slot structure is embedded in the multiplicative one. On the other hand, a student may interpret a statement like $(a+b)^2$ with the following psychological slot structure, $(__)(__)$. This is a 2-slot

structure where the slots are connected by multiplication only. This 2-slot structure yields non-normative interpretations of the expression, $(a + b)^2$. We provide this brief analysis to indicate what we mean by combinatorial reasoning and structure, but in our analysis below we do not map out in full detail the psychological slot structure that is implied by the analysis. We simply point out more informally where combinatorial reasoning can be inferred in the context of examining histories of algebra.

Examples from Histories of Algebra

The historical period between Cardano's (1545) *Ars Magna* and Lagrange's (1771) *Réflexions sur la résolution algébrique des équations* is often framed as a period of slow or no growth in histories of algebra (Stedall, 2012). This framing, in part, is because the discovery of methods for solving cubic and quartic equations, published by Cardano, occurred within a short timeframe. The short timeframe primed mathematicians to expect that the solution of higher degree equations might follow quickly (Bashmakova, 2000, van der Waerden, 1985). However, that did not happen; the discovery of whether and when higher degree equations could be solved had to await Ruffini (Ayoub, 1980), Abel (Pessic, 2003; Rosen, 1995), and Galois (Edwards, 1984, 2012) in the early 19th century. Thus, if growth is measured from the perspective of finding a solution to higher degree equations, then, indeed, little progress was made between Cardano and Lagrange.

However, Stedall (2012) convincingly argues that this time-period has both been neglected in histories of algebra, and that it was a period of *essential* growth (see also, Nový, 1973; Tignol, 1988/2016); essential in that the growth opened the way for the transformation that occurred to algebra in the 19th century (Kiernan, 1971). The developments in the 19th century changed the focus of algebra from the study of the solution of polynomial equations to the creation and study of structure more broadly (Derbyshire, 2006). The developments in algebra during the 16th to 18th century, then, should be of significant interest to mathematics educators who are aiming to design coherent algebra curricula across grades 9-16; that is, developments during the 16th to 18th century offer insight into the kinds of reasoning that supported the transformation of algebra in the 19th century. Moreover, the fact that historians of mathematics have given minimal attention to the 16th to 18th centuries means that studying it has the potential to provide new insight.[2]

One important transformation that occurred in the 16th to 18th century was mathematicians' perspective on equations and equation solving. At the beginning of the period, equations, including the solution of cubic and quartic equations, were solved within a mathematical framework of proportions. This perspective began to shift by the middle of the 17th century in the work of Girard, Descartes, and Harriot, all of whom began to treat polynomial equations as a product of factors (Funkhouser, 1930; Stedall, 2012, p. 50). This shift in perspective, coupled with mathematicians' growing use of symbolic algebra, supported mathematicians to make increasingly complex computations with symbols. As numerous mathematicians at the time commented, one impediment to progress on finding a general solution to higher order polynomials was the length and complexity of the symbolic computations (Stedall, 2012). Frequently, the structure that could be used to both simplify and organize this complexity was combinatorial structure (e.g., Cajori, 1912; St. George, 2003).

To illustrate this claim, I present three examples. The examples I provide are not intended to be a historical account of specific mathematicians' contributions to the development of algebra.

[2] Plenty has been written about Cardano's and Lagrange's contributions to algebra (e.g., Branson, 2013). It is largely the period between the two that is the focus of Stedall's (2012) book.

Rather, the purpose of the examples is to make explicit the forms of reasoning that emerged during this period, that, if developed with modern students, could serve as a basis for the design of more coherent algebra experiences across grades 9-16.

The first example is a combinatorial argument for Viète's formulas. A combinatorial argument relies on treating binomial factors as a choice between an indeterminant and parameter. So, for example, in the quartic case one can expand the binomials, $(x+a)(x+b)(x+c)(x+d)$, using the following reasoning. There is 1 way to choose the indeterminant from each binomial (blue, Figure 2), 4 ways to choose the indeterminant from three of the four binomials (grey, Figure 2), 6 ways to choose the indeterminant from two of the binomials (green, Figure 2), 4 ways to choose the indeterminant from one of the binomials (yellow, Figure 2), and 1 way to choose the indeterminant from none of the binomials (orange, Figure 2). Using this structure, the

Selection Process	Outcome of Selection Process	Selection Process	Outcome of Selection Process
$(x+a)(x+b)(x+c)(x+d)$	x^4	$(x+a)(x+b)(x+c)(x+d)$	bcx^2
$(x+a)(x+b)(x+c)(x+d)$	ax^3	$(x+a)(x+b)(x+c)(x+d)$	bdx^2
$(x+a)(x+b)(x+c)(x+d)$	bx^3	$(x+a)(x+b)(x+c)(x+d)$	cdx^2
$(x+a)(x+b)(x+c)(x+d)$	cx^3	$(x+a)(x+b)(x+c)(x+d)$	$abcx$
$(x+a)(x+b)(x+c)(x+d)$	dx^3	$(x+a)(x+b)(x+c)(x+d)$	$abdx$
$(x+a)(x+b)(x+c)(x+d)$	abx^2	$(x+a)(x+b)(x+c)(x+d)$	$acdx$
$(x+a)(x+b)(x+c)(x+d)$	acx^2	$(x+a)(x+b)(x+c)(x+d)$	$bcdx$
$(x+a)(x+b)(x+c)(x+d)$	adx^2	$(x+a)(x+b)(x+c)(x+d)$	$abcd$

Figure 2. Combinatorial expansion of four binomials

final polynomial can be written as: $(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+dc)x^2 + (abc+abd+acd+bcd)x + abcd$. This combinatorial structure can, then, be used to link the roots and coefficients of a polynomial because $r_1 = -a$; $r_2 = -b$; $r_3 = -c$; $r_4 = -d$, which yields the following: $x^4 - (r_1+r_2+r_3+r_4)x^3 + (r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)x^2 - (r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4)x + r_1r_2r_3r_4 = x^4 + Ax^3 + Bx^2 + Cx + D$. This yields that the negative sum of the roots is the coefficient of the cubic term (i.e., $A = -(r_1+r_2+r_3+r_4)$); the positive sum of all possible combinations of two of the roots is the coefficient of the quadratic term (i.e., $B = (r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)$), etc.

One important observation about Viète's formulas is that the expressions in the roots that make up the coefficients (i.e., $-(r_1+r_2+r_3+r_4)$) are all symmetric where a symmetric expression is one whose value is invariant under all permutations of the roots. For example, swapping r_1 and r_3 so that $-(r_1+r_2+r_3+r_4)$ becomes $-(r_3+r_2+r_1+r_4)$ does not change the value of the expression. The expressions in Viète's formulas are called the elementary symmetric polynomials, when they are expressed in *n* indeterminants (Figure 3). During the 17th

$$\sum_{1\le i_1\le n} x_{i_1} = x_1 + x_2 + \cdots + x_n$$

$$\sum_{1\le i_1<i_2\le n} x_{i_1}x_{i_2} = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$$

$$\sum_{1\le i_1<i_2<i_3\le n} x_{i_1}x_{i_2}x_{i_3} = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_{n-2}x_{n-1}x_n$$

$$\vdots$$

$$\sum_{1\le x_{i_1}<x_{i_2}\ldots\le x_{i_n}} x_1x_2\ldots x_n = x_1x_2\cdots x_n$$

Figure 3. The elementary symmetric polynomials

century, mathematicians worked on problems in which they generated other symmetric expressions, e.g., determining when two cubic equations shared a common root (St. George, 2003). Such problems led to the fundamental theorem of symmetric polynomials (FTSP) (Blum-

Smith & Coskey, 2017; Tignol, 1988/2016) and the Newton-Girard identities[3] (Cajori, 1912; Funkhouser, 1930). The FTSP states that any symmetric polynomial can be expressed as a polynomial in the elementary symmetric polynomials.

The second example I use to illustrate combinatorial structure is a case of the FTSP. This example continues with a quartic polynomial, $x^4 + Ax^3 + Bx^2 + Cx + D$. For convenience, I introduce the notation $\sum x_1^2 x_2 x_3$ to mean the sum of all possible combinations that take the form $x_1^2 x_2 x_3$ in four indeterminants (i.e., $(x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + \cdots + x_4^2 x_2 x_3)$). The FTSP states that it is possible to express this sum in terms of the elementary symmetric polynomials. Before accomplishing that goal, the sum itself can be understood through combinatorial structure; there are 4 choices for the indeterminant that will be squared, and $\binom{3}{2}$ choices for the remaining two indeterminants, which yields a total of $4 \cdot \binom{3}{2} = 12$ summands in $\sum x_1^2 x_2 x_3$.

Having identified a combinatorial structure of the sum, the goal is to find an expression in the elementary symmetric polynomials for the symmetric expression $\sum x_1^2 x_2 x_3$. We start with a product of elementary symmetric functions, $(x_1 + x_2 + x_3 + x_4)(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)$, that produces partial products that take the appropriate form, $x_1^2 x_2 x_3$. Here again combinatorial structure can help to identify the number and form of the partial products. There are 4 summands in each of two polynomials so the product of the two polynomials will have $4^2 = 16$ partial products. Moreover, there are, for example, 3 expressions in the second polynomial that contain the indeterminate, x_1, and 1 expression that does not contain, x_1. Therefore, when we multiply the second polynomial by the indeterminate x_1 we will get these 3 expressions, $x_1^2 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_2 x_3$, and this 1 expression $x_1 x_2 x_3 x_4$, a structure that will be repeated for each of the four indeterminants, giving the 12 summands of the form, $x_1^2 x_2 x_3$ and 4 summands of the form, $x_1 x_2 x_3 x_4$. Thus, we can write $\sum x_1^2 x_2 x_3 = (x_1 + x_2 + x_3 + x_4)(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4) - 4(x_1 x_2 x_3 x_4) = \sum x_1 \cdot \sum x_1 x_2 x_3 - 4 \sum x_1 x_2 x_3 x_4$. This example illustrates multiple ways that combinatorial structure is useful to reasoning about a case of the FTSP.

Lagrange put the FTSP to work in his *Réflexions* where he provided an analysis of known methods for the solution of cubic and quartic equations (Barnett, 2017; Stedall, 2012). All of the methods he analyzed relied on using a resolvent equation, a secondary equation that allowed for resolution of the first (e.g., Cardano's method relied on a sextic equation that was quadratic in the indeterminant). Lagrange's central insight in this process was to determine the relationship between the roots of the resolvent equation and the roots of the primary equation to be solved (Edwards, 1984). From this analysis, he showed that the solution of a quartic equation could be reduced to the solution of a cubic equation precisely because there was an expression in the roots of the quartic equation that took on *three* values (Tignol, 1988/2016, p. 127-128). One way to understand this observation is that any expression in the roots of a quartic equation that takes on three values under all permutations of the expression can be used to reduce a quartic equation to a cubic equation.[4] As part of this reduction the FTSP must be used to write the coefficients of the cubic equation in terms of the quartic equation.

[3] The Newton-Girard identities are recursive formulas that can be used to express the sum of powers of roots/indeterminates (e.g., $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3)$). Practically, one use for these identities is that it is possible to express the sum of powers of the roots in terms of known coefficients without having to find the roots.

[4] Ruffini seems to have taken this observation as a starting point for his proofs of the insolvability of the quintic. With this as a starting point, he was able to show that no function in 5 indeterminants could take on 3 or 4 values

The third example, then, that I use to illustrate the role of combinatorial structure is the reduction of a quartic to a cubic equation, using the expression, $x_1x_2 + x_3x_4$, where the four indeterminants are assumed to be the roots of the quartic equation, $x^4 + Ax^3 + Bx^2 + Cx + D = 0$ (Cajori, 1912, p. 90-91). The expression $x_1x_2 + x_3x_4$ can take on the following three values under permutation of the indeterminants, $t_1 = x_1x_2 + x_3x_4$; $t_2 = x_1x_3 + x_2x_4$; $t_3 = x_1x_4 + x_2x_3$, an observation that we consider to be a form of combinatorial structure. It is, then, possible to write the following cubic equation, $(x - t_1)(x - t_2)(x - t_3) = x^3 - (t_1 + t_2 + t_3)x^2 + (t_1t_2 + t_1t_3 + t_2t_3)x - t_1t_2t_3$. Doing so uses a similar combinatorial structure to the first example. The expressions in the t_i are now symmetric, which means the expressions in the x_i are symmetric as well, a conclusion that involves a recursion of combinatorial structure. Figure 4 shows the relationship between the t_i and x_i. The second example we provided demonstrated how the FTSP could be used to express $t_1t_2 + t_1t_3 + t_2t_3 = \sum x_1^2x_2x_3$ in terms of the elementary symmetric polynomials. The FTSP can be used to derive the other two relationships shown in Figure 4, and doing so entails using combinatorial structure. Once each of the elementary symmetric expressions in the t_i are written as symmetric expressions in the x_i the following cubic equation can be written in terms of the coefficients of the quartic equation, $x^3 - Bx^2 + (AC - 4D)x - (A^2D + C^2 - 4BD) = 0$. The solvability of the original quartic equation, then, can be reduced to the solvability of the cubic equation with the addition of a fourth equation, namely that $D = x_1x_2x_3x_4$ because that yields a system of four equations in four unknowns.

$$t_1 + t_2 + t_3 = \sum x_1x_2$$
$$t_1t_2 + t_1t_3 + t_2t_3 = \sum x_1^2x_2x_3$$
$$t_1t_2t_3 = \sum x_1^3x_2x_3x_4 + \sum x_1^2x_2^2x_3^2$$

Figure 4. The relationship between the t_i and x_i

Discussion, Contribution, and Conclusion

In Lagrange's *Reflexions*, he commented that the FTSP was "self-evident," and provided no proof of it (translated in Tignol, 1988/2016, p. 95; Lagrange, 1770, 1771, Art. 98, p. 372). Tignol interprets this statement in two ways—both that the proof is not difficult and that among 18th century mathematicians the FTSP was commonly used in the solution of a range of problems. The latter interpretation is of importance to the argument in this paper; namely, that combinatorial reasoning with algebraic symbols was a foundational form of reasoning, even if taken for granted, that became commonplace during the 16th to 18th century as mathematicians sought the solution of higher degree polynomial equations. Moreover, an analysis of historical sources shows that this kind of reasoning was much broader than its use in Viéte's formulas, and the FTSP (e.g., Derbyshire, 2006; Stedall, 2012; Tignol, 1988/2016). This foundational form of reasoning contributed to the more obvious developments that occurred in 19th century algebra. Therefore, I position it as one form of connective tissue between "algebra" as the search for a general solution to polynomial equations, and "modern algebra," as the study of structure. It defines a *kind of reasoning and corresponding structure* that could be emphasized much more intentionally in modern secondary and post-secondary textbooks. Such an emphasis has the potential to lend coherence across students' algebra experiences.

(Ayoub, 1980; Bewersdorf, 2006). This starting point was one of the flaws with his proof—taking it as a starting point assumes the roots have a certain form without proving that they must be of this form. Abel's proof of the insolvability of the quintic addresses this flaw (Pesic, 2003).

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Extending the Theoretical Boundaries of MKT to Mathematics Instruction in Service Courses for STEM Disciplines

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In this theoretical report, we argue for a need to extend the theoretical boundaries of Mathematical Knowledge for Teaching (MKT) beyond its historical applications. Our motivation is the longstanding concern regarding the ability of Calculus as a service course to effectively scaffold non-math-major students' ways of knowing mathematics to support them in transitioning to their primary discipline, particularly in Engineering. We extend the theoretical boundaries of MKT to encompass the knowledge needed to connect Calculus to its ways of use within Engineering in a way that is perceptible and comprehensible to students. Finally, we distinguish this form of MKT from existing work in MKT, and we demonstrate its utility in advancing our understanding of the knowledge required to improving Calculus instruction.

Keywords: calculus, mathematical knowledge for teaching, instruction, engineering

In many U.S. colleges and universities, introductory differential and integral Calculus courses serve students majoring in STEM client disciplines such as engineering, biology, and chemistry, rather than primarily mathematics (Bressoud et al., 2013; Rasmussen et al., 2014). These courses play two roles in tension: "filter" and "scaffold" (Biza et al., 2022). Performance in Calculus acts as a filter as it is a prerequisite to various courses (Bressoud, 2021; Törner et al., 2014) and is associated with retention (or the lack thereof) within STEM majors (Ellis et al., 2016; Rasmussen & Ellis, 2013; MAA, 2015; PCAST, 2012). At the same time, Calculus can support students by scaffolding their transition from high school to college as well as in their transition to the advanced coursework involving mathematics that is required by their major (Biza et al., 2022; Corriveau & Bednarz, 2017).

However, the STEM education community has come to question the efficacy of college Calculus in fulfilling its role as a scaffold. For instance, professionals in mathematically intensive STEM careers such as engineering have reported viewing their undergraduate mathematics experiences as irrelevant to their current work (Quéré, 2017; Härterich et al., 2012; Pearson, 1991). Likewise, student engineers have been reported to find university mathematics in conflict with mathematics use in engineering (Alpers, 2017; Bingolbali et al., 2007; Britton et al., 2005; Gainsburg, 2007; Jablonka & Bergsten, 2022; Wood et al., 2008). One hypothesis regarding the cause of this phenomenon is that Calculus courses, like other mathematics courses, are typically taught from a purely mathematical perspective. Applications to other fields often appear as optional examples that only come after relevant mathematics has been addressed (Kent & Noss, 2002) and serve to ultimately reinforce the mathematical perspective (deemed "applicationism" by Barquero et al., 2013). These concerns were one of the core motivations with the MAA's (2004) CRAFTY initiative which urged mathematics faculty to provide opportunities for students to "see mathematics in context" and to explain "the hows and whys" (pp. 3-4) in solutions over 20 years ago.

We offer a second hypothesis that builds on the first: that our current conceptualization of the *mathematical understanding* required for *mathematical knowledge for teaching* Calculus (Ball, Thames & Phelps, 2008) is premised almost exclusively on ways of knowing mathematics within

its own discipline. This is an issue for students, given that many essential concepts in the Calculus sequence are framed and understood differently by engineering faculty (Willcox & Bounova, 2004; Flegg et al., 2012). In this way, mathematical understanding for teaching students of client disciplines is related to but qualitatively distinct from disciplinary mathematics (e.g., Alpers, 2010; Diaz Eaton & Highlander, 2017). For Calculus courses to be a scaffold for engineering students, we must first understand the mathematical knowledge required for instructors to teach Calculus so that it connects naturally to STEM coursework in client disciplines.

In this theoretical report, we address the question: How does one define *the kind of mathematical knowledge needed by Calculus instructors* to support students in connecting mathematics to client disciplines? Similarly to Biza et al. (2022), we center our illustrative examples and literature review within this report on the client discipline of Engineering due to the relative wealth of research in this area regarding associations between student retention and mathematics. However, our intention is for our treatment of this conceptualization to serve as a model for those interested in examining mathematical knowledge for teaching Calculus to students in any client discipline.

Conceptual Framework

Here we conceptualize the key construct of this report: the kind of mathematical knowledge needed by Calculus instructors to support students in connecting mathematics to client disciplines. To do so, we first delineate this domain from three related domains: (1) mathematics as its own discipline; (2) mathematical knowledge for teaching (MKT) as it has been typically represented; and (3) mathematics as it is used in and for engineering (MEng). We then show how each of these three are conceptually related to the key construct, which we will refer to as MKT×MEng -- a notation that shows the intertwining of mathematical knowledge for the purposes of teaching in and for engineering.

Mathematics as its own Discipline

Mathematics can be pursued for its own sake, and the work of professional mathematicians entails its own language, reasoning, questioning, and practices (Kitcher, 1984). Mathematicians' practices often involve developing strategies for generalizing and specializing patterns and structures (Pólya, 1990), potentially to give insight into new ways of thinking (Weber, 2010). Most college Calculus instructors are enculturated to this discipline through their graduate studies, either completed or in progress.

Mathematical Knowledge in and for Teaching (MKT)

Teaching mathematics differs from *doing* mathematics for its own sake as well as doing mathematics in other mathematically intensive professions (Ball et al., 2008). We define *mathematical knowledge in and for teaching (MKT)* as the mathematical skills, dispositions, and reasoning entailed in and developed through the recurrent work of teaching (Ball et al., 2008; Rowland & Ruthven, 2011; Thompson & Thompson, 1996). MKT is a practice-oriented domain (Biehler et al., 2022; Ball & Bass, 2003; Kent & Noss, 2002). As situated knowledge (Lave, 1988), mathematical meanings may be constituted in terms of workplace concerns and interactions. In teaching, these include unpacking student thinking and constructing explanations that render content comprehensible to learners.

MKT is premised in part on the norms and expectations of mathematical practice; teaching mathematics is often conceived as teaching students to learn content in ways aligned with

mathematics as its own discipline (e.g., Stein et al., 1996; Greeno & Middle School Mathematics Through Applications Project, 1997; National Research Council, 2001). For instance, studies of instructors' knowledge in Calculus and Pre-calculus focus on mathematical constructs, and applications serve an ultimate purpose of creating mathematical meanings (for reviews, see Bressoud et al., 2016 and Thompson & Harel, 2021). Some exceptions include recent literature in mathematical modeling, but even this scholarship frames results in terms of enhancing teachers' and students' practices for mathematics as its own discipline (e.g., Anhalt & Cortez, 2016; Wickstrom & Jung, 2024). *Due to this inherent theoretical tie to the discipline of mathematics, we propose extending the theoretical boundaries of MKT to align with the norms and expectations of other STEM disciplines. This extending may allow us to better understand the kinds of mathematical knowledge required to teach Calculus effectively in service courses.*

By "kinds of mathematical knowledge" we include specialized content knowledge (SCK) and pedagogical content knowledge (PCK). SCK is defined as ways of knowing mathematics unique to teaching. In elementary mathematics, SCK includes understanding and connecting different fraction representations and the definition of fraction. In a Calculus course whose goal is only to enculturate students to the norms and expectations of math as its own discipline, it may include ways of explaining the concepts of limit, rates of change, and derivative. PCK includes familiarity with student conceptions as well as how students may need content in future courses. *As we will argue, it is reasonable to believe that SCK and PCK must be conceived differently when instruction is predicated on a domain other than pure mathematics.*

Mathematics in and for Engineering (MEng)

We use the term *mathematics in and for engineering (MEng)* to mean the mathematics entailed in recurrent work of engineering. Although this construct has been studied by a variety of researchers (see Alpers (2010) for a review), there was no common term that we found in the literature; researchers variously described "mathematical expertise of mechanical engineers" (Alpers, 2010), "mathematical dispositions of structural engineers" (Gainsburg, 2007), or "mathematical components of engineering expertise" (Kent & Noss, 2002). We use the term "mathematics *in and for* engineering" to acknowledge its practice-oriented nature.

The subject of Calculus provides several core ideas critical to solving problems in a variety of engineering domains such as continuity of functions, derivatives, and polynomial basis functions (Alpers, 2010), as well as minimizing surface area, finite element analysis, and Navier-Stokes Equations (Cardella, 2006).

However, ways of knowing Calculus that are relevant to MEng are distinct from those that would be relevant to mathematics as its own discipline; MEng is constrained and motivated by specific problems encountered by engineers (Alpers, 2010; Gainsburg, 2007; Kent & Noss, 2002). Throughout deployment of MEng, engineers using mathematical ideas may think primarily in terms of engineering objects and tools, and preference industry needs and practicality when evaluating which inputs, outputs, and options are relevant in given scenarios (Alpers, 2010; Gainsberg, 2007; Kent & Noss, 2002). MEng may differ from mathematics as a discipline in part because of the role of generalization. To illustrate, we use this excerpt from Gainsburg's (2007) ethnographic study of two engineering firms:

> The problem-solving conditions for these engineers engendered mathematical practices that often diverged from those in traditional and reform mathematics classes. Popular in reform instruction is the practice of generalizing (Radford, 2000; Schoenfeld, 1994), that is, detecting patterns in data and expressing them as mathematical rules or formulas...Generally, the engineers were expected to adhere to the rules of the industry,

> not invent their own. When invention did become necessary...no data existed from which to generalize a rule anyway. Further, the rare method or representation form that the engineers improvised was so situation-specific that it would have had no life beyond the task for which it had been developed (p. 485).

MKT×MEng: Mathematical Knowledge Needed to Support Students in Connecting Mathematics to Engineering

We now introduce the central construct of this paper: mathematical knowledge needed to support students in connecting mathematics to engineering. We use MKT×MEng to refer to this domain, as it is a form of mathematical knowledge for teaching, intertwined with the purpose of fostering learners' understanding of mathematics in and for engineering. *We define MKT×MEng as the MKT needed to connect course content to MEng in a way that is perceptible and comprehensible to students.*

MKT×MEng is distinct from mathematics as its own discipline, from MKT as it has been typically represented and examined, and from MEng. The purpose of MKT×MEng is pedagogical, which separates it conceptually from both mathematics as its own discipline and MEng. It is also different from MKT as often conceived, because the practices that it aims to ultimately support are mathematical practices within MEng. We see MKT×MEng as part of MKT, because it deals wholly with mathematical issues; but we also see it as strongly connected to knowledge outside MKT, because it is motivated by engineering contexts and concerns that are outside mathematics as its own discipline. To demonstrate one way that MKT×MEng is related to but conceptually distinct from MKT, we provide an illustration of how SCK as it is typically conceptualized within MKT does not meet the needs of MKT×MEng. A similar discussion of PCK is possible but is not provided here for brevity.

SCK within MKT includes mathematical knowledge for evaluating the viability of students' alternative algorithms for solving problems, with attention to whether proposed solution strategies are generalizable to other mathematical problems and contexts (Ball et al., 2008). Given the prior characterization of MEng, it seems unlikely that SCK in the domain of MKT×MEng would emphasize the notion of mathematical generalization. Instead, SCK in the domain of MKT×MEng may include an understanding of the contextual factors that determine the viability of various solutions strategies in different applications. For example, in the areas of statics and circuits, continuity is not treated as a property to be evaluated within a given function. Instead, emphasis is placed on how one manages discontinuity within an existing system (Faulkner et al., 2020) so that parameters can be chosen that satisfy the necessary constraints while being viable in the engineering context. In this way, many traditional assignments in single-variable Calculus that ask students to redefine functions at key points or intervals to make them continuous may unintendedly contribute to subsequent confusion for engineering students. SCK in the domain of MKT×MEng may therefore involve instructor knowledge regarding the context-dependent nature of the existence of discontinuities within a handful of real-world systems. Such knowledge could enable an instructor to do more than simply point out an application of the notion of continuity in another field; it could allow them to support and respond to student thinking on Calculus assignments that intentionally interrogate various interpretations of discontinuities and their context-dependent meanings.

We close this section on MKT×MEng with three questions we have encountered about the scope of MKT×MEng and a clarification of our views. (1) Is a focus on MKT×MEng meant to supplant any intention to teach mathematics as its own discipline? (2) Wouldn't MKT×MEng require an engineering degree? (3) Isn't MKT×MEng already encoded in curricular materials that

connect mathematics and engineering, and is therefore unnecessary to study? Our view on (1) is that there is active and robust literature on teaching Calculus (and other fields of mathematics) to represent disciplinary mathematics practices. More can be done to close gaps between research and practice, and we support the perspective that one aim of Calculus is to convey mathematics as a discipline. But in this paper, we draw attention to a domain of professional teaching knowledge that has not been extensively studied and whose examination may address the inefficacy of Calculus courses being a scaffold to disciplines outside of mathematics.

With respect to (2), we are not asserting that all Calculus instructors must be fluent in all aspects of MEng. Just as an elementary teacher does not need to know all of mathematics to develop robust MKT, we suggest that a Calculus instructor does not need to know all of engineering to develop the MKT×MEng needed to support freshmen and sophomores who are only beginning their engineering trajectories. The degree of fluency with respect to MEng that is required for such support remains an open question-one whose empirical investigation we intend to support by providing this framework.

Finally, in response to (3), MKT×MEng is *knowledge* and it is distinct from curricular *materials* that connect mathematics and engineering. We develop this argument in the next section.

How Attending to MKT×MEng Advances Existing Work

Here we review literature describing interventions in college mathematics classrooms that are meant to support student understanding of MEng. We have two purposes. First, we underscore the lack of conceptualization on the *knowledge for teaching* required to facilitate such interventions successfully, which we suggest is MKT×MEng. Second, we argue that a theoretical underpinning of MKT×MEng would provide affordances to work that seeks to improve college mathematics outcomes.

Interventions may be organized according to the depth of resources (class time, preparation, etc.) required to facilitate them. On one end lies successful resource-intensive approaches, such as those reported by Johnson et al. (2024) who increased student retention by revamping differential and integral Calculus courses to center ways of knowing Calculus that would better prepare students in client disciplines. For instance, they emphasized R to gain intuition about Calculus concepts and de-emphasized concepts viewed as necessary only to mathematics as its own discipline (e.g. l'Hopital's Rule). Similar resource-intensive approaches include team teaching between mathematicians and instructors from other STEM fields (e.g., Quintanilla et al., 2007), coordinated mathematics and engineering courses (e.g., Laoulache et al., 2001; Lowery et al., 2010; Schroeder & Carpenter, 1999), and specialized sections of Calculus courses for engineering majors (e.g., Horwitz & Ebrahimpour, 2002; Sathianathan et al., 1999).

These efforts require significant coordination and knowledge-sharing between mathematics and engineering departments. However, literature reporting on such interventions has tended to focus on the materials themselves rather than the MKT×MEng acquired by the mathematicians to decide how to sequence course topics, frame mathematical ideas, and interpret and respond to student thinking. For instance, in the area of pedagogy, Johnson et al. (2024) only report that their revamped Calculus course involves student board work with instructor feedback. One wonders, then: in what ways did mathematics faculty need to know Calculus differently to effectively respond to student board work and support connections to their client disciplines? Similarly, Horwitz and Ebrahimpour (2002) describe a variety of projects created for engineering-intensive sections of a differential Calculus and integral Calculus courses and provide more than enough information regarding the knowledge that *students* needed to know to

complete the projects. Almost no information is provided regarding the mathematical knowledge required of the Calculus instructor to support students in their investigations. Some projects required students to assess the extent to which experimental data can be approximated by mathematical formulae. The mathematical knowledge required to make such an assessment is not part of the standard Calculus curriculum and thus must have been modeled by the mathematics instructor. However, the following remains unknown: what knowledge of Calculus as it is applied in experimental fields was required for that instructor to effectively *teach* this to students? Questions such as these concerning MKT×MEng are critical to consider when thinking about how one might translate these successful interventions to other institutions.

Moderate resource-intensive approaches focus on in-depth assignments, rather than a total course overhaul. Perhaps the most notable example is the work of Schmidt and Winsløw (2021) who, over two decades, created in-depth assignments for service mathematics courses in contexts such as Halbach magnet arrays and red blood cells. In all tasks, the "central challenge" was mathematical, and context and discourse were true to engineering (p. 270). Their views are representative of other assignments with in-depth applications of mathematics to engineering (e.g., Anderson & Nguyen, 2024; Anderson & McCusker, 2019; Barquero et al., 2013; Härterich et al., 2012; Lowery et al., 2010; Peters, 2023; Rønning, 2023; Neubert et al., 2014). Given their more specific nature, literature regarding these assignments typically has the space to describe the MEng required for an individual to complete the task successfully. For instance, Anderson and Nguyen (2024) provide a rigorous, step-by-step overview to support the reader in "mastering the mathematical foundations of the linear algebraic" tool (p. 809) that underlies their in-class intervention. This information is necessary to understand the intervention, but it may be sufficient for supporting instructors in responding to the demands of teaching it. We propose that an investigation of the MKT×MEng employed by instructors, including in facilitating such projects, would provide theoretical support for implementation in other contexts.

Finally, low resource-intensive approaches focus on shorter examples infused into existing Calculus content throughout a term. For instance, Kerrigan and Prendergast (2021) solicited examples from engineering faculty for a flipped Precalculus course. Burrill et al. (2023) reported a similar effort to redesign linear algebra, with the effect that the "students appreciated the content much more and enrollment ... grew vigorously while that of the analogous class from the school of engineering dropped precipitously" (p. 800). Although these examples take up little class time, their effective facilitation may still require a particular type of mathematical knowledge to situate the examples. The relationship between resource intensity of an instructional intervention and degree of MKT×MEng required for effective facilitation remains an open and important question for future research.

Implications for Further Research

In summary, we propose a theoretical construct -- MKT×MEng -- that builds on existing work in MKT and whose further development will respond to ongoing calls for Calculus to fulfill its promise as a scaffold for STEM client disciplines. While mathematics education literature abounds with various examples, activities, and projects for use in courses such as Calculus, we are lacking in systematic studies of the mathematical knowledge required to teach with such materials in mathematics courses.

In broaching and delimiting the construct of MKT×MEng, we support researchers in undergraduate Calculus education in investigating questions such as: What ways of knowing Calculus within MEng are relevant for college Calculus instructors? By what epistemological principles (core assumptions, concepts, and norms) do engineering faculty interpret student

errors and evaluate student solutions involving Calculus concepts, and how do these principles compare to those prevalent in current work in MKT? What are features of contexts where Calculus within MEng are likely to arise, and what knowledge is needed to detect and explain these forms of MEng so they are visible to students? In investigating such questions, we advance not only our theoretical understanding of mathematical content for teaching in service courses, but also our potential for implementing the wealth of existing materials into Calculus courses beyond the institutional context in which they were developed.

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Towards a Unified Group Scaffolding Framework: Insights From MGTAs

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Supporting students' collaborative work requires more than simply placing them in groups; instructors must actively shape participation and discourse through scaffolding. Group scaffolding refers to the set of instructional strategies that help students coordinate, distribute responsibility, and build shared understanding in groups. Despite its relevance for undergraduate active learning environments, existing theories have tended to isolate central elements, including teacher guidance of student reasoning, discourse moves shaping participation, and classroom norms structuring collaboration. In this paper, we analyze episodes of instruction by mathematics graduate teaching assistants (MGTAs) to motivate a unified theory of group scaffolding. Our framework integrates four interdependent dimensions: (a) norms setting, (b) task and material design, (c) discourse moves, and (d) access distribution. We argue that this theory both advances research on collaborative learning in undergraduate mathematics and informs professional development efforts that prepare instructors to foster more equitable and effective group work.

Keywords: Group Scaffolding, Active Learning, Discourse Moves, Norm Setting

Active learning has been used to characterize student-centered instructional approaches focused on heightened engagement with mathematics through structured activities (Doolittle et al., 2023). The implementation of these approaches in undergraduate mathematics initially centered on gateway courses, but it has also expanded to upper-division courses, including linear algebra, geometry, and logic (Theobald et al., 2020). This widespread adoption implies that mathematics graduate teaching assistants (MGTAs) are now frequently required to implement these pedagogical strategies.

Typically, classroom activities in these courses are often designed as problem-solving and discussions in group settings, allowing students to investigate, communicate, and reflect together while receiving feedback from peers and instructors (Laursen & Rasmussen, 2019). Yet, mathematical discussions in groups are dynamic, constantly influenced by a number of socio-cognitive and instructional factors that could create unequal conditions for students to achieve similar learning goals. Esmonde (2009), for instance, examined cooperative structures and found that students often assumed roles such as "expert," "novice," or "facilitator," with experts dominating quiz settings while facilitator roles promoted more equitable participation. Moreover, Reinholz and Johnson (2022) provided compelling evidence that active learning, when implemented without additional instructional support for cooperative learning, can lead to performance gaps across certain student groups.

Effective instructional practices require more than simply grouping students; structured approaches, including rotating roles, designing open-ended tasks, and framing explicit group norms, can help to distribute authority and promote deeper engagement among groups (Cohen & Lotan, 2014; Esmonde, 2009; Laursen & Rasmussen, 2019). Furthermore, other strategies such as circulating to pose probing questions, monitoring participation patterns, and amplifying marginalized voices could foster both collaborative learning and sense of belonging (Kogan & Laursen, 2014; Reinholz & Shah, 2018).

While some of these practices are typically associated with the well-established notion of scaffolding—the instructional support provided by a more knowledgeable other to help learners progress beyond their independent capabilities (Wood et al., 1976), active learning settings require attention to group scaffolding. Unlike traditional scaffolding, which targets individual learners, group scaffolding refers to strategies that support the collective participation, discourse, and problem-solving of the group as a whole (Webb et al., 2025).

Some theories have identified overarching elements involved in group scaffolding, including discourse moves that shape participation (Chapin et al., 2009) and classroom norms that structure access to collaboration (Yackel & Cobb, 1996). These elements, however, have largely been treated in isolation, suggesting the need for a more cohesive framework that integrates these perspectives into a unified account.

The purpose of this report is twofold. First, we explore the possibility of a general framework for group scaffolding in undergraduate mathematics, drawing on instances of novice instructors coordinating collaborative activities in active learning classrooms. Although some studies have reported difficulties experienced by novice instructors in using group scaffolding (Winter et al., 2002), their experiences provide contrasting cases that highlight the affordances and limitations of group scaffolding in practice. Second, we consider how this framework can inform the design of professional learning opportunities for novice instructors. Research has shown that MGTAs often default to individualistic instructional practices, downplaying student-to-student interaction (Winter et al., 2002), and may prioritize procedural efficiency over conceptual learning, offering prescriptive guidance with little time for reflection ([Reference removed for blind review]). Our central research question is: how can a detailed analysis of group scaffolding in periods of MGTAs instruction reveal the critical elements required for a comprehensive theory?

Conceptual Framework

Scaffolding was first introduced by Wood et al. (1976) to describe the process by which a more knowledgeable other supports a learner's progress beyond what they could accomplish independently. Grounded in Vygotsky's (1978) theory of the Zone of Proximal Development, scaffolding involves providing temporary, adaptive support—such as prompting, modeling, or structuring tasks—that enables learners to engage in more sophisticated reasoning. As students gain competence, this support is gradually withdrawn, fostering independence and deeper understanding. Traditionally, scaffolding has been conceptualized as a one-on-one interaction between teacher and learner.

In student-centered environments, however, learning often occurs in small groups where students engage collectively in problem-solving and discussion. In these contexts, instructors need to carefully attend to the dynamics of group participation, shared reasoning, and broad access to mathematical ideas. Group scaffolding refers to instructional moves, task designs, and resources that deliberately support the group as a collective—facilitating broad participation, encouraging explanation and justification, and helping students build on one another's contributions. Unlike individual scaffolding, which centers on the dyadic relationship between teacher and student, group scaffolding emphasizes how instructors shape the social and cognitive ecology of collaborative work.

Although active learning reforms in undergraduate mathematics have increased the use of group work, research suggests that group scaffolding remains underutilized, especially among novice instructors such as MGTAs (Reinholz & Shah, 2018; Winter et al., 2002). Moreover, the use of group scaffolding has often been found to be incomplete, leading in some cases to students defaulting to unequal roles, limited participation, and superficial task completion.

Research on scaffolding group work identifies a number of strategies that could support broad collaboration. Assigning roles and setting group norms clarifies expectations and fosters equitable engagement (Cohen & Lotan, 2014; Esmonde, 2009; Horn, 2012), while structured tasks and discourse moves coordinate problem-solving and promote peer-to-peer discourse (Boaler, 2008; Chapin et al., 2009; Lotan, 2003). Reflection, self-assessment, and journaling encourage metacognition and shared responsibility (Bielaczyc & Collins, 1999; Gillies, 2016), and instructor facilitation and visual representations provide just-in-time support and anchor joint understanding (Cobb et al., 1997; Roschelle & Teasley, 1995).

Three studies have attempted to identify the most essential elements involved in group scaffolding. First, Wood et al. (1976) focused on teacher-to-student interactions, suggesting that effective scaffolding involves breaking tasks into manageable steps, maintaining student engagement, and transferring responsibility to learners (fading). This perspective highlights the teacher's role in providing just enough structure to support students' problem-solving without taking over the process entirely. Although it is not discussed by these authors, the two initial phases of this process are closely connected to instructors' learning goals. While breaking a procedural problem into more manageable steps and maintaining engagement, instructors can ask students to justify each step, providing opportunities for students to deepen their conceptual understanding.

While this framework focuses on the interactions between teachers and individual students, Calor et al. (2022, 2024) extended these ideas at the group level identifying three principles: diagnosis, contingency, and fading. Group scaffolding is a process that requires an initial assessment of the group collaboration (diagnosis), after this assessment, support is adapted to the specific needs of the group at that moment (contingency), and then support is gradually removed as the group becomes more independent, shifting the responsibility for learning back to the students in the group (fading).

Chapin et al. (2009) emphasized the role of discourse moves in shaping participation during mathematical discussions. Their framework identifies specific teacher discourse moves, such as revoicing, asking students to restate or expand on peers' contributions, and pressing for reasoning, that can sustain dialogue and promote deeper engagement with mathematical ideas. These strategies not only support individual student reasoning but also serve as group scaffolds by making ideas public, encouraging broader participation, and fostering a collective sense of inquiry. The focus on discourse highlights that scaffolding is not limited to structuring tasks but also emphasizes the role instructors play in mediating talk to create wider opportunities for collaboration.

A third perspective comes from Yackel and Cobb (1996), who theorized the role of classroom norms in shaping how students access and engage in collaborative mathematics. They distinguished between sociomathematical norms—taken-as-shared understandings of what counts as a valid mathematical explanation or justification—and more general interactional norms, such as expectations for listening to and responding to peers. From this perspective, scaffolding is inseparable from the broader social and normative structures that govern participation in a classroom. Instructors contribute to scaffolding not only through immediate interventions but also by setting norms that could enable or constrain collaboration. This framework underscores that effective group scaffolding must be understood in relation to the classroom's ecology of participation, rather than as a series of isolated teacher actions.

Methodology

This paper is best understood as a conceptual study with empirical grounding. Our aim is to theorize group scaffolding in undergraduate mathematics instruction by bringing together existing strands of research (e.g., scaffolding, classroom norms, talk moves, task design, access to opportunties to learn) into a unified framework. To motivate and situate this theorizing, we draw on a small set of illustrative episodes collected in the context of a professional development program for MGTAs. The episodes were drawn from classroom observations of MGTAs in undergraduate mathematics courses. Although MGTAs were relatively new to teaching, they all participated in a professional learning opportunity intended to provide opportunities for them to reflect on a number of instructional issues related to students' participation in the classroom, including classroom norms, students' identity, and discourse moves. We selected four short episodes that highlight variation in how instructors attempted to scaffold group work. These episodes were not chosen to be representative but to serve as eliciting cases (Flyvbjerg, 2006) that reveal different dimensions of scaffolding—norm-setting, task design, use of tools, feedback practices, and participation. We analyzed each episode through iterative discussions, asking: What forms of scaffolding are visible here (structural, discursive, participatory)? What is missing or underdeveloped? How does this episode help identify the critical components of a unified theory of group scaffolding? The methodology follows a theory-building through exemplars approach (diSessa & Cobb, 2004), in which empirical material is used to motivate and refine conceptual categories.

Conceptual Framework

Four episodes were selected to inform a unified framework of group scaffolding, representing contrasting cases of different aspects of group scaffolding. Each episode is accompanied by a brief analytic commentary. Rather than reporting outcomes (e.g., student learning), we focus on how each episode illustrates different approaches to scaffolding group work and what those approaches reveal about potential dimensions present in a unified theory.

Episode 1: Group Work at the Board. An MGTA asked students to work on the whiteboard in groups, restricting each group to a single marker. This design choice created a structural scaffold for participation by requiring shared control of a resource. The establishment of explicit norms or facilitation moves, however, was not clear, which could prompted a dynamic in which some students dominated the discussion and others were marginalized from it. This episode illustrates the importance of material tools as scaffolds but also the limits of tool-based scaffolding in the absence of norms and discourse supports.

Episode 2: Optional Group Work. Another MGTA invited students to work in groups but framed group work as an option. Consistently, when several students chose to work individually, the MGTA did not encourage further collaboration but attempted to monitor each student's work, visibly increasing the demands to attend to individual needs. Here, the absence of norm-setting undermined the possibility of collaboration. The episode shows how scaffolding is not just about offering opportunities for collaboration but about structuring access so that participation is expected and supported.

Episode 3: Prescriptive Group Monitoring. In a different class, an MGTA required students to work in groups and regularly circulated among them to monitor their progress. However, the instructor’s feedback was mainly prescriptive, telling students what to do rather than prompting them to reason collaboratively. This scaffolding emphasized teacher authority

rather than group autonomy, showing how certain forms of monitoring can unintentionally reduce opportunities for decentering authority and promoting student-to-student discourse.

Episode 4: Inquiry-Oriented Group Monitoring. Finally, an instructor also required group work but engaged in monitoring that was inquiry-based, using open-ended questions and encouraging all members to contribute. This episode reflects a more integrated enactment of group scaffolding: task structure, norms of inclusion, and talk moves worked together to promote collaborative reasoning. However, even here, isolated moments of uneven participation persisted. It is possible that this uneven participation could have faded in subsequent classes, but it could also suggest a need for other types of other group scaffolding strategies (e.g., role assignments, status interventions).

These episodes show a spectrum of developing scaffolding practices: from structural-only (Episode 1), to secondary (Episode 2), to prescriptive (Episode 3), to more fully integrated (Episode 4). An overall analysis of these episodes highlights the following central dimensions of group scaffolding closely interconnected: Norms setting (e.g., whether group work is required, how roles are structured, how ideas should be shared), task and material design (e.g., open-ended problems, tools like markers to regulate access), discourse moves (e.g., prescriptive vs. inquiry-based feedback, individual vs group questioning), and access distribution (whether the participation of each peers in a group is supported). These results lay the groundwork for our proposed unified theory of group scaffolding.

Toward a Unified Theory of Group Scaffolding

The analysis presented above reveals both the affordances and limitations of MGTAs' attempts to coordinate group work. Together, they illustrate the existence of multiple dimensions associated with group scaffolding instead of isolated elements previously identified in the literature–task design, discourse moves, or classroom norms. Effective group scaffolding emerges through the interaction of these elements that jointly support broad participation in mathematical reasoning. Drawing from prior theories (Chapin et al., 2009; Wood et al., 1976; Yackel & Cobb, 1996) and extending them through the analysis of our episodes, we propose a unified theory of group scaffolding that integrates four interdependent dimensions illustrated in Figure 1. The following is a brief description of each of these elements.

Norms and Expectations. Group scaffolding requires the establishment of clear expectations for participation, including both general interactional norms (e.g., listening to peers, giving space to contribute ideas) and sociomathematical norms (e.g., what counts as a valid explanation). As Episode 2 illustrated, in the absence of such norms, group work may fail to materialize altogether.

Task and Material Design. The design of mathematical tasks and the structuring of material tools shape opportunities for collaboration. The one-marker activity in Episode 1 demonstrates how material resources can serve as structural scaffolds, channeling participation toward collective work. Yet, without accompanying discourse supports, such tools may also reinforce uneven participation.

Discourse Moves. Instructor talk is central to sustaining collaboration. Prescriptive feedback, as seen in Episode 3, may reduce opportunities for students to engage with each other's reasoning. In contrast, inquiry-oriented talk moves (Chapin et al., 2009), such as revoicing and pressing for justification, can prompt groups to develop shared understandings and sustain mathematical dialogue.

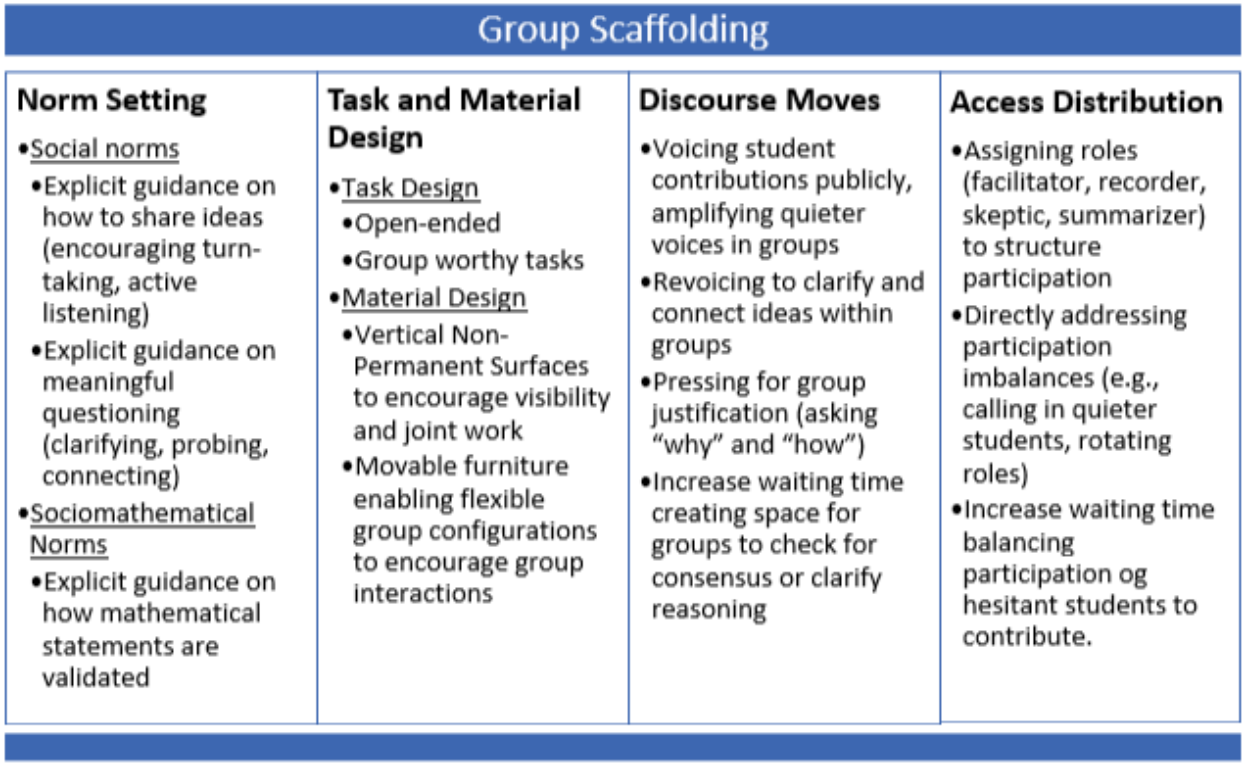

Figure 1. Central Dimensions of Group Scaffolding

Access Distribution. A comprehensive theory must also account for how scaffolding practices distribute opportunities to learn (Esmonde, 2009) across all group members, widening access to mathematical content, discourse, and positionalities to all students. As Episode 4 showed, even when inquiry-based scaffolding is present, participation may remain uneven, presenting obstacles to collaborative learning. Mediation strategies, such as assigning roles (Cohen & Lotan, 1997) or directly addressing participation imbalances (Esmonde, 2009), are necessary to ensure that scaffolding supports the learning of all students, not just the most vocal.

We argue that group scaffolding can be conceptualized as the dynamic coordination of these four dimensions, with instructors navigating among them in response to student activity. Rather than a checklist of strategies, the framework positions scaffolding as a relational and situated practice that integrates structural, discursive, normative, and broad supports. This unified perspective provides a basis for analyzing classroom practice and designing professional development that makes visible the complex work of orchestrating group learning in undergraduate mathematics.

Toward a Unified Theory of Group Scaffolding

This unified theory of group scaffolding proposed here has direct implications for the design of professional development (PD) for novice instructors. Traditional PD often emphasizes either structural strategies, such as assigning roles, or general encouragement of active learning without fully addressing how norms, discourse moves, tools, and equity interact in practice. By contrast, our framework suggests that PD should make these dimensions explicit and help instructors learn to coordinate them in response to classroom dynamics. One promising approach is the use of video-based reflection, where novice instructors analyze episodes of classroom practice—both their own and others'—through the lens of the four scaffolding dimensions. Such an approach

not only highlights the complexity of group orchestration but also makes visible the affordances and limitations of different scaffolding strategies.

In addition, PD should provide practice-based repeated opportunities for instructors to rehearse and refine scaffolding moves. For example, role-playing activities can allow instructors to practice inquiry-oriented monitoring and equity-promoting talk moves in a low-stakes environment, while structured debriefs can help them connect these practices back to the broader framework. Importantly, PD must also attend to the developmental trajectory of novice instructors. Our analysis of episodes suggests that instructors may first lean on structural scaffolds before developing the capacity to integrate discursive and practices that effectively broaden participation. PD, therefore, should be sequenced to support growth across these dimensions rather than treating scaffolding strategies as isolated techniques.

Conclusion

This study proposes a unified theory of group scaffolding in undergraduate mathematics, integrating four interdependent dimensions: norms setting, task and material design, discourse moves, and access distribution. By grounding our analysis in episodes of MGTA instruction, we illustrate how partial and emergent scaffolding practices highlight both the challenges instructors face and the critical elements required for effective group work. These episodes serve not only as empirical grounding but also as pedagogical resources for making visible the complex work that instructors must learn to do. Our contribution is twofold. First, we extend prior theories of scaffolding, discourse, and norms by showing how they can be synthesized into a coherent account of group scaffolding that better reflects the realities of undergraduate mathematics classrooms. Second, we argue that novice instructors provide a particularly generative site for theory-building and PD design, as their developing instructional practice could make visible gaps in instructional practice. For the field of undergraduate mathematics education, this unified theory provides a lens for analyzing group work and a guide for strengthening professional development opportunities that move instructors beyond fragmented approaches toward more integrated, collaborative scaffolding. Future work should further test and refine this framework in diverse classroom contexts, with attention to how it supports not only instructor development but also students' opportunities to learn collaboratively.

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The Case for Complexity Theory and Formal Science to Study the Structure of Mathematics

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Mathematical structure refers to the characteristics, topology, and relationships within and between mathematical concepts invariant to external contexts and interpretations which instantiate the concept. Research on mathematical structure is typically guided by the post-positivist and interpretivist research paradigms, which respectively emphasize the impact of human observations and interpretations on mathematical concepts, both of which are external to the intrinsic structure of mathematical concepts. This report proposes a shift to complexity theory, a research paradigm standard to modern physics and engineering which emphasizes physically necessary mathematical principles over observation and interpretation, to faithfully study structure intrinsic to mathematics itself. I propose the Tornado Theory as a theoretical framework emerging from the tenets of complexity theory and current mathematics education literature which could guide future research on mathematical structure. Finally, potential analytical techniques for the Tornado Theory based on network- and information-theoretic principles are introduced.

Keywords: Mathematical structure, complexity theory, nature of mathematics, network theory, reification

Why is the Structure of Mathematics Important for Mathematics Education Research?

Mathematical structure is a frequently occurring concept in mathematics education research which cuts across numerous domains of mathematics education literature (Gronow et al., 2022). Mason (2003) defined *structural* features as the internal characteristics, topology, and dynamics of a system which are invariant to factors external to the system. This is in contrast to *surface-level* features, which refer to the effect external contexts, interpretations, and examples have on a system (Mason, 2003). When applied to mathematical content, *mathematical structure* refers to the topological and dynamical properties of a mathematical concept independent of the contexts, interpretations, and examples which instantiate the concept (Alcock & Simpson, 2004, 2005; Dawkins, 2012; Dive, 2003; Martin, 2013; Sfard, 1991, 2003). For example, the powers of i and rotation by 90 degrees are two mathematical concepts with different surface-level features but identical structural features, because these two concepts represent the same mathematical idea (the cyclic group, i.e. oscillation, with a period of 4) but are instantiated in different contexts and facilitate different contextual interpretations.

The importance of student understanding of mathematical structure has been well-documented in mathematics education literature. Skemp (1976) differentiated between relational and instrumental understanding of mathematics, the former referring to connections between concepts and the topology of these connections and the latter referring to isolated mathematical concepts and skills. Baroody et al. (2007) and Star (2005) expanded upon Skemp's ideas, claiming deep conceptual and procedural knowledge build upon each other to generate highly connected and meaningful schema. Relational understanding of a concept therefore requires coordination of conceptual and procedural knowledge and attending to the *meaning of a mathematical concept for* connections to other mathematical ideas rather than the *meaning of* concepts in isolation (Baroody et al., 2007; Brownell, 1947). This focus on connections requires a shift in attention from surface-level features to structural features (Alcock & Simpson, 2004,

2005; Dawkins, 2012; Lehrer & Schauble, 2020; Martin, 2013; Mason, 2003; Sfard, 1991, 2003). Therefore, student attention to mathematical structure is critical in developing a relational understanding of mathematics.

In order to shift students' attention from surface-level to structural features, mathematics instructors need to consider both the structure of mathematical concepts and its relations with surface-level features which students may be inclined to focus on (Mowat, 2010; Strom et al., 2001). This enables mathematics instructors and mathematics education researchers (MERs) to develop hypothetical learning trajectories which scaffold students to shift from surface-level interpretations to structural thinking while remaining faithful to the structure of mathematical concepts (Ekici et al., 2020; Leikin & Dinur, 2007; Simon, 1995). Mathematical structure is therefore a key component to frameworks characterizing pedagogical content knowledge (PCK) (Ball et al., 2008; Gronow et al., 2022; Rowland, 2013; Vale et al., 2011).

Thus, attention to mathematical structure is critical to relational understanding of mathematics for both students and instructors. Given the importance of mathematical structure, it is valuable for the MER community to consider mathematical structure from a theoretical perspective to guide MERs in answering questions such as: what is mathematical structure? What topological properties emerge from and govern relationships between mathematical ideas and domains? How can mathematical structure be characterized and used by teachers to inform their instruction? How does mathematical structure evolve over time as new mathematical ideas are discovered or created? The *Structure of Mathematics (SOM)* refers to these general meta-mathematical explications of mathematical structure. In this report, I will first review existing literature on SOM and discuss a philosophical dilemma surrounding current SOM research. Next, I will suggest a shift in research paradigm which mitigates this dilemma by invoking complexity theory from modern physics and engineering. Finally, I will propose the Tornado Theory, a framework based on complexity theory, which could guide future research in SOM.

Literature Review

MERs have begun studying SOM by using qualitative techniques and graph theory to construct and analyze graphs representing connections between mathematical concepts (Hensley, 2025; Lapp et al., 2010; Mowat, 2010; Strom et al., 2001). These studies typically did not provide an explicit theoretical framework for SOM and implicitly assumed SOM could be extracted from student concept maps. Strom et al. (2001) created directed graphs modeling concept maps of second graders' mathematical arguments involving the area of shapes. Lapp et al. (2010) studied linear algebra concept maps constructed by undergraduate students as an undirected graph. Hensley (2025) used topological data analysis (TDA) to study concept maps created by calculus students and identified topological gaps as missed connections in students' understanding of calculus.

Mowat (2008, 2010) criticized the use of concept maps to characterize SOM, asserting concept maps are architectural structures designed by external observers of mathematics rather than originating from the connections between mathematical concepts themselves. Instead, Mowat (2008) described an explicit theoretical framework drawing on Lakoff and Núñez's (2000) theory of embodied cognition. Mowat's (2008, 2010) framework models SOM as a nested network where nodes represent conceptual domains and links represent conceptual metaphors. Most nodes contain few connections, but a few nodes (known as hubs) contain many connections, making these networks display a hub-and-spoke topology. Mowat (2008) posited this hub-and-spoke topology has a high risk of cascading failure. Thus, learners need to construct links between non-hub nodes to gain a deeper understanding of mathematical concepts. Mowat

(2010) provided a proof-of-concept of her framework by using data from semi-structured interviews with mathematicians, teachers, and MERs to extract conceptions about exponentiation from semantic connections, creating a network of concepts related to exponentiation. This framework will be discussed in further detail in the Tornado Theory section.

An alternative approach involves extracting emergent connections between concepts from texts to form a network, which has been used to study both SOM and the structure of other STEM disciplines. In particular, this approach was conducted by Chai et al. (2020) on biomedical research papers, Christianson et al. (2020) on linear algebra textbooks, and Tschisgale et al. (2023) on German Physics Olympiad questions. These studies typically used collocation analysis, where the strength of connections were determined by the location of words rather than implicit meaning. Although some of these studies are not in mathematical contexts, they leveraged network-based approaches similar to Mowat (2008, 2010) to characterize the structure of knowledge.

Mowat's (2008, 2010) critique of the usage of concept maps reflects an issue underlying current research on SOM. Because SOM fundamentally studies *structure* as internal properties independent of external contexts and interpretations, frameworks and methodologies used in SOM research should reflect the focus on internal rather than external aspects. Similarly, Hensley (2025) posited existing methodologies used in SOM research such as qualitative coding, frequency analysis, and adjacency matrix analysis focus on contextual, surface-level components of mathematical concepts rather than topological and structural relationships between mathematical concepts. In other words, current SOM frameworks and methodologies focus on the *meaning of* mathematical concepts rather than the *meaning of* mathematical concepts *for* the construction of the larger structure of mathematics (Brownell, 1947; Hensley, 2025).

In particular, Strom et al. (2001) and Lapp et al. (2010) explicitly referenced interpretivism as their high-level research paradigm, which situates knowledge and reality with respect to cognitive interpretations and social factors (Heath & Bleiler-Baxter, 2025; Stinson & Walshaw, 2017), both of which are external to the structure inherent in mathematics (Dive, 2003). Despite Mowat's (2010) and Hensley's (2025) urges for a focus on structure over context, their methodology drew upon the interpretation of exponentiation (Mowat, 2010) and calculus (Hensley, 2025) made by students, teachers, mathematicians, and MERs, which are again external to the structure inherent in mathematics. Literature outside mathematics education (Chai et al., 2020; Christianson et al., 2020) typically used post-positivist techniques such as collocation analysis to reduce the interpretive influence of textual analysis, but the post-positivist paradigm introduces factors external to mathematics in another way. In post-positivism, empirical observations hold authority on reality and knowledge (Stinson & Walshaw, 2017). Empirical observations, however, are influenced by the interpretation of the observer and thus also external to the structure inherent in mathematics itself and an unreliable source of knowledge for SOM (Davis & Sumara, 2008). Collocation analysis has a further drawback of assuming connections between words could be fully represented by their spatial relationships on a page, which is an oversimplification (Borge-Holthoefer & Arenas, 2010).

Therefore, there exists a disconnect between the ontological and epistemological foundations of the current research on SOM. By definition, SOM requires extraction of characteristics, dynamics, and topology internal to a mathematical system. On the other hand, SOM studies are informed by worldviews which extract information from unreliable sources such as empirical observations, cognitive interpretations, and social structures. This begs the question: ***how can we directly study internal properties of mathematical systems when the very nature of research is***

external? In the next section, I propose an alternative high-level theoretical worldview, complexity theory, which provides an approach to answer this question.

A High-Level Paradigmatic Shift: Complexity Theory and Formal Science

Complexity theory is a high-level paradigm arising from the scientific and engineering movements in the post-war twentieth century (Davis & Sumara, 2006, 2008). At this time, the shift from Newtonian physics to modern physics inspired scientists and engineers to study the universe and its objects as chaotic systems defined by irreducibly complex ensembles of moving parts rather than deterministic systems constructed from components. Complexity theory therefore posits the irreducibly complex nature of individual systems make it futile to study the theories and laws governing individual systems. Rather, learning about the universe entails learning about the mathematical relationships governing nonlinear, chaotic, and complex systems (Davis & Sumara, 2008). For example, Newtonian physics is grounded by observations of the natural world made by scientists before the twentieth century. Mathematical relationships were created to model and explain these observations. The mathematics of Newtonian physics could therefore be considered as *artificial interpretations* of the physical world. On the other hand, although quantum mechanics was initially inspired by observations made in the late nineteenth and early twentieth centuries, mathematicians and physicists in the mid- to late-twentieth century found quantum mechanics follow directly from the mathematical relationships of linear algebra, functional analysis, category theory, and braid theory (Baez & Lauda, 2011). Mathematical relationships are therefore *physically necessary* for quantum mechanics to exist in its current form (Baez & Lauda, 2011; Dive, 2003; Halvorson, 2011; Townsend, 2012) rather than *artificially necessary* to explain human interpretations of the universe.

In other words, in Newtonian physics, observations precede mathematical relationships, which are artificially created to explain observations. In quantum mechanics and complexity theory, mathematical relationships precede observations, and observations serve as evidence of the physical necessity of mathematical relationships for the universe and the human experience to exist (Dive, 2003; Lakoff & Núñez, 2000; Thagard, 2020). Schelling's (1971) seminal work applying complexity theory to the social sciences summarized complexity theory as follows, "[the] rich will get richer, the advantaged will gain more advantage – not because of intention, but because of the laws of nonlinear dynamics" (Davis & Sumara, 2008, p. 170). Such a worldview places the physically necessary system of mathematical relationships at the forefront of research. To a complexity theorist, external sources of authority such as empirical observations, cognitive interpretations, and social structures are peripheral to the physically necessary, internal mathematical relationships which govern complex systems (Davis & Sumara, 2008; Dive, 2003). Thus, complexity theory suggests investigations of SOM should leverage physically necessary mathematics to study internal structure to avoid external interpretations.

Formal science (not directly related to the formalist view of the nature of mathematics) is an example of applying complexity theory to research. Dive (2003), Franklin (1994), Ramoo (2024), and Thompson (2007) described the formal scientific process as based on abstraction and mathematical reasoning rather than empirical observation and statistical testing. In formal science, researchers do not test hypotheses by designing experiments and collecting data, which is the case in traditional, empirical science (Ramoo, 2024; Thompson, 2007). Rather, researchers convert the system to be studied into an abstraction using mathematical frameworks. This abstraction ideally removes external, surface-level features, enabling the researcher to focus solely on internal, structural features. Then, researchers study the system and formulate solutions by analyzing the abstraction using mathematical relationships. It is critical the mathematical

relationships used are physically necessary. This ensures the analysis process does not introduce additional surface-level features which may corrupt intrinsic structural relationships (Dive, 2003; Ramoo, 2024; Thompson, 2007). Complexity theory and the formal sciences are commonly used in engineering and industrial mathematics, where the goal is to design solutions whose effect cannot be observed at the time of design. Formal science is also commonly used in modern physics, where many phenomena such as quantum entanglement and the Higgs boson cannot be observed without precise and expensive instrumentation. Therefore, physically necessary mathematical frameworks must first be applied to formulate the initial engineering design or discover physical phenomena prior to experimentation or testing (Dive, 2003; Franklin, 1994).

Although it is impossible to confirm the physical necessity of mathematical frameworks, the robust coherence between mathematics derived from category theoretic principles (Baez & Lauda, 2011; Halvorson, 2011), the physical world (Thagard, 2020), and the human experience (Lakoff & Núñez, 2000) suggest mathematical frameworks based on category theory is, to the best of human understanding, physically necessary (Baez & Lauda, 2011; Dive, 2003; Franklin, 1994). Common mathematical frameworks used in the formal sciences include game theory, control theory, information theory, and network theory (Dive, 2003; Franklin, 1994).

In particular, network theory provides a promising direction for applying complexity theory to SOM for two reasons. First, existing literature on SOM typically leverage network theory and its derivatives such as TDA to model structure (Chai et al., 2020; Christianson et al., 2020; Hensley, 2025; Mowat, 2008, 2010; Lapp et al., 2010; Strom et al., 2001; Tschisgale et al., 2023). Furthermore, a set of network-theoretic tools have been developed to extract structural features from social scientific data such as semantic texts and social networks (Barrat et al., 2008; Borge-Holthoefer & Arenas, 2010; Scott & Carrington, 2014). Therefore, leveraging network theory situates SOM in existing literature. Second, network theory extracts topological and structural relationships, aligning with the goals of studying SOM. In the next section, I will develop an emergent theoretical framework used to abstract SOM with network theory, the Tornado Theory. Other mathematical frameworks may reveal additional information related to SOM such as dynamic evolution of mathematical structure (control theory) or the impact of human activity on the discovery, creation, and teaching of mathematical structure (game theory). A full picture of SOM and its impact on the discovery, creation, teaching, and learning of mathematics involves integrating abstractions and implications from a variety of mathematical frameworks. Thus, a framework purely based on network theory is intrinsically limited, and future theoretical work should develop SOM frameworks from other mathematical perspectives.

Mid- and Ground-Level Framework: The Tornado Theory

The Tornado Theory (TT) describes SOM from a network theory perspective by synthesizing the theoretical frameworks of Mowat (2008) and Sfard (1991). Sfard's (1991) reification theory described the structure of mathematical knowledge as emerging from an iterative process of condensation and reification. Any mathematical concept can be represented as a concrete, embodied *process* or an abstract, decontextualized *object*. The process representation of a concept, due to its concreteness, is easier to construct, but the object representation of a concept is required for the concept to be transferred to contexts independent of its original construction pathway (Dive, 2003; Sfard, 1991; Sfard & Linchevski, 1994). *Condensation* refers to the initial construction of a mathematical concept from old concepts, which requires old concepts to be transferred to the context of the new concept. Thus, old concepts can only be used in condensation if they are represented as objects decontextualized from how they were initially constructed. On the other hand, because the new concept is constructed in the context of old

concepts, the new concept is initially represented as a process (Sfard, 1991, 2003; Sfard & Linchevski, 1994). For the new concept to be used in novel situations, it must be re-represented in a form independent of the context of its construction. This decontextualization is known as *reification*, and the reified concept could then be used in the condensation process of further mathematical ideas (Sfard, 1991). Therefore, mathematical concepts could be characterized through an iterative cycle of condensation and reification known as the *Tornado*, which Sfard (2003) described as "the driving force behind the incessant growth of knowledge" (p. 359).

For instance, consider integration, which is tied to area and differentiation. During its initial construction, integration is constructed as a process in the context of area and differentiation. Integration, however, has other meanings, such as smoothing and frequency domain division. In order for integration to be used to construct further concepts such as low-pass filters (based on smoothing) and Fourier Transforms (based on frequency domain division) (McClellan et al., 2015; Oppenheim et al., 1994; Rønning, 2021; Wang, 2025), integration must be reified into an object, i.e. represented independently from the contexts of area and differentiation. Continuing the cycle, although Fourier Transforms are initially constructed as a process in the context of integration, Fourier Transforms have other meanings such as an inner product on the L_2 Hilbert Space (Ekici et al., 2020; Debnath & Mikusinski, 2005; Rønning, 2021). In order for Fourier Transforms to be used in the context of Hilbert Spaces, it must be reified as an object, i.e. decontextualized from its initial construction contexts of integration and the frequency domain. This condensation-reification cycle known as the Tornado continues iteratively to drive the discovery of new mathematical concepts.

Mowat's (2008) network theory approach to SOM could be used to operationalize the topology of mathematical concepts formed by the Tornado. In particular, mathematical concepts may be arranged as a network with nodes representing individual concepts and links representing connections between concepts. Each node contains a network structure within itself representing the structure of the concept, with nodes of this sub-network representing sub-concepts building up the greater concept. Each sub-concept is represented by another network describing the structure of the sub-concept (Mowat, 2008). This nested network continues like a Russian doll.

Inspired by existing research on network theory characterizations of semantic networks (Albert & Barabási, 2002; Barrat et al., 2008; Borge-Holthoefer & Arenas, 2010; Newman, 2003), Mowat (2010) leveraged the Barabási-Albert Model (BAM) to describe the topology of each network layer. BAM posits the number of nodes N with exactly k links in a network can be approximated with the relationship $N[k] = \psi k^{-\gamma}$, where ψ is an arbitrary constant and γ characterizes the overall structure of the network (Albert & Barabási, 2002; Mowat, 2010). If $\gamma > 1$, the number of nodes quickly decreases for higher numbers of links, meaning most nodes have few connections and few nodes have many connections. Thus, the network displays a hub-and-spoke structure, which is easily constructed but has a high probability of cascading failure. If $\gamma < 1$, the number of nodes decreases slowly for higher number of links, meaning there are a significant number of nodes with many links. Thus, the network displays a distributed structure, which are difficult to construct or interpret but are robust and adaptable in novel situations due to the low probability of cascading failure (Mowat, 2010; Strom et al., 2001).

The BAM and Russian doll structure described by Mowat (2008, 2010) reflects and provides a mathematical abstraction for the Tornado. Condensation refers to the construction of a new layer of networks from old network layers in the Russian doll. Mowat (2010) claimed these new layers follow a hub-and-spoke structure (i.e. $\gamma > 1$) due to their ease of construction. This hub-and-spoke structure is analogous to the process representation of the Tornado. A hub-and-spoke

network, however, is difficult to apply to contexts outside its initial construction process due to its high probability of cascading failure. The hub-and-spoke network needs to be converted into a distributed network (i.e. decreasing γ), analogous to the object representation of the Tornado (Mowat, 2010; Sfard, 1991, 2003; Sfard & Linchevski, 1994). Reification, which Sfard (2003) claimed to be "the driving force behind the incessant growth of knowledge" (p. 359), could therefore be represented by BAM as a decrease in γ of the network representation of the concept.

Although BAM was used by Mowat (2010), other formal scientific approaches could also be leveraged to demonstrate TT. For instance, Ferrer-i-Cancho (2005, 2006) established an information-theoretic approach to network analysis. Network construction is dependent on a fundamental tradeoff between "maximizing information transfer and saving the cost of [construction]" (Borge-Holthoefer & Arenas, 2010, p. 1285). The tradeoff is represented via the energy function $\Omega = \lambda I(N; C) - (1-\lambda)H(N)$, where $I(N; C)$ represents the amount of information intrinsic to concept C transferred via its network representation N, and $H(N)$ represents the entropy, or complexity, of the network itself. The key parameter is $\lambda \in [0,1]$, which represents the nature of the network: $\lambda = 1$ represents a fully faithful representation of the concept at the expense of ease of network construction, and λ close to 0 represents an easy-to-construct network at the expense of faithfulness to the intrinsic structure of the concept (Borge-Holthoefer & Arenas, 2010; Ferrer-i-Cancho, 2005, 2006; Ferrer-i-Cancho & Solé, 2003). Lower λ values correspond to higher γ values in BAM, a hub-and-spoke network, and a process representation of a concept. Higher λ values correspond to lower γ values in BAM, a distributed network, and an object representation of a concept. Thus, reification can be represented in Ferrer-i-Cancho's framework as an increase in λ value. Using techniques developed in information theory (Krippendorff, 1986; Shannon, 1948), it is possible to extract values for $I(N; C)$ and $H(N)$ and demonstrate reification via variation in λ values.

Conclusions and Implications for Future Research

The frameworks developed in this report provide a starting point for researchers seeking to study SOM. Complexity theory was proposed as a high-level paradigm relevant to studying SOM because post-positivism and interpretivism places observations and interpretations external to the structural components of mathematics at the forefront of research (Heath & Bleiler-Baxter, 2025; Stinson & Walshaw, 2017). The need to focus on internal structure rather than external observations and interpretations could be accomplished through complexity theory and formal science. The Tornado Theory, as proposed in this report, represents one approach to ground research on SOM based on both formal scientific frameworks (Borge-Holthoefer & Arenas, 2010) and existing mathematics education literature (Sfard, 1991, 2003).

Future research should develop and apply sub-theories of the TT such as BAM and Ferrer-i-Cancho's framework to analytically uncover topological and structural relationships between various mathematical concepts and domains. To remain faithful to complexity theory, all steps of the research process should reduce the impact of external observations and interpretations. The sub-theories should be based on physically necessary mathematical relationships such as the principles of network theory and information theory. For example, interviews and surveys may not be appropriate, as these methodologies focus on participants' interpretation of mathematical structure rather than mathematical structure itself, even if the participants are experts. Rather, researchers should use mathematical relationships to analyze the abstractions provided by these sub-theories to arrive at conclusions about SOM. Finally, frameworks based on other formal scientific theories such as control theory and game theory should be leveraged to arrive at a holistic view of SOM.

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Conceptualizing Culturally Meaningful Participation in Undergraduate Mathematics

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Participation in undergraduate mathematics courses is viewed as an important metric for assessing student learning and motivation, especially in active learning classrooms. However, engagement has been understood through a narrow lens, which overlooks the way students—especially students of color—engage with mathematics courses and positions them as disengaged. In this theoretical report, we introduce the idea of culturally meaningful participation in undergraduate mathematics to capture the organic ways that students of color draw upon cultural resources to create spaces for engagement in or with their mathematics courses. We draw upon Yosso's Community Cultural Wealth Framework and the notion of Ong's STEM counterspace to support our theorizing. We argue that CME recognizes the assets that students bring to support their engagement with mathematics and also illuminates how these assets are drawn upon. We argue that CME has the potential to support broader anti-deficit efforts in undergraduate mathematics education.

Keywords: participation, culture, community cultural wealth, language, counterspaces

> En unas pocas centurias*, the future will belong to* the mestiza*. Because the future depends on the breaking down of paradigms, it depends on the straddling of two or more cultures. By creating a new* mythos-- *that is, a change in the way we perceive reality, the way we see ourselves, and the ways we behave*-- la mestiza *creates a new consciousness* (Anzaldúa, 1987).

The undergraduate mathematics education landscape has changed significantly over the last 30 years. Driven by social progress and university commitments to DEI, mathematics classrooms now serve a more racially and culturally diverse student body. However, racially minoritized students remain underrepresented in STEM education (National Science Foundation, 2016), persist in STEM at lower rates (Whitcomb & Singh, 2021), and often experience inequitable learning experiences (Eddy et al., 2015). These patterns underscore the ongoing need for equity and inclusion efforts in both research and practice.

While there have been important efforts to improve access in undergraduate mathematics (e.g., including equity in IBL, Laursen & Rasmusen, 2019), as a field, we still struggle to understand how to center identity and culture in these efforts. Often, culture is an integral part of students' identities, shaping how they interpret the world and interact within it. As the quote above reminds us, students' abilities to straddle multiple cultural worlds and transform dominant narratives can foster new ways of imagining participation in mathematics education.

In this paper, we introduce a theoretical perspective for understanding student participation through a culturally responsive lens. This perspective, which we call c*ulturally meaningful participation,* counters deficit narratives about students of color by highlighting the cultural resources that they draw upon to foster meaningful mathematical engagement in undergraduate mathematics courses. It acknowledges that mathematics classrooms are racialized and gendered spaces, which inevitably shape how students navigate their engagement. As such, we argue that culturally meaningful participation needs to account for ways students engage with mathematics and their course outside of the formal classroom space. To build this idea, we first discuss how student participation has been conceptualized in undergraduate mathematics.

Undergraduate Mathematical Participation

Classroom Participation

The shift from lecturing to active learning has significantly reshaped the undergraduate mathematics classroom experience and how instructors and researchers perceive student engagement through different ways of participating. In lecture-based classrooms, participation generally involves sitting, listening, taking notes, and occasionally asking or answering questions. In contrast, in more active classrooms, students are expected to work collaboratively, discuss ideas with peers, and explain their thinking to the class. Active learning classrooms give more opportunities for students to speak and share their ideas, and largely this type of visible forms of classroom participation have become the main indicators (and measurements) of student engagement, learning, and motivation.

The shift in perspective on participation is consistent with the social turn in mathematics education research, which reflects an inherent change in the conceptualization of learning towards "the person-acting-in-*social- practice,* not person or their knowing on their own" (Lerman, 2000, p. 25; see also learning as participation metaphor in Sfard, 1998). Studies have examined participation as an important metric related to equity (e.g., Ernest et al., 2019; Reinholz et al., 2022). This type of work importantly highlights systemic inequities in the classroom. However, when we view their findings alongside common reactions from teachers related to increasing participation, they begin to tell a story. For example, teachers in Rios & Lozano (2025) shared comments such as, "It is hard to get them out of their comfort zone /…/ it's like an elastic band, you can try to pull them out but sometimes they want to snap back," or "At risk" students have a "tendency to hide in the background." Together they become a part of a broader discourse concerning the underparticipation of students of color in mathematics classrooms.

This deficit discourse often manifests in the classroom as deficit positioning of students who "do not" participate. For example, Rios (2024) specifically documents how a multilingual student from Guatemala felt that her teacher positioned her as not engaged or interested because she was often quiet during class time, when in fact it was a coping mechanism for the linguistic microaggressions she experienced by her group members. Other studies have also documented that active participation in the classroom is often more challenging for students of color, given the racialized and gendered nature of our society and its role in classroom dynamics. For example, Battey et al. (2022) reported that "withdrawing participation" was a common response among their participants in managing "cognitive labor from discouraging events" (p. 104).

We argue that the reliance on visible and normative forms of participation creates space for deficit interpretations of students of color's participation in undergraduate mathematics. Along with the social turn in mathematics education, deficit narratives might evolve from fixating on deficiencies in students' mathematical sensemaking (Adiredja, 2019) to perceptions of underrepresented students' "struggle" with classroom participation. As the field continues to work on understanding ways to make the undergraduate classroom more inclusive (e.g., Downing & McCoy, 2021; Jett, 2013), we now consider other spaces that might be more culturally affirming.

Participation Outside of the Classroom

Much of the learning in an undergraduate mathematics course occurs outside of the classroom, especially with lecture-based courses. Traditionally, built-in support at universities for learning outside of the classroom consists of attending office hours and going to mathematics

tutoring centers. So, another way to assess students' participation is through their use of these support services. Research has documented the (racialized) nature of attending office hours for many students of color as related to confirming negative stereotypes (Dauber & Hsu, 2025; McClain, 2014). Thus, relying on such a normative form of participation, again, leads to deficit positioning of students of color.

For this reason, research has begun exploring the importance of STEM *counterspaces* in higher education (Ong et al., 2018). Counterspaces are spaces that lie outside of traditional course structures but provide students of color with environments where their academic and personal identities align and are affirmed with others who share their identity. Examples of such counterspaces include cultural resource centers for different social identities (e.g., African American Student Center) and national organizations like SACNAS (Society for the Advancement of Chicanos/Hispanics and Native Americans in Science).

While these spaces have been shown to support identity development (e.g., Leyva, 2021; McGee & Martin, 2011), they tend to exist outside of the academic departments. A notable exception is the work of Uri Treisman (Treisman, 1992). Treisman observed the cultural practice of Chinese students forming study groups with other Chinese students, where they worked on mathematics collaboratively and built community with upper classmen. This can be seen as an example of a culturally affirming space created by students where they engaged in mathematical learning.

Building on this idea, we are curious of other ways that students of color engage in practices that support their learning that are culturally meaningful for them. In particular, if we can identify cultural resources that students use to support participation in mathematics, then we can begin to challenge deficit discourse as related to underparticipation of students of color in mathematics. To help us conceptualize what other cultural resources students might use, we draw upon Yosso's (2005) Community Cultural Wealth (CCW) framework from higher education.

Community Cultural Wealth (CCW)

The Community Cultural Wealth (CCW) framework (Yosso, 2005) challenges deficit perspectives about the achievement and college readiness of students of color (Valencia, 2010). CCW focuses on the resources and capitals nurtured by their communities. Yosso argues that cultural wealth are vital resources that help students of color successfully navigate higher education. CCW identifies seven capitals: 1) aspirational capital, the ability to maintain hopes and dreams in the face of oppression; 2) linguistic capital, the intellectual and social skills from understanding of multiple languages; 3) familial capital, "cultural knowledge nurtured among familia (kin) that carry a sense of community history, memory and cultural intuition" (p. 79); 4) social capital, resources for and from the community; 5) navigational capital, skills to navigate spaces that were not created for them; 6) resistant capital, knowledge and skills learned from opposing inequalities and injustices; and 7) spiritual capital, "a set of resources and skills rooted in a spiritual connection to a reality greater than oneself" (Huber, 2009, p. 721).

A few studies have applied this framework in the context of undergraduate STEM education (Dika et. al, 2018; Peralta et al., 2013; Luna & Martinez, 2013; Samuelson & Litzler, 2016). These studies have primarily focused on identifying different forms of capital—primarily familial and aspirational—rooted in students' communities as motivation for students in pursuing and persisting in their education (e.g., Dika et al., 2018; Samuelson & Litzler, 2016). In contrast, Rios-Aguilar et al. (2011) discuss the notion of *using* cultural capital, that is, the "the mobilization of cultural resources" (p. 170), in understanding the experiences of students of color in STEM. The authors highlighted the important distinction between the possession of cultural

capital vs. the activation of cultural capital and what contexts allow students to activate these resources. To contribute to this work, we are interested in how students draw upon these forms of capital in their mathematical practice (not just that they have them), and how these capitals impact students' engagement beyond serving solely as motivation.

Defining and Illustrating Culturally Meaningful Participation

We assert that students of color possess important culturally nourished resources that they can draw upon to support their mathematical practice at the undergraduate level. The CCW framework identifies such resources which center language, identity, community, connection, and resistance. We call this *culturally meaningful participation in mathematics*—the practice by which students participate in important mathematical activities as part of their mathematics courses as a result of drawing upon their community cultural wealth. We argue this type of practice allows students of color to creatively navigate these spaces, without leaving their identities, communities, and values behind for the sake of STEM assimilation.

To illustrate what culturally meaningful participation in undergraduate mathematics (and STEM) courses might look like by drawing from two datasets: 1) interviews with five undergraduate students who self-identified as Hispanic/Mexican and participated in a summer calculus bridge program (previously analyzed and discussed in Adiredja & Rios, 2019), and 2) interviews with 28 multilingual, undergraduate students about their experiences in precalculus and calculus courses that use active learning (discussed in Rios, 2024). Both datasets are rich in examples of how students found avenues to engage in and access supports in their courses, by leveraging what we recognize as community cultural wealth. We start with two short examples of social capital before delving deeper into examples of participation that stem from students' linguistic and resistance capital.

Illustrating social capital, Marco was a student studying STEM who identified as Hispanic and grew up in a small, "tight-knit" border town. When Marco started taking STEM courses at the university, he already had an established support system of friends and cousins from his home town who were also pursuing STEM. He described going to their dorm late at night to get help on an assignment, and he jokingly added that they were also "cooking up some chicken" for him. This form of social capital, rooted in his hometown community, became his main way of participating in his courses. In another example, Ahmad, a student who arrived in the U.S. as a refugee from Iraq, sought support from his father when he was struggling in his calculus course. His father then connected him with his own former high school mathematics teacher, who was still in Iraq, and provided Ahmad with tutoring over video chat. This became an important (social) resource for Ahmad and a way that he was engaged in the course, which was facilitated through his father (familial capital). This was a different way of participating than attending office hours that could have been easily overlooked by his instructor. Ahmad was drawing on a valuable (family-based) resource to support his learning and engagement in calculus.

"My time to shine"

To illustrate culturally meaningful participation using resistance capital, we share Lesli's experiences in calculus. Lesli identified as half Mexican and half Puerto Rican, and was a determined and accomplished undergraduate student studying chemistry. In her Calculus II course, Lesli navigated what were likely racialized and gendered interactions and to integrate into a community by joining in a high-status study group. This study group worked together on calculus in the library and were not initially accepting of Lesli:

> I walked in and they were working on some questions, and everyone was kind of skeptical, like, 'Oh, this is a new girl' /.../ I was having issues with a similar type of problem that they were having. So I asked them questions on how they did them and what not. And I could sense already that they were like, "Oh no, she doesn't know enough."

Lesli sensed their gendered skepticism, but resisted:

> So then we worked on another question, and that question for them was super hard. No one could get it, out of all the people who were there, probably six guys /…/ Thankfully that was a question that I knew like off the top of my head because I had spent a good hour working it out by myself earlier that day /.../ I was like, "This is my time to shine!" /.../ After that it was like, I had been accepted.

Lesli summarized the experience of being accepted by stating: "I feel like you kind of have to prove yourself to get in. That's why my mom calls it 'the mafia' because not everyone gets in".

Despite having to prove herself, which we interpret as drawing upon her resistance capital, Lesli attributed much of her success in calculus II to her participation in this study group. She described the members of her study group as her closest friends: "I've been able to find a group where we truly care about each other and about each other's learning because we've developed that kind of friendship."

We learned that Lesli's ability to resist was rooted in her community and cultural upbringing. As a student from the "South side," Lesli shared:

> Just coming from a high school like Central, I know when I was in high school people would talk about, 'Oh she's a central [an under-resourced school] girl who won 2nd place at the national medical spelling bee /…/ She came out of central, from the South side, where she shouldn't have, and all those accomplishments'. So I came from a place like that, and so I feel like my background helped me to be able to handle criticism and skepticism and indifference.

Here, Lesli acknowledged the narrative and lower expectations people placed on her. At the same time she reclaimed this experience by recognizing that being from her community allowed her to develop important tools for handling microaggressions:

> Indifference can pretty much kill you, if you let it kill you. And I think from the interaction that I get with other groups that may look down on me. Beside skepticism, criticism, looking down, I think it's the indifference that hurts the most I'd say. You know we as humans, we have to get over it and we have to just ask questions and you have to force yourself to interact with them /…./ it's just a matter of re-centering yourself and re-wiring yourself to believe that what you're doing is worth the sacrifice and un-comfortableness /…/ In my experiences I have had to make myself notable, like you know, I am here too. I am a thinking human being. I have a brain you can ask me too. I'm here to help you too.

Being able to navigate and ultimately resist these deficit narratives were a part of Lesli's lived experience—something that was meaningful to her and a source of pride. Thus, for Lesli, being able to participate in a high-status study group, and the benefits she gained from it, exemplified a culturally meaningful participation.

"Speak tu propia lengua, no?"

To illustrate how linguistic capital can foster culturally meaningful participation, we share examples from two students: Veronica and Sai. Veronica grew up in Mexico and moved to the United States during high school. As a bilingual STEM student, Spanish was very important to Veronica and she used this cultural capital to find community within her STEM courses.

Veronica described how Spanish helped her form a special connection with other Spanish-speaking students:

> I just came from [Mexico], I still have my Mexican culture, no? So I think language is a big plus for me. Not just academic wise, just conversation you know. You can just talk more easy. You can joke more easy /…/ In upper division [STEM courses], you mostly see Americans and when I see somebody that speaks Spanish too, I get like 'oh wow, you speak spanish? Me too!' and then we just speak Spanish from there on now. And it's kind of really nice to speak your own language, no? Cuz you can say different things.

Veronica explained that when she had a question, the first resource she turned to was these friends because she could explain her question in Spanish. Veronica also described getting help for her STEM courses from a bilingual graduate student, because he could explain in Spanish:

> Last semester I had a lab TA that-- He was Mexican. And then, I would actually go to his office hours because he could explain to me in Spanish, and um, I just think it's a-- I don't know how to explain, you just have a connection.

She often attended his office hours instead of her professors, sharing: "I feel like I ask [professors] things and they get something else for some reason and they give me that answer and I don't really like saying, 'Oh. That's not what I meant'". Instead of utilizing more conventional course supports, Veronica drew on cultural knowledge—Spanish—and the connection through shared language identities to find support systems in her STEM courses.

Similarly, Sai, an international student from India, found a "good support system" in his calculus II class by leveraging his linguistic capital as a Hindi speaker:

> So whenever I see an Indian in my class I message him personally and try to be friends with him and whenever I talk to him I use Hindi and for solving math I use English while talking to him in Hindi. So bilingualism has introduced that concept of familiarity. /…/ I've seen this happen a lot of times, just switching that mode from English to Hindi is the form of connection. We're like, we come from the same background, we share the same you know, history, same language and that ends up, you know, bonding us together.

Additionally, Sai intentionally enrolled in a section of calculus II taught by an Indian graduate student, because he knew he could connect with him and would be able to ask questions in Hindi. Sai was one of the few students that regularly attended office hours, which he attributes to feeling "more of a connection" and "more comfortable" with a Hindi-speaking instructor.

Both Veronica and Sai's course experiences offer another illustration for culturally meaningful participation. They both used their linguistic capital to find community and spaces where they could participate in mathematics in a way that affirmed their linguistic (and cultural) identities.

Discussion and Implications

In this paper, we argue that a narrow, normative lens on assessing participation misses the humanistic dimension of how each student navigates who they are in their mathematics courses. It also neglects the important role of culture and identity in how students choose to participate. To capture this, we propose the notion of *culturally meaningful participation* to highlight that students of color bring meaningful cultural resources they draw upon to facilitate participation in undergraduate mathematics courses. As the examples we share illustrate, participation often occurred through unofficial channels and spaces outside of the classroom (e.g., study groups, lab TA's office hours, tutoring with a family friend). These examples serve as counterstories capturing the dedication students have for their education and the rich ways they participate in

their mathematics courses. By leveraging their resources to participate, students do so in a way that, from our perspective, affirms their culture, community, family, and lived experiences.

Our conceptualization of culturally meaningful participation in undergraduate mathematics contributes to broader anti-deficit effort in undergraduate mathematics (Adiredja, 2019; Peck, 2021). As we noted, relying on traditional forms of assessments of participation, whether it is through verbal participation in the classroom or attending office hours contribute to deficit discourse about students of color's participation in mathematics. Our work expands the definition of participation beyond traditional measures, which recognizes students' cultural resources they use to participate in mathematics, and also disrupts such deficit discourse. Theorizing about and illustrating culturally meaningful participation has the potential to provide a lens for researchers and practitioners to think about equitable participation in mathematics.

We hope that empirical studies can begin to further document and track the kinds of participation we introduced in this paper. In teaching, addressing issues of underrepresentation in classroom participation, we can go beyond the fixation of getting students of color to speak more. Instead, we can begin discussing the potential of grouping students based on a shared cultural identity, helping students connect with others and form study groups outside of the classroom, connect students with TAs or other instructors who share an important identity, or even noticing and celebrating acts of resistance and perseverance in the classroom. We argue that centering cultural responsiveness in discussions of participation can help researchers, instructors—and mathematics departments more broadly—recognize the resources students bring to their mathematics courses and reimagine learning experiences that mirror the stories of participation rooted in culture that students engaged in and valued.

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Preliminary Reports

Epistemological Frameworks of Advanced Math Majors: A Case Study

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Although students' epistemological frameworks have been extensively studied in introductory undergraduate courses and lecture-based advanced mathematics courses, there has been less work on students' epistemological frameworks in advanced student-centered courses. The purpose of this case study is to examine the beliefs held about mathematics learning of nine students in a flipped real analysis course using Schommer et al.'s (1992) mathematical epistemological framework as a theoretical perspective. The findings suggest that although there were minimal differences in beliefs when grouping students by major or final course grade; overall, the typical participant's epistemological framework was grounded in effort, a desire for understanding, and belief that there must be at least some reliance on their instructor and peers to provide instruction and support their learning in the course.

Keywords: epistemological framework, flipped classroom, real analysis

An epistemological framework is the set of assumptions a learner holds that taken as a whole define how they believe knowledge is understood. Students' epistemological frameworks influence how they perceive the quality of a class, their motivation, and even what content they focus on learning in a course (Ji et al., 2021). Although students' epistemological frameworks have been extensively studied in introductory classes (e.g. Krupnik et al., 2018), there has been less work on students' epistemological frameworks in advanced student-centered courses. If we better understood what advanced mathematics students believed about mathematics and mathematics learning, we could help them adopt more productive beliefs and more effectively tailor our pedagogies in advanced student-centered mathematics courses. Prior literature on lecture and student beliefs suggests that although mathematics students vary widely in their beliefs of what understanding is and how to be a successful mathematics student, students agree lecture is key to learning, which makes student-centered courses with little lecture challenging to conduct (Zolt et al., 2023). Although lecture is a common instructional method, formal understanding of what is written vs informal understanding of what is said vary widely from class to class (Umgelter & Geisler, 2023). Students tend to struggle to record or recall the informal content of lectures, as it is rarely written down (Fukawa-Connelly et al., 2017).

Student beliefs about mathematics and mathematics learning affect their motivations, behaviors, and academic success (Limeri et al., 2023). However, student beliefs about learning have been primarily studied in early undergraduate mathematics students (Ji et al., 2021), even though students' interpretation of instruction is filtered through their epistemological framework and beliefs about learning, even in advanced coursework (Krupnik et al., 2017). Further, there have been calls for more research on beliefs and affect in advanced mathematics students (Melhuish et al., 2022). The purpose of this study was to investigate the epistemological frameworks of real analysis students in a flipped real analysis course, and was guided by the research question: what is the typical epistemological framework of an undergraduate enrolled in an advanced student-centered mathematics course?

Theoretical Perspective

For this qualitative study, we chose Schommer et al. (1992) as the theoretical perspective for students' epistemological frameworks. There are ten implicit assumptions in Schommer et al. (1992); although these assumptions were developed for students reading mathematics textbooks, we chose this theoretical perspective because it encapsulated all of the beliefs that have emerged as themes in prior epistemological framework literature (e.g. Limeri et al., 2023). These assumptions are: (1) learning is supposed to be quick (2) you can't learn how to learn (3) concentrated effort is a waste of time (4) success is unrelated to effort (5) ambiguity is something to be avoided (6) single answers are preferred (7) external sources authority are the preferred way to check answers (8) the ability to learn is innate (9) integration of concepts makes learning harder (10) learning typically occurs in the first attempt. We based the interview questions on these assumptions, reverse coding half of the questions.

Methods

Participants were recruited from a flipped real analysis class with extensive daily activities and class discussions at a rural research university in the southern United States. Nine students consented to a 45-minute semi-structured interview about their epistemological frameworks and perceptions of the course. All participants were mathematics majors or second majors; a second math major is larger than a math minor but contains fewer required courses and credits than a full major. Eight participants were male or male-presenting non-binary. Seven participants self-identified as white and two self-identified as Hispanic. At the time of the interview, four participants were earning A's, two were earning B's, and three were earning a C or below. Real Analysis requires a prior course in proofs as a prerequisite, so it was at least the second proof-based mathematics class for all participants, though it was the first student-centered proof-based class for most participants.

After the interviews were transcribed, the authors coded each question for evidence of one of the ten beliefs in the theoretical perspective for each individual student. The initial coding was then discussed until all differences were reconciled. Although we tried comparing participants based on achievement level in the course and whether their primary major was math or physics, there were no clear differences in belief patterns when grouping students using these variables and we did not have enough female participants to compare by gender. Instead, student beliefs were classified as universal (all nine participants held the same belief), common (six to eight participants held the same belief) or uncommon (five or fewer participants held the same belief).

Findings

There were three universal beliefs, five common beliefs, and two uncommon beliefs held by participants. Overall, the beliefs held by participants were consistent across participants, and generally positive. However, our sample was not particularly representative of a typical real analysis course, as two-thirds of our participants were taking this class as an advanced mathematics elective for a second major rather than as a requirement for a primary major.

Universal Beliefs

Although there were no clear epistemological framework patterns when considering students by achievement, there were beliefs held by all nine participants: it is possible to learn how to learn, effort aids success, and connecting concepts is important in mathematics. Maya, a math major and prospective high school teacher, believed it was possible to learn how to learn:

> Yeah. I think I would have to say yes [you can learn how to learn], just because the whole metacognition part of the taxonomy and the exercises you find to help you get to that part of that pyramid. I'm like, yeah, I could see how you would need to learn how to learn. We see this because not everybody learns in the exact same way.

Leon, a physics and math double major, gave a typical response when asked if mathematics can be learned:

> I think it's possible to learn it [mathematics]. There are people who are going to enjoy it and it'll become easier, but generally, I think it can be learned. I sure hope it can be learned. I'm tutoring people to do it.

The second belief about mathematics held by all participants was that effort aids in their success in both mathematics in general and real analysis in particular. Luca, a physics major with a second major in math, explained that without effort, he would likely not be passing, "Probably, I would say, if I didn't put forth effort I would be failing right now." Carter agreed, adding, "Yes, very much so. Highly correlated. Yeah, sometimes it's slower than other times. Sometimes it's very grinding. You've already experienced some of that with my proofs."

The third universal belief about mathematics was that making connections between concepts was very important in learning mathematics. Albert, a physics major with a second major in mathematics, gave a typical answer when he was asked how important was it to connect concepts in mathematics in general and real analysis specifically:

> Very, because then I'll know how to use them in other problems that are similar to this. Like you said earlier about how a problem could have multiple answers. I say yes, because I remember that we could solve a certain problem using different methods, and with how those methods tell us, like, the result of those methods could also tell us the same answer, but in different forms. That's what I say.

Taylor, also a math and physics major, agreed, emphasizing the importance of the building connections between his two chosen majors:

> I think it's very important. As a person who's taking and has taken a lot of physics classes, taking the math classes makes physics a lot easier and sometimes there'll be something I learned in math class, and then we talk about it in physics on a deeper level, and it's using calculus or analysis that totally makes the physical concept come quicker.

Common Beliefs

Five of the ten components of the epistemological framework were classified as common beliefs, where six to eight of the participants held the same view of mathematics in general and real analysis in particular. The first was that repetition is important to learning, a belief eight participants held. Maya, a mathematics major with an emphasis in secondary education, explained that repetition was learning:

> That's how I learn, period. A lot of the stuff is me looking at an example of how to do it, doing it, and kind of figuring out how you get from step one to B. And a lot of it is just doing it over and over again until I understand the thought process behind why I need to do it. And I'm like, okay, yeah, I can replicate it. And even if the question is not exactly like the one I just did, I'll still be able to figure it out.

Wade, a physics major with a second major in math, was the only one who disagreed with the consensus:

> [Repetition is] Not so important. I think, I don't necessarily find myself going back on all notes and rereading them. I find it more beneficial to try to use those ideas and things in different contexts. I find that to be the best strategy for learning for me.

Two common beliefs had seven participants holding a consensus belief: It is not normal to learn real analysis quickly and instructor quality is important. Taylor, a math major who switched to a physics major during real analysis, explained that he found the proof-based class challenging, and did not agree that it was normal to learn real analysis quickly: "Well, I don't know [if it's normal to learn real analysis quickly]. I've only taken one Real Analysis class. It feels like for me it's faster than what I was ready for. I'll say that." The two students who believed real analysis was relatively quick to learn were both high achieving students; as Jeremy, a mathematics major, explained: "Um, if it's just what's covered in this class, I don't think it'd be too hard to learn it at the same rate as we go through it in class on our own." The other belief held by seven students was that instructor quality was important to mathematics learning. Wade provided a typical explanation of why an instructor matters so much in advanced mathematics:

> It's very important. I think it's important in keeping passionate about a subject. Recently, I've done a lot of self-study, so I suppose in that case, quality of the book that you choose is important to keep you motivated. I think having a good teacher can inspire you to want to do more on the subject, which then is probably the best thing that they could do for you. And that's, I think, a good way.

Carter, a mathematics major, disagreed. He said that although instructor quality is nice to have, learning is possible even without an instructor:

> Again, ideally, high quality instruction is great, right? It only improves what you get. You could work with a lot less, that's for sure. You could work with nothing. I mean, these things weren't around. These weren't originally ideas. Eventually, they came about.

There were two beliefs with a six-student consensus: effort helps you understand things more and problems with multiple solutions are in general a positive thing. Interestingly, all nine students classified a proof as a problem with multiple solutions. Wade gave a typical answer, explaining that pushing past the initial urge to quit and focusing on a problem he did not immediately know how to solve was generally successful:

> Yeah. You know, a lot of problems, you look at them and think you want to just give up right away, and I think if you force yourself to sit down, like consider for however long, you're at 20 minutes, whatever, I think that some progress is better than no progress. With some progress, you do understand things better. It usually is.

Three participants partially disagreed with the consensus, stating that effort only sometimes helps, and occasionally there is no amount of effort that will move you forward in a problem. Luca, a physics major with a mathematics second major, when asked if trying harder helped him understand better, replied, "Sometimes, sometimes I just get more confused."

Problems with multiple solutions were generally seen as at least an interesting phenomenon. Carter, a mathematics major, explained, "I mean, those are the exciting ones, right? I think that's enough said." The three students who disagreed with Carter tended to be among the lowest-achieving participants. As Maya explained:

> I don't like them. Because I guess, I say as I'm a mathematics major, it's like math is very logical, and so it should only have one specific answer. And so you really cause problems. I'm like, I know this is technically right. I don't like it.

Uncommon Beliefs

The two beliefs where there was little consensus among participants were whether or not it was normal to learn mathematics quickly in general (4/9 yes, 4/9 depends on the context, and 1/9 no), and how to verify correctness on solutions (4/9 external sources such as peers or instructor, 3/9 internal sources such as intuition and logic, 2/9 a combination of internal and external sources). Lewis, a physics major with a second major in mathematics, felt that mathematics was generally something he learned quickly, "It usually doesn't take too long for me to figure out concepts or I don't feel like anything goes too fast for me to follow usually." Jeremy felt that the speed of learning was context dependent, "Like what is the math and what is quick? So it's very context-dependent." Wade, was the only student who felt that it was not normal to learn math quickly, as he explained, "No, I don't think so. I think it depends what you want to learn, I guess. If you want to learn it well, I think you need to spend a lot of time digesting topics."

Albert used external sources to verify his work, "[I use] My peers, the book, and the instructor [to verify my work]." Taylor trusted his intuition that his first answer was correct, "Often I don't check that it's right. I think I'm pretty careful in my calculations. And so, usually I'll just go with the first answer that I get, and trust my gut." Carter, one of the participants that used both internal and external sources of authority to verify his answers, discussed the value of office hours to fine tune answers, but felt that the most valuable way to check his answers was internal consistency:

> Well, I'll make sure it's consistent with other things, so... It's hard to think about this specifically, but pull any other facts, I suppose, and make sure that it doesn't lead to inconsistency. Check for consistency.

Discussion

Overall, the typical participant's epistemological framework was grounded in effort, a desire for understanding, and belief that there must be at least some reliance on their instructor and peers to provide instruction and support their learning in the course. These beliefs were remarkably consistent across participants, which suggests in part that there may be a set of adaptive beliefs students must adopt to be successful in advanced mathematics courses; it would be interesting to compare students' epistemological frameworks in an introductory proofs course to these participants. Our findings align with prior work on undergraduates' epistemological framework (e.g. Limeri et al., 2023; Krupnik et al., 2017) in that most students hold fairly productive beliefs about learning, but are still very reliant on external authority as a source of conviction for their work.

However, there are several limitations to this inquiry. The participant pool was overwhelmingly white, male, and majored in physics with a second major in math, so they may not have been particularly representative of a typical population of real analysis students since this class was an elective and not required for 6/9 participants. Despite these challenges, the consistency of beliefs the participants in this study hold provide a foil for future research on traditionally underrepresented groups in advanced mathematics classes. Future research on advanced students' epistemological frameworks should include a variety of advanced math classes as well as more diverse participant pools.

For the RUME conference, we hope to receive feedback from our peers about the theoretical framework of our study. What alternative framings might provide additional insight on the data? How does the context of a student-centered class add to the findings?

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Exploring Features of Graphs Across STEM Fields: An Emerging Theoretical Framework

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Graphs are essential across STEM fields, yet students often lack opportunities to develop graphing competencies. One challenge is that the ways graphs are used vary across disciplines. This study develops a framework for characterizing graphs in undergraduate STEM contexts by examining features such as coordinate systems, reference frames, and the objects of interpretation. Preliminary analyses of STEM resources highlight differences—such as spatial versus quantitative coordinate systems—and similarities that can inform cross-disciplinary instruction. Our goal is to support more coherent approaches to graphing instruction, preparing students to interpret and construct graphs in ways that align with disciplinary practices.

Keywords: Graphs; STEM; Undergraduate Education

Graphical representations are common in STEM fields for modeling, communicating, and analyzing phenomena. Graphs are tools used for understanding and communicating scientific phenomena (Larkin & Simon, 1987; Meira, 1995). Researchers (Knorr-Cetina & Amann, 1990; Tabachneck-Schijf et al., 1997) have shown how scientists and engineers require graphs to complete certain tasks. Relatedly, graphs are ubiquitous in STEM textbooks and practitioner journals (e.g., Lemke, 1998; Roth et al., 1999). Hence, supporting students in developing meanings for graphs is critical for supporting the next generation of scientists.

Despite the importance of graphs, results from research (e.g., Lai et al., 2016; Ruf et al., 2024) and assessments (e.g., Nation's Report Card, 2005) indicate U.S. students are not being provided sufficient opportunities to develop graphing meanings relevant to STEM. One possible explanation for this is that there are fundamental differences in the ways graphs are created, used, or interpreted within and across STEM fields (e.g., Lee & Guajardo, 2023; Paoletti et al., 2020) and that these differences are uncategorized or under-described in STEM education. Our primary goal in this study is to develop frameworks that explain the various ways graphs are used and expected to be interpreted across undergraduate STEM fields. To accomplish this goal, we intend to address the question: *How can we characterize graphs across STEM fields?* In this preliminary report, we characterize a graph based on the types and number of quantities represented, the underlying coordinate systems, and the way information is intended to be represented. By creating such a framework, we intend to develop and support a more coherent, connected, and holistic approach towards supporting students' developing graphing meanings.

Examining Several Graphs Across STEM Fields

We use the six graphs in Figure 1 to exemplify differences in graphs across STEM fields. We acknowledge that interpreting aspects of these graphs requires domain-specific knowledge (i.e., knowledge about the particular quantities in the graph), as such knowledge is important for

interpreting graphs (e.g., Altindis et al., 2024; Roth et al., 1999; Roth & Bowen, 2003; Zimmerman et al., 2023). Our goal is not to present a complete understanding of each graph but instead to present some high-level differences across the graphs. Additionally, we highlight similarities across these graphs that could be leveraged in undergraduate STEM courses.

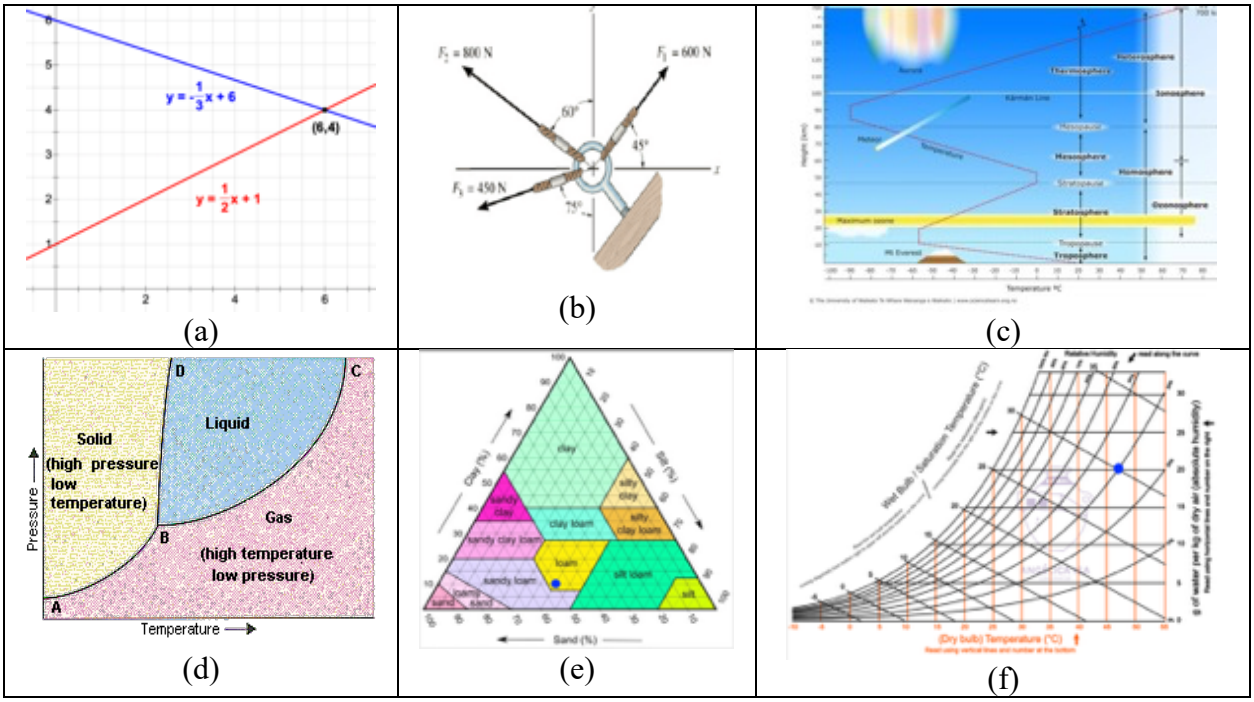

Figure 1: Graphs from a (a) Mathematics, (b) Engineering Statics, (c) Astronomy, (d) Chemistry, (e) Soil Science, and (f) Meteorology.

Figure 1a represents a solution to a system of equations, typical in an algebra class. Figure 1b is a graph from an Engineering Statics textbook (Hibbeler, 2013) used to mathematize the direction and net force exerted on a screw eye by ropes pulling in different directions with different forces. Figure 1c, from a unit on the Atmosphere (University of Waikato Te Whare Wananga o Waikato, 2025), uses the vertical height axis to classify the atmosphere based on altitude. Additionally, the dotted line shows the relationship between height and temperature in different atmospheric regions (e.g., in the troposphere, as height increases, temperature decreases). Figure 1d shows a phase diagram used in Chemistry (Bodner Group, 2025) to represent the effect of pressure and temperature on a hypothetical substance. The graph is divided into distinct areas, representing solid, liquid, and gaseous states of the substance. A person can determine the state given any (pressure, temperature) combination.

Figure 1e shows a soil texture triangle (Hennion, 2025) from the Environmental Science field relating the percent of soil comprised of clay, silt, and sand. Each point totals to 100% of the soil composition. For example, the blue point corresponds to a loam soil that is 10% clay (moving from the point left to the clay axis), 50% sand (moving from the point down-and-to-the-right along the lines to the horizontal sand axis), and 40% silt (moving from the point up-and-right to the diagonal silt axis). Finally, Figure 1f shows a psychometric diagram from a meteorology context (such diagrams are also used in medicine via growth charts for children and in ecology via graphs with isoclines). The graph is intended to show the relationship between temperature, absolute humidity, and relative humidity. For example, the blue point shows that if the

temperature is about 47 degrees Celsius (moving from the point down to the horizontal axis) and there are 20 grams of water per kg of dry air (moving from the point right to the vertical axis), then the relative humidity is at 30% (moving up-and-to-the-right along the curve).

At first blush, there are major differences across each of these graphs, given the different domain-specific information they are intending to convey. However, using our preliminary set of frameworks, we can identify commonalities. First, most graphs rely on a Cartesian coordinate system (all except Figure 1e). Second, some graphs are used to represent a physical space or phenomena (Figure 1b, the vertical axis of Figure 1c), whereas other graphs are representing a space distinct from physical space (Figures 1d, e, f). Some graphs are explicitly coordinating two quantities (Figure 1a, b, c, d), whereas others are explicitly coordinating 3 quantities (Figures 1e, f). Further, we note the graphs in Figures 1d and 1e have regions representing distinct states.

We also note curriculum developers likely intend these graphs to serve different purposes. For instance, some graphs are likely intended to help students understand or visualize some aspect of a phenomenon (Figure 1b, 1c). Several graphs could be intended to make inferences about a single moment represented by a point in the graph (e.g., Figures 1a, c, d, e, f) or could be used to make inferences about how two quantities change together in context (e.g., the red line in Figure 1c, following curves in Figure 1f). Hence, while the graphs may seem extremely different at first glance, there are features of these graphs that can be categorized both by using existing theory and by developing new theory. After providing a brief overview of relevant research, we describe our preliminary analysis. Throughout, we return to these six graphs to highlight ways we are leveraging theory to characterize similar and distinct features across these six graphs.

Background, Analysis, and Theoretical Constructs Explaining Aspects of Graphs

Researchers have shown that students of all ages experience persistent difficulties as they engage in creating and interpreting graphs in their STEM courses (e.g., Glazer, 2011; Lai et al., 2016; Potgieter et al., 2008; Ruf et al., 2024). We make two observations from this body of research. First, many of these studies have identified alternative conceptions of graphing and provided insights into varying difficulties students might experience in producing graphs, resulting largely in deficit accounts of students' mathematics (e.g., Leinhardt et al., 1990; Mevarech & Kramarsky, 1997; Moritz, 2003; Ruf et al., 2024). Second, most of this research examined students' interpretations of pre-constructed graphs or students' graph constructions on pre-constructed axes; these researchers did not pay particular attention to the ways students understood the underlying coordinate system (CS) and/or the quantities represented by the CS. Our ongoing research exploring students' developing meanings for graphs is novel relative to this work, as we have begun to create theoretical constructs to help explain nuances in students' conceptions for CSs and the reference frames (RFs) they intend to represent. These constructs can be divided into categories based on (a) the types of CS, (b) the number and types of RFs, and (c) the object(s) students are expected to interpret within the graph space.

Preliminary Analysis

To explore our research question, we have begun preliminary analysis of several STEM textbooks and resources. To analyze the graphs, we conducted a content analysis, which "takes texts and analyses, reduces and interrogates them into summary form through the use of both pre-existing categories and emergent themes in order to generate or test a theory" (Cohen et al. 2007, p. 476). Consistent with this methodology, we used pre-established frameworks, as described above, to categorize, code, and analyze graphical representations in STEM textbooks and resources. Following the main steps of content analysis, we defined a sample of texts, the

unit of analysis, and the categories and codes for analysis. We then coded and categorized data. We acknowledge that codes were attributed to each graphical representation based on *our* interpretations of the author's intention of the graphical representation.

Coordinate Systems

A CS is a representational space an individual uses to systematically coordinate quantities (Thompson, 2011) to organize a phenomenon. This space typically involves coordinating two (Figure 1a–d) or more (Figure 1e–f) quantities. Across STEM fields, there are a variety of commonly used CSs, such as Cartesian (Figure 1a, c, d), polar, and logarithmic CSs.

Depending on the purpose the coordination serves, we have distinguished two types of CSs, *spatial* and *quantitative* (Lee, 2017; Lee et al., 2020). A *spatial CS* is a system of measurements through which an individual organizes or coordinates spatial elements in conceived or experienced space. Figure 1b is an example of a 2D spatial CS intended to mathematize the physical space of the screw eye, and Figure 1c contains an example of a 1D spatial CS if we only attend to the different categorizations of atmosphere by height. In contrast, a student may abstract quantities from a phenomenon and coordinate them in some new representational space, different from the space in which the quantities were originally conceived. We call this new structure, used to coordinate abstracted quantities, a *quantitative CS*. Figure 1d–f are all examples of quantitative CSs, because each represents quantities in a space that is different from the physical space in which the phenomenon is occurring.

We note there are often graphs in math classes that do not clearly fall under either of the two types (e.g., Figure 1a), as there are no obvious situational quantities involved. However, we have found the distinction between quantitative and spatial coordinate systems useful in characterizing the types of graphs used in middle school textbooks (Lee & Guajardo, 2023) and STEM textbooks and journals (Paoletti, et al., 2020). Further, this distinction proved explanatory in describing different ways students construct and interpret graphs (Paoletti et al., 2025). We note we have also explored other types of CSs in our work, including polar CSs (Moore, Paoletti, et al. 2013; 2014). Hence, we conjecture these distinctions are likely to be important in exploring different uses of CSs in undergraduate STEM education.

Reference Frames

A CS comprises RFs. RFs are mental structures through which an individual gauges the extent of various attributes of objects or phenomena (Lee, 2017; Levinson, 2003). RFs have been central to studies in spatial cognition (e.g., Levinson, 2003; Taylor & Tversky, 1996), children's conception of space (e.g., Piaget & Inhelder, 1967; Piaget et al., 1960), and quantitative reasoning (e.g., Joshua et al., 2015; Lee et al., 2019). RFs relate directly to the reference points and organizational structure (e.g., axes) an individual uses in a CS. For example, if we only focus on the categorization of atmosphere based on height, we can interpret Figure 1c as representing a single RF (height). If we use the dotted-red graph to consider the relationship between height and temperature, we can consider the graph as coordinating two RFs.

It is typical for CSs to represent two RFs (e.g., Figure 1a, b, d). There are various goals curriculum designers may have when tasking students to interpret a graph representing two RFs. They may ask students to interpret the meaning of a single point in a moment (Moore et al., 2025). For example, they may ask students to look at the dotted-red line in Figure 1c and ask them what the average temperature is at an altitude of 15 km. They may also ask students to make inferences about how two quantities change together, such as asking students to interpret the dotted-red line to describe how height and temperature change together in the mesosphere

(i.e. as the height increases, the temperature decreases). Reasoning about two quantities changing together is called covariational reasoning (Thompson & Carlson, 2017).

There are also examples across STEM fields of graphs intended to represent three (or more) RFs in quantitative (Figure 1e–f) and spatial (e.g., a 3D analog of Figure 1b) CSs. When students are tasked to coordinate three (or more) quantities, they are asked to engage in multivariational reasoning (Jones 2002; Panorkou & Germia, 2024). Researchers have noted that students as young as middle school and high school (Bagossi, 2024; Panorkou & Germia, 2024) can engage in multivariational reasoning to interpret graphs such as those in Figure 1e. By exploring the prevalence of such graphs in undergraduate STEM courses, we can develop curricular materials that better prepare students to interpret such graphs in STEM fields.

Types of RFs. In addition to attending to the number of RFs represented in a graph, our research has identified novel ways students interpret graphs as representing continuous versus ordered-discrete RFs (Olshefke-Clark et al., 2024, accepted). Whereas students are often tasked with interpreting graphs as representing continuous quantities (typically using covariational or multivariational reasoning), other times we expect students to attend to distinct regions of a graphical space (e.g., Figure 1d, e). In these latter cases, curriculum designers may expect students to attend to these discretized chunks represented within the graph space when interpreting the graph—either regions as a whole, or to individual points within these regions, rather than considering lines representing changing quantities (e.g., the type of soil in Figure 1e or the state of the matter in Figure 1d). However, the extent to which curricula in STEM fields are expecting students to interpret graphs based on points, lines, or regions is an open question.

Implications and Feedback Questions

We anticipate the types of CSs (spatial v. quantitative; Cartesian v. non-Cartesian), the number of RFs, the types of RFs (continuous v. ordered discrete), and the object(s) students are expected to interpret (points, lines, regions) within the graph space will prove explanatory when examining the ways STEM curricula use graphs to represent phenomena. By exploring the prevalence and function of these different features of graphs in STEM curricula, we can develop curricular materials that math teachers can use to better prepare students to interpret (or construct) graphs in their future undergraduate STEM coursework. Further, by comparing the different types of graphs used across STEM fields, we may identify tendencies for certain fields to use certain types of graphs and develop curricular materials specific to those fields to better prepare students to interpret graphs as curricula developers intend.

Although we believe our criteria will successfully characterize many types of graphs we encounter in STEM textbooks, we also expect there will likely be other significant features of graphs of which we are currently unaware. Thus, we have several feedback questions related to our current framework: Are there important features of graphs across STEM fields that our framework may not capture? If so, what are they? Are there other extant frameworks characterizing graphical features that may be useful in this study?

We also conjecture that our framework may be useful in analyzing graphs beyond curriculum. For instance, we are curious if RUME attendees have insights into the question: How might our current (or a revised) framework be used in analyzing graphs used in STEM instruction? Would this framework be useful in analyzing the ways in which instructors expect students to interpret or construct graphs? Hence, we hope to obtain feedback that will inform both the theoretical framework and new avenues we can extend this work into to better explore the use of graphs across undergraduate STEM fields.

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Persistence in the Margins: An Autoethnographic Study of a First-Generation, Low-Income Mathematics Major

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Although research on persistence and attrition in the undergraduate mathematics major remains limited, existing studies point to structural, curricular, and identity-related barriers that shape student trajectories. This autoethnographic study extends that conversation by exploring the lived experience of a first-generation, low-income undergraduate math major in the United States, using digital journaling to explore persistence, barriers, and systems of support. Guided by Tinto's theory of student departure, the study highlights the impacts of financial uncertainty, faculty interactions, and institutional and social demands on students' academic integration, sense of belonging, and self-image as a mathematician. Persistence emerges as an act of resistance. This study contributes to the literature on equity in undergraduate mathematics education by highlighting the complex interaction between identity and institutional structures. In doing so, the study offers insights for the creation of inclusive advising and pathways that support the success of students from disadvantaged backgrounds in mathematics.

Keywords: Persistence, First-Generation, Low-Income, Mathematics, Autoethnography

Introduction and Relevant Literature

Research in undergraduate mathematics education has long emphasized the gatekeeping role of mathematics courses in science, technology, and engineering (STE) pathways (Olson & Riordan, 2012; Seymour et al., 2019; Seymour & Hewitt, 1997). This emphasis has led to a plethora of research on the topic of how mathematics courses serve as barriers to persistence and retention of a wide range of STEM majors (Enderson & Ritz, 2016; Leyva et al., 2022; Olson & Riordan, 2012; Seymour et al., 2019; Seymour & Hewitt, 1997; Whalen & Shelley, 2010). Because of this dominant narrative, there has been a large focus on Calculus 1 (D. Bressoud et al., 2015; Johnson, 2016; Rasmussen et al., 2014). While this focus has been crucial in reimagining undergraduate mathematics instruction, it positions mathematics as a service discipline. This focus risks misrepresenting the experiences of mathematics majors, who differ significantly from other STE students. Only 2% of students enrolled in Calculus 1 intended to major in mathematical sciences (D. M. Bressoud, 2021), and unlike some STE majors (e.g., biological science, engineering) who declare during year 1, most math majors (64%) declare after (Chen, 2013). This misalignment suggests that while research on calculus 1 has generated valuable insights for STEM persistence broadly, it does not adequately capture the distinct pathways, challenges, and experiences of students intending to major in mathematics in the United States. These statistics underscore the need to look beyond calculus 1 to understand the persistence and attrition of undergraduate mathematics majors.

Although research specifically centered on undergraduate math majors in the US remains scarce, a small body of researchers has begun the exploration within this population. First, Ellington and Frederick (2010) focused their work on Black, high-achieving mathematics majors, finding that family support, program resources, and prior math experiences shaped persistence and identity as high-achieving. Later, Bahi et al. (2015) employed a time-series analysis to estimate a 23% probability of switching by the end of the first semester, noting no differences among genders. Zhuhadar et al. (2019) used survival analysis techniques with n=272,

showing that attrition risk increases later in the major and indicates that 3000-level courses better predict dropout as opposed to introductory calculus. And, more recently, Hubbard (2023) made an attempt at 'dispelling the myth' that there is a specific course that deters students from completing a math major and instead investigated persistence in between math courses within a specific institution. The author found that while median pass rates were 73%, persistence varied across demographics. Of worthy note, females are passing at higher rates but persisting less than males, and Black students are passing at lower rates but maintaining a higher persistence rate. Collectively, the present literature underscores the complexity of persistence and attrition of the undergraduate mathematics major. This literature supports the assumption that the factors that support a student's persistence (or lack thereof) are equally complex. What has not been investigated are factors such as structural, curricular, and programmatic barriers alongside identity, external factors, motivation, and holistic experiences in the major. The current study aims to fill these gaps.

Given these gaps, my socioeconomic identity (first-generation and low-income), and the call-to-action previously mentioned, an autoethnographic approach seemed fitting to investigate the complexities and the whole undergraduate journey of a mathematics major. Understanding the factors that influence persistence and attrition of the mathematics major is critical for the RUME and the larger mathematics community as we seek to address equity, access, and the definition of success in mathematics. By centering the lived experience of a mathematics major, this autoethnographic project aims to contribute a more nuanced understanding of how disciplinary identity, institutional structures, and personal identity interact to shape persistence. I seek to reopen dialogue and expand conversations within RUME beyond mathematics as a gatekeeper and towards mathematics as a discipline whose majors deserve focused scholarly attention.

Theoretical Framework

To answer this call-to-action, a theoretical framework is needed that considers the complex natures of equity, access, and the definition of success in mathematics. This study is grounded in Tinto's Theory of Student Departure (Engle & Tinto, 2008; Tinto, 1975, 1988). Tinto theorizes student attrition as the result of the interplay between characteristics of the individual (the student) and the institutional environment. His framework categorizes departure factors into three dimensions: academic and social integration, commitment, and external influences.

Academic and Social Integration

Persistence is shaped by the extent to which students engage with academic structures, achieve in coursework, and the degree to which students develop meaningful social connections with peers, faculty, and the broader campus/institution community.

Commitment

A student's commitment to educational goals and the institution is a key predictor of persistence. Higher levels of commitment correspond with greater likelihood of degree completion.

External Factors

Persistence is also mediated by factors beyond the institution. These factors include, but are not limited to, socioeconomic status, familial responsibilities, and aspects of personal identity.

While Tinto's framework serves as the foundation for this study, my application of it is situated within a constructivist epistemology and informed by a critical theoretical lens. From this perspective, knowledge is understood as socially constructed, and research is positioned as a means of interrogating systemic inequities in education. Thus, I extend Tinto's model by considering how inequities (tied to identity, access, and belonging) shape students' opportunities to persist in the undergraduate mathematics major. Consequently, external factors are more accurately conceptualized as factors beyond the locus of control of the student, not beyond the institution.

The decision to use the autoethnography aligns with this theoretical framing and understanding. Autoethnography situates lived experiences as a valid and rigorous form of knowledge (Bochner & Ellis, 2022; Creswell & Poth, 2016). In doing so, the autoethnographic methodology allows for the disruption of dominant narratives about success in mathematics. Within this study, the method also serves as a form of healing, allowing for critical reflection on the ways my lived experiences as a first-generation, low-income mathematics major interacted within the broader community to impact persistence. Therefore, healing is not conceptualized as separate from inquiry but as integral to the process of making sense of persistence in undergraduate mathematics. This framework informed the design, including the research questions and interpretation of this study.

This study is guided by the following research questions.

1. What personal and systemic factors influence my persistence as a first-generation, low-income undergraduate mathematics major?
2. What barriers did I encounter during my undergraduate journey, and how did they impact my academic and personal development?
3. What types of programmatic and institutional support could have better supported my persistence and success within the mathematics major?

Methodology

As stated by Bochner and Ellis (2022), evocative autoethnography emphasizes the personal narrative as a means of creating resonance and connectedness for readers. In alignment with this approach, I engaged in structured, digital journaling over a period of seven days, writing for approximately twenty minutes each day. These writing sessions were guided by prompts developed with both the literature and theoretical framework in mind. The aim was to elicit memories and reflections on persistence, barriers, institutional support, and self-identity changes. The use of digital journaling provided a space to revisit and reconstruct my undergraduate journey while connecting my experiences to systemic patterns identified in prior research.

Data analysis occurred through iterative, open coding (Creswell & Poth, 2016) of the journal entries, with attention to both emergent themes and connections to the theoretical framework. I sought to highlight how individual agency, social and institutional constraints, and factors outside my locus of control interacted to shape my trajectory through the mathematics major.

Validity was addressed through transparency and reflexivity. Throughout the journaling process, I documented my decisions, acknowledged my positionality, and considered the ways my interpretations were shaped. Autoethnography does not aim for generalizability in the traditional academic sense. But it does offer transferability through resonance and connectedness, inviting readers to draw connections between my lived experiences and broader patterns in undergraduate mathematics education.

Preliminary Findings

In this section, I present the preliminary findings from the analysis of this autoethnographic study. The purpose of this autoethnographic study is to explore and gain a better understanding of the nuances of the factors that led to my persistence and shaped my navigation through the undergraduate math major here in the United States. I wrote, *"I had to finish this degree. But I didn't want to be in math spaces anymore. I gave up on my dreams of medical school or an advanced degree in math. I believed what I was told that I did not belong in this community… that I did not know enough math… that I guess I will just teach instead… I could not go back to poverty. This sentiment stays with me even to this day, but I'm healing."* This reflection highlights the treacherous navigation of my academic journey and its impact on my sense of belonging. Through open coding of the digital journals, four major themes emerged: (1) persistence as resistance, (2) navigation of higher education, (3) identity and belonging in math spaces, and (4) external factors. At the time of this submission, my analysis has focused on theme four, external factors.

Financial Uncertainty and Institutional Constraints

Financial uncertainty, which led to institutional constraints, significantly shaped both my academic choices and relationship to the institution. I write, *"I had acceptances at many places... Yet, I felt like I had no actual choice… Without financial support, I was not going to be able to go to college."* These institutions varied by location (both in- and out-of-state) and level of institution (private, public, liberal arts, state schools, new-ivy, and ivy leagues). Yet, it became internally apparent that the decision of a specific institution was not mine to make. Instead, it was my lack of financial security that was going to make the decision. Ultimately, the institution that was chosen was a small, liberal arts college, located in the Appalachian region of the country. I wrote, *"For better, for worse, I don't know. I am giving up a lot of resources and access to a larger world with this decision. But I also have no other real option."* This exemplifies a critical point in Tinto's (1993) theory, where he argues that external obligations, lack of institutional integration, and decreased resource level increase the likelihood of departure. The implications of this forced decision limited my access to robust faculty options and resources.

Faculty Gatekeeping and Limited Academic Support

The institution had only two faculty during my undergraduate timeline, one of whom was my primary advisor, and the sole instructor for any course beyond calculus, limiting my academic experience to their problematic approach. This faculty member questioned my belonging and ability in mathematics. I recalled, "*You just don't get it... Someone like you doesn't belong in math."* These were common occurrences, according to the journal entries, and "*were confirmation of insecurities I had worked to overcome."* Tinto (1993) emphasizes that academic integration is essential for student persistence. I understand academic integration to be the meaningful and validating relationships with faculty and institutional academia at large. In direct contrast, I experienced integration as a gatekeeper. I wrote, *"I was made to feel minuscule and disposable."* Any option to request another advisor was not available. I wrote *"No path to escape [their] authority."* Efforts were made to report the pattern of mistreatment. However, those efforts were met with indifference and redirection. I recalled the frequent, *"Maybe you should just change your major."* But this did not exonerate my feelings. These experiences created a climate in which seeking support felt dangerous or risky, while fostering doubt about my ability to learn mathematics, turning persistence into an act of endurance rather than growth.

Overworked and Social Demands

Tied directly to financial constraints and a desire to be involved in campus activities, affairs, and institutional structures, I navigated undergraduate life dominated by work, constant social demands, and mathematics. I held a federal work-study position every semester. I worked, yearly, as a campus (math, chemistry, physics) tutor and peer mentor. I was a collegiate athlete. I was elected Vice President for the student government association. I even held off-campus work at a local department store and as a server at a local restaurant. I wrote, *"I had to. I wanted to feel like part of the college, like I was making it better for those like me. I also had to pay for school. I had to pay to live."* Balancing these responsibilities left little room for recovery from setbacks. Tinto (1993) argues that external demands that reduce time and academic engagement can lead to attrition. Though I chose to persist, I found that my identity as a first-generation, low-income student was pushed aside by the institution's lack of mechanisms to reduce these external and social demands. *I wrote, "This was saddening, infuriating, considering the institution prided itself on its large (about 50%) portion of first-generation and low-income students 'success."* Internally, I understood this to reinforce that the institution of academia was not built for students like me. Tinto (1993) suggests that departures often occur due to insufficient institutional integration, rather than lack of motivation; my experience affirms this and illustrates how persistence can function as an act of resistance.

Discussion and Conclusion

This study highlights the ways institutional structures, financial pressures, and faculty dynamics shaped the persistence of a first-generation, low-income mathematics major. My experience underscores that support structures, or the absence of, directly impacts students' academic engagement, sense of belonging, and ultimately their identity as mathematicians.

The current findings align with prior research emphasizing the importance of institutional agents and structures shaping outcomes for first-generation students (Burnett, 2017; Engle & Tinto, 2008; Jehangir, 2010; McCain et al., 2024). However, my findings add nuance by showing how unchecked faculty authority and institutional rigidity can transform structures of support into barriers. This connects to survivor narratives among low-income students, suggesting that persistence often requires enduring trauma rather than accessing support (Cota, 2024; Hahn, 2023; Wallace, 2018). Furthermore, my analysis contributes an additional layer to the complexity of persistence by showing how deeply power dynamics (faculty-to-student and lack of options of teaching faculty) can function as a structural condition that limits students' capacity to learn and enhances harm.

I am interested in thoughts from the audience about the following: (a) feedback on my analysis that would inform my exploration of the other three themes, (b) possibly expanding this work through a larger quantitative or mixed methods study to examine systemic factors across mathematics courses and majors, or (c) employing a duoethnographic methodology (Burleigh & Burm, 2022) with another first-generation, low-income mathematics (or STE) student. The latter two studies could explore the variance among institutions, regions, and the complexity of self-identity and mathematical identity formation. These studies could further illuminate the intersection and interplay of identity, systemic (academic and social) constraints, and persistence.

Lastly, this work makes clear the importance of inclusive advising, support structures, and pathways that reflect the realities of students from disadvantaged backgrounds (*Disadvantaged Background*, n.d.). Not only recognizing but addressing these systemic and academic barriers is critical to fostering persistence and furthering the conversation of equity in undergraduate mathematics education.

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"…it's more of a healthy skepticism…": Descriptions of Undergraduate Students' Mathematical Convictions and Proclivities

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This study used a survey containing two open-ended questions and a ranked choice question. I investigated undergraduate students' mathematical convictions relating to whether they experienced a lack of conviction. Students were asked about their proclivities when an instructor presents a proof to the class. Preliminary findings show that few students experienced a lack of conviction with either theorems or proofs. For theorems, a lack of understanding was often cited. For proofs, students cited doubts about the rigor of those proof types such as picture proofs. Students' proclivities seemed to show that their focus was on understanding a proof's content. Further evidence provided by follow-up interviews corroborated this interpretation and that students may in fact reflect on matters of conviction in mathematical settings.

Keywords: Mathematical conviction, mathematical proof

Mathematical convictions are described as the ways in which one is persuaded about the truth of a theorem given a mathematical argument or statement. Nebulous concepts in mathematics education, like mathematical convictions (Brown, 2018), continue to challenge researchers in the ways we investigate them. This study contributes to the growing field of knowledge around mathematical convictions by investigating undergraduate students' mathematical convictions and proclivities. The purpose of this study was to determine whether and how students consciously operate in a way that foregrounds mathematical convictions. I also investigated the roles that theorems, proofs, and instructors have in matters of students' convictions.

Background Literature

I note that mathematical proofs are often defined as "…valid arguments that establish the truth of a mathematical statement. By an argument, we mean a sequence of statements that [begin with an assumption and] end with a conclusion" (Rosen, 2012, p. 73). Indeed, many textbooks describe mathematical proofs in this way which often do not mention matters of conviction. So, this leads to questions regarding how and why students may be convinced.

Aspects or characteristics that students often cite as contributing to whether they are convinced by a mathematical proof or argument are familiarity with the content and method (Brown, 2018; Chazan, 1993; Healy & Hoyles, 2000), example-use (Healy & Hoyles, 2000; Weber et al., 2022), and whether one understands that proof (Brown, 2018; Selden & Selden, 2003). A contentious aspect reported in the body of research relating to mathematical convictions is that of authority. For undergraduate students, this may be observed by students referencing that the instructor said that the mathematical statement was true or the mathematical proof was valid (Brown, 2014). For mathematicians and educators, a classical perspective is that students are convinced that a theorem is true if and only if they have a mathematical proof. However, prior research described students as being too difficult to convince (Fischbein, 1982) or too easily convinced by mathematical objects that were not proofs (Harel & Sowder, 1998). Interestingly, Weber et al. (2022) described mathematicians' behaviors like students rather than the typical mathematician behavior described in previous research.

Methods

I used a survey intended to capture characteristics of undergraduate students' mathematical convictions and their proclivities as they engage with mathematics. In this preliminary report, I report parts of that survey and the follow-up interviews pertaining to those parts of the survey.

I contacted 231 instructors across the U.S. teaching undergraduate algebra or analysis courses in Fall 2024. Fourteen (6%) agreed to use the survey in their course(s). Two instructors implemented the survey as an assignment, four implemented it as extra credit, and nine forwarded the survey to their students to complete voluntarily. There was a total of 212 students enrolled in these instructors' course sections ($M = 15.14$, $SD = 8.72$). There were 95 students (45%) who responded to the survey. A student's response was removed from analysis if: 1) the duration to complete the survey was less than five minutes, 2) the student was a graduate student, 3) the student did not consent to the study, or 4) the student did not sign the consent form. After this, there were 71 responses (33%) remaining. The average time to complete the survey was 33 minutes while the median time to complete the survey was 11 minutes.

Research Instruments

There were two open-ended questions. The first asked, "was there ever an instance where your professor presented a theorem, and you were not convinced the theorem was true?" If the student responded with "yes," the question continued by asking for further details such as the course in which it occurred, the theorem itself, and why he or she was unconvinced. The second open-ended question asked, "is there a type of proof that you find unconvincing?" If the student responds with "yes," the question continued by asking for further details like type of proof, examples of when it was used, and why he or she was unconvinced by this type of proof.

A ranked choice question had students indicate behaviors they engaged in and their importance during a scenario. These behaviors were generated from assumptions about students' proclivities and prior research regarding students' mathematical epistemologies as it relates to standard practices in the classroom. The prompt was, "when my instructor presents a mathematical proof to me or the class, I personally focus on…." They then rank the following six behaviors on whether they engage in them and how important they are: 1) Trying to understand the content of the proof, 2) Checking whether the proof is valid, 3) Becoming convinced that the theorem is true, 4) Writing what the professor writes or says, 5) How this would be useful for the exam(s) or assignments, and 6) Anticipating the next line of the proof.

Follow-Up Interviews

Students were invited for a follow-up interview if they met two of three criteria: 1) described a theorem that they were not convinced was a mathematical truth, 2) described a proof type that they were not convinced proves, or 3) indicated that they focused on becoming convinced that a theorem is true when the instructor presents a proof. I conducted five follow-up interviews with five different students. The average interview time was 10.8 minutes ($SD = 3.47$ minutes).

I refer to these students as S, K, B, U, and C. Student S was beyond their fourth year with a major in mathematics and completed five proof-oriented courses. Student K was a third-year mathematics major and had completed one proof-oriented course. Student B was a second-year majoring in applied mathematics and had completed one proof-oriented course. Student U was a fourth-year majoring in mathematics and had completed three proof-oriented courses. Student C was a fourth-year majoring in mathematics and had completed three proof-oriented courses.

The questions in the interviews were tied back to responses to the criteria for why they were invited. For example, asking about the role(s) an instructor played or plays in becoming (un)convinced. Also, each student was asked whether they found themselves wondering about whether they're convinced in courses and whether not being convinced is ever an issue.

Results and Preliminary Analyses

Results of the Open-Ended Questions

The results for the open-ended questions are presented below in Table 1. Thirty-eight students (53%) said that they did not experience an instance where their instructor presented a theorem and they were not convinced that theorem was true. Twenty-four students (34%) said that instances of not being convinced by an instructor's presented theorem were due to not understanding. Four students (6%) said that they did not believe that theorem they encountered to be a mathematical truth. These instances included an instructor presenting a theorem relating to converging sequences without justification, the infamous invalid claim that "all horses are the same color," and the reverse triangle inequality. These students described stated their lack of conviction stemmed from the statements not being "obvious" or "fundamentally incorrect."

Most students (83%) said that there was no type of proof that they did not think proved. Four students (6%) said that they were not convinced a proof type proved. They described proofs by picture, proofs by examples, and proofs of uniqueness. For proofs by picture, two students said that pictures can be interpreted differently and drawn in misleading ways. For proofs by examples, the student said having an artificial intelligence program attempt to prove a theorem only used examples which "did not cover all cases." For the student who mentioned uniqueness, they said the proof of "square roots are not rational" and that "when we just name the same variable two different things, it makes sense that it will always come together."

Table 1. Results of the Two Open-Ended Questions

Was there ever an instance where your instructor presented a theorem and you were not convinced the it was true?*				Is there a type of proof that you find unconvincing?*			
	Yes.				Yes.		
No	Did not believe the theorem to be true	Believed it was true, but did not understand why	Other	No	Do not believe that that type of proof proves	Believed it proves but do not understand why	Other
38	4	24	4	59	4	4	2

*There were three instances of a student declining to answer either question.

Results of the Ranked Choice Question

The results of the ranked choice question are presented below in Table 2. I note that a rank of 6 indicated the most importance, a rank of 1 indicated the least importance, and a 0 indicated no importance (i.e., not ranked at all). These rankings were tabulated and then weighted according to the students' rankings. The table is organized from most important to least important. These results show that students indicated understanding the proof's content as frequent behavior and notable importance while checking the proof's validity was infrequent and less important.

Table 2. The results of the Ranked Choice Question

Rank	Students' Focus During Instructor Proof Presentation					
	Trying to understand the proof's content.	Writing what the instructor writes or says.	Becoming convinced that the theorem is true.	Anticipating the next line of the proof.	Usefulness for the exam(s) or assignments.	Checking whether the proof is valid.
6	40	10	3	4	7	3
5	13	14	14	14	6	4
4	5	9	13	13	9	8
3	3	7	9	4	11	12
2	0	5	4	5	11	8
1	1	6	2	5	6	4
0	9	20	26	26	21	32
Weighted	4.72	2.86	2.50	2.44	2.38	1.77

Results from the Five Follow-up Interviews

Four students mentioned that they viewed the instructor as a support mechanism in the process of becoming convinced or resolving issues of certainty. Student C said that instructors "provide tools necessary to be able to understand the proof by giving examples or other theorems." Student S described an instructor's role as engaging with the class by asking whether the theorem or proof makes sense. They emphasized that it was important for the instructor to engage students' thinking with questions relating to specific justifications in a proof. Student K described the instructor's role as a "kind of… comforting thing. I [can] ask them any question, even if it's very complicated." Student U described the instructor's role as "presenting the pieces that you need to follow to be convinced that you or one of the ways you could be convinced a theorem is true." They continued by mentioning that probing questions were especially helpful to "start actually building the theorem."

Few students mentioned strong feelings toward the role of becoming convinced as they engage with mathematics. For example, Student K said, "generally speaking, if it's a complex theorem, I'll break it down into a million pieces and I won't trust it at face value…. I just… can't move on." They continued by saying, "I'll be second guessing myself [otherwise]. Just something I can't help." I asked Student U if being unconvinced is ever an issue, they said that "it's [never] stopped me from trying to engage with a topic…. Perhaps it's more a healthy skepticism that I…probably would benefit from having more than blindly [believing] in the book every time." In other words, "…not being convinced of a theorem just by reading it." Student U believed they developed healthy skepticism by saying, "it's gone in a bit of a weird shape graphic because I started off just blindly believing everything…" and gestured a graph like Figure 1. The reason Student U gestured like this was due to their introduction to proof course instructor who helped develop a sense of skepticism, but as Student K progressed through subsequent mathematics courses, they had rebuilt "blind trust" in textbooks "because…the [questions] we use [in courses] are simply prove this is true or prove this is not true."

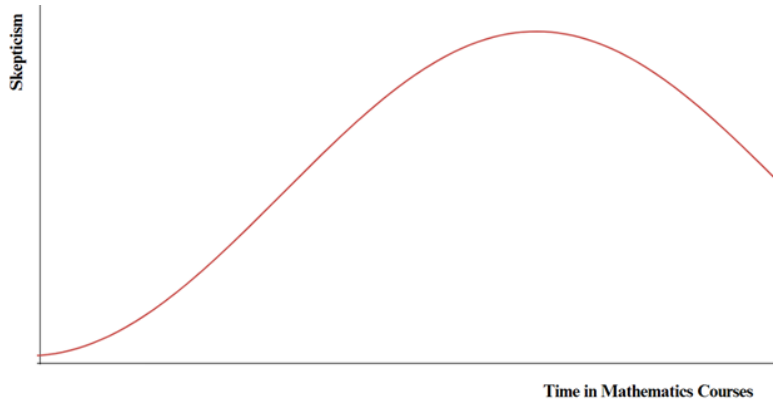

Figure 1. Recreation of Student U's Gestures about Skepticism and Mathematics Courses

Discussion and Conclusion

The open-ended questions revealed that most undergraduate students in this study are convinced that the theorems their instructors present are true and that the proof types they encounter are valid. The instances of students not being convinced that the instructors' theorems were tied to a prevailing desire for more justification and reasoning—a mathematical proof. The instances of students not being convinced by a proof type included picture proofs, proofs by examples, and proofs by uniqueness. Notably, students were articulate about their reasons for why proof types were unconvincing. These reasons relied on skepticism, or doubts, about the rigor of the proof type rather than a misunderstanding.

The ranked choice question revealed that students primarily focused on and find importance in understanding the content of a proof while checking whether the proof is valid was often less important and frequent. This provides some evidence about the role of an instructor. The students in this study seemed not worried about whether an instructor was providing invalid proofs. This is reasonable given the standard structure and practices of mathematics. Interestingly, becoming convinced that the theorem is true was third when the rank was weighted. So, becoming convinced is among the list of proclivities but other mathematically relevant behaviors take priority. In other words, there is evidence that students knowingly think about becoming convinced in their courses. This provides evidence that some students' mathematical epistemologies encompass mathematical convictions given that the scenario was written to exclude specific details such as content, location, and instructor.

The results from the follow-up interviews provide some evidence that students indeed consciously reflect on matters of conviction. For some students, it appeared as ways they come to know and understand mathematics. This reflection was shown to be fostered in the class for one student but not others. Nevertheless, the instructor's role for these students provides understanding which has been observed greatly influencing whether students are convinced (Brown, 2018; Selden & Selden, 2003).

Preliminary Nature of This Report

I do not anticipate having further analysis developed because the analysis is limited by the research instruments' wording. Therefore, I wish to avoid making higher inferences than warranted by the data. However, I wish to develop further research instruments, avenues, and questions pertaining to how reflections on whether one is convinced is taught or enculturated in courses and how these mechanisms influences students' epistemologies as they engage with mathematics (proofs and theorems or otherwise).

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Students' Challenges in Double Integration Over Non-rectangular Regions

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Double integration is challenging for many students. It involves the interrelationship between three variables x, y, z within an object in 3D space expressed in both algebraic and graphical representations. When the integral domain is not a rectangular region, values of x and y along its boundaries may become dependent. However, how students perceive interdependency of variables and what mental actions are involved in double integration have not been fully explored. To answer these questions, multivariational reasoning framework is used in this study to analyze students' cognitive activities. Interview data collected from a university-level calculus enrichment course is analyzed to explore student's reasoning. The findings suggest that students demonstrate several mental actions of multivariational reasoning when performing double integration over different regions. They also suggest some possible directions of extending the theoretical framework of multivariational reasoning to adapt some specific concepts such as range and change of variables in double integration.

Keywords: multivariate calculus, double integral, multivariational reasoning, visualization

Introduction

Double integral is typically taught in multivariate calculus, and it is widely applied across STEM disciplines (Jones & Dorko, 2015). Despite the prolific research in students' understanding of single variable calculus (Thompson & Harel, 2021), relatively less attention is drawn to multivariate calculus (Martínez-Planell & Trigueros, 2021).

Students' thinking towards double integration over rectangular region has been studied as the generalization of definite integration to bivariate function, regarding the two independent input variables (Jones & Dorko, 2015). However, the case of non-rectangular regions has been under-explored until recently (Craig & Pehlivan, 2025). Such problems appear in vector calculus and engineering applications, regarded as a common challenge for students (Khemane et al, 2023).

The difficulty of transitioning from single to double integration may be explained by the complexity of multiple variables and the challenge of visualizing their interrelationship in 3D space. With non-rectangular regions, these two factors are further amplified. The cognitive activities involved when handling multiple variables has been studied by Jones (2022) as multivariational reasoning. Jones categorized types of multivariation structures and induced some mental actions from students' reasoning. This provides a suitable theoretical framework to analyze students' thinking processes in double integration. In this paper, task-based interview data is analyzed to explore students' mental actions of multivariational reasoning involved in double integration over different regions, and to inform potential development of the framework.

Literature Review

The concept of double integration is rooted in bivariate functions and definite integration. A bivariate function is commonly represented in the form of $z = f(x, y)$. With inputs (x, y) as a point in a 2D region, the graph of a bivariate function is a simple surface in 3D space. Trigueros and Martínez-Planell (2010) used APOS theory to study how students transition the concept of single variable functions to bivariate functions in different representations. They found that this process is not straightforward especially when graphical representations are involved.

Definite integrals are associated with a bounded region for the input variable. In the case of positive single variable function, a definite integral can be constructed as the area under the graph over an interval. Jones and Stevens (2022) combined several frameworks into a more general concept of "integral with bounds". Their approach attends to different layers of thinking, involving numeric, graphical, and symbolic representations.

The transition from definite integrals of single variable functions to bivariate functions is indirect. Jones and Dorko (2015) studied how students generalize the concept of multivariate integrals based on single integral. They found that for most students, the generalization is based on the concept of "area under the curve". Martínez-Planell and Trigueros (2020) explored how students relate Riemann sums of bivariate functions to double integrals.

The above studies involve rectangular regions only. Non-rectangular regions are essential to mathematical applications for engineering problems, which makes them research-worthy. Khemane et al (2022, 2023) identified students' common difficulties in distinguishing between rectangular and non-rectangular domains, and swapping the order of integration. Craig & Pehlivan (2025) analyzed students' conceptualization of the bounds of non-rectangular regions. However, these studies did not explicitly delve into students' perception of the interrelationship between variables, and its role in the reasoning process. This study aims to bridge these gaps by exploring students' use of multivariational reasoning (Jones, 2022) in double integration.

Conceptual Analysis

Double integration involves three variables x, y, z being interrelated to each other given a bivariate function, and some constraints on the boundary of the integral domain. To analyze how students perceive and represent this interrelationship, multivariational reasoning by Jones (2022) is adopted in this study. Based on the idea of covariation (Thompson & Carlson, 2017), Jones studied students' cognitive process in coordinating multiple variables. This framework begins with a categorization of multivariation structures into independent, dependent, and nested types, followed by several mental actions involved in students' reasoning processes. Based on this categorization, the conceptual analysis (Thompson, 2008) is described. Double integral over a rectangular region $\int_{y_1}^{y_2}\int_{x_1}^{x_2} z(x,y)dx\,dy$ is associated with independent multivariation since the output variable z varies with two independent input variables x, y. The case of non-rectangular region can be regarded as a special type of dependent multivariation. If the region is vertically simple, the double integral can be written as $\int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} z(x,y)dy\,dx$ where the range of the inner variable y depends on the outer variable x. If the region is horizontally simple, the double integral can be captured through $\int_{y_1}^{y_2}\int_{x_1(y)}^{x_2(y)} z(x,y)dx\,dy$, when the range of x depends on y. Figure 1 illustrates these three cases and their associated multivariation structure.

Jones' (2022) framework is further extended to conceptualize students' cognitive activities associated with multivariational reasoning when performing double integration:

1. *Recognition of in/dependence* is involved when students identify the interrelationship between variables given an object before conducting double integration.
2. *Coordination of simultaneous changes* is involved in setting up iterated integral with inner and outer variables, with the range of inner variable dependent to the outer one.
3. *Switching variables/constants* occurs when students perform inner integration, regarding the outer variable as constant. It also occurs when swapping the order.
4. With one variable regarded as a constant, students can regard the bivariate function as a single variable function, hence *reduce into an isolated covariation*.

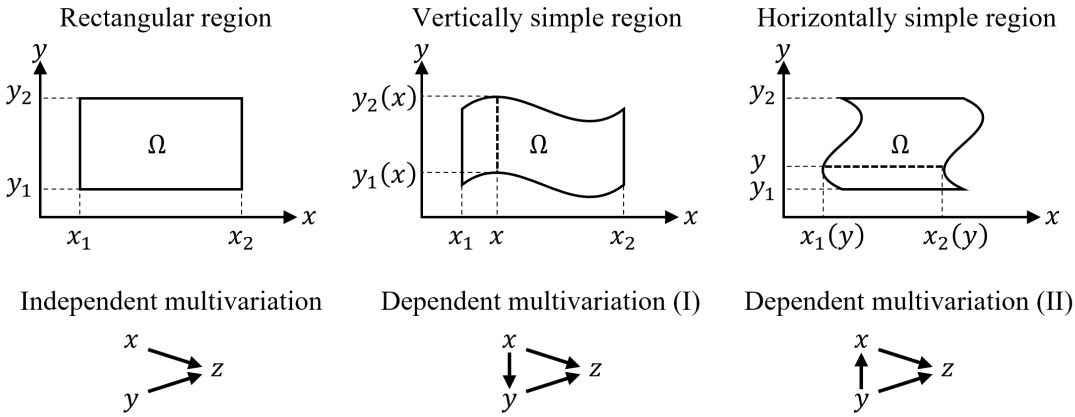

Figure 1. Typology of integral domains and associated multivariation structure.

Method

The data for this study was collected in a university-level enrichment course offered to high-achieving secondary students in Hong Kong. This five-weeks course conducted in summer 2025 by the first author covered single and double integration. The pedagogy involved lectures on function, calculus and visualization, along with interactive activities using dynamic geometry software. After completion of the course, 11 students were recruited for individual task-based interviews, each lasting between 1 to 2 hours. The students were given a bivariate function $z = x + y$ and regions on the xy-plane: square, triangular, quarter-circular (Figure 2). The first two regions correspond to independent and dependent types of multivariation, respectively. The typology of the third region depends on the coordinate system used. Hereby, it can become dependent when x, y are used or, independent for r, θ in polar coordinates.

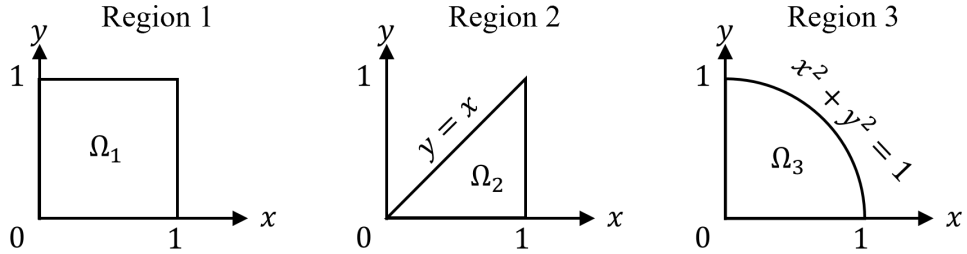

Figure 2. Integral domains used in interview task.

The students were asked to imagine and sketch the graph of the function, express the range of values and dependence of the variables, and eventually evaluate the volume of the solid objects, namely D_1, D_2, D_3, underneath the graph of function and lying on the three regions. These solids and regions were described by the interviewer in words without showing the figures in the beginning. At different stages of the interview, students were provided with dynamic computer graphics illustrating the objects in 3D space virtually to assist their visualization and problem solving. Transcription of the interview recording with students' writing, drawing, hand gestures and usage of computer graphics were analyzed based on Jones' (2022) framework.

Preliminary Findings

The findings below are presented for each task, based on the responses from three students (in pseudonyms), who demonstrated different levels of multivariational reasoning. Due to the preliminary nature of this report, the data from other interviewees has not been included.

Ranges of variables and their interrelationship given a solid object

After provided with the computer graphic of D_1, students were asked to identify the ranges of values of x, y, z within the solid. Henry recognized the base of D_1 as a square and responded that $x = 1, y = 1$ without addressing the ranges of x, y. Consequently, he said z is a line segment going up to a point of 2 and thereby replied $z = 2$. He admitted that z is determined by x, y with the relation $z = x + y$, as a single value. This case indicates a confusion between value and range of variables, which may be rooted in the perception of length when discussing a line segment, rather than the varying positions of a point on it. This can be regarded as the level of "*no variation*" (Thompson & Carlson, 2017) which is before co- or multi-variation take place.

By contrast, Carl claimed that "x is 0 to 1, y is 0 to 1, and z is 0 to 2". After the interviewer asked "if $x = 0.3, y = 0.4$, can z go up to 2?", he realized that the ranges of the variables might not be independent. Then, he stated that the solid is "sandwiched" between two planes $z = 0$ and $z = x + y$, hence modified his answer as z being from 0 to $x + y$ dependently. This graphical interpretation allows a range of z dependent to the position with x, y coordinates within a region. This demonstrates the mental action of *recognizing dependence* of the range of z on x, y, as he realized that values of x, y restrict the allowed values of z. It is notable that Carl did not address this dependence relationship in the beginning, until he applied the graphical representation. This suggests that visualization can enhance students' understanding of interrelationship of variables.

Double integral over triangular region

In the discussion about D_2, Henry thought it would look like a square considering the equation $y = x$, consequently he said that when x increases, y and z also increase. After being asked to consider the solid's projection on the xy-plane, he drew the line of $y = x$ and claimed that the region is a triangle as the line cut through the previous square region into two halves, without drawing the whole triangle with other boundaries. Henry could setup the iterated integral with $dydx$, without explaining why x and y goes from 0. He regarded "starting from 0" as a common habit in integration. When being asked to integrate with a different order $dxdy$, he directly swapped the integral domain $[0, x]$ of y to the outer integral (Figure 3) without rewriting it as an independent interval $[0,1]$. The mental actions of *switching variables/constants* and *coordination of simultaneous changes* seemed to be missing from his reasoning.

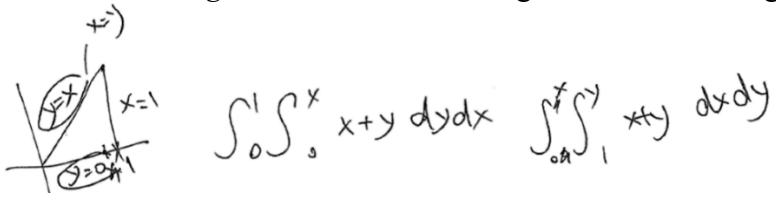

Figure 3. Graph and integrals written by Henry.

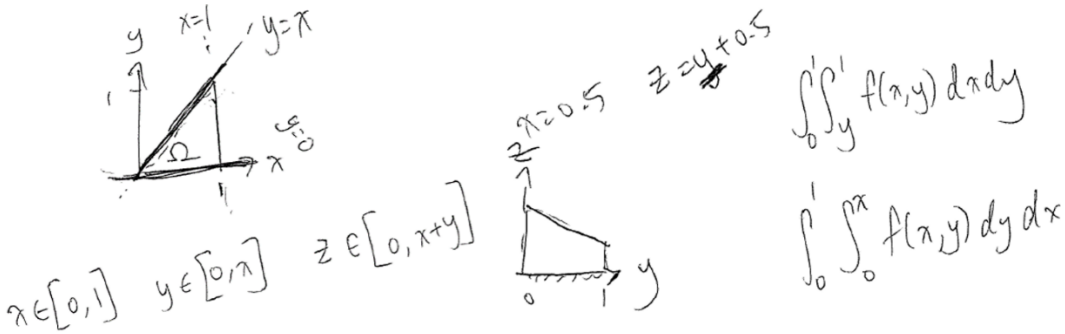

Figure 4. Graph and integrals written by Carl.

Carl, in comparison, could describe the ranges of variables in D_2 using intervals and drew the region accurately. He suggested that y depends on x while z depends on both x, y in this solid. He also rejected the saying that "z depends merely on x" with a visual counterexample: when $x = 0.5$, the solid would have a trapezium cross-section on the yz-plane with $z = y + 0.5$ on top, along which value of z varies with y. Referring to the region, he could write down iterated integrals in both orders of integration correctly (Figure 4). From the framework, Carl could mentally *switch the variable x as a constant* and hence *isolate y, z as a covariation*.

Double integration over quarter-circular region

Dave was asked about D_3. He realized that the base of D_3 is a quarter-circular region with boundary $x^2 + y^2 = 1$ and the axes. When asked about the dependence of variables in D_3, he divided it into two cases with x, y being dependent on the boundary due to the equation, and they are independent inside the region. He then claimed that z is independent to x, y, yet he could clearly describe that z always lie between 0 and $x + y$ within the solid. This response shows a diverse thinking when *recognizing dependence relationship* within non-rectangular region.

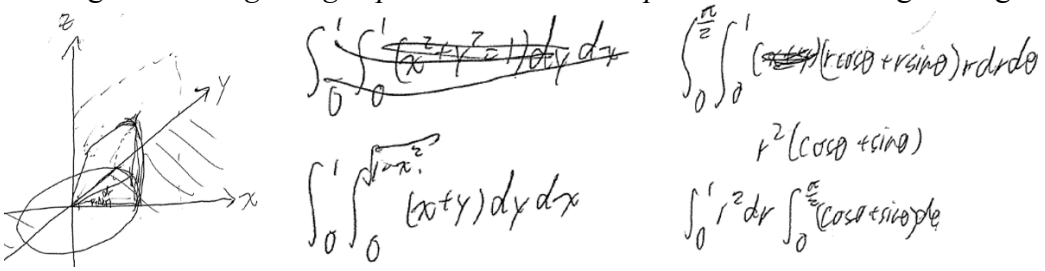

Figure 5. Graph and integrals written by Dave.

When asked to evaluate the volume of D_3, Dave first wrote an iterated integral with both x, y from 0 to 1, and then deleted it. He then wrote another iterated integral with the upper value of y dependent on x, although he agreed that this way is tedious in calculation. Instead, he thought of using polar coordinates. He figured out the ranges of r, θ to set up the iterated integral, then evaluated it with separation method (Figure 5). This case illustrates *recognition of dependence relationship* is involved when students determine their approach of integration.

Concluding Remarks

With the conceptual analysis derived in this study, multivariational reasoning framework (Jones, 2022) is useful for analyzing students' understanding of interrelationship of variables within bounded regions. Students who could complete the tasks with clear explanation of their approaches demonstrated a few mental actions in the framework. However, the conceptualization and expression of double integrals require an understanding of the dependence among the ranges of variables, which is not covered in the framework. The terms "dependent" or "independent" are defined based on values but not ranges of variables. A discussion of range seems necessary for extending the framework to explain students' cognitive process prior to double integration. Compared to rectangular regions, non-rectangular regions with boundary equations impose dependence relationship between ranges of x, y, hence require a higher level of multivariational reasoning associated with visualization. Moreover, the framework can be extended to explain the reasoning processes involved in the transformation of variables and coordinate systems.

In this preliminary report, only three cases have been analyzed due to time and page limits. Although some conceptual analysis has been done, the new extended framework has not been fully developed. Analysis of other interviews in future would validate the new framework.

Acknowledgement

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When We Say "Rigor" in Undergraduate Mathematics, What Do We Mean?

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One equity-oriented Networked Improvement Community (NIC) aimed to interrogate their use of "rigor" and its potential ramifications for the undergraduate mathematics program. In this preliminary report, we use critical discourse analysis on journal and interview data from this NIC to uncover how overloaded the word "rigor" is in mathematical spaces. These preliminary results reveal that rigor is conflated with difficulty, deep understanding, and other notions associated with teaching and learning mathematics. Findings suggest that there is a need to clarify and agree upon what we mean when we say "rigor" within our mathematical circles. We end with discussion and implications for practice.

Keywords: rigor, power, discourse analysis

"Mathematics is a language" is often an adage said by mathematics instructors to frame their course as one where students will be tasked (or challenged) to learn how to "speak math" (Wakefield, 2000, p. 272). As mathematicians, we may often obsess over the clarity or specificity of our phrasing and word choice when we speak or do mathematics, especially when writing proofs. This obsession perhaps demonstrates a desire to be understood exactly as we intended. Often there is an unsaid trust within mathematical circles: that when mathematicians talk, especially when using their mathematical register (e.g., Herbel-Eisenmann et al., 2015), our message is interpreted how we intended. Discourse beyond mathematical proofs, though, is not as clear-cut as mathematical logic. We should examine the assumptions we make when this unsaid trust spills into other discourses. Do we agree on what we mean when we use certain words? Many phrases, even in the RUME community like 'active learning,' are interpreted, conceptualized, and operationalized differently depending on who and where phrases are used (see Williams et al. (2022) for an analysis of various conceptualizations of 'active learning').

Amongst all the fascinatingly complex, intellectually-stimulating words to unpack, we highlight "rigor" in this study for the alarming ways rigor has been used to justify a gatekeeping logic in undergraduate mathematics (Hagman et al., accepted; Leyva et al., 2021) and the drastically different experiences with and operationalizations of "rigor" even amongst the authors. As we discuss in this report, there are also positive aspects to rigor; this report is not meant to vilify rigor. However, if mathematics departments and instructors place a large value on enacting rigor, then it is paramount that we, educators and instructors, interrogate the use of the word "rigor". Therefore, we ask the following research questions: When individuals in mathematical spaces use the word "rigor", what other words are used in relation to it and what do these other words reveal about the diverse ways in which rigor is conceptualized?

Literature Review

At the heart of this report is a desire for *transparency* in the sense of Gutiérrez (2013) where we interrogate taken-for-granted constructs and dominant notions in our mathematical culture, such as mathematical rigor. This does not mean we aim to construct a singular definition; as Riley (2017) argued, definitions of rigor are socially constituted and thus reflect a myriad perspectives on and experiences of rigor. As evidence of this myriad of conceptualizations, Draeger et al. (2013) proposed a model of academic rigor from the point of view of college faculty which posits meaningful content, active learning, higher-order thinking, and expectations comprise the "anatomy" of academic rigor. Later, Draeger et al. (2015) drew upon student data to develop a model of academic rigor from the perspective of students whose anatomy of rigor involved grades, difficulty, high workload, and interest.

As early as the 1920s, mathematicians like Pierpont (1928) have discussed that "absolute rigor"—mathematicians' goal as they scrutinized the foundations of mathematics in light of recent paradoxes at the time—is unattainable (p. 23). Almost a century later, our understanding of rigor has grown from a topic of concern to theoretical mathematicians to a central idea shaping classroom practices and education policy. Leyva et al. (2021), Marsellas (2021), and McNeill & Gresalfi (in press) interrogated rigor in relation to inclusion, care, and equity (or lack thereof) in the mathematics classroom. In policy making, matchmaking K-12 students into different course sequences comes with language that highlights the varying levels of "rigor" present in sequences (Oaks & Guiton, 1995; Herbel-Eisenmann et al., 2018).

Our goal for transparency around *rigor* is to reveal the ambiguity surrounding this word's meaning; when someone says rigor, they may not refer to the same things as others in the same spaces. We contribute to the emerging body of work on rigor and its continued presence in mathematics circles through uncovering the connections "rigor" has with other concepts and notions popular in mathematics instruction discourse. Our overall goal is to start conversations, so that when we say "rigor" in undergraduate mathematics, we can know what we mean.

Conceptual Framework

We conceptualize rigor as part of some mathematical cultures, where culture is broadly composed of the behaviors and values which are learned, shared, and exhibited by a group of people (Hagman et al., accepted; Yosso, 2005). In our view, rigor is a central cultural value within many mathematical cultures in postsecondary contexts. Rigor informs the expectations mathematics instructors have about their students' learning (Draeger et al., 2013) and what teaching practices are valued or necessary for the course to be mathematically rigorous (Leyva et al., 2021). It also informs the interpretation of workload and difficulty of the course for the students (Draeger et al., 2015). Broadly, we approach rigor as deeply involved in different aspects of mathematics teaching and learning—from the perspectives of faculty and students.

Data

This analysis is within a larger study that investigated Networked Improvement Communities (NICs) under ACT UP Math (EES #2201486). Through this project, education researchers with equity specializations partnered with three mathematics departments to initiate equity-oriented changes, supported by equity-oriented professional development, quantitative and qualitative data, and a broader network of support across the three departments. Hagman et al. (accepted) focused on one NIC at Kappa University and investigated how their NIC's goal of addressing rigor served as fertile ground for reckoning with tensions between the perceived value and operationalization impacts of rigor. The present report uses interview transcripts and journal

reflections to investigate the variation among meanings of rigor that were unearthed through our analysis of rigor as it relates to equitable instruction.

The Kappa NIC was composed of four faculty affiliated with the introductory mathematics program, two graduate students who had taught in this program, and two undergraduates who had taken classes within this program. Research participants opted into the NIC with interests of contributing to equity-oriented change and were all equally paid by ACT UP Math grant. The two NIC faculty co-leaders (Tiersa, Skylar) invited faculty (Anna, Senna) and purposefully selected out of an applicant pool the graduate (Parker, Kiara) and undergraduate students (Ricardo, Timothy). Pseudonyms were chosen by participants. The NIC identified as predominantly white women, and included one woman of color and two men.

Preliminary Nature

In Hagman et al. (accepted), we focused our argument in demonstrating how discourses around rigor reveal a culture of power (Delpit, 1998) in undergraduate mathematics education. This focus resulted in some analyzed data not being appropriate for that argument. We realized there is bountiful discussion to be had with focusing on the data centering only on how the word "rigor" itself was used, how other words served as a synonym or antonym for "rigor", and how "rigor" made distinctions in mathematics education. Therefore, we aim to share our current findings from Kappa, gather feedback from the RUME community towards a more complete manuscript using data from all three NICs, and spur motivation for others to investigate how "rigor" is used in their mathematical spaces.

Analysis

We selected a subset of data for analysis by locating instances of the word "rigor," taken together with relevant context. We leveraged critical discourse analysis—a method that examines how ideologies related to societal power relations shape discourse, both implicitly and explicitly (Fairclough, 1989, 2013). Our multi-stage analysis was methodologically aligned with reflexive thematic analysis, which generally includes six phases: familiarizing yourself with the dataset; coding; generating initial themes; developing and reviewing the themes; refining defining and naming the themes; and writing up the analysis (Braun et al., 2023). First, we read through all responses to the journal prompt "What does mathematics rigor mean to you?" as a basis for our preliminary codebook which included synonyms and antonyms as categories. The second stage consisted of developing our codebook using the subset of data which mentioned "rigor" and its surrounding context. We found "distinctions" as another category which referred to things that are described as distinct from rigor but could not necessarily be interpreted as an antonym. Each excerpt was coded by at least two authors, wherein interpretations of an excerpt were reconciled within partner-to-partner or full author group discussions. In the final stage, we focused on each category of synonyms, antonyms, and distinctions to find themes within the discourses.

Findings

We organize our findings by giving empirical illustrations of words used in relation to rigor: synonyms, antonyms, and distinctions. Distinctions refer to phrases used in relation to rigor not necessarily as similar or as in opposition. For synonyms and distinctions, similarly-used words or phrases were grouped together. There was only one instance for antonym.

Rigor as a Synonym

Deep/true understanding, justification, analytical/logical. When speaking about rigor, all

faculty in the NIC highlighted the educative benefits of rigor for students from their perspective. Senna, Skylar, and Tiersa each said that deep or true understanding is itself rigor or is required for rigor in their responses to "what does math rigor mean to you." For example, Senna shared, "I would say that something is mathematically rigorous if it requires a student to truly understand why something is true, and to be able to effectively communicate their ideas behind the concept." In her journal, Anna focused more on rigor's place in argumentation saying that rigor "means clearly constructed arguments toward a finding or conclusion. It means clearly constructed analytical/logical thought about the concepts over completing mechanics." As was in Draeger et al. (2013), faculty members of the NIC understood rigor through a positive lens of benefiting students in providing higher-order thinking or meaningful learning. This is not to say students are also not making these positive connections. For example, Ricardo shared,

> … [A] rigorous class would normally have expectations that students can extrapolate based on what they've learned, reaching conclusions that might go beyond what's being explicitly taught in lectures, which is a positive, challenging, and potentially rewarding experience.

However, overall, the faculty tended to speak about rigor more positively than students.

Difficult(y) and challenge. As for the students in the NIC, "difficult(y)" and "challenge" were used in relation to rigor. As a representative example, Timothy shared in his journal,

> I think that mathematical rigor is like difficulty and challenge in mathematics. And when I think of that rigor a lot of different things come to mind. I picture myself struggling in some of my math classes, not really understanding the content, um, feeling challenged, feeling like I don't, um, know what's going on, feeling like I'm not doing well.

As was in Draeger et al. (2015), students in the NIC understood rigor through a lens that they are being challenged and their experiences in class are made difficult for the mission of rigor.

Unnecessarily hard. Related to difficulty and challenge was the phrase "unnecessarily hard" which many NIC members picked up on from their department's ACT UP Math data. This had been particularly moving for the faculty members of the NIC who had not realized that mathematics at Kappa had such a negative reputation with the undergraduate students. During an interview, Skylar shared her dream with respect to the NIC's goal of transformation,

> My dream would be to change the narrative of "Kappa Math," so that it is no longer perceived as "unnecessarily hard." I think we are working towards this as a NIC, and that key steps will be interrogating our own perceptions of rigor and students as experts, as well as trying to better understand the source of this narrative for students so that we can address it.

We highlight here the importance of the descriptor "unnecessarily" and how Skylar (and also Tiersa) adopted the phrase from students. The sobering effect of "unnecessarily" gave a clear target for the NIC to strive for in their change efforts.

Clarity and precision. We found that students and faculty aligned in their perception that rigor is (or adds to) the clarity and precision that is required in discussing mathematics. Tiersa, for one, highlighted how rigor is linked with mathematical notation, stating that "I also feel that notation and precision in language are important." Kiara understood rigor as "being very focused on precision and careful detail checking." She added that, "it's all done meticulously and within a very set way of communicating in a set way of doing the things, um, that's like agreed upon." Again, we see that rigor is discussed in relation to clearly communicating ideas to others with a pronounced focus on precise notation as it relates to proof writing.

Rigor as an Antonym

Manipulating formulas. The idea that rigor is in diametric opposition from the manipulating formulas was the only instance of rigor being described using antonyms. When describing what

mathematical rigor meant to her, Skylar said it required students to "truly understand why something is true, and be able to effectively communicate their ideas behind the concept" and added that "**by contrast,** most of my students come from high school very adept at manipulating formulas." By contrasting manipulating formulas with rigor, Skylar echoed the associations mathematicians and faculty make with rigor and higher-order thinking (Draeger et al., 2013).

The two undergraduate students Timothy and Ricardo also echoed this diametric opposition. Timothy, in his response to "what does mathematics rigor mean to you," described experiences in high school as different from those at Kappa, "[rigor is] also **not like math in a high school** sense… where you're just given formulas and you have to plug them in and call it a day." For Ricardo, he understood that mathematical rigor is associated with the depth of content "**as opposed to** being explicitly told how to solve specific types of problems." Both students understood that the rigor they are experiencing at Kappa not only goes deeper but is disjointed from manipulating formulas. So, while faculty in the NIC began to understand the association of rigor with "unnecessarily hard" as a synonym, students also began to understand the association of rigor with "deep understanding" by viewing manipulating formulas as an antonym for it.

Distinctions

Excerpts coded as distinctions clearly described something as different from rigor, but not framed in opposition, i.e., an antonym. For example, the question of inclusivity in mathematics classroom was brought up in the context of rigor, "How can we keep rigor in the classroom, but in a way that doesn't feel exclusionary or daunting to the students?" (Senna). With regards to the NIC's change efforts, Skylar further discussed in an interview that she hoped to make "Kappa Math" a "kinder, gentler, more welcoming" department after hearing students' perspectives on rigor. And in her journals, Tiersa reflected on care and rigor (challenge),

> I think that before spending time with focus group data, I might have argued that "rigor" in our courses is part of what makes them rewarding. After looking at the data, I think that in rigorous courses with high expectations, students may need additional care and support in order to appreciate the challenge.

Thus, what we saw here are faculty members of the NIC making a distinction between rigor, inclusivity, and care—that both can exist simultaneously in a way that they hope would improve the undergraduate mathematics experience.

Implications & Discussion

As we aimed to demonstrate, "rigor" is an ambiguous word in our spaces. Draeger et al. (2013) and Draeger et al. (2015) already showed the different perspectives on rigor from a faculty perspective versus from a student perspective, and here we confirm this difference in the context of one undergraduate mathematics program. As demonstrated by this NIC, it is imperative for mathematical spaces to discuss these differing perspectives—wherein faculty in the NIC learned that their "Kappa Math" had a negative reputation while students in the NIC learned that rigor could be positively enacted and rewarding. Moreover, as in Hagman et al. (accepted), the interrogation of rigor can lead to an interrogation of power. In the case of Kappa, their interrogation of rigor led to student perspective and voices to be valued as their change and improvement efforts emphasized rewriting the "unnecessarily hard" narrative of Kappa. Thus, we urge the RUME community to interrogate our use, conceptualization, and operationalization of "rigor," so that when we say "rigor" in undergraduate mathematics, we know what we mean. So, our initiating discussion question for RUME is: *what does mathematics rigor mean to you?*

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Investigating GTA Beliefs That Might Reinforce Inequity in Pre/Calculus Courses

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In this preliminary study, we examine the beliefs of graduate teaching assistants and investigate how seemingly identity-neutral beliefs may reinforce deficit narratives in precalculus and calculus courses. Drawing on the Attributions of Mathematical Excellence (Jacobson et al., 2022), we report the analysis of one participant's attributional beliefs and how they may explicitly and subtly reify deficit discourses. Our findings underscore the need for critical conversations about instructor beliefs regarding success and struggle in undergraduate pre/calculus. We recommend training programs provide graduate teaching assistants the opportunity to engage in reflective dialogue to identify opportunities to disrupt harmful narratives in undergraduate mathematics education.

Keywords: Equity, Beliefs, Calculus, Precalculus, Teaching Assistants

Deficit narratives are built around mathematics being perceived as "objective" and "universal"; thus, all students are expected to experience mathematics in the same way. When students, especially those in marginalized groups, do not ascribe to the master narrative of mathematical experiences, educators often hold learners personally responsible for their struggles and underachievement while also refusing to make visible the mathematical understandings the students do hold (Adiredja & Louie, 2020). With this narrative in mind, instructors can often tout a racial- and gender-neutral perspective in mathematics. While this mindset can be seen as an avenue to mitigate harm based on identity, Leyva et. al. (2021) argues that students can still interpret their experiences in the classroom as racialized or gendered, even when race and/or gender is not explicitly mentioned in their classroom. The classroom environment is shaped by the instructional practices instructors utilize, and as prior research has empirically shown, teacher beliefs have a significant impact on their instructional practices (Schommer, 1990; Pajares, 1992; Wilhelm et al., 2017). Therefore, our research seeks to explore instructors' beliefs regarding mathematical success and struggle in college precalculus and calculus classrooms. Our research goal is to shine a light on instructors' beliefs that might explain why "seemingly neutral behaviors and comments" (Leyva et al., 2021) might reinforce or disrupt racist and sexist disparities in precalculus and calculus, especially among new and early career instructors in higher education. The goal of this report is to share preliminary results of our work towards our broader research goal, by sharing our analysis of one participant's beliefs along with a discussion on how those beliefs explicitly and subtly reify deficit narratives about students. We focus on precalculus and calculus because those undergraduate courses tend to serve as gatekeepers and impact interest and career choices within the STEM fields (Riegle-Crumb & Humphries, 2012; Levya et al., 2021).

Empirical Background and Theoretical Framework

Deficit discourses in mathematics education attribute faults in students' academic success and intellect to perceived defects of the student or their community (Adiredia & Louis, 2020). For example, some deficit discourses in mathematics education claim that Black and Latinx students are responsible for their own underachievement (Nasir & Shah, 2011), that "women and

Latinos are not good at mathematics" (Leyva, 2016, p. 109), or that Mexican and Black families and cultures do not value schoolwork (Solórzano & Yosso, 2002; Adiredia & Louie, 2020).

Recently, Jacobson et al. (2022) used theories from social psychology to characterize these deficit and anti-deficit discourses in mathematics education with their attributions of mathematical excellence (AME) framework. Within this framework, an attribution is a belief about a perceived cause to an effect. The AME framework allows researchers to characterize the attribution beliefs teachers hold about students' mathematical success or struggle into types informed by larger deterministic beliefs that tie back to the deficit discourses in mathematics education. There are four categories of attributions in the AME framework: genetic, personal, social, and educational. The genetic category, informed by the theory of Genetic Determinism (Keller, 2005), describes attributions with factors related to a student's genetics, like biological sex. The personal category, informed by the theory of School Meritocracy (Wiederkehr et al., 2015), describes attributions with factors related to students' personal characteristics, like effort. The social category, informed by the theory of Social Determinism (Rangel & Keller, 2011), describes attributions with factors related to students' upbringing and socialization. Lastly, the educational category describes attributions with factors related to students' prior educational opportunities. AME are thought to be productive in that AME-E challenges deficit discourses, or unproductive (all other types) which contribute to deficit discourses +. Importantly, the AME framework highlights how seemingly identity-neutral attribution beliefs correspond with negative racial and gender stereotypes (Jacobson et al., 2022). For example, the explanation that greater effort begets academic success (AME-personal) corresponds with the discourse in mathematics education about Black and Latinx students being underachievers. We seek to answer the research questions: *What are graduate instructors' beliefs around mathematical success and struggle in the context of precalculus and calculus?* and *In what ways do graduate instructors' beliefs about mathematical success and struggle reify or challenge deficit discourses in the context of precalculus and calculus?*

Methods

We employed a basic qualitative research design for this project (Merriam, 2015). We purposefully selected graduate teaching assistants serving as instructors for precalculus and calculus courses, because they are typically responsible for leading recitations, which generally involve greater interaction between students and instructors. In total, we interviewed four graduate students who were all enrolled in a doctorate program in mathematics, with different concentrations. The data for this research was collected using semi-structured zoom interviews designed to investigate teachers' attributional beliefs about the causes of students' successes and struggles in mathematics. Interview questions focused on what each participant believed to be the underlying cause of their students' successes or struggles in (pre)calculus courses. For example, the interview included questions like, "Why do some students succeed in (pre)calculus while others struggle?" Based on our prior work, (Roan et al., 2025), we anticipated that our participants' beliefs would be interconnected so we used follow-up questions such as, "Why do you think this happens?" which aimed at determining clusters of influential factors. To analyze the data, we categorized the attributions made in the interview based on the categories in the AME framework: Genetic, Social, Personal, and Educational. To explore the interconnectedness of our participants' attributions, we conducted causation coding (Saldana, 2021), which was used to create a visual model of our participants' attributional beliefs. Working together, the research team coordinated to ensure that their individual coding efforts standardized. This type of collaboration provides what Saldana (2021) calls "crowd-sourcing reality checks" that increase

the interrater reliability and interpretive congruence ensuring code agreement. This process also allowed for the consensus in coding "dialogical intersubjectivity" (Saldana, 2021) of unclear causes provided by the participants. As Saldana (2021) contends, "this dialogic exchange of ideas in a workshop and collegial atmosphere attunes the research team to new and varying perspectives before more intensive scrutiny" (p. 53) as analysis of the data progresses. To identify chains of influential factors, we applied three aspects of causal coding during analysis based on Saldana (2021) recommendations. Following a structure that maps antecedent or starting conditions → mediating variables → outcomes, where the arrow (→) denotes what a causal relationship "leads to". Given that multiple causes can result in multiple outcomes, subcodes were employed to represent the complex, multidimensional nature of belief system networks. These codes were then translated into causal diagrams, which visually illustrate the intricate relationships that are often difficult to fully convey through text alone.

In this report, we share preliminary results of our analysis by sharing some results from the analysis of one participant, pseudonym Frida. Frida was a graduate teaching assistant in a mathematics department at a large midwestern university. She identified as female and white European. Frida was an instructional assistant leading recitation for calculus one and calculus two multiple times, but she had never explicitly taught Precalculus. Frida was an instructor of record for a course designed to be a brief introduction to calculus one time.

Results

In this report, we will highlight two prominent clusters of Frida's attribution beliefs, one that explicitly places a deficit on students, and one that does so subtly. We note that the attribution clusters as reported are not self-contained. These beliefs are connected to a larger belief system present in the data, but we do not share this larger system due to space.

Frida said that mathematical success is performing well in the course, learning the material well, learning to study mathematics, and applying oneself. When asked what factors influence students' mathematical success in calculus, Frida said

> . . . some of them, probably genetic environment, like genetic factors. Some people just have a higher IQ than others. I guess environmental influences. What level of math exposure have they had to in previous courses. What's the level of education their parents have had, like these environmental sociological factors. Genetic, obviously, too. And then, I guess where it boils down more to the student is, have they learned maybe efficient ways to think about math . . .

In this quote, Frida attributed students' mathematical success to a) genetic factors, such as baseline intelligence and lineage, b) educational factors, like students' prior educational experiences, c) social factors, like education level of the parents, and c) personal factors, like efficient learning, and experiences. For this report, we focus on the first two attributions from Frida's quote. Frida connected genetic factors and success in college calculus as follows:

> . . . the genetic influence, the IQ, there's probably very little you can do to change like that baseline intelligence. . . . I've seen that also in myself and classmates like at a certain point, like even with work, even with trying to like, apply good study strategies. There's that certain point where IQ just does play that significant role.

In the quote, Frida explained that students who are very successful in college calculus, from her perspective, have a very high "baseline intelligence". For her, baseline intelligence is a genetic trait. She noted that this does not necessarily mean that students with a "low IQ" cannot be successful in mathematics, only that they will always be less successful than their peers with a "high IQ". When asked to describe the students with a high IQ, Frida explained "I've kind of

noticed males either tend to be really high on the IQ spectrum, or a bit low, and females tend to be more average in the middle. That's just from what I've observed." In this quote, Frida made the connected attributions that biological sex influences whether a student has a "high IQ", which influences mathematical success.

Frida explained prior educational experiences and success in college calculus as follows "If some of them have say, taken AP calculus before. They're also able to get to that next level." In this quote, Frida acknowledges that students who have taken AP calculus in high school are more likely to "get to that next level" of understanding, an indicator she had previously stated as her evidence for if a student was successful. When asked to describe the students are who have taken AP calculus before, Frida said they "have parents that are maybe in STEM themselves" or are "students that know pretty early on in high school, they're going into STEM, or they're going to be a math major". Based on these quotes, we inferred that Frida made two attributions, that taking AP calculus in high school is influenced by 1) whether a student's parents are in STEM, and by 2) a student's interest in STEM. Due to space, we only detail the first attribution in this report. When asked to explain why the students' parents being in STEM influenced the students taking of AP calculus, Frida said

> When you grow up, what you see is the norm you observe. So, I guess, maybe just seeing that, being familiar with it [math], [makes math] probably less of an obstacle and barrier. . . maybe some natural advantage. [pause] They have the, I guess, personal help at home if they need it.

In this quote, Frida makes three different attributions. First, Frida made the attribution that having parents in STEM influences how STEM is normalized for students, which influences whether the student sees STEM as a barrier, which influences whether they take AP calculus in high school. Second, Frida made the connected attributions that having parents in STEM influences whether students have a natural advantage in STEM which influences whether they take AP calculus. We infer that Frida meant natural advantage to be genetic in nature. We made this inference based on her prior statements regarding the connection between students' mathematical success and their parents. Specifically, she said that the parents were related to "environmental sociological factors. Genetic, obviously, too". Lastly, Frida connected the attributions, having parents in STEM influences the quality of help received at home, which influences whether students take AP calculus in high school. Together, these attributions formed an attribution beliefs cluster where each outcome was attributed to a prior attribution outcome see Figure 1.

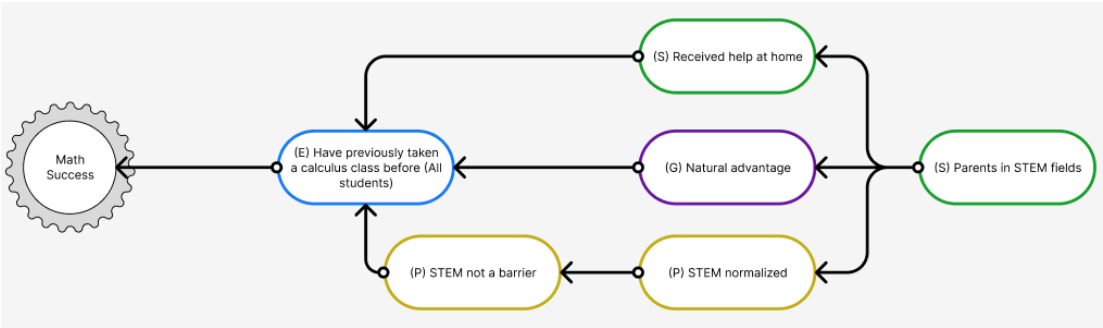

Figure 1 Part of Frida's attribution diagram that shows an anti-deficit explanation based in deficit explanations. The colors correspond to the categories of AME: Genetic (Purple), Social (Green), Personal (Yellow), and Educational (Blue).

Preliminary Nature of Proposal and Discussion

In this proposal, we presented an in-progress analysis of graduate teaching assistants' beliefs about the nature of success and struggle in mathematics education. Specifically, we reported two attribution clusters of one of our participants, Frida. In the first attribution cluster, Frida attributed biological sex to innate intelligence, which directly influences success in college calculus. From our interpretation, this attribution cluster shows beliefs that explicitly place a deficit on the student. She further explains that males tend to be high on the IQ scale while females tend to be average. This reifies the deficit narrative that women are innately less intelligent than men. In the second attribution cluster, Frida attributed success in college calculus to prior experience with calculus. This could be seen as an anti-deficit explanation of student success because Frida attributed students' success in college calculus to experiences in AP calculus in high school. That does not explicitly put a deficit upon the students nor their families. However, Frida connected experience in AP calculus to factors like having support for mathematics at home, feeling that STEM is not a barrier, and having a natural advantage, all of which stem from having parents in STEM fields. Those are unproductive explanations because Frida is attributing students' success to traits belonging either to the student or to their families. From our theoretical framework, this attribution cluster shows that productive beliefs based around educational opportunities can be traced back to deficit narratives, such as who belongs in STEM. Here, seemingly neutral attributions (i.e., parents in a STEM fields), when viewed through our theoretical lens, become linked to beliefs in genetic (Keller, 2005) and social determinism (Rangel & Keller, 2011). When put into the context of the ways in which women and people of color have historically been marginalized in STEM fields, Frida's belief becomes deficit in nature, steeped in both racist and sexist ideologies. This indicates that anti-deficit beliefs can be shaped by deficit narratives

Our findings also connect to the idea of "objectivity discourse", in which instructors consider experiences to be race- and gender-neutral, while concurrently acknowledging students' experiences differ (McNeill et al., 2022). McNeill et al. (2022) noted that holding conversations to address these perspectives can be useful. We agree with this notion and respond to the call by Jackson et al. (2017) to shift teachers' perspectives on their students' capabilities by recommending that instructors should be given the opportunity to reflect upon their beliefs regarding students' success and struggle. Given the population of our study, we focus our suggestion for graduate preparation programs. Prior research indicates that the level of pedagogical training for graduate students varies vastly per institution (Harris et al., 2009). Additionally, research on graduate teaching assistant training has focused on more cognitive aspects of teaching (e.g. Speer, Gutmann, & Murphy, 2005; Ellis, 2015). While there is research that target equity in math education through graduate assistant training (e.g., Taylor et al., 2025), we see a vital need for graduate programs to hold conversations around student success and failure, especially for instructors teaching gateway courses like precalculus and calculus.

We intend to continue data analysis and gather feedback from the RUME community. We welcome any feedback on the presentation of the data, particularly regarding the diagram in Figure 1. We also welcome feedback on connections to literature to support situating our study with the larger conversation regarding equity in undergraduate mathematics education. Lastly, we recognize that connections between the participants' beliefs and deficit narratives revolve around race and gender. Thus, we ask the reader: in what ways might we add a more critical lens within the research design and analysis?

Acknowledgments

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Developing Position From Velocity Through Playful Exploration and Feedback

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Kinematics graphs offer potential for developing relationships between derivates and antiderivatives, but students experience challenges with recognizing the meaning of the area under a velocity graph as change in position. We designed a series of playful math tasks offering structured exploration, in which students worked in virtual environments to develop relationships between speed, time, and distance in tabular and graphical formats. Preliminary analysis suggests that students can use rate reasoning to develop accumulated distance under constant velocity graphs, which then can be developed under linear graphs via Riemann sums.

Keywords: Mathematical play, calculus, kinematics, velocity, position

Graphs related to kinematics, such as position, velocity, and acceleration, play an important role in calculus instruction (Barniol et al., 2024; Doorman & Gravemeijer, 2009), offering a rich potential for addressing the relationships between derivatives and antiderivatives (Doorman et al., 2022). An understanding of the kinematics concepts requires students to make sense of the relationships between the graphs of position (the function), velocity (first derivative), and acceleration (second derivative), as well as conceptualizing the antiderivative as the area under the curve between, for instance, the velocity and the change in position (Dominguez et al., 2017). Despite the centrality of these ideas, research indicates student difficulties in making connections between kinematics concepts and their graphical representations (McDermott et al., 1987). In particular, it can be challenging for students to recognize the meaning of the area under the curve for a velocity or acceleration graph (Bajracharya et al., 2012; Beichner, 1994; Martínez-Miraval & García-Cuéllar, 2020), much less to construct meaning for why the area under a velocity curve, for instance, represents change in position.

We became curious about the viability of helping students meaningfully develop the area under a graph (such as a velocity graph) as change in position (i.e., distance traveled) through playful exploration. In working with a group of pre-service secondary mathematics teachers, all who had successfully completed a calculus sequence, we implemented a task to promote an understanding of the relationship between position, velocity, and acceleration in terms of integrals and derivatives (Ely et al., 2025). The task required students to move an astronaut 10 miles in 10 minutes via a constant acceleration (or a series of accelerations) for specified time periods. For instance, if one uses an acceleration of $a(t) = 0.2$ miles / min^2 from $t = 0$ to $t = 10$, the astronaut's velocity is the antiderivative, $v(t) = 0.2t$, $0 \le t \le 10$. One could then find the total distance traveled by integration, either by reasoning with the area under the velocity graph or symbolically as $\int_0^{10} v(t)dt = 0.1t^2 \big|_0^{10} = 10$ miles. We found that our participants struggled to distinguish between acceleration and velocity and did not draw on their understanding of integration to determine exact distance values, findings that mirrored the literature. This led us to

wonder whether we could we support students to reason about change in position from velocity graphs. Specifically, could we help students meaningfully consider the area under a velocity curve as a representation of distance traveled by leveraging structured exploration through playful math tasks? Our preliminary findings suggest that students can use their rate reasoning to develop accumulated distance under constant velocity graphs, which then can be developed under linear graphs via Riemann sums with instructor scaffolding.

Background and Theoretical Framing: Area as Distance and Mathematical Play

Areas Under Velocity Curves

We anticipated that students might be able to leverage both geometric reasoning and rate reasoning to develop area as a measure of distance. Regarding the former, the Mean Speed Rule could be relevant to students' reasoning. The rule states that if, over a period of time, Object A has a constant velocity and Object B constantly accelerates (or decelerates) with the same average velocity as A, then the two objects will travel the same distance. Mathematicians at Oxford developed this in the early 1300s, three centuries before coordinate geometry and without referring to area under a velocity curve. One of these scholars, William Heytesbury, reasoned with situational symmetry to justify the rule: If one object constantly accelerates from a velocity of four to eight (no units specified), and another decelerates from four to zero, the latter will offset the former at each moment, so they together must traverse twice the distance of a third object moving at a constant speed of four (Katz, 1998).

Regarding rate reasoning, students may be able to leverage their unit rate understanding to reason about why one can multiply a height by a width to yield an area in the first place (beyond simply relying on the trick of 'canceling units', e.g., ft/sec · sec = ft). Izsák (2025) suggested a general multiplication image that treats velocity as a unit rate. From this perspective we can see a y-value on a velocity graph, such as 20 ft/sec, not just as a one-dimensional height, but as a rate of 20 ft in 1 sec, corresponding to an imagined two-dimensional stack of 20 squares in a 1-sec-width column. With this image, each square is a foot, so one can imagine each second on the x-axis accumulating 20 ft in distance as represented by the area of a two-dimensional column with a width of 1 sec.

Mathematical Play, Exploration, and Feedback

We define mathematical play as entailing exploration, self-selection of goals, immersion, and enjoyment (Ellis et al., 2025). Of particular importance to our focus is exploration, which is a key element of play (Gresalfi et al., 2018) and which can occur in a physical or virtual environment, or even in an imagined realm in which students can run mental simulations. Exploration often starts with free play (Mason, 2019), in which learners familiarize themselves with the task, the environment, and its affordances. It entails asking questions such as "what happens if I try this?" and noticing the results of one's actions. As learners' actions become more goal-oriented, they will attend to the feedback from those actions and modify their subsequent actions in response (Williams-Pierce & Thevenow-Harrison, 2021). Over time, one's actions can then become increasingly anticipatory, with students ultimately able to make mental simulations without having to carry them out in the environment (Simon, 2011).

Researchers have discussed mathematicians' playful activity in terms of exploration and feedback (e.g, Holton et al., 2001; Mason, 2019; Su, 2020). For instance, Su (2020) described an inquiry phase, in which one engages in pattern exploration and reasons from empirical feedback to build conjectures from specific examples. Holton and colleagues (2001) similarly described a

process in which mathematicians produce tentative examples without a clear direction to make sense of the problem space before then becoming more systematic and intentional in generating specific cases and conjectures. Mathematicians are adept at engaging in exploration and generating their own feedback, often relying only on paper and pencil or even engaging in purely mental simulations. Students, however, may need more structured environments in which they can try out different actions and observe empirical feedback. For that reason, we designed a series of playful math tasks that encourage exploration while also providing a method for intrinsic feedback. These tasks place students in a virtual environment in which they can explore what happens when they input different speed/time values or acceleration/time values to achieve various goals, such as making an astronaut reach 10 miles in 10 minutes. By providing visual feedback, our intention was to help students structure their exploratory actions so that they could shift to more anticipatory actions over time, specifically by developing their own velocity graphs to make objects travel a specified distance in a specified time.

Methods

We collected data with 7 high-school students (2 9th-grade students, 5 10th-grade students) during an 8-session summer design experiment (Cobb et al., 2003). The sessions took place over a two-week period and each lasted 90 minutes. The students worked in small groups using iPads, and 6 members of the author team were present to observe, engage students in questioning, and run the cameras. We collected video data and screen recordings on each group's iPad as well as all written work. We designed a series of playful math activities to structure students' exploration of constant rate via speed situations, in which students could input various speed and time values to achieve goals such as helping a character reach a specified destination at a particular time. This report is based on the engagement of one trio of students, Adam, John, and Zilar (pseudonyms), during two activities called *Go Karts* and *Describe My Journey*.

Go Karts is a Desmos activity that shows a red go kart that moves at a constant speed of 10 ft/sec, traveling 200 ft in 20 sec. Students can input up to five velocity/start time/end time values in a table to make a blue go kart reach the finish line at the same time. For instance, one could input the following four entries to make the blue go kart tie with the red go kart: 40 ft/sec from 0 sec to 1 sec; 0 ft/sec from 1 sec to 10 sec; 15 ft/sec from 10 sec to 18 sec; and 80 ft/sec from 19.5 sec to 20 sec. Once they have chosen values, students can hit play and watch both go karts run the race to observe the outcome of their actions. We then shifted from tabular inputs to a graphical input, providing students with a blank graph velocity / time graph. Students could then draw graphs to make their go kart tie with the red go kart. Our intention was to support students in creating constant graphs (such as a constant function at 10 ft/sec) and to use their rate reasoning to consider how much distance would be traveled after 1 sec (10 ft), 2 sec (20 ft), and so forth, relating distance traveled to the rectangular area under the constant velocity function. We then anticipated that students might explore linear and piecewise linear graphs, using geometric reasoning similar to the mean speed rule to calculate the areas of triangles under the velocity graph to determine distance traveled. The second activity was *Describe My Journey*, in which groups of students created and traded velocity/time graphs. Each group's task was then to describe the go kart's journey based on the provided graph.

Our coding process is preliminary and ongoing. We began by transcribing the video of the students' activity and enhancing the transcript with relevant images, pauses, and descriptions of the students' gestures. We then completed an initial reading of the data noting where in the tasks students reasoned about velocity and distance. Once we identified those data portions, we examined the supports for students' reasoning, including the use of feedback from the playful

math tasks as well as more direct scaffolding from the teacher-researchers (TRs). Our continuing process will next shift to analyzing the interrelated nature between students' exploration, empirical feedback, and emerging conjectures and strategies.

Results

The students explored different graphs with the *Go Kart* activity, developing first a constant graph at 10 ft/sec for 20 sec, and then reasoning geometrically about symmetry above and below the $v(t) = 10$ line to develop piecewise constant velocity functions that yielded a total distance of 200 ft. For instance, they created a graph for the function $v(t) = \begin{cases} 12, & 0 \le t \le 10 \\ 8, & 10 < t \le 20 \end{cases}$ and were able to predict that for the first 10 sec the go kart would travel 120 ft and then it would travel 80 ft for the remaining 10 sec. They also created the piecewise constant function seen in Figure 1a, predicting before running the simulation that their go kart would tie with the red go kart. Adam explained, "The velocity's 30, and that's feet per second, so 30 feet per second and that's 5 seconds, so 30 times five that's 150 feet. So, 150 feet, so that's, and you're trying to go 200. So, you have 150, and, it stops twice so that's not adding anything. And the last one, it's 5 seconds, or, 10 velocity, so five times 10 is 50, and that's 200." The students could also predict that the graph provided by a TR (Figure 1b) would beat the red go kart because it would travel a total of 216 ft in 20 sec (14 ft/sec for 12 sec and 8 ft/sec for 8 sec). However, during the creation of these graphs, the students did not produce linear functions. It was unclear to us whether they would be able to develop area under a linear velocity function to determine total distance traveled.

During the next activity, *Describe My Journey*, Adam, John, and Zilar worked with a graph produced by their classmates (Figure 1c). They could calculate slope values and make sense of the graph in terms of the go kart's journey. They interpreted the constant portions of the graph as constant speeds and they interpreted the portions with negative slope values as periods of deceleration. They could also interpret the portions of the graph below the *t*-axis. For instance, for the portion of the graph from $t = 5$ to $t = 15$, the slope is -1.1. Adam described the go kart as going backwards for 10 seconds, accelerating at a rate of "1.1 feet per second per second." Similarly, for the portion of the graph from $t = 19$ to $t = 25$, Adam explained, "So you decelerate at 1 feet per second per second for six seconds. This one isn't even that hard!" Describing the final portion of the graph (below the *t*-axis with a positive slope), John explained, "You slowly go back to zero. So, you're slowing, you're backing up until you're at zero."

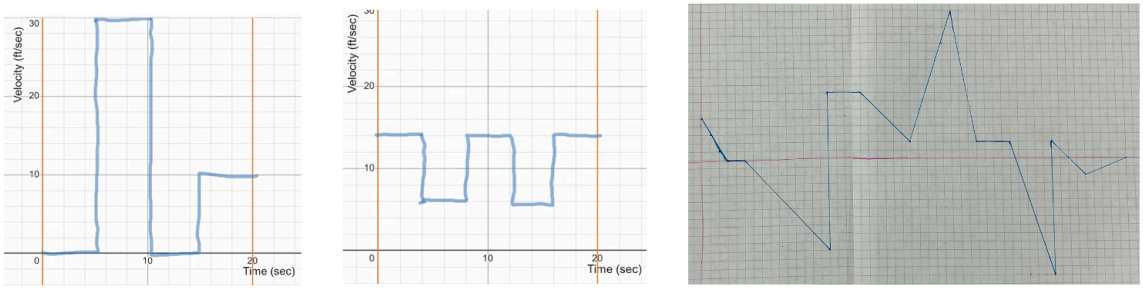

Figure 1. Student-produced velocity/time graph of a go kart's journey (a), instructor-produced graph (b), and student-produced graph for the Describe My Journey activity (c).

At no time during their interpretation of the graph did the students determine distance traveled. A TR therefore asked the students how far the go kart traveled during the constant portion of the graph from $t = 15$ to $t = 19$ where $v(t) = 8$. Adam responded, "You're going 8 feet

per second for 4 seconds. So, you're going 32 feet." After the students then determined the distance traveled for the other constant portion of the graph, the TR asked about the linear portion from $t = 19$ to $t = 25$. John reasoned, "You de-accelerate at a rate of one foot per second per second per six seconds" and Adam replied, "So, you go six feet?" John replied, "Well, you start at eight and you end at two for velocity." Adam responded, "You're going 1 feet per second for six seconds," and John objected, saying, "Well, you're *de-accelerating* at a rate of 1 foot per second," and Adam and Zilar both agreed. The TR asked, "Tell me more about the speed. So how is it changing?" John responded, "It's decreasing at one foot per second, per second. From eight feet per second, per second, to two."

The TR then suggested breaking that portion of the graph into a finer picture and asked the students to consider what would happen in the first second. John said, "You go down by one foot per second, per second." The TR asked, "Now, if I ask you how much you travel in that one second, what would you say?" John hesitantly responded, "Seven point five?" The TR asked why that would be the case, and John traced a horizontal line across the portion of the graph from $t = 19$ to $t = 20$ and hesitantly said, "Because if you're going at eight and seven, would it average out to these 7.5 straight line?" We interpret John as imagining a constant line at $y = 7.5$, and the TR suggested that the students pursue this idea further. Adam said, "Like, six point five," indicating the amount traveled in the next second, and John said, "five point five," and Adam continued, "Four point five plus three point five plus two point five…" and John concluded, "Which is thirty."

Preliminary Questions and Discussion

We had anticipated that the students would make a connection to area under piecewise constant velocity graphs by perceiving the area of rectangles as in Figures 1a and 1b. However, there was little mention of area in the students' conversation. Instead, they often appeared to be reasoning with rates but not directly with area. The students understood that x ft/sec meant traveling x ft in 1 sec, $2x$ ft in 2 sec, and so forth by iteration. Our initial observation, both from the data reported here and from our similar work with pre-service teachers, is that students with this reasoning can then approach the problem of finding the distance traveled with a linear velocity with a Riemann sum, leveraging reasoning similar to the mean speed rule; this seems to be a natural concept that emerges from reasoning with rates to determine accumulated distances. This suggests that certain cognitive activities related to calculus are shared by secondary and undergraduate students.

Our initial analysis indicates that the exploration and feedback students engaged in when creating and analyzing graphs for the *Go Kart* activity supported their ability to predict distances using constant and piecewise constant velocity graphs, as over time they began to be able to make those predictions without needing to run the simulations. We would now like to better understand how to leverage structured exploration and feedback to make the area connection more salient for students, or even to perhaps necessitate it (Harel, 2013) through the task structure. We will therefore engage the audience in the following discussion questions: (a) What is the role of exploration and feedback in supporting students' development of area as distance traveled under velocity curves? (b) What adjustments to task structure could we make to support the transition from constant velocity graphs to linear and piecewise linear velocity graphs?

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Embedded Math Tutor's Noticing: A Collaborative Exploration and Change of Practice

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Embedded tutor (ET) programs have increased significantly in recent years to respond to new state requirements to phase out remedial math courses and increased need for tutoring support brought about by the COVID-19 pandemic. Despite this increase, ETs have remained understudied. ETs can face limitations in the classroom when instructors are underprepared to integrate ETs into their lessons. In particular, instructors can be unaware of the physical limitations ETs encounter in their practice, which can prevent them from delivering needed support to students. To better understand how ETs develop their practice this study uses the construct of teacher noticing to analyze interviews with an ET as they reflect on video recordings of their instruction. Findings demonstrate how an ET identified and addressed classroom features that limited their access to students. This study presents implications for faculty to ensure the presence of a collaborative work relationship between ETs, Instructors, and faculty.

Keywords: Embedded Tutor, Teacher Noticing, Tutoring

Embedded tutors (ETs)—who provide in-class support and out-of-class tutoring (Chaves et al., 2023; Racchini, 2020)—are understudied despite their increasingly important role in undergraduate education. Colleges and universities have leaned on ETs as alternative support systems to remedial math, which have been shown to decrease retention rates (Bahr, 2008) and to have disproportionately negatively impacted the success rate of minoritized students (Bahr, 2010). State legislation that began a phasing out of remedial mathematics, like in California, prompted some universities to begin ET programs as an alternative support system for students (Duffy & Burkander, 2023). The COVID-19 pandemic was another motivating factor that prompted universities to place ETs in virtual classrooms, at a time with remote learning had created distance between students and traditional tutoring (Chaves et al., 2023). While ET programs have increased in popularity over the past decade, there is evidence to suggest ETs' teaching assets are being underutilized because faculty and instructors are not prepared to shape instruction to include the ET (Cawley et al., 2022). However, existing research on the practices of ETs' is not yet extensive (Channing & Okada, 2020), providing limited guidance for how instructors can better leverage ETs' strengths in the classroom.

Evidence based training programs could provide a space for collaborative learning for both the instructor and ET. Training programs could provide information the instructor needs to best integrate ETs into their classroom and lessons. Training programs could also help ETs voice principles they draw on in their practice and recognize the assets they could be utilizing in the classroom. This study hopes to provide evidence that could inform ET and instructor training programs. This study uses the construct of teachers noticing to analyze a conversation between an ET and I as we reflect on classroom video recordings of their practice. In particular my research questions is: How can we collaboratively work through limitations to an ET's practice that are imposed by the classroom environment?

Theoretical Perspective

For this study, I build upon teacher noticing, and in particular I deploy Star and Strickland's (2008) classroom feature framework as an analytical framework to investigate what ETs attend

to and interpret as important features of the classroom. Although there are multiple definitions of teachers noticing, these definitions share three common components (Wei et al., 2023). First, teachers draw their attention to what is noticed, which they label as *attending*. Second, teachers then interpret what they are attending to, which they label as *interpreting*. Finally, some definitions of teacher noticing have a third step related to the teacher's *response* after they have interpreted what they have attended to. The classroom feature framework was developed to evaluate preservice teachers' development of teaching practices (Star & Stickland, 2008).

Teachers' connections between broader teaching principles and the classroom environment are mediated by what they notice (Star & Strickland, 2008). Star and Stickland claim that the most fundamental component of noticing is the identification of "important or noteworthy" classroom situations. To inform a survey for preservice teachers, they identified five classroom features: Classroom environment (e.g., desk arrangements, materials, and class size), classroom management (e.g., teacher physical presence, changes of pace, and moves to deal with disruption), tasks (e.g., warm-ups, taking notes), mathematical content (e.g., representation of the math) communication (e.g., student to student interactions or teacher to student interactions). While Star and Stickland acknowledge the limitation of their framework for its lack of exhaustion, possibility of category overlap, and the arbitrary choice of important and unimportant features, I argue it is sufficient for this study as it was for theirs. Similar to teachers, ETs knowledge and experiences are mediated by what they notice. The framework captures a variety of classroom features "in order to assess the full range of what novice teachers do and do not notice" (p.112). ETs are also novice educators in the classroom, and the framework serves to provide a foundation of what is recognized as a limitation.

Literature Review

ETs have a measurable positive impact on retention rates (Channing & Okada, 2020) and grades (Hayes, 2021). While retention rates and grades are important markers of success for students, there is a social component of success within mathematics spaces that are important for long-term student engagement with mathematics. When ETs are in the classroom, students can "gain a familiarity with the tutor" (Racchini, 2020, p.49). Tutors can leverage this relationship to provide psycho-social support (Henry et al., 2011).

While it is important to make clear the demands of an ET in the classroom, those demands must be supported by training instructors in addition to ETs. Specifically, instructors must know the purpose of the ET program, and the function of the ET in their classroom so that everyone is "held accountable for the success of the program" (Chavez, 2023, p.170). Instructors must integrate ETs into classroom activities, introduce and emphasize their role in the classroom, and communicate with the ET and their supervisor (Chavez, 2023). When instructors are not trained to integrate an ET in the classroom or emphasize their role, that ET can struggle to build rapport with the classroom (Cawley et al., 2022). For example, when an ET is relegated to inaccessible parts of the classroom, then opportunities to impart their wealth of knowledge or to act as a role model in the classroom for students are limited. An incomplete understanding of ETs practices can contribute to the underutilization of an ET in the classroom and limit the efficacy of classroom support for students.

Research that could inform the development of ET training programs is sparse. Even when scholars have identified characteristics of a successful tutoring program, their grounding in student or faculty perceptions—rather than ETs' experiences—limits their applicability to improve training programs. For example, Chavez and colleagues (2023) generated their model of ET from a survey sent to a listserv of Learning Center Professionals, group discussions, and

reports from four participating colleges that analyzed their own program. The ET program best practices construct provides principles for ETs to draw on during their practice, like ETs should get to know students, or that ETs should not enforce classroom discipline. The framework, however, does not propose a model for an effective classroom environment for ETs, or a model for ETs practices. For example, while the framework promotes the importance of frequent communication between instructors, students, and ETs, the framework does not adress how a classroom should be constructed to promote communication or how an ET could navigate a classroom layout that creates space between an ET and a student, as seen in Cawley and colleague's (2022) investigation. To better support ETs in their practice, research exploring ETs' classroom practices, and how they may be improved, is needed.

Methodology

Context

This study took place at a research-intensive Midwest state university during the 2025 spring semester. The participant of this preliminary report, Lucy (pseudonym), was an ET assigned to college algebra stretch A. College algebra stretch is a sequence of two courses that break up the college algebra curriculum into two semesters --- college algebra stretch A and college algebra stretch B. Students enrolled in this course during the spring semester were students who did not pass the same course in the previous semester or who just started the sequence. Classroom instruction included direct instruction, independent activities, and group activities. Lucy engaged with students primarily during individual or group activities. Lucy, an undergraduate student, was responsible for grading, proctoring exams, holding tutoring hours, and providing in-class tutoring support.

This study employed a purposive sampling method; ETs were recruited using a mix of a priori criteria and contingent criteria (Bryman, 2012). A priori criteria restricted the sample to ETs for college algebra stretch because I had the opportunity to build rapport with these ETs through my role as a lead TA. As a lead TA, I facilitated group meetings and discussions to address common issues that emerged during their practice. The contingent criteria selected for scheduling compatibilities, as well as consent from the instructor and students. From the four studied participants, Lucy was selected as a focal case because preliminary coding revealed strong alignment with the framework and their practice uniquely responded to classroom limitations.

Data Generation

Data generation began during the 10^{th} week of the semester, at which point I started taking video recordings of the classroom and audio recordings of ET-student interactions. Lucy and I reviewed these recordings from the classroom to motivate both of our 30-minute conversations. When I emailed Lucy the link to the recordings of the classroom, I asked who they were attending to in the classroom and what moments stood out to them. My first conversation with Lucy took place in the same classroom environment where she served as an ET. The second conversation was held in a room nearby due to scheduling conflicts.

The goal of this study was to investigate how Lucy and I could collaboratively move through limitations on her practice. I made the methodological decision to center Lucy' experiences, since an over reliance of student and instructor perspectives produced findings that are limited in their potential to improve training programs. I began the conversation by asking Lucy how she felt reviewing her video recordings. She responded that she felt uncomfortable but also indicated

she had attended to the space she took up in the classroom. At the beginning of the conversation, Lucy set the stage for what she wanted to talk about, and I followed her down this line of conversation. Lucy's responses aligned with Star and Strickland's (2008) classroom features (i.e., classroom environment), and Cawley and colleagues (2022) findings that suggest desk layout limited ET-student interactions. My follow-up question aimed to have Lucy consider whether the space she took up in the classroom was limited by the desk layout. The conversation that ensued is presented in the findings below.

Data Analysis

Data collection and analysis overlapped and mutually influential (Holmes, 2014). During classroom recordings I took analytic notes to document how ETs navigated the classroom environment. Preliminary analytic themes guided the data collection conversations. Once data was collected, I deductively coded the transcripts (MacLure, 2013) from both of our conversations to identify moments when Lucy was attending to the classroom environment (Star & Strickland, 2008), and where our conversations focused on working through environmental limitations.

Findings

The answer to the research question "How can we collaboratively respond to classroom characteristics that limit an ET's practice?" came from Lucy's and my conversation about classroom layout. In Lucy's opening response in our first conversation, she indicated that she was aware she is primarily "walking up and down the aisles staring people down". In response, I offered my opinion that the room is "awkwardly structured" and asked whether they felt they were influenced by the layout "to be in certain parts of the room". While I made the decision to explore the topic of classroom layout, Lucy responded, "that was one thing I was sort of thinking about as I was watching back the videos". By being in the room shown on the left side of Figure 1, Lucy was able to point to specific rows that she walked down "because that one has the most space". She elaborated that being able to walk into aisles enabled her to "catch people's eye more". Furthermore, Lucy hoped that by walking down the aisle more often, even though some aisles were "skinny", she would be able to "get more people to reach out if they have questions". Lucy's solution to the limiting "skinny" aisles was to just find a way through them, but I offered an alternative solution. While some aisles were made wider through an intervention with another ET, I also proposed that we could make a third column of tables as shown on the right side of Figure 1. Lucy responded positively to the suggestion and believed that it would "get more people to talk to us". At the end of our conversation, I summarized three major points of discussion, but in response Lucy only reiterated her "love" for the "idea of rearranging desks" or "just trying to walk up and down the aisle a bit more". In summary, prompted by the video recordings, Lucy attended to her position in the classroom, and I guided the conversation towards an investigation of the classroom environment. Lucy attended to the limitations of "skinny" aisles on her practice, and through both of us suggesting solutions, we arrived at a plan of action to respond to the limitation.

After our conversation, I arrived early at Lucy's class, asked the instructor for permission, and proceeded to rearrange the tables. During our second conversation, when asked about the impact of moving to a three-column table layout, Lucy celebrated the change. Lucy felt that the classroom felt more "active" and "loved" the change but proposed that there were other factors at play that could have contributed to the "active" classroom. Later in the conversation, however, we were able to identify a moment in the classroom recording where she was interacting with a

student who had been difficult to reach in the past. Lucy proposed that they did not frequently interact in the past because their seat was "in a row and not at the edge of the row". Lucy was referencing that in the previous classroom layout, there are only 10 students who sit at the edge of a row, but the new layout places twenty students at the edge of the row. At the end of our conversation Lucy shared that she felt she has "really grown as a [ET]" by "having these talks and conversations". The second interview validated Lucy's interpretation that the table orientation was limiting her interactions with students and showed the impact that a relatively minimal change to the classroom limitations could have on an ETs' practice.

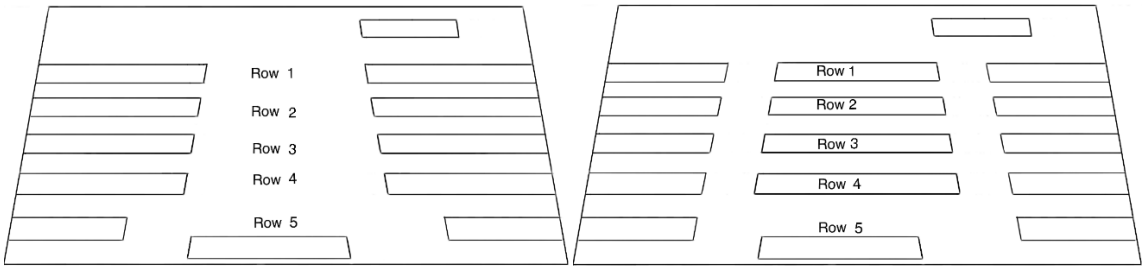

Figure 1: Sketch of the classroom before and after rearranging desks

Discussion

This preliminary report develops knowledge of ET practices that may provide needed evidence-based strategies for ET training programs. While the instructor was able to shape tasks of the class to allow the ET to gain familiarity with some of the students, as expected by Racchini (2020), the instructor had an incomplete understanding of what Lucy was attending to in the classroom. Reflecting the findings from Cawley and colleagues (2022), Lucy was relegated to parts of the classroom that were inaccessible to some of the students. While Lucy showed a desire to adhere to principles of getting to know the students (Chavez, 2023), there was a breakdown of communication between the ET and the instructor. These findings suggest that two-way communication between ETs and instructors could lead to collaborative problem solving to maximize the potential reach of the ET.

This preliminary report is a product of initial deductive coding using the classroom feature framework of one participant of the study. Future research will include the deductive coding of three other participants' conversations, and thematic analysis of these conversations, identifying aspects of ET practices that both align with and depart from Star and Strickland's (2008) framework for noticing of classroom features to expand our understanding of ETs' assets.

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Indifference as a Key Aspect of a Global Perspective on Equivalence Relations

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Equivalence is a central concept of mathematics and is a fixture in the K-16 curriculum. Though equivalence has been very well-studied overall, very little is known about how students reason with formal equivalence relations in advanced mathematics. Recent research has demonstrated that one component of productive reasoning about equivalence relations entails the ability to flexibly move between viewing equivalence from both an elementwise (local) perspective and a set-oriented (global) perspective. Building on this work, we report on a pilot study intended to aid the development of a conceptual analysis of equivalence in advanced mathematics - in which we seek to refine the notion of a global perspective on equivalence. In this preliminary report, by showcasing excerpts from task-based clinical interviews with two advanced mathematics students, we propose 'indifference' as a beneficial way of reasoning about equivalence from a global perspective.

Keywords: Equivalence Relations, Indifference, Global Perspective

Equivalence is an idea that transcends many areas of mathematics, from the relatively concrete equality of representations of rational numbers and algebraic expressions to the more abstract notion of decontextualized equivalence relations. Though equivalence has been extensively studied in the mathematics education literature, nearly all of it has focused on students' conceptions and uses of equivalence in K-12 contexts (e.g., Kieran, 1981; Knuth et al., 2006). At the advanced level, the existing research has focused on how students reason with the standard properties that define equivalence relations (e.g., Chin & Tall, 2000; 2001; Asghari, 2004b; Hamdan, 2006; Brandt, 2013). At the moment, the literature is not positioned to address how advanced mathematics students might productively leverage notions of equivalence relations themselves to help solve a problem. Payne & Cook (2025) highlight a distinction between an element-focused local perspective and a set-focused, global perspective on equivalence relations, arguing that the flexibility to transition between these two perspectives is productive for students. These perspectives are broad categories of student reasoning; as defined, their grain-size does not detail how students may identify two elements as equivalent, or the ways in which students might reason about sets of related elements. In the case of a local perspective, this gap has been filled by research that has, for example, examined how students might reason about equivalent fractions (e.g., Smith, 1995) and numerical expressions (e.g., McNeil et al., 2006). Other work has examined general ways of reasoning about equivalence that are relevant across multiple types of objects (e.g., Cook et al., 2022). The same cannot be said for a global perspective, as little has been done to investigate students' reasoning with sets of equivalent elements. Put another way, since a global perspective requires attention to a set of related elements, most equivalence literature has a local focus (e.g., Kieran, 1981) leaving reasoning about equivalence from a global perspective understudied. Building in part upon the work of Payne & Cook (2025) with respect to equivalence relations, we work to answer the question: What aspects of a global perspective on an equivalence relation are productive for students? Based on the equivalence relations literature, and data collected from pilot interviews with advanced undergraduates, we hypothesize that a productive feature of reasoning from a

global perspective is the ability to adopt an indifference towards the choice of element within an equivalence class.

Literature and Theory

This is a pilot study to inform a dissertation project aimed at developing a conceptual analysis (Thompson, 2008) that describes ways of knowing equivalence relations that are propitious for students. The literature offers a few insights regarding such understandings. Several authors have previously used the term *global* to describe a perspective that considers sets of equivalent elements, though authors use this term in slightly varying ways. With respect to equivalence relations, Payne & Cook (2025) articulate a *global* perspective on an equivalence relation as being characterized by reasoning about collections of equivalent elements (such as equivalence classes). These authors indicate both that advanced mathematics students reason about equivalence relations in a way that matches this description, and that it is productive. Adopting this definition, a global perspective will refer to reasoning that involves sets of equivalent elements, and a *local* perspective will involve particular equivalent elements. To illustrate, in Arnon et al. (2001) students reasoned about lines with rational slopes as proxies for equivalence classes of fractions (which would require a global perspective) *prior* to learning algorithms to transform fractions into their equivalent forms (which would require a local perspective). However, *most* equivalence literature at the K-12 level has focused (implicitly) on local perspective. Since equivalence classes emerge as a fundamental aspect of equivalence in advanced mathematics, there is a need to study the global perspective as well. Moreover, many studies which focus explicitly on equivalence relations at the undergraduate level concern how students work with their reflexive, symmetric, and transitive properties, and on how these properties relate elements to each other, (e.g., Asghari, 2004a; 2004b; 2005), but do not investigate the specific ways in which students use equivalence relations as a problem-solving tool.

A productive approach that emerged from some earlier work (Payne & Cook, 2025) with students working on equivalence relations tasks involved the students' awareness that when reasoning about equivalence classes, the existence of equivalence classes often made it unnecessary to specify a choice of element from within an equivalence class. This observation warranted further study, though most of the original tasks were not positioned to investigate the strategies that students used when reasoning about an equivalence relation as a global property of a set. In addition to this empirical example, we also find some hints at the notion of indifference in the literature. For example, Piaget et al. (1977, p.155) describe elements of an equivalence class as "indistinguishable", and one of Hamdan's (2006) students describes equivalence relations as unable to discriminate between related elements. Similarly, in the context of teaching homomorphisms one instructor in Rupnow (2021) observed that they could "take any [pre-image of a coset of the kernel] as a representative" (p. 11) due to the induced equivalence relation, though these are comments made in passing, leaving indifference yet to be investigated directly. Since these strategies remain under-investigated, and since we hypothesize that this a productive understanding for students working with equivalence relations tasks, this study was geared towards eliciting more instances of articulated indifference towards particular elements.

Methods

Since the ultimate purpose of this project is to develop a conceptual analysis (e.g. Thompson, 2008) of how students reason about equivalence from a global perspective, the first author conducted task-based clinical interviews (e.g., Clement, 2000) with advanced mathematics

students. One goal of conceptual analysis is to build models of student's thinking (Thompson, 2008). Hence, we conducted task-based clinical interviews (e.g., Clement, 2000) intended to explore the mental processes of the participants. Since literature informing a global perspective on equivalence relations remains somewhat sparse, pilot interviews were conducted to be largely exploratory, and to guide the development of a more targeted interview protocol in a future dissertation study. The first author individually interviewed three undergraduate Introduction to Abstract Algebra students three times each in 60-minute sessions and recorded all verbal and written responses on an iPad. We will relate episodes from two of these students, Andrew and Brian, in which they used indifference as a problem-solving tool when reasoning about equivalence relation tasks. The tasks are reproduced below:

Table 1. Two of our tasks, on which we report in the results section.

Task A - Even and Odd Functions - Andrew (pseudonym)	
Task: Suppose f(x) is an even function and g(x) is an odd function. Let $$h(x) = \frac{f\left(\frac{g(x)}{2}\right)}{x^2}$$ If $h(5) = 2.2556$, is $h(-5) = -2.2556$? Answer: No, since $h(x)$ is an even function	Rationale: Function parity partitions the set of all real-valued functions into four subsets: Functions that are even only, odd only, functions that are neither even nor odd, and the function that is both ($f(x) = 0$). The task can be answered by deciding if $h(x)$ is odd, and does not require specification of $f(x)$, $g(x)$, or $h(x)$
Task B - Concentric Circles – Brian (pseudonym)	
Let $(x, y)R(w, z)$ if and only if $x^2 + y^2 = w^2 + z^2$ be a relation that partitions the plane into curves. What is the maximum distance between two points on curves containing $(1,2)$ and $(0, 4)$? Answer: The maximum distance is $4 + \sqrt{5}$	Rationale: If the student observes the equivalence classes are circles, this becomes a matter of finding the distances between points on the opposite sides, but this does not require that students specify the locations of either point.

Tasks were designed to require students to reason either about the partition structure induced by an equivalence relation, or about sets of equivalent elements to observe students' productive use of a global perspective on equivalence relations, and begin to identify useful strategies to approaching these tasks, one of which was hypothesized to be indifference towards the choice of element.

Following Clement (2000), the first author conducted an *interpretive* analysis, which was generally in agreement with Thematic Analysis (e.g., Braun & Clarke, 2006). The notions of a global perspective and indifference both served as initial, a priori codes. Other codes emerged during the analysis that extend beyond the scope of this preliminary report. After each interview was transcribed, the first author coded each transcript for instances of both a *global perspective* and *indifference* towards the choice of element within an equivalence class or a subset of a partition, based on student's observable behaviors. Student's reasoning was inferred to be from a global perspective if they referred to, or made representations of, sets of elements that were

related by an equivalence relation, or alternatively if they reasoned about subsets of the corresponding partition. Once a student's reasoning was inferred to involve a global perspective, student's speech was coded as indicating indifference if it contained words or phrases such as "regardless", "independent of", "no matter", or "I didn't specify" in reference to some element of an equivalence class that featured as part of their productive problem solving.

Results

In task A, Andrew quickly recognized that this problem was related to determining whether $h(x)$ was an even or an odd function, though he could not recall at first precisely what it meant for functions to be even or odd. Citing examples of functions that he knew to be even or odd, he was able to recover a definition that he found useful:

> *Andrew*: for $f(x)$ to be even, that means that say $f(1) = f(-1)$, or in more general terms like $f(x) = f(-x)$, like how cosine works.
>
> *Interviewer*: Okay. So then what about odd functions?
>
> *Andrew*: So well basically, $g(1)$ should equal $-g(-1)$, or in general terms $g(x)$ should equal $-g(-x)$. [...] like how sine works [...] that's why I felt like it was saying, "Is it odd?" is because, now I see that's the definition that we're working with there. [...] So going back to the question, 'is this part odd or even', is kind of how I'm looking at that.
>
> *Interviewer*: How would that information help you answer the question?
>
> *Andrew*: still going off of this… if $h(5) = 2.2556$, is $h(-5)$ negative? That feels like asking if h is odd. So I think I'm kind of trying to work through and see if I can say that h is odd, and then that will definitely include that being true. And if it's even, that would definitely disqualify it from being so.

Here, we claim that Andrew is demonstrating a global perspective about a partition of the set of functions by parity, since his goal has become to assign the function $h(x)$ to one of the subsets of that partition, and based on the properties of that subset, he will be able to solve the task. However, Andrew goes on to use indifference as a tool in his problem solving, based on an approach to building $h(x)$ by dividing an even function by x^2, and deciding that it would still be even, independent of a particular choice of $f(x)$.

> *Andrew*: So I was able to determine [$h(x)$ was even] using the fact that $f(x)$ was an even function [...] I really only used that $f(x)$ was an even function and made a determination that an even function divided by x^2 would still be an even function, and that's what got the answer. [...]
>
> *Interviewer*: Okay. So even though you didn't have a specific function for $f(x)$, you were able to determine that $h(x)$ was an even function?
>
> *Andrew*: Yes. [...] Based on just the reasoning through the character of a general […] even function [...] Like how I talked about an even polynomial. Reducing it, like dividing by x^2, would always retain its identity as an even function, [...] So regardless of the specific identity of $f(x)$, the specific function, as long as it's even, which it is, then $h(x)$ will be even, regardless of the specifics. [...] the reasoning solely ended up relying on $f(x)$ being even.

We infer that Andrew reasoned from a global perspective because he used a partition of the set of functions. In particular, he used the assignment of elements to certain subsets of that partition to assign other elements to the appropriate subsets. Andrew did not initially recall the formal definitions of even or odd functions, but he used examples to jog his memory and recover the characteristics common to all even and odd functions, which let him proceed without a

specific choice of f(x) or g(x). As Andrew noted, his reasoning relied only on the fact that f(x) is even, "regardless of the specific identity", which we argue indicates indifference towards the choice of representative.

Brian adopted a similar indifference in his work on task B, solving the task without determining the exact locations of two points on opposite sides of the circles. After he determined the radii of the circles and added them to find his answer, the interviewer followed up with questions about his reasoning:

Interviewer: Did you ever find the coordinates for, uh, either of those points in your problem-solving to find that distance? So what I'm asking is not, [..] "can you tell me what they are?" What I'm asking is "did you ever find them?"

Brian: I guess I didn't find $(0, 4)$, it's given to me. [...] And then I didn't exactly determine this coordinate. Like I never wrote out, uh…, $(0, -\sqrt{5})$. But it's kind of what was going on in my head. Like I knew I was looking for [the opposite] y-value.

Interviewer: Okay. So were you thinking about it in terms of like where that coordinate was or were you thinking about it more in terms of like the diameter and radius in the circle?

Brian: I think more, I was, initially I was like, okay, I need to find this point on the graph, but the properties of the circle will tell me what that exact value is. [...] So once I started thinking about the circle, I kind of left the coordinate idea behind.

Here, we infer that Brian reasoned from a global perspective; he indicated that his reasoning relied on the circles, which are the equivalence classes under this relation. We also consider his reasoning to indicate indifference towards the particular representative, since he notes that in his thinking about the circles, he "left the coordinate idea behind". Here, the symmetry afforded by the equivalence classes enables any choice of point on either circle to be as good as any other, and it is the consideration of the entire equivalence classes that enabled Brian to disregard the coordinates.

Discussion & Questions

We seek to address two gaps in the literature about equivalence relations: the lack of information on the global perspective on equivalence relations, and how equivalence relations are used by students as a problem-solving tool. Our hypothesis addresses both gaps, in the sense that from a global perspective on an equivalence relation, a productive strategy that becomes available is indifference towards the choice of a representative from an equivalence class. In our excerpts, Andrew and Brian productively used indifference to reason about equivalence relations from a global perspective supporting this hypothesis.

Preliminary Nature of the Report

Due to the unexplored nature of indifference as an idea linked to equivalence relations, this pilot study was conducted with the intention of developing a model to be subsequently refined, gaining viability through a cycle of interpretive analysis (Clement, 2000). There remains work to be done to document and articulate other productive forms of reasoning that accompany the use of equivalence relations. To the audience, we pose the following questions: As mathematicians and educators, what are some of the productive strategies that you would consider to be associated with the use of equivalence relations, particularly as global statements about equivalence? and in what mathematical settings do you observe or implement these strategies?

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Preservice Teachers' Reasoning About Area and Volume Measurements

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This preliminary research report examines how undergraduate elementary preservice teachers (PSTs) reason about area and volume measurements prior to receiving instruction in mathematics content course in algebra, geometry, and measurement topics. This is a pilot study for a larger project. The preliminary analysis results from the pre-assessment data of 27 PSTs indicate that they lack productive reasoning about area measurements and their conversions, and they also employ direct linear dimensions conversion strategies on the volume task. This highlights the need for instructional unit to focus explicitly on developing PSTs' productive reasoning about area and volume measurements.

Keywords: area measurement, preservice teachers, quantitative reasoning, volume measurement

In most countries' curricula, area and volume are fundamental measurement concepts that students must learn and apply. Both the Common Core State Standards for Mathematics (CCSSM, 2010) and researchers (Battista & Clements, 1996; Huang & Wu, 2019; Panorkou, 2021) have emphasized the importance of deep and progressive meanings for these concepts. From early elementary grades through college level, students are expected to develop productive meanings for area and volume measurements. Such a foundation, which emphasizes visualization and reasoning over rote memorization (CCSSM, 2010; Huang & Wu, 2019; Panorkou, 2021), is crucial for more complex geometric reasoning and problem-solving topics like optimization, integrals, related rates, volumes of solids of revolution in their later school years (Dorko & Speer, 2013, 2015).

Research has consistently shown that K–12 students are not being provided sufficient opportunities to develop more productive meanings for area and volume measurement. Students frequently struggle to visualize and mentally structure 3D arrays, leading them to count visible faces in an image instead of imagining the total number of unit cubes (Battista, 1999; Battista & Clements, 1996; Eames et al., 2013; Sisman & Aksu, 2016; Vasilyeva et al., 2013). Further difficulties include misapplying or confusing volume formulas with those for surface area or height, and conflating volume with weight (Curry et al., 2006; Ergene & Isiksal Bostan, 2022; Ho & McMaster, 2019; Kim & Oláh, 2019; Seah & Horne, 2018; Sisman & Aksu, 2016; Vasilyeva et al., 2013). Given the challenges faced by K–12 students, there is an expectation that preservice teachers (PSTs) will be equipped to support students in developing productive meanings for these concepts. However, research on PSTs (e.g., Chamberlin & Candelaria, 2018; Hong & Runnalls, 2022; Zembat, 2010) shows significant similarities between the challenges they encounter in developing meanings for area and volume measurements and those faced by children. These findings collectively emphasize the importance of providing instructional opportunities to help PSTs (and students generally) develop more productive meanings for area and volume measurements. Considering the challenges faced by PSTs, it is essential to implement targeted instructional strategies in teacher education programs that foster a deeper conceptual grasp of these mathematical concepts. By closing these knowledge gaps, we can better prepare PSTs to support their future students in gaining more productive meaning for area and volume measurement. In this preliminary report, we address the research question: *How do PSTs reason about area and volume measurement before and after receiving instruction focused on developing meanings for*

area and volume as quantities? Data collection is on-going in the Fall 2025 semester, so in this paper, we only present results on PSTs' meanings for area and volume by examining their responses before the short instructional unit. We will have additional data and further analysis completed prior to the 2026 RUME conference. Further, we note that the results of this study will serve as a foundation for the first author's dissertation work.

Theoretical Framework

Quantitative reasoning entails conceiving situations in terms of quantities and relationships between them. To conceptualize a quantity, one must identify a particular measurable attribute of an object, recognize an appropriate unit of measure, and conceive of a measurement process (Smith III & Thompson, 2008; Thompson, 2011, 2023). For area and volume, students must first conceive of a 2D or 3D space as a measurable attribute, respectively. With respect to area, students can use either standard units (e.g., square units; Cullen et al., 2018) or non-standard units (e.g., rectangles; Simon & Blume, 1994) to cover the measurable 2D space or a dynamic sweep of a line segment in a perpendicular direction through a distance to generate a rectangular area (Panorkou, 2020). The process of covering the space with the unit is the measurement process described by Thompson and colleagues. With respect to volume, students could also use standard or non-standard units. This may entail conceiving of a 3D object with an array of 2D cubic units iterated in the third dimension (Curry & Outhred, 2005; Curry, Mitchelmore, & Outhred, 2005, 2006) or a dynamic sweep of an area through a length (Panorkou, 2020, 2021).

Research on PSTs' Reasoning about Area and Volume Measurement as Quantities

Research on PSTs' reasoning about area and volume measurement reveals significant difficulties in relation to the three criteria Thompson described for quantities construction. With respect to conceptualizing a measureable attribute, several researchers have argued that PSTs often define area as "length × width" rather than the amount of space within a region, thus lacking meanings for these procedures that connect to a process of measuring an attribute (Livy et al., 2012; Ma, 2020; McNeal et al. 2024; Simon & Blume, 1994).

Related to the importance of units, researchers have found that many PSTs could not accurately identify the units involved in area measurement tasks because they lack experiences that give meaning to the concept of "square units," (Browning et al., 2014; Ghosh Hajra et al., 2016; McNeal et al. 2024). Similarly, Dorko and Speer (2015) examined students' understanding and use of units when calculating area and volume. They found that only 26.6% of students used the correct units for all tasks, while 73.4% made at least one mistake in applying or omitting units in area and volume calculations. The most common error was using length units (such as cm or m) instead of area units (cm^2) or volume units (cm^3).

The findings from these studies highlight the importance of supporting PSTs to develop more productive meanings for area and volume as quantities (as defined by Thompson) units. Ultimately, strengthening their meanings for area and volume measurements will not only enhance their mathematical reasoning but also enable them to teach these fundamental concepts more effectively to future students.

Methods

In this section, we describe our participants and the setting of the course, followed by an overview of the data sources used.

Participants and Setting

The participants for this study are elementary PSTs taking a required undergraduate mathematics content course within a four-year teacher education program at a university located in the Mid-Atlantic region of the United States. The participants were sophomores and juniors. We invited all 30 participants in the cohort class to take part in a pre- and post-assessment test. The goal of the course is to provide PSTs with the opportunity to develop productive meanings for algebra, geometry, and measurement topics and to learn how these topics are developed throughout the K-8 curriculum. All students in the class will participate in the same instructional activities. Additionally, three PSTs are actively taking part in a short teaching experiment (Steffe & Thompson, 2000), of which the first author is primary teacher-researcher, to explore more in depth the efficacy of the task sequence.

Data Sources

PSTs will take the same assessment both at the beginning and end of the semester, administered online through a Desmos Activity. The pre-assessment consisted of several tasks. Of relevance to this report, the first two tasks (Figure 1) required converting between area measurement units, one involving standard units (Task 1) and the other involving non-standard units (Task 2). Task 3 (Figure 2) required finding the volume of a box involving a non-standard unit and performing unit conversions.

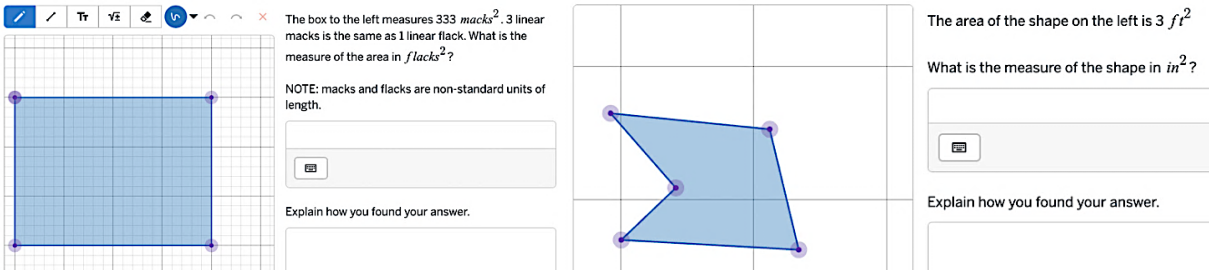

Figure 1. Tasks 1 and 2

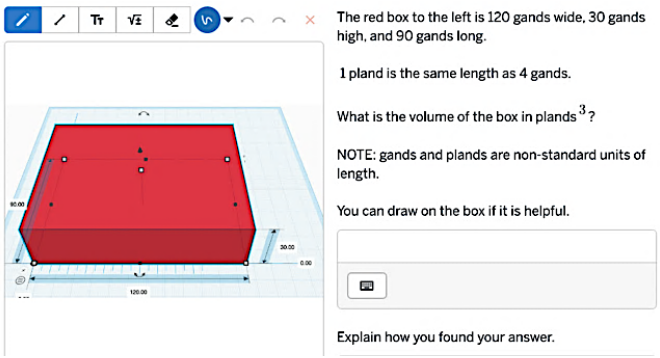

Figure 2. Task 3

Analysis

To code the PSTs' responses to pre- and post-assessment items, we will begin by individually reading through the PSTs' responses and making notes about ways they reasoned about area and volume in each problem. We will then meet to compare observations and develop codes that could apply across problems. We will likely use both a priori codes (e.g., relying on formulas) and emergent codes (e.g., converting linear units before determining area or volume). Once we agree

on features for coding, each author will independently code several pre- and post-test responses for each problem, and we will calculate interrater reliability. Whereas we will present the results of this analysis at the conference, due to the time constraints for preparing this report, we will focus on PSTs' numeric responses to the tasks to highlight the ways such analysis is likely to provide powerful insights into their reasoning.

Preliminary Analysis and Results of the Pre-assessment Test

We first present a summary of the PSTs' responses to the three tasks. For each task, we present the PSTs' numeric values, regardless of whether they included units in their response. We note that for the two tasks in Figure 1, only 2 PSTs obtained the correct numeric values for the tasks (37 and 432, respectively); these were the same two PSTs. The majority of PSTs provided a value that was likely determined by multiplying or dividing by the linear conversion fact (e.g., dividing 333 by 3 and multiplying 3 by 12).

Table 1. PSTs' Responses to Pre-Assessment Tasks on Area and Volume

Task 1 (n = 27)		**Task 2** (n = 27)		**Task 3** (n = 24)	
Answer	№ of ans. (as %)	Answer	№ of ans. (as %)	Answer	№ of ans. (as %)
37	2 (7.4%)	432	2 (7.4%)	5,062.5	13 (54.1%)
111	22 (81.5%)	36	23 (85.2%)	81,000	6 (25.0%)
1110	1 (3.7%)	32	1 (3.7%)	1,296,000	2 (8.3%)
100	1 (3.7%)	9	1 (3.7%)	972,000	1 (4.2%)
99.9	1 (3.7%)			5,062.3	1 (4.2%)
				32	1 (4.2%)

In contrast, a majority of PSTs (54%) obtained the correct response (5062.5). The most common incorrect answers were probably obtained by multiplying the three given dimensions (120×30×90), then dividing or multiplying this value by the given linear conversion factor, thus obtaining 81,000 or 1,296,000, respectively.

We were initially surprised by the large discrepancy between the number of PSTs who correctly answered Task 3 (Figure 2) compared to Tasks 1 and 2. However, as we preliminarily reviewed the PSTs' written responses (which we will more formally code once both pre- and post-assessments are complete), a possible explanation became apparent. Rather than determining a conversion factor to determine an answer (i.e., there are 4×4×4 plands^3 in 1 gland^3), we found that many PSTs directly converted the three given linear measures before multiplying; this strategy was not available to them in the area problems (Tasks 1 and 2), as no dimensions were provided. For example, one PST on Task 3 justified her correct answer by saying:

> I divided each measurement by 4 to convert gands into plands, because if 1 pland is the same as 4 gands, there will be a smaller amount of plands to take up the same amount of space. Then I multiplied the length times width times height to get the volume.

On Task 2, the same student explained how she determined there were 36in^2 in 3ft^2 said:

> There are 12 inches in a foot, so for every 1 foot^2, there are 12 inches^2. There are 3 ft^2, so 3 times 12 inches per 1 foot is 36 inches^2.

Hence, the PST was able to determine a correct solution to the volume problem by converting linear measures and then multiplying, a strategy not available in Task 2, as no linear measures

were provided. We observed numerous other similar responses from PSTs who successfully converted linear dimensions in Task 3 but not in Tasks 1 and 2.

Discussion

Our preliminary findings corroborate what other researchers have noted. Namely, prior to intervention the PSTs' meanings for area and volume seem to rely on procedures (Chamberlin & Candelaria, 2018; Hong & Runnalls, 2022; Livy et al., 2012; Koçak et al., 2017; Ma, 2020; McNeal et al. 2024; Simon & Blume, 1994; Zembat, 2010). With respect to Tasks 1 and 2, these procedures do not seem to reflect a robust meaning of either area as entailing a process of using a unit to measure an attribute (e.g., reasoning about using different units to cover a space; Thompson 2011, 2023). However, our prompt for Task 3 may be limiting our ability to explore PSTs' meanings for converting between units of volume. We plan to add an additional volume question to the post-assessment test without linear dimensions provided to better explore their meanings for volume conversions.

As we continue to code the pre- and post-assessment responses, we will further analyze the data as described in the methods. We anticipate several codes as being relevant based on the literature (e.g., Dorko & Speer, 2015), such as "omission of unit, but correct numerical value," "use of wrong unit, but correct numerical value," "omission of unit and wrong numerical value," and "use of wrong unit and wrong numerical value." This coding process will provide a clearer picture of the changes in their PSTs' meanings for area and volume measurements, and after the course. We are hopeful that the instructional unit will support them in developing such productive meanings, and these meanings will be reflected in students' post-test responses.

Implications and Questions for Feedback

Based on the preliminary analysis and results of our pre-assessment test, we highlight two main implications that can be drawn from these preliminary findings. First, a significant number of PSTs answered Tasks 1 and 2 incorrectly, indicating that the PSTs' meanings for area measurements and their conversions are not robust enough to address a variety of tasks related to area. This implication highlights the need for instructional unit to focus explicitly on developing productive reasoning related to area and volume measurements. Second, the findings revealed that PSTs employed direct conversion strategies in the volume task (Figure 3), which were not easily applied in the two area tasks due to dimensions not being apparent. This highlights the need to further explore and develop PSTs' meanings for both quantities as measurable attributes entailing a unit and measurement process.

Based on these preliminary findings and implications, we intend to ask the audience for the following in their feedback:

1. In what ways do you think this study can be expanded upon? Are there other critical ideas to explore that support PSTs in developing productive meanings for area and volume measurements?
2. In what other directions can the analysis focus, especially when both pre- and post-assessment data sets are available after the course?
3. Apart from quantitative reasoning, are there other theoretical frameworks to consider grounding the study in?

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Student Agency in the Collaborative Development of Novel Interactive Textbook Questions in Calculus and Linear Algebra

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In this preliminary report, we present results of examining the collaborative process used by teams of students, teachers, authors, and researchers to develop novel interactive questions for free, open calculus and linear algebra textbooks. Of concern is student agency as both a temporally-based social process and an ecological phenomenon as collaborators understand and construct the role of student. Discursive practices of the collaborators contributed to student positioning. In some cases, this resulted in an absence of student voice in the development of content, while in others, student agency yielded meaningful pedagogical changes in the development of questions.

Keywords: role construction, positionality, agency, undergraduate mathematics textbooks

In this time of rapid development of free, open curriculum materials, particularly textbooks, effective methods are needed for generating content that supports both student engagement and teacher pedagogical needs. Attending to the reality that teachers and students interpret and mediate the use of curriculum resources to produce an enacted curriculum different from the intended one (Rezat et al., 2021), the PROTEUS project positions Students and Teachers alongside textbook Authors and educational Researchers, in a "STAR team," to propose, revise, and evaluate novel textbook questions that will engage a wide range of math learners. The success of this collaborative approach to generate content differently than the author would generate in isolation depends upon the various stakeholders drawing on their lived and practice-based experiences in mathematics classrooms. We are interested in the agency of the students in this unique collaboration fraught with differentials in content expertise and echoes of classroom power dynamics.

Theoretical Framing

Discursive moves can position students as agents in the context of the undergraduate mathematics classroom Mesa and Chang (2010). While the PROTEUS project uses collaboration outside the classroom, we acknowledge that participating teachers' and students' positioning is inevitably linked to social structures forged in the classroom. Thus, our conception of student agency in this context draws upon Emirbayer and Mische's (1998) definition of agency as a process shaped by past, present, and projected (future) social engagement. While our design has attempted to position the students as experts of their own (past) experiences, the students navigated uncertain positioning and power dynamics as they worked alongside content experts who in other contexts hold positions of authority relative to students. We furthermore rely upon Biesta and Tedder's (2007) expansion of agency as an ecological phenomenon wherein the interaction between an individual and their context-for-action leads to the transformation of environmental structures. Our design aspires to make equitable space for all stakeholders, students in particular, to shape the very processes engaged in by the team. Power differentials among stakeholders naturally make such a realization of student agency troublesome.

Methods

PreTeXt-Runestone: Open Textbooks Engaging Undergraduates in STEM (PROTEUS) is a collaborative project led by the four organizations, one of which serves as the independent evaluator. Together, these organizations coordinate the development of interactive questions for PreTeXt-authored textbooks (https://pretextbook.org/) in calculus and linear algebra, ensure their integration into the XXX online textbook platform, and evaluate their effectiveness in authentic classroom settings. In the content development stage, we seek to answer the question: *What are the strategies and heuristics used by teams of stakeholders to create content for three types of interactive questions (Reading Questions, Parsons Problems, Matching Exercises), for calculus and linear algebra that will engage students with learning the material?, w*hile in this report, we address the subquestion: *How is the role of the student constructed by the teams?*

This study uses qualitative research methods within a phenomenological paradigm to explore how students' roles are constructed and understood during collaboration (Ernest, 1997; Wood, 2017). To capture authentic experiences and perspectives, we employed bottom-up research methods that allow codes and themes to emerge directly from the data rather than imposing a priori categories (Blair, 2015). Specifically, we investigated students' utterances in team meetings to develop interactive questions to be added to four textbooks for calculus (*Active Calculus* [AC], Boelkins, 2022; *APEX Calculus* [APEX], Hartman et al., 2018) and linear algebra (*A First Course in Linear Algebra* [FCLA], Beezer, 2022; *Understanding Linear Algebra* [ULA], Austin, 2022).

Participants

Each STAR team was composed of one or two students, two teachers, the author of the respective participating textbook, and one or two researchers. The eight teachers were recruited from the authors' networks because of their familiarity with the textbooks and their experiences teaching with them. Three educational researchers, who specialize in mathematics education at the undergraduate level, participated in two STAR teams each. Researchers coordinated the team meetings and oversaw the design-based research process (Bakker & van Eerde, 2015) of iteratively developing and testing new textbook questions.

There were seven students across the teams; each was pursuing an undergraduate degree and had recently completed or was currently taking calculus and/or linear algebra. Students varied in terms of race and ethnicity identity (including Asian, Black or African American, White, or Other with two students identifying as Hispanic/Latinx), gender identity (4 women, 1 men, 2 agender), and prior mathematical experiences. Six of the students were between 20 and 21 years of age, with one over 23 years old. Several of the students had experience tutoring and some completed their teaching practicum concurrently with their participation in the study. They were primarily recruited through the teachers in our teams and through networks of instructors in the state of Michigan. Each student was assigned to either a linear algebra team or a calculus team depending on their background and the needs of the study. Students were asked to contribute their authentic perspectives on how learners engage with interactive questions, to represent the kinds of difficulties students encounter, and to provide feedback based on their experiences.

Outside of their activities developing textbook questions as a team, the STAR members contributed their feedback through surveys and interviews with the external evaluation team. All participants provided informed consent and were compensated for their participation.

Data Collection and Analysis

Data for this analysis were drawn from 51 STAR team meetings during the 2024-25 academic year, which were video recorded through Zoom and subsequently transcribed verbatim. To familiarize ourselves with the data, the first three authors read the transcripts multiple times to understand the meeting content and participants' discursive patterns. Using NVivo (https://lumivero.com/products/nvivo/) qualitative data analysis software and guided by the central research question of this study, we engaged in iterative cycles of independently coding transcripts and comparing results until we achieved strong agreement on the discursive segments that contributed to the construction of the student role. We concluded that it was important to distinguish between participants' references to the *General Population* of students and to the *STAR Students* themselves. Furthermore, we distinguished between STAR student *References* and *Utterances* in team dialogue*,* as utterances *made by* students, *directed to* students, or *about* students contributed in different ways to role construction. Finally, we attended to *Critical Events* (e.g., a decision to revise a question) in which to examine these categories of utterances and how they positioned students (Author & Colleague, 2010). We are in the early stages of textual and discourse analysis to identify how references to students were framed, how student voices appeared (or were absent), and how discursive practices shaped the construction of the student role within meetings. This approach has been chosen to move our analysis beyond surface-level categorization and examine the underlying meanings, assumptions, and institutional logics embedded in the conversations (Bloome et al., 2008).

Findings

Critical events were identified during team processes of norm setting, topic selection, proposing and vetting of questions, posing of conjectures about thinking the questions may elicit, and evaluating responses to the questions in both interview and classroom settings. As team members interacted in a variety of ways with students' articulation of difficulties, representations of textbook use, and descriptions of personal classroom experiences, students were positioned by both discourse and team processes. By authentically representing their own experiences, students surfaced conceptions that influenced team decisions about question design.

Positioning Students Through Discourse

Utterances contribute to the constructed position of participants in dialogue (Author & Colleague, 2010), and as the teams held regular meetings, utterances from each of the four STAR roles positioned students and delineated the (often implicit) scope of their expected contributions. Student utterances were predominantly embedded among those of other roles, such as teachers and authors, and occurred most often as responses. Thus, teacher, author, and researcher utterances shaped the discursive environment in which student utterances were elicited, or into which they were injected. For example:

> *Author*: I think there, there are the basic [questions] about understanding how composition works that have no reading at all associated with them. And then the, you know, read the chain rule formula, and then try to use it, and then kind of the more synthetic ones at the end, where you're trying to put a few things together.
>
> *Student*: I think still having a reading section like the very first section where it discusses a little bit about composition, might be useful for those questions, instead of just having no reading section.
>
> *Researcher*: I agree, I agree.
>
> *Author*: So what, what it would be? What do you think it would be? Let me see.

Researcher: …Do we want to have two batteries of questions, one that is contextual and the other is more related to the, to the actual symbols? That might be a good way.

Student: I like that idea. It's first having them look at it as symbols and then translating it into an actual context. I think it's, it's a lot more difficult to apply it to a, to a full context. So I do think it's important that we actually add a question that speaks on that.

This sequence from a calculus team highlights the "transactional" achievement of agency (Biesta & Tedder, 2007). Both author and researcher position the student as authoritative by being solicitous of and responsive to the student's experience with learning the chain rule topic.

Student Agency Through Team Processes

Participation in structured team processes both supported and evidenced student agency. All team members, students included, took turns taking meeting notes, allowing the documentation of team discussions and decisions to reflect the differing focus, perspective, understanding, and voice of the team member whose turn it was. In this excerpt from a linear algebra team, the student offers a clarification of terminology as the team makes decisions about their development process:

I defined what a cycle is right here, but if somebody would like to change what the definition of a cycle is, from—I was explaining that one cycle is that, like, it's written or edited, and then it has gone through the, through the interviews. That is one cycle. And then, and if we edit it again, and then it goes through a cycle, or an interview, that's a second cycle.

The responsibility of note taking served to elicit a glimpse of the STAR student's perceived capacity to take action ("I defined") and their sense of immersion as a team member ("we edit"), empowering them to shape the team's operational environment (Biesta and Tedder, 2007).

Student Agency in Surfacing Conceptions

One of the linear algebra teams had a single student member, (alias) Melody. During the team's work to develop novel questions for the eigenvector section of their textbook, Melody proposed the short-answer question: *Describe the relationship between Eigenvectors and Eigenvalues.* A practice promoted by the researchers within each team was to listen to the question proposer's rationale for their proposed content. Melody explained their inspiration:

I know there's a formula, and I can't name it off the top of my head right now, for the relationship between eigenvectors and eigenvalues. But I sometimes think that it's forgotten, the why behind what is happening. And it's just memorized what you have to do.

Their proposed question differed categorically from those of the author and teachers, each of whose proposed questions in the same topic probed a relationship between an eigenvector and a *matrix*. Furthermore, their verbal justification brought to the team's attention the student experience of fixating on a process without internalizing a relationship. The author on the team suggested that Melody's proposed question begin the question set for the textbook section. After team discussion and revision, the altered form of the question read: *What does it mean for a vector* $\boldsymbol{v}$ *to be an eigenvector of a matrix* $\boldsymbol{A}$*?* Notably, all members of the team continued to refer to this fundamentally altered question as "Melody's question," positioning the student as agent not only of surfacing student conceptions, but as an agent influencing the textbook content that would address those conceptions.

Agency as Social and Ecological Phenomenon

Evidence of the highly contextual nature of the constructed student roles came at the very end of the 2024-25 academic year, when the four separate STAR teams joined together in a week-long workshop in Ann Arbor. The students' roles, previously constructed in the liminal environment of the Zoom meeting and delineated primarily through discourse (or its absence), were given an opportunity for physical embodiment, such as spatial positioning of students with respect to other members of the teams, behaviors and mannerisms accompanying the distribution of tasks, and differentials in the enacted use of students' time and resources and content experts' time and resources. Perhaps most importantly, interpersonal norms were drastically rewritten in the in-person setting. Marked changes in student roles emerged during the workshop, underscoring both the sensitivity of role construction to environmental factors (Biesta and Tedder, 2007) as well as issuing from activities of social engagement (Emirbayer and Mische, 1998).

Implications and Future Work

While the findings presented here are preliminary in nature, early stages of analysis provide evidence that discursive practices positioned students differently across teams. Some students were positioned so that they exerted agency in developing novel content and providing feedback to questions that others produced, whereas others were positioned so that they only provided feedback for questions that others produced. In this latter situation, it appeared as if students' agency was made invisible relative to how the teams were expected to operate.

Limitations of the study include the less consistent attendance of STAR team meetings by students than by other members, inevitably diluting student voice in the process and impacting the construction of students' roles. Furthermore, one might argue that the first act of role construction was the decision by the research team to associate members of the STAR teams with their experience-wise labels (Student, Teacher, Author, Researcher) as they were emplaced in their team environments. However, beyond this structural necessity, specific roles, in the functional sense, were not imposed on members of the teams. The design choice to ask students (relative content novices) to participate as equals alongside authors, teachers, and researchers rather than to replicate classroom positioning (Wagner & Herbel-Eisenmann, 2008) was, to varying extent, a source of tension in the teams.

Our larger investigation of question development heuristics will be informed by what we learn about how teams used student expertise and it will allow us to see how the work in the teams helped subvert the social construction of institutionally-constructed roles of teachers and students. We suspect that in this setting the students' perception of the textbook's authority (Wagner & Herbel-Eisenmann, 2014) is being challenged as well.

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The Humanity in Mathematicians' Writing

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Mathematical writing is often portrayed as objective and authorless, yet prior work shows that mathematicians' writing contains nonobjective and authority-related features (Burton & Morgan, 2000). We build on this foundation by analyzing 51,325 single-authored mathematics papers from arXiv, focusing on proofs and surrounding narratives. Using curated dictionaries of hedges and evaluative adjectives and mathematics-specific rules, we document their frequency and explore whether gender and publishing output of the author relate to their use. Preliminary results indicate that both features appear to vary with author characteristics. These findings suggest that linguistic choices in mathematics are tied to identity and status, complicating the myth of purely objective discourse.

Keywords: proof genre, mathematician practice, language analysis

The discourse about mathematical proof and writing has often espoused values of author independency and its absolute nature (Dawkins & Weber, 2017). Davis and Hersh (1998) are often quoted for describing the ideal mathematician's writing as "conceal[ing] any sign that the author or the intended reader is a human being" (p. 2). Yet, others point to the fact that mathematical writing and proof are very much the product of humans both in terms of communication styles (Tanswell & Rittberg, 2020) as well as the infusion of authority markers (Burton & Morgan, 2000). That is, a product created by humans cannot be truly dehumanized. Students come to mathematics with diverse linguistic communication styles and rhetorical patterns, some of which align more closely than others with the conventions of proof and mathematical writing. For those whose styles diverge, there may be additional cognitive load in adapting to the ways of writing that have become normative in mathematics. Thus, it is important to understand the features of mathematical writing beyond the idealistic dehumanized products.

This preliminary work builds on Burton and Morgan's (2000) analysis of mathematicians' papers and interviews, which showed that presenting writing in research articles is not dehumanized. They identified markers of authority, such as positioning something as "obvious" (positive authority) or expressing doubt and qualification (negative authority). They suggested that status shapes their use. For example, women may avoid negative authority because their belonging in the field is less secure. In the 25 years since, little work has examined how author identity relates to linguistic choices in mathematical writing. With large corpora now available, we can investigate such questions at scale. In this preliminary report, we focus on two authority markers (hedging and evaluative language) asking: (1) What forms of hedging and evaluative language appear in proofs and surrounding narratives? (2) How do author characteristics such as gender and publication record relate to their use?

Literature Background on Mathematician Writing

Many scholars have focused on the proof product as a particular genre of writing with many distinct linguistic conventions (e.g., Bowers & Küchle, 2020; Selden & Selden, 2013) with

slightly different norms (Lew and Mejía-Ramos, 2019; 2020) depending on context (e.g., student proof or proof in a research article). If we turn to studies specifically on mathematician writing in publications, we find few studies with the majority in linguistic journals. Studies have explored publication patterns in terms of genders of authors and relationship to journal quality (Mihaljević-Brandt, et al., 2016), revision processes (Andersen et al., 2021; McGrath, 2016), and corpus analyses of particular features such as explanatory language (Mejía-Ramos et al., 2019) and use of imperatives (Tanswell & Inglis, 2023). Furthermore, scholars have established that mathematical research articles are quite distinct from standard forms lacking traditional structures such as "Methods" or "Discussion" (Graves et al., 2013) and features such as establishing research gaps (Graves et al., 2014) as well as different types of phrase frames (e.g., "is [also, fairly] easy to see that", Cunningham, 2017). Mathematics articles can be deconstructed into two types of writing, mathematical (e.g., proofs, definitions) and meta-mathematical (the surrounding narrative) (Kuteeva & McGrath, 2015). Andersen et al. (2021) studied the revision process of a graduate student and his advisor writing an article identifying how the mathematical and meta-mathematical have different roles and ways of justifying and engaging with the reader. This suggests any analysis of research articles should attend to these types of writing separately.

If we turn to our focus, we identified Burton and Morgan (2000), McGrath and Kuteeva (2012) and Alasmary (2022) as empirical studies that address authority in mathematical research articles in ways that are relevant to language choices such as hedging and evaluating. They all support the larger claim that mathematical writing is not purely objective in mathematical or meta-mathematical sections. McGrath and Kuteeva used Hyland's (2005) stance and engagement framework to provide counts of words in a set of 25 research articles finding frequent use of imperatives, infrequent use of hedges, but frequent use of boosters ("really") and attitude devices ("obviously"). Alasmary compared graduate level texts, doctoral dissertations, and a set of published research articles (34 articles) via analysis of four-word lexical bundles finding that participant-oriented (affective and engagement with reader) were found the least, but they were found more often in journal articles than in the dissertation set. Burton and Morgan's (2000) qualitative analysis of 59 articles similarly found establishing positive authority (affective judgments such as "obvious") was more common than negative (e.g., hedging). Paired with interviews, they also provided the conjectures we discussed in the introduction.

Theoretical Orientation

We situate this work as a direct extension of Burton and Morgan (2000), who used Halliday's (1973) systematic-functional grammar to analyze 53 mathematical papers. They considered both ideational (presentation of mathematics) and interpersonal functions (relationships between author, subject, and audience). Our interest lies in the interpersonal function, particularly authority. Burton and Morgan showed how authors positioned themselves with positive or negative authority: deferring to others, signaling incompleteness, or hedging (negative), versus asserting expertise through evaluative words such as clearly or obvious (positive). Such markers disrupt the notion that mathematical writing is purely objective.

Burton and Morgan documented variations across their sets of papers in terms of both positive and negative authority markers. In their sample, men more often used markers of negative authority. They conjectured that, women are positioned as not fully belonging in the field and therefore cannot risk using the traditional language of mathematical power. They further suggest that negative authority provides a risk and that, "risk is, of course, relative to the author's preexisting status within the community" (p. 441). They supported this conjecture with

the fact that their outlier in terms of many positive and negative authority markers was a well-established male mathematician in the field. High-status likely means someone's authority is not going to be questioned regardless of using negative authority or even positive authority (which can be perceived as too self-assured for a novice mathematician.)

Building on this foundation, we focus on two authority markers well-studied in other fields: hedging and evaluative language. Hedges (e.g., might, suggests) serve to "express tentativeness" and to qualify "confidence in the truth of a proposition." (Hyland, 1996, p. 433). They serve as a means of negative authority. The author is claiming they do not have complete knowledge of the field. In contrast, evaluative language (e.g., clear, important) positions the author with positive authority. These are adjectives that serve for to assign a value-judgment (Stojanovic, 2015) on a noun and tend to be gradable (scaled between two polarities). Following Burton and Morgan (2000), we make several conjectures. First, hedging language will be more prominent in papers written by men and well-established mathematicians. Second, evaluative language will vary with experience in the field.

Methods

To explore this question, we make use of SIGMathLing's 1.58 million document data set from arXiv.org (Ginev, 2020) years ranging from 1990 to 2020. As we were interested in automatically extracting proofs, we removed all papers that did not use the LaTeX "proof" environment or which had poorly formatted proof environments, yielding a set of 277k papers. For the scope of this paper, we then narrowed this data set to papers whose subject area was research mathematics, excluding papers whose primary subject was not prefixed with "math" and papers whose subjects were in computer science or philosophy. We also filtered to papers that were single authored with identifiable gender in a final dataset of 51,325 papers. To identify gender, we ran author first-names through gender-guesser.py, a python library based on a list of 40k names alongside genders collected by Jörg Michael (2007). We note the limitations of this binary approach and making such strong inferences from names. We then organized authors based on how many papers they had on arXiv into deciles to document output as a proxy for establishment in the field of mathematics. The lowest decile was labeled "Vlow," the next two as "low," deciles four through seven as "mid," deciles eight and nine as "high" and the 10th decile as "Vhigh." For each paper, we then subdivided the portions into "proof" and "nonproof" based on the usage of the proof environment in LaTeX. We also removed any mathematical symbols and replaced them with the generic "formula."

To measure "hedging" and "evaluative" language, we began with existing dictionaries from other projects (Ulinski and Hirschberg, 2019 and Rockladge, et al. 2018, respectively). Initially, we used these sets of words to calculate their frequencies in the entire set of mathematics papers. From this process, it became apparent that words that may have served as hedges or evaluative language in non-mathematical contexts were not always adaptable directly. For example, "estimate" may be a hedge in some contexts, but in mathematics papers was often referring to a precise output of an algorithm. Similarly, a word such as "ideal" would be an evaluative adjective in most settings but is more frequently an algebraic structure in mathematics papers. This necessitated a careful look at each hedge and evaluative adjective. For each word, we selected a sample of papers that used the word to evaluate the contexts. The research team then indicated if the usage seemed to be meeting the definition of hedge or if it had a more precise mathematical meaning. This led to developing and testing several rules. For example, "most" was identified as a hedge when it was part of the phrase "most of" rather than in contexts where

it was describing something precise. In other cases, simple rules did not function well such as was the case for "may." When "may" operates like "might" it is a hedge. Yet, it commonly operated like "can" where something is mathematically permissible. For this hedge, we used semantic embedding to encode the words around each usage in a vector and to compare its distance to the average vectors of "might" and "can". If "may" was much closer to might, it was maintained as a hedge. Presently, we are in the middle of this process reporting a set of 13 hedges and 12 evaluative adjectives (see Results) that we are successfully picking up as intended in our corpora. For each of the hedges and evaluative adjectives, we counted their frequency for each paper in the proof section and the narrative section. We also calculated relative frequency by dividing the frequencies by the total number of words in each respective section. Each of these are treated as outcome variables where gender and publication level were predictor variables. For the scope of these preliminary reports, we provide descriptive statistics and a preliminary logistic regression on the binary of existences of hedging or evaluative adjectives.

Results

In our analysis of 51,325 papers, we found that 66.8% of them had at least one hedge (of our thirteen) in the narrative sections, and 8.6% of the papers had at least one hedge in the proof sections. We found that 88.3% had evaluative adjectives (of our twelve), and 62.7% had at least one evaluative adjective in their proof sections. Figure 1 provides counts of the evaluative adjectives and hedges (meeting our rule requirements) from the analysis.

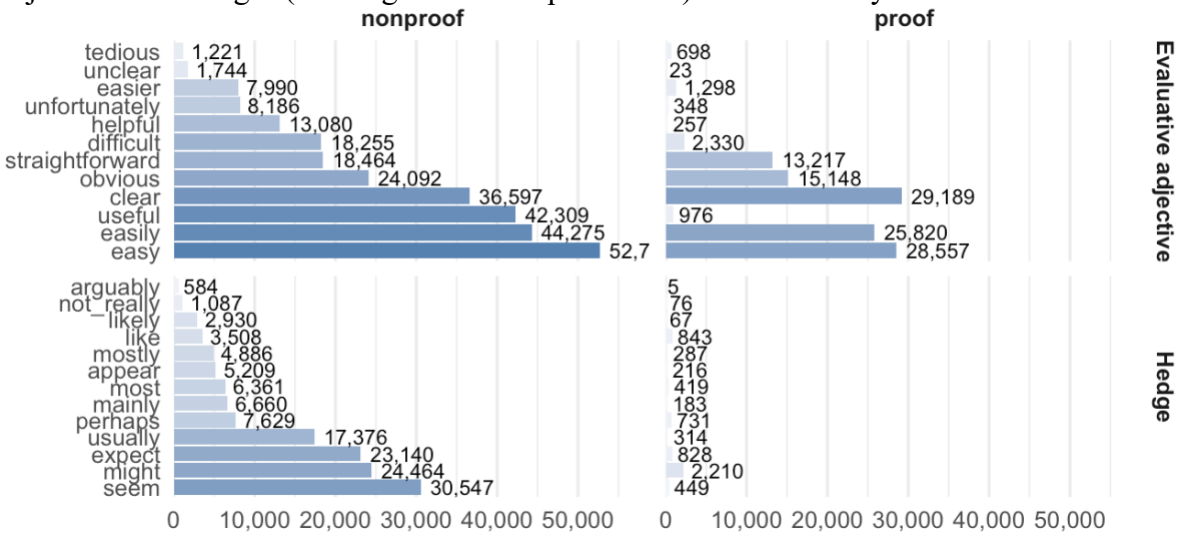

Figure 1. Frequencies of a Preliminary Set of Hedges and Evaluative Adjectives

In Figure 2, we present boxplots representing the relative hedge use (relative to wordcount) broken out by gender and then publishing category for non-proofs. Note that the proof section has so infrequent hedges that these are not boxplots but outliers. In Figure 3, we provide the boxplots for Evaluative Adjectives Relative Frequencies in both proof sections and on proof sections.

Presently, we have run only preliminary models looking at the relationship between gender, publishing output, and the focal language. For each proof and each nonproof, we gave a set of binary variables (has hedge; has evaluative adjective). We found that gender and publishing output were significantly (at the p=0.05 level) related to whether we identified evaluative adjectives in the mathematical (proof) and metamathematical (non-proof) parts of a paper. We

found that both were also predictive of the existence of hedge language in the metamathematical (non-proof) section, but not significantly predictive of hedge language in the mathematical (proof) section. To provide a brief illustration, we share the results for existence of evaluative language in proof whose equation is: $\text{odds}(eval)=1.227\times(1.248)^{GenderMale}\times(1.039)^{publishing_group}$. According to this model, a woman who is in a low output publishing group would be estimated to have a probability of 56% of using evaluative language in the proofs of her paper. A man in the same publishing group would be estimated at 61%. At the other end, a woman in the high publishing group would have a 59.8% chance of using evaluative language in their proof while their male counterpart would have a probability of 65%.

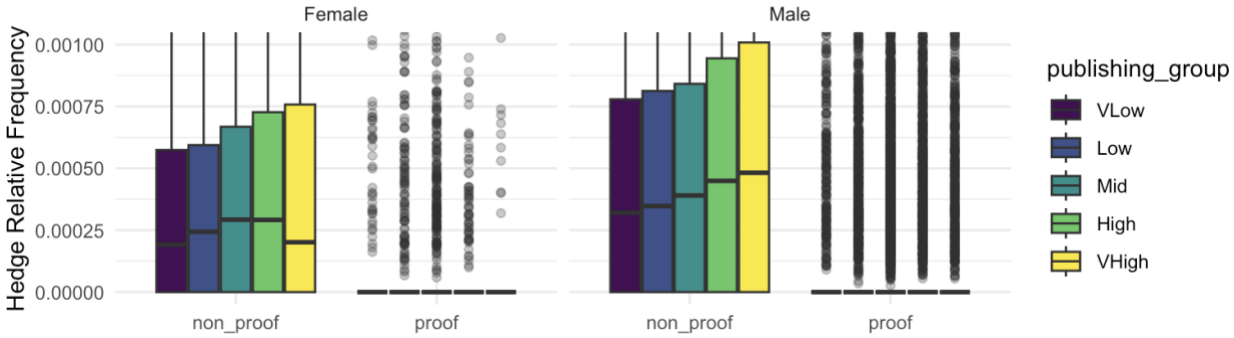

Figure 2. Relative Frequency of Hedges Across Gender and Publication Output Group Narrative Sections.

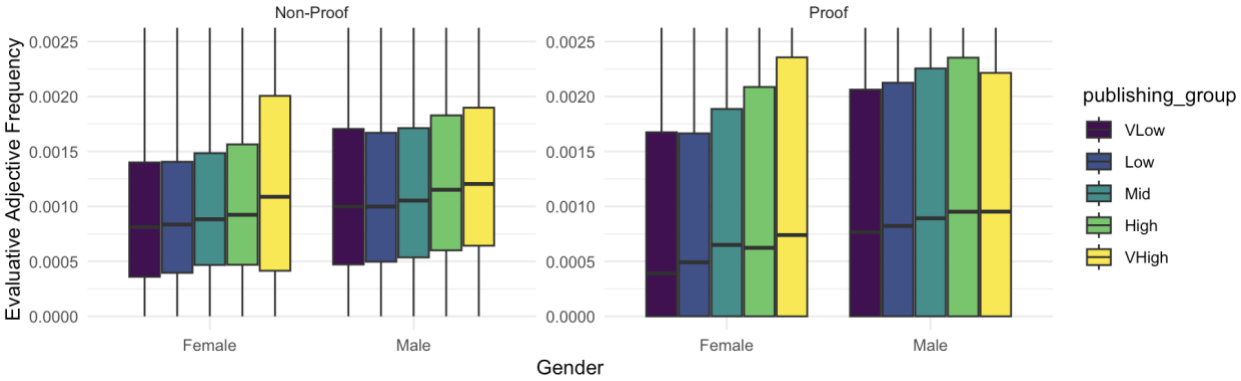

Figure 3. Relative Frequency of Evaluative Adjectives Across Gender and Publication Output Group in Narrative (left) and Proofs (right).

Preliminary Nature of Proposal and Discussion

In this proposal, we are presenting an in-progress analysis of a large corpus of mathematician writing. Our goal was to directly build on Burton and Morgan's (2000) results and conjectures about authority in mathematical writing. We found that the types of markers mentioned by Burton and Morgan did appear in the larger data sets at rates high enough to conduct further analysis. Furthermore, some of their conjectures about status in the field are proving viable. Currently, we have only processed a total of 25 words for this analysis. We intend to continue this process and gather feedback from the RUME audience about how we might consider other factors related to status and address authority markers beyond the set of hedges and evaluative adjectives. We believe the community will find the methods involved interesting. Furthermore, as both scholars and instructors, these results can challenge how we discuss proof and its values based on the reality of how proofs are communicated in published journals.

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"I" Find it Difficult Because...: Identifying What Features Students Report as Affecting Difficulty to Understand a Proof

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The student perspective provides valuable information into what students find to be difficult when reading proof. Such information can be used to productively support students as they face these difficulties. In this preliminary report, we investigate what students report as contributing factors of how difficult a proof is to understand. For this report, we focus on data regarding two proofs from the introduction to proof setting. Our findings indicate that the density and width of a proof are key contributors towards the difficulty rating given to the proofs.

Keywords: Reading proof, student perspective, difficulty

Scholars have identified various ways students encounter difficulties with proof (Moore, 1994; Selden & Selden, 2008), typically focusing on students constructing or validating proofs. Meanwhile, understanding a written mathematical proof is also a difficult task. Proofs are full of challenging features, such as the level of abstractness (Rowland, 2001; Weber, 2004) and various writing norms and conventions (Dawkins & Weber, 2014; Lew & Mejia-Ramos, 2019; Melhuish et al., 2025). These norms and conventions include minimizing unnecessary warrants (Dawkins & Weber, 2017, Selden & Selden, 2003), excluding references to people (Dawkins & Weber, 2017, Melhuish et al., 2025), and following the rules of academic language writing (Lew & Mejia-Ramos, 2019). Overall, scholars have identified that students have difficulty with the abstract nature of proof (Alcock, 2010; Weber, 2004), with the complexity of proof (Alcock, 2010), in understanding the mathematical content (Moore, 1994; Selden & Selden, 2008), and having to make implicit warrants and information explicit (Samkoff & Weber, 2015). All of which is taken from an expert's perspective. Though, one may wonder what do students find to be difficult in understanding proof and what impacts this perception of difficulty?

Few existing studies investigate students' perspectives in the reading and understanding of proof (Guajardo & Lew, 2024; Guajardo, Lew, & Melhuish, 2025), particularly regarding what students find difficult in understanding proof (Selden & Selden, 2008). We argue that knowledge of students' perspectives on proof and proving can provide valuable insight into both the teaching and learning of mathematical proof. Namely with students' perspectives, researchers can explore student interactions with proof with greater depth. Additionally, instructors can leverage findings about student perspectives to better understand what students take from class and how to adapt instruction to build on to students' perspectives and existing knowledge base.

In this paper, we offer some preliminary insight into what students find to be difficult in understanding proof. The study presented is part of a larger dissertation study, focused on understanding students' perspectives about various aspects of proof. Here, we present a subset of the data to answer the research question: What do students indicate as factors that contribute towards a higher rating of difficulty to understand a proof?

Theoretical Framing

To situate our study, we conceptualize our thinking of abstraction and complexity of proofs. For abstraction, we take on Hazzan and Zazkis' (2005) three levels of abstraction:

> (a) abstraction level as the quality of the relationships between the object of thought and the thinking person, (b) abstraction level as reflection of the process–object duality, and (c) abstraction level as the degree of complexity of the concept of thought. (p. 103)

In using these levels of abstraction, we consider abstraction to be an individual's perception of the separation between the concept and a concrete model. Separations may be perceived as larger due to the student having less experience with the concept (first level), having a process conception (second level; Dubinsky, 1991), or in dealing with a compounded concept (third degree). With this definition, we can identify how participants view the abstract nature of the concepts throughout the study by considering how they talk about their experiences with the concept, the utility of the concept, and any compounded nature of the concepts.

In terms of complexity, we refer to complexity here differently than Hazzan and Zazkis' third level of abstraction above. For complexity, we use Hanna's (2014) discussion of the width of a proof, "the number of distinct ideas one has to keep in mind" (p. 31). We expand this idea by including the number of moving pieces in a proof one must keep track of (high cognitive load) rather than just the memorization of key ideas or crucial steps to construct a proof.

Methods

Eight undergraduate students, who completed an introduction to proof (ITP) course the semester prior, participated in this study. Table 1 provides participant demographic information.

Table 1. Participant Information

Participant	Race/Ethnicity	Gender	Grade Received in ITP
Adrian	White or Caucasian	Non-binary	B
Claire	White or Caucasian	Female	B
Erik	White or Caucasian	Transman	A
KTRM	Latin/Hispanic	Female	C
Marie	Latin/Hispanic	Female	A
Olivia	Mixed Race	Female	B
Tifanni	Latin/Hispanic	Female	A
Zeus	Latin/Hispanic	Male	B

Participants met with the first author for seven semi-structured interviews. Four of the interviews were task-based interviews where participants read a provided proof (GCD, Partition, Function, and Cardinality) and attempted to understand it to the best of their ability. Participants then answered a series of questions, some of which asked how difficult the proof was. For this analysis, we consider participants' responses to the following questions from the task-based interviews: (a) How well do you feel like you understand this proof?, (b) If you could [rate] this proof in difficulty from 1-10 (ten being very difficult, 1 being not difficult at all), how would you [rate] this proof?, and (c) If given the chance to rewrite the proof, would you change anything about it? We also consider data from the exit interview which included discussion on what participants find to be difficult with understanding proof and reflections on the ratings they gave

the proofs throughout the study. For this report, we focus on the two proofs rated most difficult to understand: the Cardinality and the Partition proofs (see Table 2 for the theorem statements).

Table 2. Theorem statements for Cardinality and Partition proofs

Proof	Theorem
Cardinality	The sets $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality.
Partition	Suppose R is an equivalence relation on a set A. Then the set $\{[a] \mid a \in A\}$ of equivalence classes of R forms a partition of A.

To determine the two highest rated proofs, we conducted a points allocation method of ranking based on participants individual rating of each proof: 4 points was awarded to the highest rated proof (most difficult) and 1 point was awarded to the lowest rated proof (least difficult). The Cardinality proof was rated highest by the participants, and the Partition proof was rated second highest. Deductive coding was applied to data from the relevant task-based and exit interviews. The data relevant to the Cardinality proof were coded independently by both authors. Coding was discussed to resolve any discrepancies and reach agreement. Data relevant to the Partition proof was then split equally amongst both authors and were independently coded.

Six codes related to features of proof identified through the larger dissertation study (Guajardo, 2025) were used: abstractness, density, length, width, and writing. Using our definition of abstraction from Hazzan and Zazkis (2005), abstractness was coded when participants explicitly voiced how abstract the proof was or when they discussed visualizing the content in a more concrete way. Density focuses on participants discussing the amount of information hidden from the reader, to which they must deduce on their own. Length was coded when participants referenced the length of the proof as contributing towards the difficulty. For width, we used our definition that was adapted from Hanna's (2014) discussion of width to code when participants were stating difficulty due to having to keep multiple distinct ideas in mind as they were reading. Lastly, the writing code refers to participants indicating the language, notation, or structure of the proof caused difficulties.

Results

Participants considered features of proofs that contributed to the difficulty (or ease) in understanding a given proof. For example, a feature of a proof could be the proof method used. Table 3 shows the features we identified for each participant as contributing towards the difficulty of the proof. When discussing the features of the proofs that affected their difficulty ratings, participants did so independently of other proofs (bolded in Table 3; typically during the task-based interview) and comparatively between proofs (standard font in Table 3; typically during the exit interview). When talking about the features of proof independently of other proofs, participants expressed abstractness, density, length, width, and writing as playing a role in the difficulty rating of the two proofs. While nearly every feature was mentioned for each proof (Table 3), we focus on the density and width features due to length restrictions. Participants primarily expressed density and width as key factors for the difficulty of these two proofs – both in their discussions of the proof individually and their comparisons across proofs.

Table 3. Features discussed by Participants. Bold font indicates feature was discussed individually. Standard font indicates feature was discussed only comparatively. Symbols +, - indicate if that feature was comparatively reported to increase or decrease the difficulty (respectively).

Proof	Features of Proof					
	Abstractness	Density	Length	Line Structure	Width	Writing
Cardinality	**Erik Zeus -** + Adrian + Claire +	**Adrian Claire** + **Marie** + **Olivia Tifanni**		KTRM +	**Adrian** + **Erik +** **KTRM + Marie + Zeus** Claire +	**Marie** Adrian -
Partition	**Adrian Zeus** + Olivia -	**Adrian Claire Olivia Tifanni - Zeus** +	Erik -	**Olivia** KTRM -	**Claire** KTRM - Marie - +	**Adrian Claire**

For the cardinality proof, most participants indicated density and width played a role in their reasoning for why the proof was difficult to understand. For example, with density participants difficulties with the proof were due to having to unpack a lot of information. Claire explained her rating the proof as having a 4 out of 10 difficulty because the "super specific things of like *why* f, OK, I don't really know [emphasis added]." In this quote, Claire is focusing on the line-by-line arguments made in the proof and unpacking why these arguments are being made. For width, participants expressed having to handle a large quantity of information. Marie expressed the proof was not easy to understand because the proof "gives us like 4 variables in total. The more variables there are, the harder it is to mentally keep track. So even though it's in front of you, that can just be a little harder to keep up with." Here, Marie is having difficulty keeping up with the information presented in the proof. All other features were discussed by at most two participants.

For the partition proof, most participants indicated that density played a role in their reasoning for why the proof was difficult to understand. Similar to their comments regarding the Cardinality proof, participants expressed having to unpack or link information. Olivia explained that she gave the Partition a rating of 9 (and not lower) because she does not "understanding why they're saying what they're saying. Like, at least, when I'm trying to prove an if-then I know what it's supposed to look like. […] with this I'm like, what are we even trying to do here?" Similar to Claire above, Olivia is focusing on having to unpack a lot of information in the proof. As with the cardinality proof, all other features were discussed by at most two participants. This indicates that density and width play a larger role in how participants discuss the difficulty of these proofs.

When looking at how participants compared between proofs, the width of the Cardinality proof was used as justification for a higher difficulty rating than other proofs. In comparing the Cardinality proof to another proof in the study, Adrian explained the Cardinality proof was harder because "it has a lot more stuff […] it's got these different functions and all these different sets that this other one doesn't have." We attribute Adrian's discussion of "a lot more stuff" to having to keep up with more concepts, increasing the width of the proof. Most other features were not mentioned by many of the participants for either proof in the exit interviews. When they were discussed regarding the Cardinality proof, the features were almost exclusively used to explain why the proof was more difficult to understand than others. Participants did not

frequently compare the Partition proof to others and when they did, most participants expressed which features made the proof easier to understand than the others. The participants commonly compared the Partition proof to the Cardinality proof as they were typically rated similarly (either hardest to understand or second hardest to understand).

Discussion and Next Steps

In this paper, we discussed the different ways in which undergraduate students rate proofs in difficulty to understand. Participants expressed various features of the proofs as reasons for their ratings. In our data, the features density and width play a critical role in determining how difficult it is to understand these two proofs. This supports the existing literature that suggests that students have difficulty with unpacking information (Samkoff & Weber, 2015) and have trouble with the amount of information that can be present in a proof (Alcock, 2010), but more importantly contributes student voices to the discussion. In particular, students may use these features to compare between proofs: considering which proofs are denser or wider to determine the difficulty of understanding a proof.

Overall, these preliminary findings reveal the complexities of how difficult a proof is to understand. Additionally, it is important to recognize that these features of proof identified by the participants are inherent in proof and cannot easily be hidden or removed. Rather, it is important for researchers and instructors to work toward better supporting their students. One way this may be achieved is by helping students use strategies to overcome these difficulties caused by the features of proof. Guajardo et al. (2025) describe proof comprehension strategies most used by undergraduate students as they attempted to understand proofs. This repertoire of comprehension strategies, paired with our findings on students' perspectives on difficulties, can be used to help students mitigate the difficulties caused by the features and better comprehend the proofs. More research is needed to better understand how to support students in mitigating these difficulties.

Next steps for this study are to continue analysis of the proofs and difficulties identified by participants. Important factors for these participants were not only features as discussed above, but also non-features (aspects of the reader that impact the difficulty rating of the proofs). Two codes were identified through the data that represent non-features; effort to understand and current understanding. With effort to understand, participants reflected on their actions they did or the actions it would take to understand the proof. For their current understanding, participants reflected on how they viewed their own skills to understand (self-efficacy) or their understanding of the content of the proof. From a brief initial analysis, a non-feature that participants find to be a key factor in the ratings give not the proofs is their own understanding of mathematical concepts and arguments in the proofs.

Preliminary Nature

Analysis is still ongoing and theorical framing is still being developed to help situate the discussion of the results. As we discussed above, there are more factors to the difficulty attributed to a proof than just features of the proofs themselves. We welcome any comments and suggestions in moving forward with the analysis of non-features or in any theoretical components that could be productive to consider for our study. In particular, are there any socio-cultural or cognitive theories that could help in analyzing and situating the non-features?

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Using Large Language Models for Qualitative Coding of Structured Data on Post COVID Research Based Instructional Strategies

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In the spring of 2025, introductory Physics, Calculus, and Chemistry instructors were surveyed about their usage of active learning techniques and use of Research Based Instructional Strategies. Using open-source large language models running on consumer hardware, responses to open ended survey items data were analyzed to better understand the usage of specific strategies during and after the Pandemic Response Teaching period resulting from the COVID-19 pandemic, as well as for reasons how and why instructors adopted, maintained, or stopped RBIS. While the large language models were able to apply qualitative codes to the data, their effectiveness was significantly varied based on the dataset, initial prompt, large language model used, code specificity, and code prevalence.

Keywords: Large language models, Qualitative Analysis, Research Based Instructional Strategies, Pandemic response teaching

In 2025, we conducted a national survey of instructors of undergraduate chemistry, mathematics, and physics courses. The timing and selection of instructors were designed to understand if there were any lasting changes to how these courses are taught following the COVID-19 pandemic. In part, this survey asked instructors of first year Calculus, Chemistry, and Physics about their awareness and use of research based instructional strategies (RBIS), while providing an opportunity for respondents to explain how and why they use (or do not use) RBIS. The nature of this data offered us information into instructional trends and provided an opportunity to experiment with large language models in qualitative analysis.

AI, particularly in the form of large language models (LLMs), has become more common and powerful. Within education research, these tools have been applied to various forms of qualitative analysis, with varied success. However, their use has raised important questions about the nature of qualitative research, privacy, bias, and transparency. We attempted to address several of these questions by using an open source, locally run LLM to complete the first two steps of Braun and Clarke's (2012) thematic analysis with a graduate student researcher.

Here, we address two research questions. First, to what extent, and why, are instructors in first year STEM courses using, or shifting away from RBIS following the Pandemic Response Teaching (PRT) period? Second, to what extent can an open-source, locally hosted LLM effectively code a highly structured dataset using Braun and Clarke's thematic analysis?

Background Literature and Perspective

By comparing survey responses from a 2019 survey and this one from 2025, Apkarian & Johnson (2026) investigated the extent to which use of class-time and instructional practices changed following the COVID-19 pandemic. Analysis of that quantitative data revealed, overall, no significant changes in reports of how class time is spent with regards to listening to the instructor lecture or solve problems, working individually, working in small groups, and engaging in whole class discussion. Additionally, while there was increased usage of online office hours, supplemental videos, and recorded lectures, starting during PRT and continuing afterward, there were no major shifts to in-class instructional practices (e.g., think pair share,

clickers). Our analysis here of the open-ended survey response data seeks to provide additional nuance to the quantitative analysis completed by Apkarian & Johnson (2026).

Traditionally, analyzing open ended survey responses is carried out with any variety of qualitative analysis methods by an individual or a team. For example, thematic analysis, as developed by Braun and Clarke (2012), prioritizes six key steps, with each working towards identifying overarching themes across a dataset. By carrying out these steps, researchers can identify broader patterns key to understanding how, when, and why instructors prioritize and implement RBIS. That thematic analysis is a well-defined, commonly used approach to analyzing a well-structured dataset makes it both a practical choice for researchers and potentially a good fit for data analysis carried out by LLMs.

The robust and diverse capabilities of LLMs makes them increasingly more prevalent, including in qualitative research. Researchers have long used stochastic models for natural language processing (Chen at al., 2018; Gamieldien et al., 2023; Li et al., 2021; Rietz et al., 2021). While use of LLMs in qualitative research is still new, work on applying them within new methodologies and frameworks is growing rapidly (Naeem et al., 2018; Naeem et al., 2023; Xiao et al., 2023; Zhang et al., 2025). However, qualitative researchers have shown reasonable skepticism about their use. Some studies indicate researchers are generally concerned about hallucinations (when a model invents data and "facts"), data privacy, transparency, and explainability (Schroeder et al., 2025; Feuston & Brubaker, 2021; Springer & Whittaker, 2020; Zhang et al., 2025). Furthermore, results are generally mixed in using large language models for qualitative research. Many studies find significant variation in performance based on prompts used (Wang et al., 2024; Zamfirescu-Pereira et al. 2023), inductive or deductive methods (Bijker et al., 2024; Siiman et al., 2023), and researcher knowledge of LLMs (Chen et al., 2018; Zamfirescu-Pereira et al, 2023; Zhang et al., 2025).

Even with these challenges, LLMs can be a powerful resource. Many highlight the potential for LLMs to serve as an additional coder, where a human researcher and LLM each code the data and compare (Bijker et al., 2024). Other uses include academic writing (Tu et al., 2024), developing a codebook (Bijker et al., 2024), coding data (Marathe & Toyama, 2018; Tai et al., 2024), identifying themes (Castellanos et al., 2025), and supporting new researchers in developing qualitative analysis skills (Gao et al., 2024). In addition, several processes have been developed for using LLMs, usually ChatGPT, for qualitative analysis, like thematic analysis (Dai et al., 2023; De Paoli, 2024; Naeem et al., 2018; Naeem et al., 2023; Zhang et al., 2025).

Methods

The 2025 survey was sent to roughly 17,000 names, the same roster that was used for the 2019 survey. The goal was to capture those who taught before and after the pandemic-induced disruption from 2020-2023. To be eligible, instructors must have taught one of the target introductory courses in the 2023-24 and/or 2024-25 academic year. In total, 933 participants provided usable responses to our survey: 339 chemistry (99 new; 240 repeaters), 279 mathematics (112 new; 167 repeaters), and 315 physics (86 new; 229 repeaters) instructors. Included in the 2025 survey were the following free response questions:

1. What, if any, instructional practices did you start using during the pandemic response period (roughly March 2020-June 2022), that you had not used previously? Include anything that you tried, even if you did not maintain that practice throughout or afterward.
2. Of the instructional practices you mentioned in the previous question, which (if any) do you continue to use?

3. You may use the space below to add any additional comments about your experiences during the pandemic response period, if you wish.
4. You may use the space below for any additional comments about the context and climate of your department, if you wish.

Anonymized data was stored in a Google Sheet. Of the 933 participants, a random sample of 450 were selected to have their responses coded manually by a human coder (the first author). Instructor responses were considered holistically, even if answers to any of the four questions above were missing. Then, the data was coded using three different large language models: Gemma3:12b, Qwen3:14b, and Deepseek-R1:14b. The large language models were run locally using the Python programming language and the Ollama software package. All models were run on the researcher's consumer desktop with an AMD RX 6900 XT GPU. Each model was run using default Ollama settings, and data was input as 30 sample chunks to ensure the context window of each model was not exceeded. The prompt used was carefully designed using common prompt engineering techniques outlined in the literature and was revised 3 times before the final implementation. The prompt carefully defines the role of the LLM, explains the task in multiple ways, and provides an output template. However, the authors specifically did not extensively tune the initial prompt or implement specific techniques to improve performance.

Each run of the models output a text file consisting of codes for each row of data. This text file was then converted to a CSV using Python. This conversion expected consistent output from the LLMs and is likely undercounting codes from the models. If the model provided a synonymous code, instead of a code from the codebook, we would not count that towards our results. Afterwards, Cohen's Kappa and Krippendorff's Alpha were calculated using Python for each run, and an average for each model was calculated for each code. After this programming pipeline was created, we revised our initial prompt to better understand why instructors implemented, maintained, or moved away from RBIS in their instruction. This prompt used a "oneshot" technique, providing examples and non-examples for how the data should be coded. Only the LLMs coded the dataset for this research question and analysis is still ongoing.

Results and Discussion

Each LLM coded the entire dataset, while the first author only coded a random sample of responses from 450 instructors. We present the percentage of rows coded with each code in table 1. Each LLM coded the data 3 times, and an average for each code is reported. Since each row of data could have more than one code, the total percent frequency adds to more than 100%.

Table 1. Percent frequency of each code by "coder"

Code	Deepseek-R1	Qwen3	Gemma3	Human
Adapted existing strategies to Pandemic	14.29%	21.33%	40.09%	1.33%
Alternative Exams	1.61%	3.00%	3.47%	2.67%
Did not use RBIS	0.75%	1.46%	0.43%	0.22%
Experimented	1.57%	2.57%	15.18%	0.22%
Flipped	11.08%	10.86%	9.54%	8.22%
Group Work	17.97%	20.40%	19.08%	9.11%
Increase in tech use	14.22%	20.04%	9.50%	5.11%
Inquiry/Modeling	6.72%	4.86%	3.14%	1.11%
Lab Kits	1.04%	0.54%	1.32%	0.89%

Learning Assistants/Peer to Peer Teaching	0.82%	2.57%	1.68%	1.33%
None	15.93%	31.80%	30.01%	17.11%
Online Assignments	5.72%	8.54%	18.65%	7.56%
Online discussion tools	1.04%	2.64%	7.18%	1.78%
Online Exams/Proctoring	3.39%	6.47%	8.40%	7.78%
Online Lecture	8.82%	9.97%	24.01%	12.22%
Online Office Hours	4.22%	8.29%	9.72%	8.89%
Online simulations	2.00%	6.15%	6.36%	7.56%
Polling/Clicker	6.29%	5.93%	7.43%	4.44%
Posting resources online	1.04%	6.04%	10.97%	5.78%
Recording/Posting Videos	23.94%	25.76%	27.05%	26.22%
Reflection/Exam Wrappers	1.04%	0.39%	2.29%	0.44%
Used new RBIS	5.79%	1.54%	0.32%	0.00%

We calculated Cohen's Kappa and Krippendorff's Alpha for each of the codes, comparing the interrater reliability between each LLM and the human coder (Table 2). While the averages for Kappa and Alpha were between 0.27 and 0.47, we see significant differences in interrater reliability between various codes and across models. Qwen3 had the highest interrater reliability, while Deepseek-R1 had the lowest. Furthermore, we see a significant range in potential Kappa and Alpha values. For example, when using Qwen3, the codes Online Office Hours and Recording/Posting Videos had a Kappa and Alpha value above 0.815, while the codes Experimented and Did Not Use RBIS had a Kappa and Alpha value of approximately 0.

Table 2. Average Cohen's Kappa and Krippendorff's Alpha for each LLM across three runs

Model	Kappa	Alpha
Deepseek-R1	0.285	0.278
Qwen3	0.463	0.454
Gemma3	0.363	0.344

The prevalence of "online" codes aligns with instructors of these introductory courses transitioning from in-person lecture to Zoom. Additionally, we see fewer codes related to the instructional features of the class, such as "group work," "flipped," "polling/clicker," and "learning assistants/peer to peer teaching" than for "recording/posting videos." These relative prevalence indicators broadly match the quantitative analysis from Apkarian & Johnson (2026).

The AI framework developed was meant to be a proof of concept, involving as little programming or AI specific knowledge as possible. This simplicity, coupled with less than perfect human coding (i.e., data was first coded by a graduate student new to qualitative analysis with no second coder, the initial codebook was quite large) could have resulted in specific struggles for the LLM to code the data. However, overall, the models did relatively well with those codes used frequently by the human coder. When removing codes that were used fewer than 10 times by the first author, we find Kappa increased to 0.362, 0.576, and 0.508 for Deepseek, Qwen, and Gemma respectively. In addition to challenges around infrequently used codes, the LLMs struggled to apply the code "None." The "None" code was used relatively frequently by the human coder, 77 times, but was consistently the worst for Kappa and Alpha.

While the interrater agreement between a human coder and the LLMs used varied greatly in the first two phases of Braun and Clarke's Thematic Analysis, our data and learnings suggest, with software and hardware adjustments and technological improvements, open source, locally hosted LLMs can provide valuable support to a qualitative researcher and should be further explored.

Conclusions

We are not arguing for a total reliance on AI or LLMs for qualitative research (Christou, 2023), especially given our interrater reliability results. However, to completely avoid LLMs is likely misguided. Qualitative research has used stochastic tools (Tausczik & Pennebaker, 2010), natural language processing (Bryda & Sadowski, 2024; Gamieldien et al., 2023), and digital tools (MaxQDA, nVivo, etc.) for a long time now. However, LLMs raise questions about bias, accuracy, transparency, and privacy to address within the context of qualitative research.

With regards to questions about bias (Barany et al., 2024; Chen et al., 2018; Pangakis et al., 2023; Schroeder et al., 2025), by relying on open-source models that reveal their weights, training data, and more, we can better understand what potential biases may exist within the model and its training data. With regards to accuracy concerns, many studies have found a benefit and necessity for human checks and interaction with any AI tool in qualitative analysis (Feuston & Brubaker, 2021; Pangakis et al., 2023). For our work, instances of inaccuracy were mostly the model incorrectly assuming the data contained more information than was actually present and thus applying a code where one did not belong. We believe we can minimize this by improving prompts with one-shot techniques, averaging several runs, and other formatting techniques. With regards to concerns about data privacy (Byun, 2023; Schroeder et al., 2025), the use of local models avoids sharing IRB protected data with private companies who are explicit about their use and storage of user chats (e.g., chatGPT). Finally, with regards to transparency concerns, using an open source model allows researchers to both understand and choose the model best suited to their work, as well as better communicate to others what is happening "behind the scenes" with the models used.

Preliminary Nature of this Report

This work is still preliminary, as we believe we can further tune our use of LLMs in thematic analysis to better understand our dataset. While the models struggled with certain types of codes, our analysis suggested a second phase of coding focused on why instructors implemented, maintained, or moved away from RBIS in their instruction. We have completed an initial round of coding using the LLM pipeline, but need to compare and validate with a human coder to determine how well LLM's handled a narrower codebook consisting of more nuanced codes.

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Daily Affirmations: Strengthening Marginalized Student Identity in First Year College Calculus Courses

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This study investigates how marginalized students experienced affirmation of their mathematical identities in first-year calculus. Thematic analysis revealed thirteen forms of identity affirmation, clustered into internal (understanding and utility) and external (peer instruction, community, instructor recognition) categories. While understanding and utility were universally affirming, most affirmations were socially mediated. Findings highlight the critical role of community and instructional practices in shaping identity, suggesting that affirming environments are essential for supporting persistence in STEM.

Keywords: mathematical identity, identity affirmation, calculus

Introductory courses disproportionately drive minoritized students out of STEM pathways (Ellis et al., 2016; Hatfield, 2022). This attrition is especially troubling given the persistent underrepresentation of these students in postsecondary mathematics and STEM fields more broadly (Greene et al., 2008; Riegle-Crumb et al., 2019). Understanding the mechanisms by which early mathematics courses contribute to inequitable outcomes remains a critical concern for both researchers and practitioners.

One promising lens for examining this issue is that of student identity. Identity shapes how students perceive themselves and their relationship with mathematics and is increasingly recognized as a central construct in mathematics education research (Darragh, 2016; Fellus, 2019; Li et al., 2020; Radović et al., 2018; Rodriguez et al., 2018). How students come to see themselves—such as whether or not they identify as "math people"—has been shown to strongly influence their success, persistence, and long-term engagement in mathematics (Miller-Cotto & Lewis, 2020). As these authors argued, this self-perception is not merely internal but is deeply intertwined with the social and cultural contexts of learning, shaping how students engage with mathematical tasks, how they respond to challenges, and how motivated they feel to continue in mathematics.

At the same time, research has highlighted how mathematics learning environments can marginalize students. Solomon (2007), for instance, documented how students often experience disconnection from the mathematical community even when they retain confidence in their abilities. Such findings, along with results about the impact of mathematics learning environments on learning and community (Leyva et al, 2020; Oppland-Cordell & Martin, 2015) suggest that affirming and supporting students' mathematical identities is essential, particularly for those who are under-represented or marginalized in STEM.

This paper reports on an investigation into what was specifically affirming to marginalized students' identities in first-year college calculus courses. By centering student perspectives, we aim to illuminate practices and experiences that strengthen students' sense of belonging and mathematical identity, offering insight into how introductory courses might be reimagined as spaces of affirmation rather than exclusion.

Literature Review & Research Questions

An extensive body of research has examined identity in mathematics and STEM education, and consequently, the conceptual and methodological foundations of this work are varied. Simpson and Bouhafa (2020), in their synthesis of empirical research on identity across STEM published between 2000–2018, highlighted both the increasing prominence of identity as a construct and the persistent challenges in defining and operationalizing it. They note that while identity has been shown to be a powerful lens for understanding persistence and participation in STEM, the literature is fragmented in how it conceptualizes identity and how its dimensions are applied in research.

Radović et al. (2018) further emphasize this complexity, identifying five main categories through which the literature has operationalized identity. The most prominent approach conceptualizes identity as narratives, drawing on Sfard and Prusak's (2005) framework of identity as a collection of stories told by the individual and by others about them. This narrative lens has been especially influential in studies that seek to capture the lived experiences of students navigating mathematics learning. However, as Radović et al. argue, a persistent challenge lies in the relationship between the defining features of identity (e.g., recognition, positioning, affect, agency) and their operationalization. Identity research often combines multiple dimensions, but the lack of clarity about how these interact leaves the field with conceptual fuzziness and limited cumulative insight.

Another pattern in the existing literature is a tendency to emphasize the negative aspects of identity—how deficit narratives, marginalization, and exclusionary practices undermine students' sense of belonging and persistence in mathematics (Nasir & Cobb, 2007; Solomon, 2008). *While this body of work is critical in highlighting systemic inequities, it also creates a gap*: less attention has been paid to the affirming aspects of identity, or to the pedagogical practices that can strengthen students' mathematical identities in positive and sustaining ways.

Taken together, the gaps in the literature suggest an urgent need for research that attends to affirming aspects of identity in mathematics and connects pedagogical practice directly to identity formation. *This study seeks to address these absences by examining the specific ways marginalized students experience affirmation of their identities in first-year college calculus courses.* The study is guided by the research questions: (1) What experiences do minoritized students say are affirming to their identities? and (2) How have experiences in first year mathematics courses affirmed minoritized students' identities relative to mathematics?

Theoretical Framework

While most people have an intuitive sense of identity, defining it rigorously has proven challenging (Wetherell, 2010). In mathematics education, this has led to a proliferation of approaches and definitions, often inconsistent with one another, that has made the field of mathematics identity research difficult to synthesize (Darragh, 2016).

To bring clarity to this project, I draw on two complementary perspectives: Gee's conception of identity as Discourse and Sfard and Prusak's framework of identity as narrative. Radović et al. (2018) define five categories of how identity has been operationalized all align with James Paul Gee's (2000) notion of identity as being recognized as "a certain type of person." For Gee, identities are situated within Discourses—socially and culturally patterned ways of speaking, thinking, acting, and interacting. A Discourse is more than language: it encompasses values, beliefs, practices, gestures, and assumptions that convey belonging to a community. From this perspective, identity formation is not merely an individual psychological process but a fundamentally social one, emerging in relation to cultural norms and institutional structures. This

emphasis on the cultural and social contexts of learning mathematics makes Gee's framing particularly relevant to studying marginalized students in college calculus courses, where recognition and belonging play crucial roles.

To make this theoretical perspective analytically useful for qualitative research, I follow Radović et al.'s (2018) recommendation to be explicit about operationalization. I adopt Sfard and Prusak's (2005) conception of identity as a collection of stories about a person, told by themselves or others. For a story to be part of identity, it must be (1) reifying—transforming a fleeting behavior into a stable trait; (2) significant—meaningful in a person's social or cultural life; and (3) endorsable—validated or accepted by the person as at least partially true. This framework emphasizes how particular narratives come to matter for students, both those they claim for themselves ("I'm the kind of person who struggles in math") and those assigned to them by others ("She's the kind of student who always gets it right").

Identity as narrative is especially useful in connecting learning with its sociocultural context. Narratives like "only certain people are good at math" can be seen not just as individual beliefs but as socially circulating stories that shape students' opportunities to belong in mathematics. By attending to the stories marginalized students tell about themselves and those told about them in classroom contexts, this framework enables me to examine how classroom practices, teacher feedback, and peer interactions contribute to affirming—or undermining—mathematical identities. In this way, the framework provides both a broad sociocultural grounding (via Gee) and a concrete operationalization for analysis (via Sfard & Prusak).

Methods

The population for this study consisted of undergraduate students enrolled in either Calculus I or Calculus III, first and second year mathematics courses with the highest equity gaps in terms of DFW rates at the research site. A list of eligible students was generated by the Mathematics Department. From this list, fifteen students were recruited at the beginning of the semester via email invitation. All participants consented to take part in the study, which focused on their experiences navigating their calculus courses.

To capture students' evolving perspectives over time, each participant completed three semi-structured interviews (Kvale & Brinkmann, 2009), conducted at evenly spaced intervals throughout the semester. Interviews ranged from thirty to sixty minutes and invited students to reflect on their experiences in the course, their perceptions of themselves as learners of mathematics, and the personal and institutional factors shaping those experiences. This design allowed us to explore both immediate challenges and longer-term shifts in students' mathematical identity.

All interviews were transcribed and analyzed through an iterative process of thematic coding (Braun & Clarke, 2006) Each transcript went through systematic reading, with passages flagged when they were broadly relevant to the research questions. The first cycle of coding (Saldaña, 2016) identified broad, overarching categories related to students' academic and identity-related experiences. Particular attention was paid to structural, instructional, and relational factors.The second round refined these initial codes, shifting the focus from individual cases to cross-case analysis (Miles et al., 2014). This involved comparing codes across participants to identify recurring patterns, as well as attending to variations and unexpected insights. Holistic themes resulted that reflected commonalities in students' experiences while also preserving the complexity of individual narratives. These themes provided the foundation for interpreting how institutional and classroom practices affirmed students' mathematical identities in introductory calculus courses.

Results

Analysis of the interview data revealed thirteen distinct experiences that students described as affirming their mathematical identities. These experiences clustered into two broad categories: internal/intrinsic affirmations and external/social affirmations. While internal affirmations, such as achieving understanding and perceiving the utility of the material, were important, the vast majority of identity-affirming influences were grounded in social interactions. For brevity, a detailed illustration will only be provided for the theme of social affirmation here; more illustrations will be provided in the presentation.

Dominating Prevalence of Social Interaction

Students' mathematical identities were shaped most prominently through their relationships with peers, instructors, and broader classroom communities. Peer instruction emerged as a particularly powerful source of affirmation. Several students described how helping classmates not only reinforced their own understanding but also disrupted internalized narratives of deficiency. In the language of Sfard and Prusak (2005), these interactions actively challenged previously "endorsable" aspects of negative identity. As one student explained:

> "I'll have friends who are actually wanting to be math teachers, and they'll turn to me like, how do we do this again, and I'll be like oh, I'm the one that's supposed to know this… because I look like I know what I'm doing… oh, I do know what I'm doing."

Beyond peer instruction, opportunities to engage in community, such as through recitation sessions, study groups, and academic support offices such as CAST, were consistently described as affirming. Even experiences students framed as "internal," such as achieving understanding or recognizing utility, were often facilitated through these external supports.

Understanding and Utility

Although fewer in number, internal affirmations were universally present in students' narratives. Two themes dominated: understanding the material and recognizing its utility. Utility was described as transforming mathematics from an abstract requirement into something meaningful. One student put it succinctly: "I like [math] a lot more now I can actually understand the use."

Understanding was even more prevalent. Many students directly tied their enjoyment of the course to whether they could make sense of the material: "Once I understand it, I will enjoy it… [t]he only reason I'm feeling positive right now is just because I'm understanding it a lot better." These narratives suggest that comprehension and relevance remain central to identity affirmation, though they are often scaffolded by external supports.

Influence of Instructor

Instructors played a moderating role in shaping students' mathematical identities, either affirming or disaffirming students' identities. Positive interactions with professors were frequently reported as affirming, boosting confidence and cultivating a sense of belonging. One Calculus I student reflected: "[He] just makes you like… Oh, I feel seen in this class. So we just want to put more effort in, and at least really kind of enjoy math." In contrast, negative experiences with lecturers undermined students' sense of themselves as capable mathematical learners. Unlike other affirming experiences, the instructor's influence was notably volatile, with the potential to either strengthen or erode mathematical identity. In this study, positive

interactions with instructors were only reported by Calculus I students. There may be a variety of explanations for this, including particular instructors or subject matter, but these comparative factors were not made explicit by any students.

Overall Trends

Taken together, these findings highlight the centrality of social interaction in affirming students' mathematical identities in first-year calculus. While understanding and utility served as universal affirmations, they were frequently mediated by external supports and relationships. For some students, experiences in these courses reinforced pre-existing self-perceptions, while for others, the classroom environment catalyzed meaningful shifts in how they saw themselves as mathematical learners.

Although mathematics is often imagined as a solitary pursuit, this study underscores the essential role of community, interaction, and recognition in shaping marginalized students' mathematical identities. Affirming practices within peer groups, support structures, and classroom communities appear to be crucial levers for fostering persistence and belonging in STEM pathways.

Preliminary Nature of Report

At the time of writing, data collection and initial cycles of coding are complete, and emerging themes have been identified. The primary area of the project still under development is the articulation of the theoretical framework. While the analysis draws on Sfard and Prusak's (2005) conception of identity as narratives, further alignment with broader frameworks (e.g., Radović et al., 2018; Simpson & Bouhafa, 2020) is needed to clarify how different dimensions of identity are being operationalized in this context.

We seek feedback from the RUME community on the following questions:

1. How might we better connect our themes with existing theoretical lenses on identity?
2. What cautions or affordances should we consider in interpreting identity through both internal (understanding, utility) and external (peer interaction, instructor influence) dimensions?
3. Are there underutilized frameworks in the mathematics education literature that might strengthen the positioning of this study?

This feedback will help refine the theoretical framing and situate our analysis within the larger body of work on mathematics identity.

Conclusions

This study sought to identify what affirms marginalized students' mathematical identities in first-year calculus courses. The results suggest that while intrinsic factors such as understanding and utility consistently affirmed identity, the vast majority of affirming experiences were social in nature. Peer instruction, opportunities for community, and supportive instructors emerged as especially powerful drivers of affirmation, capable of disrupting deficit narratives and creating space for students to see themselves as "certain types of people" in mathematics.

In short, this work shows that affirmation is possible and impactful, but not yet systemic. Understanding how to reliably cultivate identity-affirming practices in introductory mathematics remains an area for future research and instructional design.

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More Than the Sum: A Four Frames Analysis of Mathematics Departmental Teaching Culture

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Understanding departmental teaching culture has been recognized as a useful starting point for implementing instructional and institutional change. However, there is a lack of consistency in the literature with respect to how teaching culture is operationally defined and systematically measured. In order to begin addressing this gap, we highlight findings from a series of focus groups conducted within a math department in the Southeastern United States. Utilizing Reinholz and Apkarian's (2018) Four Frames model, we characterize the department's culture through the four interrelated frames of structures, symbols, people, and power. These frames provide a snapshot of the unique math departmental teaching culture, which can be leveraged not only to assess the effectiveness of future change initiatives, but also to individualize such change initiatives to meet the unique needs of the department.

Keywords: Teaching culture, institutional change, departmental change, Four Frames model

Introduction and Background Literature

The teaching culture within an academic department is "hard to define, understand, and measure," yet plays a critical role in departmental change (Reinholz & Apkarian, 2018, p. 3). Changes in departmental teaching culture have the potential to improve instruction and learning, which in turn can positively impact student learning and persistence, especially within STEM fields (Couch et al., 2024; Henderson et al., 2011). Departmental teaching culture can also be leveraged as a metric through which educational reform can be guided and measured (Lau et al., 2022). Furthermore, teaching culture can be used as an evaluative tool for assessing teaching effectiveness (Dennin et al., 2017).

Despite the utility of teaching culture in departmental change initiatives, research examining departmental teaching culture is relatively inconsistent in both how teaching culture is operationally defined and systematically measured. A large proportion of the research relies on implicit characterizations of culture that draw on everyday language rather than from rigorous and established theoretical frameworks (Mohr et al., 2025). Without a clear definition of culture, these change initiatives leave little room for generalization to other contexts. This minimizes the potential for research in this domain to have broader impacts for institutional change.

This report seeks to address this gap by using an established framework – Reinholz and Apkarian's (2018) Four Frames model – to characterize the teaching culture of a mathematics department in the Southeastern United States. The Four Frames model serves as a powerful

methodological tool that can be leveraged not only to obtain a snapshot of a department's teaching culture, but also to facilitate targeted transformation within a department. We more specifically address the following research question: How do individuals perceive the roles of *structures*, *symbols*, *power*, and *people* within the math departmental teaching culture?

Theoretical Framework

We leveraged Reinholz and Apkarian's (2018) Four Frames model to characterize mathematics departmental teaching culture. This model uses four interrelated frames – *structures*, *symbols*, *people*, and *power* – as a lens through which to obtain a snapshot of culture (see Table 1). It is of particular utility for promoting change in the departmental change context, as it provides insight into both the product of change as well as the process of change.

Table 1: Description of the Four Frames (Reinholz & Apkarian, 2018; Couch et al., 2024).

Frame	Description
Structures	Roles, responsibilities, practices, routines, and incentives that organize how people interact.
Symbols	Cultural artifacts, language, knowledge, myths, values, and vision that department members use to guide their reasoning.
People	The uniqueness and diversity of the individuals in a department with respect to their goals, agency, needs, and identities.
Power	The interactions between people are mediated by power, status, positioning, and political coalitions.

Previous work using the Four Frames model includes analysis of department chairs' perceptions of teaching culture as it pertains to instructional change (Couch et al., 2024). Other work has compared culture across STEM departments (Reinholz et al., 2019) and examined the role of certain groups as potential agents of change, such as course coordinators (Williams et al., 2022).

Methods

Data were collected from focus group interviews conducted with math instructors at an R2 university in the Southeastern United States. In total, 10 instructors participated in a focus group, each containing between 2 and 5 participants as well as 2 facilitators. During each one-hour focus group, participants reflected on and discussed results from a prior departmental survey on teaching culture. In particular, discussion centered around three central themes: the perceived relatedness, competence, and autonomy of the department with respect to teaching. For each theme, participants were shown preliminary findings from their department's survey responses and prompted to discuss how their department's teaching culture could explain the findings. Then, additional data summarizing their overall college's responses to the same survey question was displayed, with prompting for comparison with the mathematics department's data. While the prior survey provided quantitative data related to perceptions of teaching culture, the focus groups provided an opportunity to further explore these data.

Data analysis was conducted by a team of six researchers. After cleaning the focus group transcripts, each transcript was deductively coded by a researcher using the Four Frames model (Reinholz & Apkarian, 2018). After this initial round of coding, the team discussed difficulties and revised the codebook to include more robust examples and details. Then, each coded

transcript was reviewed by a second researcher, with revisions made where appropriate to ensure consensus across the transcripts. Finally, the first two authors inductively coded each of the resultant four frames, focusing on identifying prevalent themes within each frame.

Results

In this section, we will present common themes which emerged from the focus group interviews. These themes will be organized along the Four Frames components.

Structures

Structures refers to the formal roles, responsibilities, practices, routines, and responsibilities which guide the actions and interactions of departmental members. Focus group participants reported on several departmental *structures* which impact instructors' teaching roles within the department, including time constraints and presence of mathematics education research faculty.

Instructors highlighted a lack of time to plan for their courses, contributing to a lack of autonomy and control. Jo, a tenured faculty in mathematics education, reported course schedules are often "not finalized until ... a week or two before the semester begins ... [sometimes] even later than that." Furthermore, instructors feel "a lot of pressure to have everything in its details [for their courses] ready by the first day." The lack of preparation time before the first day of class combined with pressure to finalize all details of a course by the first day resulted in instructors feeling rushed to prepare for their courses.

These time constraints were also experienced throughout the semester. Tenure-track faculty expressed the desire to support all students, but noted that it may be unrealistic to do so: "I want to be confident [in supporting all students], but can I realistically get there? No … I have other obligations." Such time constraints were reinforced by full-time temporary instructors (FTTs) and adjunct professors. Emmy described "it begins to get more and more difficult to find time… I teach 16 hours a week, which is a full load. So between planning, teaching, grading, it's hard to find time [to observe colleagues teach and discuss the observation]." Overall, instructors found it difficult to reflect and modify their teaching due to heavy teaching loads and other duties.

Another structural factor impacting instructors in their teaching role is the presence of mathematics education research faculty. Debbie, a graduate teaching assistant, described the university as being unique due to having "a heavy education presence ... especially in the mathematics department." Further, "there is a lot of ... collaboration amongst educators and mathematicians that does not always happen." Jo described the mathematics education group as not feeling "separate from the mathematics department in any way." The interviewed instructors hypothesized the unique collaborative relationship between mathematicians and mathematics educators resulted in a strong desire for instructors to consistently modify their course structure and teaching techniques to support all students.

Symbols

Symbols refer to the cultural artifacts and values – typically informal – which members of a department use to guide their reasoning. Common teaching *symbols* reported by members of the mathematics department include the nature of mathematics and the value of hard work.

The nature of mathematics refers to perceptions about what mathematics is and how mathematical knowledge is discovered or created. Some instructors described their perspective on the nature of mathematics and how it guides their teaching. For instance, Sophie, a lecturer, described some forms of mathematical knowledge as fixed, citing the Fundamental Theorem of

Calculus. Sophie therefore described difficulty performing iterative modifications of course content and structure due to most of the content taught being unchanged. Similarly, Cathleen, a tenured faculty, described mathematics as "more like music and art than ... the other sciences." Therefore, Cathleen teaches mathematics in a similar way to how she would teach music, as a "skill building discipline." These perceptions of the nature of mathematics sometimes created conflict with instructors' understanding of pedagogical practices. Debbie discussed department members needing to "learn how to balance what is recommended by other extant literature with ... what [we] believe mathematics is ... and what it looks like to effectively learn it."

Instructors also described hard work as a key value in the department. Dorothy, a tenured faculty in statistics, described "people in the [mathematics department] care and they are ... striving" to make a difference through their teaching. This deep level of care is reflected in Dorothy's observation in which members of the mathematics department are always "working hard," sometimes "working too hard." Dorothy elaborated by discussing "hard is not always smart" and that she may "not always [be working] smart to make the most efficient type of changes [which make impacts using] the shortest path." Despite the critique, Dorothy described the culture of hard work as a virtuous and encouraging value about the department.

People

People refers to the uniqueness and diversity of instructors which make up the department. In general, the sentiment regarding a diverse department is positive: "people [in the mathematics department] are different, and that's what makes it so great." Two ways the uniqueness and diversity of instructors are highlighted in the department include differences in teaching assignments and differences in background, training, and teaching preferences.

Dorothy described the mathematics department as having an "incredibly wide range of different mathematical disciplines that people are working in" such as "algebra, statistics, education, combinatorics" and much more. This naturally results in instructors teaching a broad range of courses. For instance, Spencer, a tenure-track professor who teaches theoretical math courses, described the content as "very difficult ... to teach" and it being "very challenging to get across certain material to students." Therefore, sticking to pedagogical suggestions from instructors teaching general education mathematics courses may not be effective for Spencer. Furthermore, being a large university, courses are typically offered in a wide variety of modalities. Dorothy, who teaches "a lot of online [courses]," found it difficult to reach students in the same way her colleagues teaching in-person courses do due to the nature of online courses.

The wide range of mathematical background is also reflected in the diversity of training and preferences with respect to teaching amongst instructors. The large education research presence in the department results in many instructors such as Debbie who have "read so much literature about what it means to be an effective teacher [by supporting] a student in learning." Some instructors also associate different styles of teaching with different types of people. For instance, Emmy described one of her colleagues as being "a math teacher [in the K-12 setting] for a long time and so she has taught in a particular way." In addition, Dorothy described some faculty in the department as being "more ... introverted in working on their projects and therefore less interested in being collaborative in their classes."

Power

Power refers to the political relationships between members of the department which may impact instructors' teaching roles. Aspects of *power* which emerged during the focus groups

include lack of autonomy due to power differentials between faculty based on position and experience as well as a chasm between mathematics and mathematics education researchers.

One FTT, Emmy, described FTTs as unable to "serve on committees or do research work or anything." Furthermore, "while … some [FTTs] have been here for over 25 years, [we are] still referred to as temporary." This power differential between FTTs and tenure-track faculty is further exacerbated by FTT's lack of instructional control and autonomy. For example, Emmy described FTTs not being "voting members of the faculty" and not being "decision makers, but ... get a lot of decisions made about [them]." Emmy reported "mandates" FTTs receive regarding teaching, such as implementing a common final exam in the college algebra, precalculus, and Calculus I courses. Although Emmy reported feeling "free to ask questions [and] challenge policies" which "might lead to modifications," she reported uncertainty in whether her suggestions are "actually changing anything at the departmental level."

Although the relationship between mathematics and mathematics education researchers were reported to be largely collaborative, power dynamics and implicit roles between these two groups impact how instructors view their teaching roles, especially mathematicians. Some mathematicians viewed mathematics education researchers as holding the power and authority with respect to pedagogy, deferring teaching-related judgements to these. For example, Cathleen described thoughts about making changes to the curriculum and her pedagogy, but because she is "not [a researcher in] math education," she does not seek to implement her thoughts.

Discussion

This study examined math instructors' perceptions of their departmental teaching culture through the four interrelated frames of *structures*, *symbols*, *power*, and *people.* Such a snapshot of a department's current teaching culture can be leveraged to individualize future change initiatives to meet the unique needs of the department. For instance, a change initiative in this department would need to require minimal time commitment in order to be responsive to the structural time constraints of instructors (Bleiler-Baxter et al., 2021). Further, due to the strong presence of and reliance on math education faculty within this department, such education-based faculty may be leveraged as early adopters and change agents that can support the diffusion of innovation to the department (Eddy et al., 2022). This snapshot also establishes a baseline measurement of departmental teaching culture that can be used to assess the effectiveness of a change initiative. While these findings are unique to the studied institution, they could serve as a roadmap to guide other institutions in characterizing departmental teaching culture within their own context. Further, the stories of these instructors may resonate and inspire other instructors, promoting relatedness and collaboration across institutions.

Preliminary Nature of Report

This report is exploratory in nature. Future work will dive further into unifying themes across the data. Future exploration of the data will also leverage additional theoretical frameworks such as Self-Determination Theory in order to broaden the impact of our findings. Further, we possess additional parallel data from other STEM departments (e.g., biology, aerospace, geosciences) that can be unified with the math departmental data reported here to form an overarching narrative spanning STEM departmental teaching culture.

During the discussion portion of the presentation, questions for the audience will include: (1) What other analytical frameworks might be useful lenses to explore this data? and (2) Which of these findings do you anticipate can be generalized to mathematics departments at other institutions and which do you think are unique to this math department?

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Investigating Student Understanding of the Infinitesimal vs. Limit Definition of Integration Through the Curious Placement of dx

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Integrals can be conceptualized with an infinitesimal or limit definition, and holding a certain conceptualization may change how someone frames and solves an integral problem. We present excerpts from two different interview studies where we asked second-semester calculus-based intro physics students integral questions. Student utterances and differing answers to these selected questions led us to define two conceptual resources concerning infinitesimals, which correspond to the different definitions of integration. This work highlights subtle differences in students' interpretation of integration and areas for future study.

Keywords: integration, resource theory, qualitative, interview

Discussions of the infinitesimal vs. limit definition of integration have been on-going in mathematics education for some time (Ely 2020 provides an excellent summary). While many mathematicians prefer a limit definition of integration, applied sciences, such as physics, tend to rely on the infinitesimal definitions of integration (Redish & Kuo 2015). Research also shows that students can hold multiple conceptions of differentials and treat them differently based on these interpretations (Artigue et al. 1990). Students may also favor infinitesimal conceptions of integration despite textbook focus on limits (Nilsen & Knusten 2023). The extent to which this difference in interpreting integrals acts as a barrier for students is unclear. The goal of this paper is not to demonstrate which definition is correct, or which should be taught, but rather to present nuanced findings on this topic and a number of challenges and inconsistencies for these two definitions that we've identified, and the impact they may have on student success in different courses contexts.

The idea that this may be a point of interest came to our attention while conducting interviews using math questions written by a mathematician alongside physics questions written by a physicist. A question we asked, from Jones 2013 which was also written by Jones, was the question $\int_1^2 2/x^3 - x^2 dx$. We observed that this problem does not have parentheses around the integrand. Consistent with a limit definition of integration, integrals have the form of ∫INTEGRAND *dx*, regardless of whether there are parentheses around the integrand terms or not. The implicit notion is that the *dx* is acting as punctuation, and a marker for the variable over which the integrand will be integrated. Within this model, these parentheses are superfluous. Due to the lack of parentheses, the limit definition of integration does not imply that the *dx* is being multiplied across the terms within the integrand. However, within applied contexts the differential (or infinitesimal) *dx*, or *dm*, or *dt*, represents a small bit of that physical quantity and has units associated with that quantity. The *dx*, for example, is multiplied across every term of the integrand, and therefore parentheses are necessary to ensure that the units work out appropriately. Throughout this paper we refer to the differential, *dx*, as an infinitesimal with the qualities just described. We acknowledge that this is not a rigorous mathematical definition, but we hope it's sufficient to understand our findings. We report preliminary findings about the difference in the interpretation of *dx*, and how it impacts student responses when solving integration problems. The main goal of this paper is to answer the Research Question: "To what

extent do students in a second-semester physics course use conventions consistent with infinitesimal definition vs. limit definition of an integral when given a question with a unique placement of *dx* within the integral?"

Theoretical Framework

For this study, we employ the Resources and Framing Theoretical Framework (Hammer et al 2005). Resources and Framing models thinking as consisting of fine-grain "nuggets" that a person has learned and used successfully somewhere previously that are unconsciously brought to bear on a novel task (Hammer, private communication). These resources may have connections to other resources, some of which may be stronger or weaker connections depending on the experience of the person who has used them together in the past (Hammer et al 2005). The accessibility of these resources are influenced by how the individual frames the particular task. Therefore, if someone doesn't articulate a particular resource, this does not mean that they don't possess that resource, rather, the resource was not activated. Within this framework, we as researchers attempt to identify the "good idea" that the individual is attempting to use for a particular problem and if it is successful (i.e., achieves a correct answer), we deem that resource to be productive (Barth-Cohen et al 2023). If the individual is not successful (i.e., gets an incorrect or incomplete answer) we deem that resource to be unproductive. Here, we identify two resources that students may activate in the provided problem, and how these resources connect to either an infinitesimal definition or a limit definition of integration.

Methodology

This study recruited students from the end of a second-semester introductory calculus-based physics class at a large R1 in the upper-Midwest. Students in this course, at minimum, were completing Calculus II in this semester and many were completing Calculus III and/or Differential Equations. Two different sets of interviews were conducted. The first set consisted of three (3 white men) and the second set consisted of four (3 white men, 1 white woman) semi-structured, think-aloud interviews that were conducted and video recorded (Browner 1988). The interviews were transcribed and analyzed through a lens of Resources and Framing (Hammer et al 2005). The protocols featured an array of mathematics and physics questions. The results of one particular question from each interview protocol are presented here. We report on these results combined with textbook observations.

Results & Analysis

Integral without parentheses

We first consider an integral question from Jones 2013. This problem, one among a series of math/physics questions, was given to three individuals. The question was given as was originally written, $\int_1^2 2/x^3 - x^2 dx$, which did not include parentheses. None of the three students articulated any issue with how this question was written. All three students successfully evaluated the integral. None of the students put parentheses around the integrand; they simply used their knowledge for integrating polynomials and achieved the correct result. We claim this demonstrates students' activation of a "*dx* as punctuation" resource, that is consistent with the limit definition of integration. This may seem relatively light evidence for such a claim, but as stated in the introduction, within a model of infinitesimals, the *dx* is a term that is being multiplied by each of the terms in the integrand. None of the three interviewees commented on

the lack of parentheses or drew parentheses before doing the integral. Each of them simply performed the integral. Therefore, we claim that they are treating "*dx* as punctuation."

In discussion with other PER and RUME researchers there was disagreement as to whether or not the lack of parentheses was appropriate, and the extent to which introductory calculus textbooks used parentheses for integrands that had more than one term. An investigation into a number of introductory calculus books showed that all instances of addition and subtraction within an integrand had parentheses around the integrand (Larson & Edwards 2017, Rogawski & Adams 2015, Stewart 2016). It's unclear how prevalent a lack of parentheses is among Calculus instructors and students, and we plan to investigate this further. In discussions with a RUME researcher, they believed the parentheses to be superfluous and were surprised that the introductory Calculus books had parentheses throughout. Additionally, this researcher identified at least one upper-division mathematics textbook that did not use parentheses for integrands that had addition or subtraction within them (Pugh & Pugh 2002). Furthermore, within the same textbook and an additional one there were integrals where the *dx* term was not written within the integral at all (Pugh & Pugh 2002, Abbott 2015). This raises further questions about the practice of experts in mathematics and how that practice is communicated to students.

The curious case of *dx* in the middle of the integral

Within this second interview protocol, four students were given the prompt $\int 2dx(x^2 - 3x)$. The purpose of this question was to investigate how students would contend with this unique placement of *dx*. The interview protocol prompts "What do you think about this integral?" and then after their explanation, a follow-up of "Could you evaluate this integral?" was asked. The following are the four responses given by our interviewees, with analysis to follow.

Student 3. Student 3 (S3) responds to the initial prompt with "The integral of 2*dx* [and makes a motion with their hands to look like air parentheses] times $x^2 - 3x$, so **I think of this as basically a parentheses being here**, that's how I read that." S3 drew in the parentheses on their page as seen in Figure 1, enclosing the integral symbol, the 2 and the *dx*. And then evaluated the integral of 2*dx* to get $2x + c$, and then wrote their final result as $(2x+c)(x^2 - 3x)$.

$$\left(\int 2dx\right)(x^2 - 3x)$$

Figure 1. Student 3's annotation of the question prompt. The parentheses indicate the integral of 2 dx as a separate quantity, the result of which they later multiplied by (x^2-3x).

Student 2. Student 2 (S2) states, "It looks out of order. ***dx* should be at the back.**" S2 then moved the *dx* behind the $(x^2 - 3x)$ term, and evaluated this integral correctly. See Figure 2. Note that the answer of S2 is different from S3.

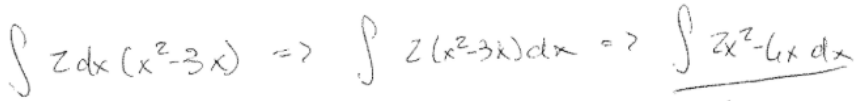

Figure 2. Some of Student 2's work on the given integral problem.

Student 4. Student 4 (S4) states, "Well, **the *dx* could be moved back to the side. So it essentially becomes the integral of two times the quantity $x^2 - 3x$.**" They move the constant 2 out in front of the integral and then write down the expression as seen in Figure 3, moving the *dx* to the right side of the expression, but they do not put parentheses around what is now their integrand. They then evaluate the integral. The interviewer asks, "So I think you kind of already

spoke to this. But is there anything wrong about the way that this is written, initially?" Student 4 replies, "I wouldn't say it's wrong. **It's certainly not what I'm used to, but you can put the *dx* anywhere** in that expression and have it still be true."

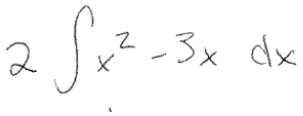

Figure 3. Student 4's first step on the given integral problem.

Student 1. Student 1 (S1) is significantly more descriptive in their thinking about the integral.

S1: "Um I just say it looks weird, but mathematically I'd say–

INT: "Why does that look weird?"

S1: "Because the d– because the *dx* is in front. Um is it, is it the last term? Because what, I was always taught like in high school– was like your– **your *dx* is like**, that's like **the period of your sentence** like that closes it off. But I know mathematicians get mad at you with this, but like when you like, just multiply differentials like you can't like you can't treat them as fractions, but you– you kinda can… **So I would think that it's perfectly logical to just move it [*dx*] to the end based on, uh, one of the properties of multiplication.**"

After this consideration of the *dx*, and, sounding as if they intend to move the *dx* behind the parenthetical term, S1 is asked to evaluate the integral. S1 immediately changes their mind, draws parentheses around the integral symbol and the *dx* and evaluates the integral the same way as S3. S1 states,

S1: "Ooh. Uh actually um. Now I think I'm gonna go back on myself. I– I actually think that this integral is– is done. Because the *dx* would end the integral."

S1: "So, so I, I– looking at it so now I would, I would integrate the 2*dx* to be 2*x* and then this part is almost treated as in parentheses and I have 2*x* multiplied by $x^2 - 3x$, and then we distribute, $2x^3 - 6x^2$, and that would be your answer."

When the interviewer asks for further explanation of the S1's response, S1 states,

S1: "Yeah. Cause, I don't know if this is the case, but like kind of mathematical like ambiguity. Um, **I'd imagine that you could just multiply the *dx* but I think conventionally the *dx* is at the end**. Like I said, with the period in the sentence. Um yeah, but if I saw this, I guess initially I saw it the other way, but if I– if I saw this, I would imagine that the ***dx* is like the ending of the integral, like the integral starts with the integral sign and ends with the differential.**"

When looking at their final results, two students (S2 and S4) treat *dx* as a quantity (infinitesimal), and move it to the right hand side of the integral before evaluating. The other two students (S3 and S1) ultimately leave *dx* where it is and evaluate the integrand of 2, both being specific and drawing parentheses to make it more clear. These two answers are quite different. Depending on the interpretation of the integral, one arrives at one of two different results.

We identify S3 and S1 as using the "*dx* as punctuation" resource. This resource is very consistent with a limit definition of integration. This is the same kind of resource necessary to interpret $\int_1^2 2/x^3 - x^2 dx$, without parentheses. The *dx* serves as the period at the end of a sentence which begins with the integration symbol, and everything in between is the integrand, the terms to be integrated. S2 and S4 identify the placement of *dx* as "looking weird" and "not

what I'm used to," respectively. However, they both move the *dx* term to the end of the integral before evaluating it. We identify this as "*dx* as a quantity (infinitesimal)" resource. This resource is the idea that *dx* within an integral has qualities, such as being subject to multiplicative properties, similar to a variable or a constant and therefore can be moved.

Three of the four students answered this question rather quickly and confidently without a great deal of context or description. S1, however, gave a rather articulate answer that bears further analysis. S1 explicitly states that they were taught in High School that *dx* acts like a period of a sentence, a statement that we would identify as "*dx* as punctuation". They then state that mathematicians don't like it when you multiply differentials like fractions, but that you still can. S1 continues by stating that they can treat *dx* like a term that can be multiplied and that it's logical to move it to the end. These are strong statements that suggest activating a "*dx* as an quantity" resource. However, when asked to evaluate the integral, S1 proceeds with the "*dx* as punctuation" resource and ultimately gets the same result as S3.

A final note on the work of S4 as seen in Figure 3. S4 does not include parentheses around the $x^2 - 3x$, even though there were parentheses around this quantity in the original prompt (See Figure 1.) We connect this lack of parentheses to the "*dx* as punctuation" resource for the "Integral without parentheses" question data. So, we could say that S4 holds and activates "*dx* as a quantity", but then uses notation conventions consistent with "*dx* as punctuation", as they do not deem it necessary to include parentheses on an integral they felt comfortable manipulating. This demonstrates that this individual holds elements of both resources.

Discussion

We have identified two different resources: "*dx* as punctuation" vs. "*dx* as a quantity," and the markers for these resources. We also demonstrated how the activation of one resource, or the other, or both, impacts a student's thinking when solving the two integration problems presented. The activation of one of the two resources can result in different answers to the same problem. While these examples are limited, they demonstrate instances where these two resources could be activated. We highlight a case where activating "*dx* as punctuation" presents no issue for students; there are instances where both interpretations of integration may be necessary, especially in the applied sciences. Although more research in this area is needed, it would seem prudent for both mathematics and physics instructors to be aware of these subtle, yet important, interpretations and distinctions in student thinking.

Future Work

Two questions that we seek to answer are: 1) to what extent is the thinking presented here prevalent among a larger sample of students and 2) to what extent do math and physics instructors think in similar or different ways pertaining to these integral questions. For question 1, we intend to distribute these questions in free-response format to students in Calculus II and Calculus-based Physics II courses in the coming semesters. For question 2, we will be collecting interview data from math and physics instructors to understand not only how they interpret these questions, but also their thinking about the use of parentheses in integrals as was discussed.

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Context Dependence of Student Responses and Explanations on Dot Product Magnitude Tasks

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Preliminary findings from a study exploring student understanding of vector products are presented. We discuss multiple-choice and free-response data from 118 students who completed four dot product magnitude tasks across two semesters of introductory physics instruction. Each semester had a physics-free dot product along with a physics context relevant to the content of each course (mechanical work and electric flux). Performance ranged from 28% to 59% and students do not always answer consistently between pairs of tasks. Coding of explanations revealed four main computational strategies used by students along with two more conceptual reasoning categories. The most common strategy was students using a trigonometric definition of the dot product with varied results depending on the context.

Keywords: vector products, student thinking, physics

Vector quantities are ubiquitous in physics, appearing very early in an introductory physics course and continuing to be used extensively throughout an undergraduate physics curriculum. Since mathematics curricula do not usually cover vectors until the third semester of calculus, it is necessary to explore the use of vectors in a physics setting. A body of work in physics education has explored student understanding of vectors (e.g., Knight, 1995; Nguyen & Meltzer, 2003; Flores et al., 2004; Gire & Price, 2014). Our research questions for this report are: How does context impact student approaches to vector product tasks? What aspects of physics contexts influence student reasoning? While this work is part of a longitudinal study of how student understanding of vectors evolves over time, only introductory-level data is presented here.

Preliminary Nature of the Report

We frame this work from a mathematical modeling perspective, looking at different steps in a modeling process that at its broadest includes connecting the physical scenario and the associated mathematical entities as well as performing mathematical operations (e.g., Redish & Kuo, 2015; Uhden et al., 2012; Zbiek & Conner, 2006). We are interested in the role of context on student understanding of vector products and on how students reason about the relevant mathematical operations. A more fine-grained analysis of student reasoning and modeling with our data has not been completed. In addition to framing via modeling, we believe that our findings are well suited for analysis via dual-process theories of reasoning (Evans, 2006; Heckler, 2011; Kryjevskaia et al., 2014; Spiers et al. 2021), in particular the impact of visual and contextual features on student reasoning. A broad question for readers: What other perspectives and/or frameworks might provide additional insight into student reasoning and especially the consistency of their reasoning across multiple tasks?

Background

The dot product (or scalar product) of two vectors, $\vec{A} \cdot \vec{B}$, can be determined in two ways, either as $AB\cos\theta$, where θ is the angle between the two vectors when placed tail-to-tail, or as the sum of the corresponding components, $A_xB_x + A_yB_y + A_zB_z$.

Work on student understanding of vectors has explored both computational and conceptual issues. On the computation side, Mikula & Heckler (2013) investigated students' ability to

calculate vector components and found that angle placement impacted performance. Heckler & Scaife (2013) compared different notations and saw better performance on algebraic problems than problems using arrow notation. Van Deventer (2008) and Carli et al. (2020) developed isomorphic mathematics and physics questions and observed context dependence of student strategies. Carli et al. reported that the vector product tasks were most challenging among vector concepts in both the math and physics contexts with a maximum performance of 41%. Barniol & Zavala (2014) developed the Test of Understanding of Vectors, a multiple-choice assessment on procedures and properties of vectors without physics contexts. On a more conceptual level, Zavala & Barniol (2013) asked students about the best interpretation of the dot product in three different contexts. They found no significant difference between the results for the two physics contexts (work and flux) but did see a significant difference between the "no context" question and the physics contexts, with better performance in the physics contexts.

Several of the dot product items in the literature have answer choices that are a mix of scalars and vectors, which could conflate evidence of student understanding of magnitude and direction. To disentangle calculation errors from reasoning about the vector or scalar nature of the results, our tasks had separate questions about the magnitude and direction of each product. In matched data on the dot product direction tasks we found no significant shift between correct and incorrect answers, but there was a significant shift from the common incorrect answer to other choices depending on the context and aligned with the physics (Molinari & Thompson, 2025). While the studies above demonstrate that context influences responses, not all these studies have matched data, and we are interested in not only how entire populations complete different tasks but also how individuals respond in different contexts.

Methods

Online surveys were administered after relevant instruction in calculus-based introductory physics in both semesters of the same academic year at a medium-sized R1 institution. Each survey asked separate questions about the magnitude and direction of a vector product (dot or cross) in a "math" context with no physical scenario and a physics context. The physical dot product contexts were *mechanical work* in the first semester (PHY1) and *electric flux* in the second semester (PHY2). *Work*, a common application context in calculus, is the dot product of force and displacement; *electric flux* is the dot product of the electric field and an area.

Each problem statement included a diagram for the situation (Figure 1) and values for the magnitudes of the vectors and angles in each diagram. The question posed was, "What is the magnitude of the dot product $\vec{A} \cdot \vec{B}$ / $W_{BR} = \vec{T}_{BR} \cdot \Delta\vec{r}$, the work done on the block by the rope / $\Phi_E = \vec{E} \cdot \vec{A}$, the electric flux through the area?" Multiple-choice options were numerical values.

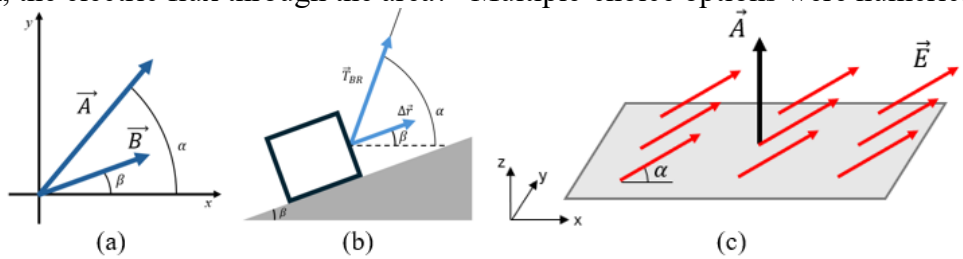

Figure 1. Diagrams provided for (a) math, (b) work, and (c) electric flux tasks.

All three contexts had enough information for students to find the angle between the two vectors ($\alpha - \beta$ for *math* and *work*, and $90 - \alpha$ for *flux*). The *flux* scenario was not a perfectly isomorphic scenario, instead presented in a form more aligned with problems given in a physics

class. Since *math* and *work* had two given angles there were more distractors in those contexts than the *flux* question to allow for options involving the cosine of each given angle.

There were 191 dot product survey responses in PHY1 and 181 in PHY2, with 118 students completing both surveys serving as longitudinal data points via matched responses. We also conducted 4 free-response, think-aloud interviews at the end of PHY2 for additional insights.

For both the magnitude and direction of each dot product task, after participants submitted their multiple-choice answers, the next page of the survey re-presented the questions, reminded students which choice they selected, and prompted them to briefly explain their reasoning for each answer. The final code book (Table 1) is the result of an iterative process in which explanations were coded – independently from the multiple-choice selections – to identify the mathematical approach students used to reach their answer. The first four rows are the main computational strategies used by students; projection/alignment and limiting cases (#5 and #6) were codes that appeared both in conjunction with the main strategies and stand-alone. Codes are independent of correctness: within each main strategy, sub-codes were used to distinguish approaches leading to correct or incorrect conclusions, such as using an inappropriate angle or function. Inter-rater reliability achieved Krippendorff's α coefficients of agreement of .92, .87, and .96 for the *math*, *work*, and *flux* contexts, respectively.

Table 1. Table of dot product task reasoning codes and descriptions. Starred categories (*) could be sub-codes of the first four codes.

Code	Description
1. Trigonometric definition	Multiplying the magnitudes of both vectors by a trig function (predominantly sine or cosine) of a correct or incorrect angle
2. Sum of component products	Decomposing, or attempting to decompose, the vectors and taking the sum of the products of the corresponding components
3. Scalar multiplication	Multiplying the magnitudes of the two vectors
4. Addition	Using strategies of either scalar or vector addition, such as tip-to-tail
5. Projection/alignment*	Discussing vector projections or how aligned the two are
6. Limiting cases*	Using maximum and/or minimum values to support their selection
7. Restating	Stating the answer selected is what they calculated without elaboration
8. Other	Does not fit into any of the above categories (also blank, unclear)

Results & Discussion

Multiple-Choice Results

The overall performance on the four dot product magnitude tasks ranged from 28% to 59%. In the *PHY1 math* context, 52% of students selected the correct response, compared to 59% in the *work* context, 42% in the *PHY2 math* context, and 28% in the *flux* context, as shown in

Figure 2. Distractors of similar types are combined for presentation. "Trig errors" includes the answers that could be reached by using sine of the angle between the two vectors or cosine of any given angle. "Addition" includes both the scalar and vector addition choices.

Consistency between tasks shows evidence of the broader challenge. Table shows the proportion of students who answered consistently, correct or incorrect, between four pairs of tasks. Between the *PHY1 math* and *work* contexts, 58% of students selected consistent responses, and only 30% were consistent between the two semesters in the *math* contexts. Each comparison to the *flux* task has less than 30% consistency between response pairs. Across all four tasks, 10% of the 118 students correctly determined the magnitude of the dot product.

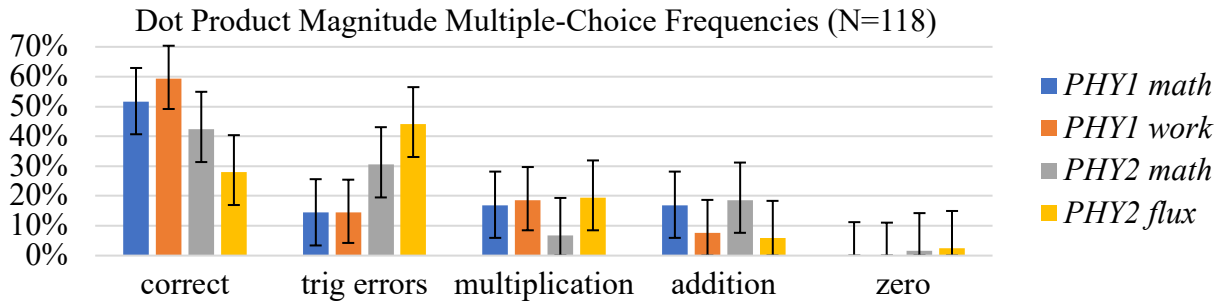

Figure 2. Response frequencies for all four dot product magnitude tasks across the year, for matched students.

Table 2. Multiple-choice response consistency across two tasks by percent of the population, N=118.

Categorization of paired responses	*PHY1 math* vs. *work*	*PHY1 math* vs. *PHY2 math*	*PHY2 math* vs. *flux*	*Work* vs. *flux*
Consistent and correct	44%	22%	16%	19%
Consistent but incorrect	14%	8%	11%	5%
Inconsistent	42%	70%	73%	77%

Explanation Results

To understand why students are selecting the answers they are, we go to the explanation codes. The trig definition approach was the most common strategy explained by students in all four contexts, with about half of the free responses coded as such in each context (shown in Figure 3). Of these students we grouped their multiple-choice responses as correct, trig errors, and other errors (multiplication, addition, and zero). The sum of "restating" and "other" codes (#7 and #8 from Table 1) were more common than the remaining specific approaches, with between 20% and 31% of responses coded as such.

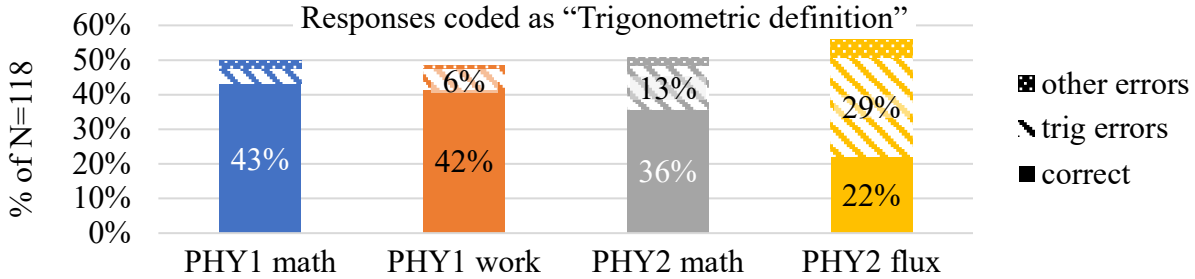

Figure 3. Breakdown of multiple-choice selections of students coded as using trigonometric definition explanations. Percentages not shown are ≤ 5%.

Due to the angles given in each context (see Figure 1), the *math* and *work* trig errors account for three multiple-choice options while there is only one *flux* trig error answer, corresponding to $EA\cos\alpha = EA\sin 90-\alpha$. Within these *flux* responses, the majority explained by saying they calculated cosine of "alpha," "30," "theta", or "the angle between them," effectively taking the dot product of two vectors that are an angle α apart instead of $90-\alpha$. This could be for a variety of reasons, ranging from simply using the angle provided to a conceptual confusion about the area vector (i.e., placing it in the plane of the surface). Most of the trig errors in the *PHY2 math* context were from using sine instead of cosine, which is likely a recency effect due to the coverage of the Lorentz force, which is determined by a cross product, the week prior.

Regardless of approach, only up to 50% of students gave explanations coded in the same category (#1-6 in Table 1) across any task pair. Furthermore, the same approach does not

necessarily lead to the same answer. For example, 43% of students who were coded using a trig definition for both *PHY2 math* and *flux* and 32% of the students who were coded using a trig definition for both *work and flux* gave consistent responses in the respective task pairs.

The projection/alignment code was not prominent but was more common in the physics contexts than in the math context. One student in *PHY1 math*, 8 in *work*, none in *PHY2 math*, and 4 in *flux* used these ideas either as their sole explanation or in tandem with the trig definition approach. Additionally, during the *flux* portion of the interviews, follow-up questions about why students used cosine in their calculations or about the significance of the angle elicited responses about the electric flux measuring how much of the field is caught by the area, with analogies like a sailboat in the wind (which was generated by the instructor), whereas the same follow-up questions in the *math* context were met with answers such as "that's what I'm remembering the equation as" and "'cause dot products use cosine, cross products use sine." Students seem to attribute the operation of a dot product with more physical meaning in physics contexts than when the physics is pared away. Our findings agree with Zavala & Barniol's (2013) results where students were more likely to describe the dot product as a projection in physics contexts than with no context. However, the very low count of the projection/alignment code among our explanations suggests that students are not likely to offer this reasoning unprompted.

A final note on student reflection. Three of the four interview participants started doing their calculations for *flux* by using cosine of the angle given, α, then either by catching it themselves or realizing after a request for an explanation that the angle they needed was actually $90 - \alpha$. Additionally, across all four dot product magnitude tasks, 11 students said they would change their answer after looking at the problem again or thinking about it more.

Conclusions & Implications

From a modeling perspective, the *math* task drops students into the mathematical world, while the physics tasks provide the physical situation and a mathematical model (a dot product) for the quantity in question that should use the same mathematical operations. However, our data suggest that students perform different operations in the two scenarios and thus may not make consistent mathematical choices across contexts, supporting the idea that physicists tend to load meaning onto symbols while mathematicians generally do not (Redish & Kuo, 2015). In our study, the meaning being loaded by the physics seems to influence the mathematical operations performed, an aspect that may not be accounted for in modeling frameworks.

Visual representations of each context and the visual features to which students attend influence what values students use in their calculations. The fact that the survey responses are susceptible to new content being introduced and the difference in explanations across semesters for the same (*math*) task suggests that models for dot products are still dynamic at that later time.

Further exploration of this work will involve utilizing dual-process theories of reasoning as a framework both for task design and data analysis that may help explain inconsistencies across contexts. Further application of findings to modeling frameworks may illuminate specific conflicts between the frameworks and student reasoning.

The presence of physical situations may cue students to reason differently as they do not often call upon the formal geometric interpretation of the dot product to perform calculations. By the time students see the formalism of vector products in mathematics courses, they have likely already seen these operations in a physics class(es), so mathematics instructors may be able to draw on physics contexts to help solidify understanding of vector operations. But these contexts come with their own conceptual overhead and thus need to be introduced with care.

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Community College Students' Perspectives on Classroom Environments in College Algebra

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We present a preliminary report on community college students' perspectives of classroom environments in college algebra. Drawing on nine student interviews from four states, we developed and calibrated a coding framework of 34 codes organized into nine categories. Our analysis highlights three broad dimensions of classroom experience: individual factors, relational dynamics, and structural or contextual elements. Students emphasized the role of supportive and responsive instruction, including instructors who re-explained concepts, offered second chances, and adapted to diverse learning needs. They also described participation structures that shaped collaboration with peers, as well as classroom climates that encouraged or at times discouraged active engagement. Beyond their current experiences, students voiced aspirations for more inclusive and collaborative practices. These early findings provide insight into how community college students experience college algebra and point toward future directions for exploring diversity, equity, and inclusion in mathematics classrooms.

Keywords: Community college, classroom environment, equity, diversity, inclusion

Although equity research in mathematics education has emphasized instructional practices and teacher perspectives (Gutiérrez, 2002), far less attention has been paid to how students perceive the classroom environments that support or hinder their learning. Prior work highlights the importance of classroom environments in shaping opportunities to learn, but student voices are rarely centered in the literature. As noted by recent research, we have limited perspectives from students underscoring the need for more studies that foreground their voices (Wickersham & Nachman, 2023). We focus on college algebra, a gateway course that can influence whether students persist in STEM-related fields (Walkington, 2025). Our study takes place in community colleges, where many students are first-generation, low-income, or racially minoritized (American Association of Community Colleges, 2025), and where equitable access to mathematics is especially consequential. By centering students' voices, our goal is to examine how they describe classroom environments and to consider how their accounts affirm, complicate, or extend existing research on equitable practices in mathematics education. Guided by this perspective, we ask: How do community college students describe classroom environments in college algebra?

Conceptual Framework

In mathematics education, equity-focused research emphasizes the importance of classroom environments that provide fair opportunities to learn and supportive conditions for participation

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(Esmonde, 2009a, 2009b). The classroom environment in mathematics education refers not only to instructional practices but also to the social and affective climate shaped by peer and instructor interactions. For example, Cobb and Hodge (2002) highlight a relational perspective on equity, in which classroom norms and interpersonal dynamics determine whether participation feels fair, inclusive, and respectful. Gutiérrez (2002) argues that equity in mathematics education requires not only access to content and achievement but also meaningful opportunities for participation and recognition of diverse student identities. Understanding interactions in diverse classrooms**,** where students vary in race, culture, language, socioeconomic background, ability, gender identity, and more, is essential but complex. Mathematics educators have sought to understand instruction as interactions that occur between teachers, students, and the content, and that take place within a classroom environment that can determine the nature of those interactions (Cohen et al., 2003). By understanding instruction in this way, and the classroom environment as either supporting or constraining the interactions, we seek to uncover individual, relational, and structural or contextual elements that may affect students' participation in mathematics.

Methods

The study follows a grounded theory design that calls for theoretical sampling and multiple iterations of data collection. Participants from eight community colleges were invited to participate in an interview study (four interviews over an eight-month period). The students had to be familiar with college algebra at their college (i.e., were taking or had taken the course). For this preliminary report, we analyzed interviews from nine participants (out of 38) across seven institutions. The sample included five women and four men (ages 18–50) with varied commitments outside school (e.g., full-time jobs, caregiving). Participants identified as African American/Black, White, West Asian, Hispanic, Kenyan, and multiracial (e.g., half Peruvian and half Guatemalan). This purposive sampling aimed to capture a wide range of perspectives while keeping the dataset manageable for early analysis. The first interview asked about general feelings toward mathematics and memorable experiences, peer and instructor interactions (including moments of support and lack of support), and definitions of diversity, equity, and inclusion. These questions provided a baseline for subsequent interviews and explored how students understood classroom interactions in relation to their participation and motivation. Interviews were semi-structured, with open-ended and probing prompts (Roulston, 2013).

Codebook development proceeded in four phases. First, one transcript was coded in Dedoose by three researchers, followed by discussion to clarify meanings and expand initial codes. Second, four additional transcripts were independently coded to seek new codes and refine definitions, producing 41 codes with examples. The transcripts were selected based on the impressions of the interviewers that they could bring different perspectives to the ones already captured. Third, the 41 codes were organized into 11 categories framed by the instructional triangle (teacher, student, and their interactions). For example, the codes *Supportive and Responsive Instruction* and *Commitment to Teach* were subsumed under the category *Instructor*, because they both describe how students perceived their instructors' teaching practices, attitudes, and dedication, rather than focusing on students' own actions or broader classroom conditions. As no new codes emerged, we moved to the fourth phase, using four transcripts from students at colleges not yet represented, to broaden the codebook. These were assigned to two new coders paired with original team members. They coded three transcripts each. The goal of this step was to confirm the definitions of the codes, make sure that they were applied consistently, and determine whether the codes were correctly placed under the existing categories. During this phase, one additional code emerged "Small class size limits opportunities for group

collaboration" which we placed under Factors. During this calibration, some codes were merged and definitions refined. The process concluded with nine categories and 34 codes, stabilized for use in this preliminary analysis (see Figure 1), and pseudonyms are used throughout.

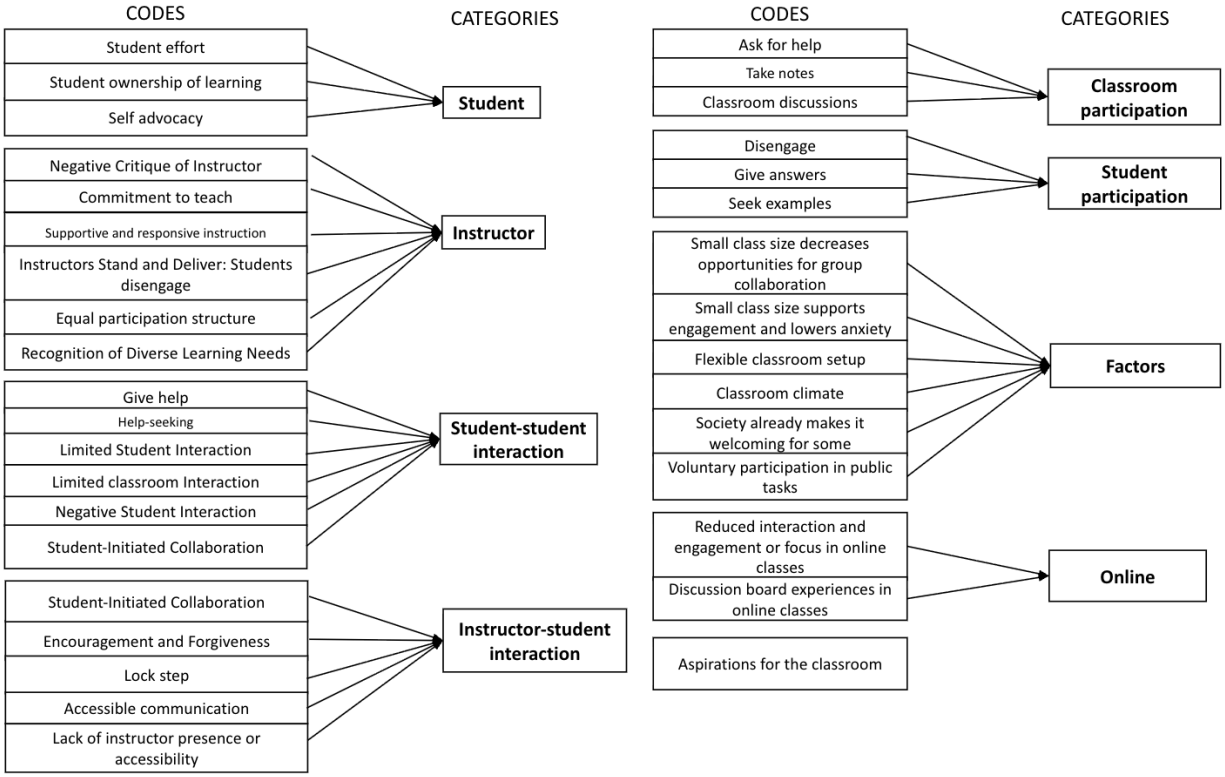

Figure 1. Preliminary codes and categories based on nine transcripts

Preliminary Findings

Our analysis draws attention to the ways students talk about instructors, peers, participation, and classroom environment. In organizing the codebook, we grouped codes into three broad dimensions: individual factors that capture how students described their own role in learning (e.g., student-focused codes such as effort and self-advocacy); relational dynamics that encompass interactions with instructors and peers, highlighting how teaching practices, responsiveness, and student–student collaboration shaped opportunities and constraints for participation (e.g., instructor, student–student, and instructor–student interactions); and structural or contextual elements that encompass broader features of the learning environment (e.g., class size, classroom climate, online formats) (see Figure 2).

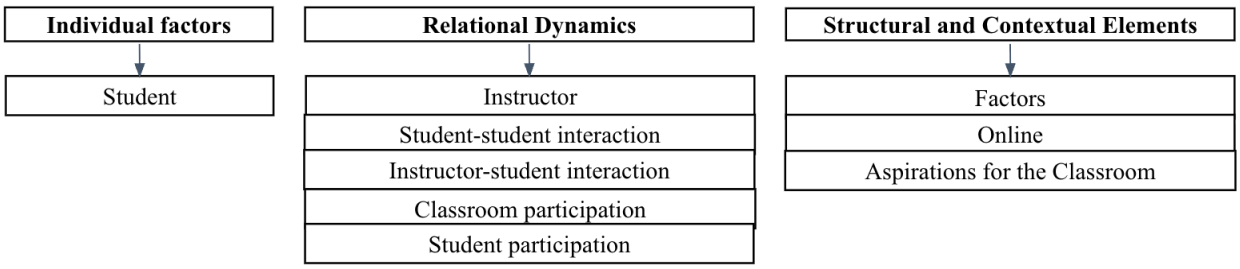

Figure 2. Preliminary categories and dimensions

These dimensions show that students' learning was shaped not only by their own effort and strategies, but also by how supportive instructors were, how much students interacted with peers, and how the classroom itself was structured. Viewed through an equity lens, these individual, relational, and structural or contextual elements show how classroom environments supported or constrained students' opportunities to learn and participate. In the following preliminary findings, we place particular emphasis on individual factors, relational dynamics, and structural or contextual elements, as these codes emerged most prominently in students' accounts.

Individual Factors. Within this dimension, we focus on two features of the classroom environment: Student Ownership of Learning and Self-Advocacy. Through **Student Ownership of Learning**, students described becoming more aware of how their own choices and persistence affected their outcomes. George reflected: "I also helped with a lot of the questions that [the instructor] asked… and at that time I actually got decently good with math." Jacob similarly noted, "I have to put in a lot of time outside of class… when I don't get that done, it really piles up after a while." These reflections show students making connections between their actions and their academic growth, underscoring their sense of responsibility for learning. With **Self-Advocacy**, students highlighted how voicing their needs and seeking support enabled them to navigate challenges. Elena explained: "I think just being more vocal about needing help instead of sitting in that frustration has just helped me." Pilar described this as a personal strength: "I think personally, I'm really good at advocating for myself and asking questions, and I've never had a teacher refuse to answer." Such accounts reflect how students exercised agency in creating equitable opportunities for themselves.

Relational Dynamics. Within this dimension, we highlight four features of the classroom environment: Supportive and Responsive Instruction, Equal Participation Structures, Recognition of Diverse Learning Needs, and Limited Classroom Interaction. **Supportive and Responsive Instruction** refers to instances in which students highlighted instructors who re-explained concepts, offered second chances, or encouraged persistence. Elena said that her instructor "made it very clear that his goal is for us to understand… he will not move on until he's like, 'Are you sure you get it?' and he tries to explain it from a different standpoint." Jacob similarly emphasized that his professor consistently checked for understanding, "going around the classroom… answering our questions, making sure that we're getting it." These accounts align with Gutiérrez's (2002) view that equity requires meaningful access to resources and opportunities. **Equal Participation Structures** collects students' descriptions of opportunities to engage simultaneously in problem-solving or to collaborate informally with peers. Jaime explained how her instructor ensured that "…so, like, people that need the extra extra explanation, it assists everyone. She's including the people that don't understand the same assignments as the people that do." Similarly, George noted the value of simply allowing time for collective work: "…So just let everyone get a chance to study together in class that would help a lot of people trying to get assignments done." Such practices reflect Esmonde's (2009a, 2009b) definition of equity as the fair distribution of opportunities to learn and participate, not only access to content, but also mathematical discourse and identities as knowers and doers. **Recognition of Diverse Learning Needs** captures students' comments about the importance of instructors adapting to varied learning styles and providing multiple entry points into mathematics. As Ayami explained, "We kind of need to branch out and explore different learning styles. Right now, I'm getting tested for neurodivergency, and it's a little bit difficult to just go through the way my current professor is teaching." Similarly, Althea said, "He does a survey… seeing how people, like, learn… 'email me if you need more explanation… and I'll

make a video about it,' because he just did a general video for the whole class." These reflections highlight students' desire for instruction that acknowledges diverse learning needs and aligns with equity-oriented calls for culturally and linguistically responsive pedagogy (Cho et al., 2022; Felton-Koestler, 2019). Finally, **Limited Classroom Interaction** refers to statements about how peer-to-peer communication was minimal or absent. Elena explained, "we all keep to ourselves… it's not too much talking, but there is some limited [interaction]." Althea, reflecting on her online algebra course, noted that "the only time I talked to another peer about math [was] in the tutoring center." Pilar similarly described the atmosphere as "kind of isolating… it almost felt like no one else was in the classroom." These accounts show that when opportunities for peer collaboration were limited students experienced learning as individuals and were disconnected. These experiences resonate with Cobb and Hodge's (2002) relational perspective, which emphasizes that equity also depends on the classroom norms and interpersonal dynamics that shape how participation is defined and enacted. When interaction is minimal, students have fewer opportunities to build mathematical identities and to experience learning as a shared endeavor.

Structural and Contextual Elements. Within this category, we highlight two features related to this dimension: Classroom Climate and Aspirations for the Classroom. **Classroom Climate** captures comments students made about their willingness to contribute to the emotional tone of the classroom and their perceptions of whether student mistakes were treated as learning opportunities or as sources of embarrassment. As Elena explained, "creating that space and that tone that mistakes are okay… we can solve the problem even though it's wrong." Jacob similarly emphasized that his instructor was "not judging… not going to us with a judging tone… not going to be super hard on us for getting it wrong." As Asa explained, her instructor interacted with all students "in the same and respectful" way and kept "trying to explain again and again so the sign translator could translate for those students". These accounts echo Cobb and Hodge's (2002) relational perspective, which argues that equity depends on whether norms and interpersonal dynamics create inclusive and respectful participation. With **Aspirations for the Classroom** we captured students' hopes for more inclusive and collaborative classroom practices. Elena suggested for instructors to check in with all students, not just the ones who frequently participate: "I would just make sure that I'm asking everyone if they understand. I am not just going to the person who always raises their hand… especially to the ones who don't answer or who don't raise their hand." Similarly, Pilar envisioned more group-oriented work: "Maybe like more problems or projects that make people work together and make more groups." These aspirations resonate with Esmonde's (2009a, 2009b) definition of equity as the fair distribution of opportunities to participate.

Next Steps and Questions

Ongoing analysis with all the interviews will identify common themes and better understand perspectives students bring to what an equitable math classroom environment looks like for algebra at community colleges. Questions for the audience, currently we have the following:

1. How might the categories we developed (e.g., student effort, classroom climate, instructor–student interaction) be grouped into broader themes that address equity?
2. What strategies could strengthen triangulation and support stronger claims about students' experiences?

Acknowledgments

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A Preliminary Investigation into Multivariate Calculus Students' Understanding of Trigonometry

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This report describes a preliminary investigation into how multivariate calculus students understand trigonometric functions. Trigonometric functions are an essential component of understanding single-variable calculus which in turn is generalized into multivariable calculus. Therefore, an understanding of trigonometry is not only expected but an important component for success in multivariable calculus courses. Replicating a Weber (2005) study investigating trigonometry students' understanding of trigonometric functions, we surveyed multivariate calculus students to see how they understand trigonometric functions. Similarly, we leveraged the idea of procepts (Gray & Tall, 1994) to aid in interpreting students' descriptions of trigonometric concepts and properties.

Keywords: Multivariate Calculus, Trigonometry, Procepts, Student Understanding

Multivariate calculus is an intense course that often introduces students to both 3-dimensional space and vector notation at the start of term so that by the end of the term they can engage with various fundamental theorems of multivariate calculus. Needless to say, there are a lot of expectations on students' prior knowledge and understanding of various mathematical concepts, one such idea is trigonometry. By the time students are in multivariate calculus instructors assume that students have been working with trigonometric ideas for a number of terms. Furthermore, the fundamental theorems of multivariate calculus are often expressed using vector multiplication which can be defined using trigonometric relationships. However, working with a concept and understanding a concept are often different feats and our study aims to start to investigate how students who are expected to "know" trigonometry actually understand it.

Background and Literature

While multivariate calculus has seen considerably less attention than the teaching and learning of single variable calculus, over the last decade researchers have begun to shift focus to investigating student learning of multivariate topics and offering insights on pedagogical approaches. It comes as no surprise that literature on student understanding in multivariate calculus overwhelmingly focus on the understanding of topics introduced in a multivariate calculus course. A recent meta study that surveyed research studies on the learning of multivariate calculus (Martinez-Planell & Trigueros, 2021) found that much of the research deals with the idea of generalization and how students' knowledge of one-variable calculus influences their understanding of multivariable calculus. This research includes investigations into the student understanding of topics such as generalizing single variable functions to multivariable functions (Kabael, 2011), how student's meaning of function properties generalize to multivariate functions (Dorko, 2017), and students' understandings on definite integral of multivariable functions and how they relate to understandings about integrals of one-variable functions (Jones & Dorko, 2015).

An understanding of trigonometry is an important aspect of calculus readiness, as indicated by the Calculus Concept Readiness (CCR) (Carlson et al., 2015). The CCR is an instrument that assesses foundational understandings and reasoning abilities that have been documented to be essential for learning calculus. Five of the twenty-five assessment items on the CCR specifically

address student understanding of trigonometry. However, research also suggests students struggle with trigonometry, specifically with trigonometric comparisons and the recollection of commonly used formulas might be a real hurdle to their understanding of concepts in Calculus II (Ruslimin et al., 2019).

Our work is motivated by one particular study (Weber, 2005) that examined college students' understanding of trigonometric functions informed by Gray and Tall's (1994) notion of procept. Weber used the idea of a procept, described in detail below, to investigate and describe students' conceptual grasp of trigonometric functions, their ability to justify mathematical properties, and their use of multiple representations. Weber found that students did in fact view trigonometric expressions as procepts, often encapsulating processes related to right triangles or the unit circle.

Theoretical Perspective

Building on Weber (2005)'s work, we consider the notion of *procept* as introduced by Gray and Tall (1994). A *procept* is a mental collection of processes along with a concept that represents the result of those processes, summarized using symbolism. For example, Gray and Tall (1994) posit natural numbers as procepts, each symbol, such as 6, denoting the result of a child's process of counting, which later, after additional exposure to counting problems, mask other sophisticated procedures, such as 6 representing variously 4+2, 3+3, 7-1, and so on. Similarly, in calculus, the derivative often denoted $f'(x)$ can be seen as a procept. It both stands symbolically for the result of differentiation of the function $f(x)$, and as shorthand for the procedure itself. In fact, there are two procedures familiarly employed in a first course: the limit definition $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and alternatively the collection of computational rules, which contains the power rule, the chain rule, the product rule, and so on. Gray and Tall (1994) emphasize that the use of a procept, once internalized by a learner, can allow a delayed computation, by "using symbols both as manipulable objects and as triggers to evoke mathematical processes" (p. 138).

The focus of this study is to revisit Weber (2005, p. 93-94)'s research questions with a new population, multivariate calculus students:

(1) *How do students in a [multivariate calculus] course understand trigonometric functions? In particular, do these students understand trigonometric functions as processes and/or procepts? If so, what types of processes do students think of when reasoning about trigonometric functions?*

(2) *Can students in a [multivariate calculus] course use their understanding of trigonometric functions to justify why trigonometric functions have the properties they do?*

In this preliminary report, we report on the first research question due to space limitations but will explore both questions in the RUME presentation.

Research Methods

Participants

The student participants in this study were enrolled in a semester long Multivariate Calculus course at a small public 4-year institution in the Rockies; taught by the first author. The pre-requisite for the course was Calculus II with a minimum grade of C-. Thirty students were enrolled in the course, 28 of whom participated in the study. Most participants were engineering majors (21), including computer engineering (10), engineering (9), and mechanical engineering (2). Mathematics (2) and secondary mathematics education majors comprised just over 10% of the participants. Lastly, there was one physics major (1), one geology major (1), and one

participant left the question blank (1). When asked when they were first introduced to the sine function and how long ago, on average participants had known of the sine function for at least 5.5 years. The least amount of time reported was 3 years, however two participants stated 'since high school geometry' which was conservatively estimated to 2 years. The most amount of time reported was 15 years.

Data Collection

Unlike the Weber (2005) study which included a pre-test, post-test, and interviews, our study consisted of a ten-question survey administered in class at the end of the term. Three of the questions were demographic in nature (reported above), one question asked participants to provide known applications of sine, and the remaining six, taken from the original Weber (2005) study, are listed below.

1. Describe sine in your own words.
2. Why is sine a function? What is its domain and range?
3. Approximate; $\sin 340^\circ$ and $\cos 340^\circ$
4. What is $\sin 270^\circ$ and why?
5. When is $\sin \theta$ decreasing and why?
6. Why is $\sin^2 \theta + \cos^2 \theta = 1$?

The first three questions were meant to gain a general sense of how participants think about the sine (and cosine) function and whether they view the functions as operations applied to angles (Weber, 2005). The purpose of the last three questions was to test students' recollection of specific trigonometric properties and elicit a justification for each. This preliminary report will discuss the analysis methods and results of Q1 and Q2 due to space constraints, while the remaining methods and results will be shared in the presentation.

Data Analysis

The first pass of analysis aimed to capture evidence of procepts used by the participants as they described the sine function. The original study exclusively referenced two processes leveraged by trigonometric students. The first approach involved reasoning about trigonometric functions using a unit circle model, and the second described trigonometric functions in term of right triangles. Weber (2005, p.93) described these processes in the context of a student's estimate for $\sin 20^\circ$ as:

- Mentally constructing a standard right triangle containing an angle of 20°, and then comparing the length of the side of this triangle opposite to the 20° angle with the length of its hypotenuse. Then, realizing that sine is equal to opposite over hypotenuse, approximating $\sin 20^\circ$ to be about 13.
- Mentally constructing a unit circle on the Cartesian plane and then constructing a ray emanating from the origin of the plane that forms a 20° angle with the positive half of the x-axis. Then, focusing on the point of intersection between the ray and the unit circle, noting that the y-value of this intersection will be a small positive number, such as 0.3.

When analyzing responses to Q1 we coded references to such approaches as "unit circle" and "right triangle" respectively.

Two new ways of describing sine were also found multiple times in our data which we named "wave" and "vertical." Participants who described sine in terms of a periodic or sinusoidal function were coded as "wave." An example of which would be, "Sine is the name of a sinusoidal function that oscillates, depending on the given angle" (P2). Responses that used a

vertical component devoid of a unit circle, such as "Sine is the trig function that is typically used for y-axis related parts of a function and whatnot" (P13) were coded as "vertical." All other responses were coded as "other." Each member of the research team first coded the data individually and came together to discussion any disagreement before reaching complete consensus.

Results

As previously mentioned, in this preliminary report we offer results to the first two questions from our survey. The data set contains responses from 28 participants.

Responses to Question 1: Describe sine in your own words

Most participants described a single strategy (21). Three students offered two approaches, and two mentioned three approaches. Two students did not provide meaningful responses.

Table 1. Student approaches used to describe the sine function

Student descriptions of sine function	
Approach	# of responses
Unit circle (exclusively)	3
Unit circle (total)	6
Right triangle (exclusively)	4
Right triangle (total)	7
Wave (exclusively)	12
Wave (total)	16
Vertical (exclusively)	2
Vertical (total)	3
Other	3

Most participants (16) referenced the "wave" approach, which again was not discussed in Weber (2005). Many of these students mentioned the graph of the function (in Cartesian coordinates), by either illustrating $y = \sin x$ or describing 'waves,' 'periodic functions,' or 'sinusoidal functions.' Twelve of these students only referenced this single strategy. For instance, participant P5 described sine as, "A wave function with period 2π and amplitude 1, starting at the origin."

Seven students described the sine function as constructing a right triangle and measuring the 'opposite' side and the hypotenuse to compute the sine. Five of these students solely mentioned this approach, for example (P16, Figure 1):

Figure 1. P16's interpretation of sine in terms of a right triangle.

Six participants mentioned the y-coordinate from a point on the unit circle. Two of these students only mentioned this approach. For example, student P1 explained, "Sine is the vertical 'axis' so to speak of the unit circle."

Three students were coded as "vertical" and seemed to reference a y direction or vertical component, devoid of a unit circle. For instance, P8 described sine as: "Sine is a direction in the y or vertical direction."

Five students offered multiple approaches. For example, one student used three approaches by describing sine as a "wave" along with right triangles and the unit circle (P3, Figure 2):

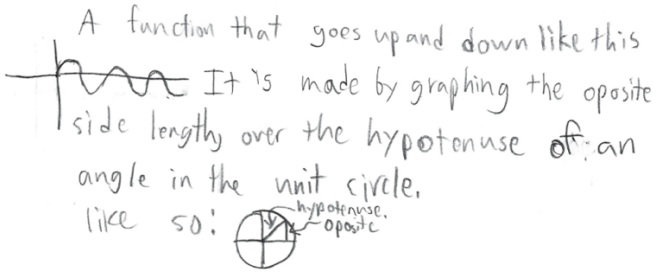

Figure 2. P3's description of sine in terms of wave, right triangle, and unit circle.

Responses to Question 2: Why is sine a function? What is its domain and range?

Many participants (7) ignored the first question, while fourteen wrote responses which could either not be coded or were off topic. Seven responses addressed the question directly. Two of these gave a completely correct answer, emphasizing that a single output was associated with each input. Four more gave a partially correct answer, stating that to each input there is "an" output associated. Lastly, one participant (P27) described a relation, saying "not exactly sure why its [sic] a function, but it relates $\angle A$, X, and H", where they had described sine earlier as relating sides of a right triangle using "the relation $\sin A = \frac{X}{H}$".

When asked about the domain and range of the sine function, a few students (4) gave restricted domains, such as $[0,2\pi]$. Three students skipped answering the question. Two students swapped the domain and range. The slim majority (15) were correct for both domain and range.

Discussion

In this study, we explored multivariate calculus students' understanding of trigonometric functions, in particular the sine function. As expected, this population used more approaches than were found in Weber (2005)'s study. However, we are hesitant to describe the new found approaches ("wave" and "vertical") as procepts since it is unclear to us what process specifically is being encapsulated. We hope further analysis of the remaining interview questions begins to shed light on this and we invite the audience to consider with us.

Consistent with the function literature, most students in our study were unable to, or unwilling to, offer a complete or correct justification of what makes sine a function. We are interested in considering how our current study could serve as a pilot for a follow-up study which would include interviews with participants. We hypothesize that multivariate calculus students have a more robust understanding of function than our survey elicited and if prompted differently, we think the findings would be more sophisticated. We invite feedback from the audience on further ideas for such a follow-up study.

Due to the preliminary nature of this work, we would like suggestions from the audience on the following: (a) What process might lead to a "wave" or "vertical" procept of the sine function? (b) When considering a follow-up study, how might we revise the context of the questions to include multivariate calculus, and would that be advantageous?

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A Quantities-based Task for Simultaneously Constructing Limits-at-a-point and Differentials

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The limit is a key calculus concept, but research has clearly shown how important differential-based reasoning is, as well. Thus, calculus education would benefit from paradigms that construct both in a way that they can coherently co-exist. In this preliminary report, we discuss a small piece of a larger project meant to build one possible paradigm. Here, we describe a single quantities-based task that is meant to develop both limits and differentials together, in the same activity. We also provide preliminary results on two pairs of students' emerging understandings.

Keywords: calculus, limits, differentials, quantities

Much work in calculus has examined the limit as a fundamental concept both for calculus, and for further mathematical study (Cornu, 1991; Davis & Vinner, 1986; Oehrtman, 2009; Viirman et al., 2022). Yet, much work has also demonstrated the benefits that reasoning about differentials, such as *dx* and *dt*, as legitimate quantitative objects affords for first-year calculus (Ely, 2021; Jones & Ely, 2023; Oehrtman & Simmons, 2023). Based on this research record, it appears that the best approach to first-year calculus may be to construct paradigms that can simultaneously, and coherently, include both limits *and* differentials. While a couple of potential approaches exist (Thompson & Dreyfus, 2016), these particular approaches tend to be at odds with the way that differentials are described in the research literature, and how differentials are typically used in STEM fields. Thus, continued efforts are needed in this area, to build differentials-and-limits paradigms that better align with the research record and calculus' client STEM fields. We are currently engaged in building one such possibility, though we readily state others could create distinct paradigms. In this *preliminary* report, we focus on a piece of this larger project, namely a single task meant to introduce limits and differentials simultaneously, in the same activity. Our contribution is to begin a conversation on how these ideas might be constructed in tandem, but in ways that they still match the growing body of research in this area.

"Preliminary" Nature of this Report

Our project has previously constructed a theoretical framework for defining differentials inside of a broader limits structure. However, the tasks we have constructed for learning differentials-within-limits are only beginning to be examined. At the very start of our learning trajectory, we have two quantities-based tasks meant to build both ideas simultaneously, and we are only now testing these with students. We discuss one of these tasks in this report. Our analysis plan and our results are therefore still preliminary, and we seek feedback in these areas.

Background Literature

A major theme in limits research is that a limit consists of both a *process* and an *object* (Cottrill et al., 1996; Dubinsky et al., 2005; Gray & Tall, 1994; Kidron & Tall, 2015; Sfard, 1991). A process is a dynamic action that one can imagine happening. There are actually two main distinct processes one can imagine for limits. First, for the limit $\lim_{x \to a} f(x)$, one can imagine x sweeping through the real numbers on its way to becoming closer and closer to a (Cornu, 1991; Davis & Vinner, 1986). Second, one could instead imagine indefinitely zooming in on the graph of f in the neighborhood of $x = a$ (Ely & Ellis, 2018; Ely & Samuels, 2019). These processes can

lead to some outcome that can now be perceived as its own, self-existing entity, which is called the limit *object* (Sfard, 1991). That is, if we see that $f(x)$ hones in on (or remains at) some value L, we can return that value as the outcome and consider that value as its own object. In fact, we can even see the limit expression $\lim_{x \to a} f(x)$ as representing that value (Gray & Tall, 1994). One problem that students tend to have with the limit process and object, however, is thinking that the quantities in the limit process "collapse" to nothing as the limit "reaches" the object (Oehrtman, 2009). For example, a student might think that a shrinking time interval has vanished to nothing.

Next, a growing body of research has identified many benefits in reasoning with differentials (e.g., *dx* or *dt*) (Dray & Manogue, 2010; Ely, 2021; Jones, 2015; Oehrtman & Simmons, 2023). In this work, a differential is a quantitative object that represents a "tiny bit" of a quantity. While this definition is fairly informal, it is precisely its informal nature that is powerful for calculus education, and it is typically how differentials are understood and used across STEM fields (Amos & Heckler, 2018; Hu & Rebello, 2013; Nguyen & Rebello, 2011; Von Korff & Rebello, 2012). Such understandings allow improved quantitative reasoning for derivatives and integrals, and are also necessary for making sense of scientific relationships stated directly in terms of differentials, such as $dU = TdS - PdV$. Attempts at formally defining differentials can bring in too much theoretical baggage for first-year calculus, just as the formal ε-δ definition of limits has done. Consequently, our framework is based on these *intuitive, informal* ideas for differentials.

Research on differentials also describes that there are two distinct types of differentials. Von Korff and Rebello (2014) described that some differentials have a "change" nature, such as the differential *dt* representing some tiny change in time. On the other hand, some differentials have an "amount" nature, such as *dm* representing a tiny bit of mass (and *not* a change in mass).

Because of the power of differentials in calculus and related STEM fields, some researchers have discarded limits altogether, using only differentials (Dray & Manogue, 2010; Ely, 2021; Keisler, 2011). The issue for us here is that these paradigms essentially require non-standard hyperreals, which are not generally utilized in the mathematics community. Further, limits are an important foundation of further mathematics study, and we think calculus students should develop an understanding of them. Also, doing away with limits makes it harder to convince the wider undergraduate teaching community to adopt differentials in curriculum. Thus, we seek to build a paradigm where limits and differentials comfortably coexist in the same classroom. While some paradigms do exist that treat both limits and differentials (Oehrtman & Simmons, 2023; Stevens & Jones, 2023; Thompson et al., 2013; Thompson & Dreyfus, 2016; Weber et al., 2012), there are two ways in which work is still needed. Some of these paradigms have not explicitly articulated exactly how a differential should be defined in relation to the limit process, nor what the exact relationships between a limit and a differential is. Other paradigms have defined differentials in a way that does not align well with the large and growing research work on differentials. We want a paradigm that not only allows differentials and limits to co-exist, but where the differentials have the meaning and reasoning power described in the research record. Our project is meant to supply one possible approach for doing so.

Theoretical Framework: Differentials-within-limits

Our project previously developed a theoretical framework for conceptually defining differentials within a broader limits architecture that matches the research on differentials. In our framework, a differential is defined as a secondary object that results from *within* the limit *process*. In other words, we do not simply see our approach as differentials-*and*-limits, we see it

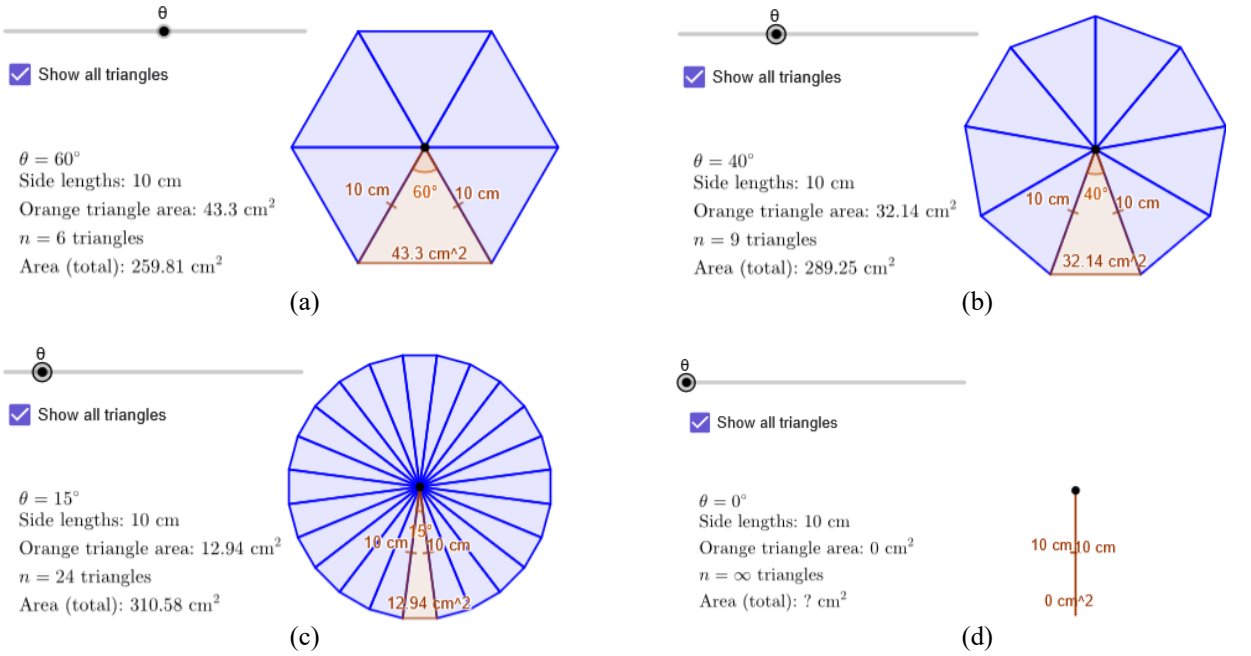
as differentials-*within*-limits. Table 1 provides an abbreviated version of our framework, including some basic definitions and some properties that are relevant to this particular report.

Table 1. Abbreviated version of the framework, with basic definitions and properties.

Differential, based on approaching	In the "approaching" limit process, if one perceives that a variable has become incredibly close to its target (i.e., x approaches close to a), they can "pause" the limit process at that moment and consider the remaining difference between x and a. This difference is defined as a *differential, dx*.
Differential, based on zooming	In the "zooming" limit process, if one zooms to tiny scales, they can again "pause" the limit process at some moment and consider the very tiny amount of a quantity remaining in the zooming window. This tiny bit is an alternate way to produce a *differential, dx*.
Definition #3: Output Differential	One can also create a differential for the *output* variable. In $\lim_{x\to a} f(x) = L$, as x gets close to a, or as one zooms into a graph near $x = a$, one can also imagine a change/amount differential for the output variable, *df* or *dy*.
Property #1: Subjectivity of Closeness	What is "close enough" in the limit process for it to be paused, and for a differential to come into existence, is subjective both on context and the reasoner's perspective. There is no specific point where this must occur. This subjectivity allows differentials to retain the desired informal nature.
Property #2: Lack of Specific Size	As a consequence of the previous property, differentials do not have a single, specific size or value. The differential object is *not* defined by a numeric value, as opposed to the limit object. Rather, the differential object is defined by the conceptual idea of a tiny change/amount.
Property #3: Non-hyperreal	While a differential does not have an inherent size, it still resides in the realm of very small real numbers, and is not a non-standard hyperreal.
Property #4: Contextual Assignment of Values	While differentials have no inherent size, for pragmatic reasons one can still assign a numeric value to a differential in a given situation, when context dictates, by "pausing" the limit process at a point deemed close enough or small enough and using that current state to assign a temporary value to a differential. This is a common practice in STEM fields.
Property #5: Differential as a "model"	A differential is a *conceptual model* of what happens at small scales. They represent what we might expect in the relationship between quantities, when measured or observed at small scales.

The Task

This preliminary report focuses on a single quantities-based task toward the beginning of a larger learning trajectory. The task uses a dynamic Geogebra applet in which the areas of regular polygons converge to the area of a circle (Figure 1). Of course, this context is certainly not new, and has a long history in mathematics (Katz, 2009). Yet, the use of it to simultaneously build both the limit-at-a-point concept and the differential concept is new, as far as we are aware. The task starts with a regular polygon made up of triangles with a certain interior angle, θ (Figure 1a). The students can change θ with a slider to see that smaller internal angles create polygons with more sides (Figure 1b, 1c). The main question is: "What happens to the area as we let θ go toward zero?" It is true that because θ can only take on certain values to create regular polygons, this task technically is the limit-at-a-point of a *sequence*. However, we believe that the ideas built in this task readily extend to the limit of a continuous function.



The tasks allows the students to slide θ all the way to zero, to see that the area is undefined there, since $\theta = 0$ correspond to no triangles, and hence no polygon (Figure 1d). Doing so helps address the collapse metaphor. Once the students identify that the area should converge to the area of a circle (100π cm^2), they are asked to explain the relationship between the shrinking θ and the converging area, A. Afterward, the specific limit notation is introduced, $\lim_{\theta \to 0} A = 100\pi$ cm^2. While this is technically a right-hand limit, we do not attend to that detail at this point.

Next, the students are asked: "When does the area actually become equal to 100π cm^2?" This question is meant to lead to the construction of differentials. When the students identify that there is no such point, they are asked how they decided on the limit. As the students describe very small angles, and an area very close to 100π, we explain that such small quantities can be termed "differentials." The students are then asked to describe the differentials they see in the task. The intention is for them to identify (a) a tiny θ as the (amount) differential $d\theta$ and (b) the discrepancy between the area given by a tiny angle and the assumed "limit" area as the (change) differential, dA. The students are asked how these differentials relate to the limit, and the limit value, and how they serve as a *model* for the context, even though we cannot reach $\theta = 0$.

We enacted this task within a larger set of interviews with two pairs of students recruited from the end of a Pathways College Algebra course that focused on quantitative reasoning. Thus, none of the students had taken calculus before, nor seen limits before, but they were familiar with reasoning about quantities. One pair consisted of two males and the other of two females. This task was one of two that began the interview sequence. While data analysis is ongoing, our initial analysis has focused on using our framework's definitions and properties as starting codes. We are in the process of identifying other themes to investigate further. The analysis and possible resulting outcomes is what we hope to receive feedback on in this preliminary report.

Preliminary Results

Because our initial analysis has centered on our framework's definitions and properties, our preliminary results have been organized around those themes. The following are early pieces of data that provide some evidence of understandings related to aspects of our framework.

Defining a differential: Several statements suggested an understanding of a differential as tiny bits of quantities that exist within the limit process, as exemplified in Claire's statements.

> *Claire*: If the angle is really close to zero, then it would be a differential. And that's just how we would like… that's how we'd like represent it. Because, like, you can't have an angle of zero, but you can have something really, really close to zero.
> *Claire*: So I think it would be, like, the differential of the area would be like the space around the circle that we haven't filled yet.

Change versus amount differentials: Notice that Claire's statements also potentially suggest both an "amount" differential for the angle and a "change" differential for the area. Her imagery of the angle seemed to be based on zooming to small angles, while her imagery for the area seemed to be based on the difference between the polygon's area and the circle's. As another example, Austin stated, "You make the angle so small," suggesting a zoom into tiny angles. His partner, Logan, later said, "So you take 100π minus whatever that [polygon] area is, and that's our differential for area," suggesting a difference between the two areas.

Avoiding collapse: The quantities-based nature of the task seemed to be extremely helpful in avoiding the collapse metaphor. While some students, at the start, thought of the triangles becoming straight lines, they rather quickly realized the issue in doing so. For example, Austin stated, "Because if it hits zero, then… you don't have, like you don't have anything." In the other group, Claire explained, "You can't make a triangle with a zero degree angle."

Subjectivity of closeness and lack of specific size: The students themselves appeared to build meanings for the differential that matches well with these two properties in our framework.

> *Claire*: Uh, yeah, just that we were talking about a differential being anything that's like approaching, or like a number really close to zero. I don't know if we ever went like, how, said how close "close" would be. But if the angle is really close to zero, then it would be a differential… Because, like, you can't have an angle of zero, but you can have something really, really close to zero.

Contextual assignment of values: The students appeared comfortable with occasionally using a very small number to represent a differential as a model for the limit process.

> *Logan*: I mean $d\theta$ would be like 0.0001 or something like that, kind of small number… Which isn't zero, but it's close to zero.

Conclusion and Questions for the Audience

Our early results favorably indicate that this task is successfully helping students construct differentials within a broader limits architecture. They constructed both types of differentials, and gave evidence of thinking of the several of the intended properties for differentials. Because this work is still in its early stages, the following questions will be used to solicit feedback:

1. What are your thoughts on this task as a way to simultaneously construct both limits-at-a-point and differentials? Are there any issues with it you see?
2. Do you agree that these student excerpts are evidence of the definitions/properties?
3. What other aspects of the students' thinking or understanding would you want to see?
4. What kind of disconfirming evidence would you want us to be looking for?

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Instances of First-year Calculus in a Calculus-based Physics Textbook

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We examine the presence of calculus concepts and skills in the portions of a standard textbook that are generally covered in the second and third courses of a typical introductory calculus-based physics course. Using the Calculus Concept Framework (Sofronas 2011) three researchers independently coded calculus concepts and skills in eighteen chapters of a calculus-based physics textbook covering electricity and magnetism, geometrical and physical optics, and modern physics. Along with prior work on the first twelve chapters covering mechanics, this analysis suggests that the calculus content is uneven and not well aligned with prerequisites.

Keywords: physics, calculus in the disciplines, instructional practices, textbook analysis

In previous work [redacted], we described analysis of the classical mechanics portion of a common introductory physics textbook using the Calculus Concept Framework (Sofronas 2011) to examine the prevalence of calculus skills and concepts in the first semester of introductory physics. In this paper, we continue this analysis to cover the portions of the textbook that cover electricity and magnetism, geometrical and physical optics, and the introduction to modern physics.

Our goals in this project have been to document and characterize instructional practice in introductory physics with respect to the use of mathematics in general and calculus in particular. We have sought to understand the extent to which calculus is used to scaffold student understanding, following the framing of Biza et al who contrasted calculus as a scaffold vs a filter (Biza 2022). As a scaffold, calculus is used to support learning and understanding; as a filter, it serves to prevent students from progressing without successful completion of calculus.

Introduction and Previous Research

As a means of characterizing calculus practice in introductory physics, we have followed the work of several other scholars in adopting the Calculus Concept Framework [CCF] (Sofronas 2011). The CCF was developed through analysis of interviews with a set of disciplinary experts who were asked to characterize the goals of first-year calculus. One element of the CCF identifies a list of concepts and skills that are essential to calculus understanding and that are generally taught in a calculus course sequence.

Several researchers have used the CCF to guide their analysis of the calculus content in later courses, including later courses in mathematics as well as those in so-called client disciplines. Czocher et al. used the CCF to describe the concepts and skills from calculus were used in a course in differential equations (Czocher et al., 2013). Subsequent work adapted this analysis in disciplines that require calculus, including introductory engineering (Faulkner et al., 2020), and general chemistry (McAfee and Rodriguez, 2024). Until our prior work, there was no similar analysis in introductory physics.

Research questions and methods

The primary research question driving this study has been: What concepts and skills from calculus are encountered by students in introductory calculus-based physics, and when are they encountered? A secondary question that has emerged is, what additional mathematical concepts

and skills are encountered by students? In previous submissions we have reported initial results from the mechanics portion of the text. For this paper we focus on the portion of the text that is generally covered in the second and third semesters of a typical introductory sequence. In addition, we reflect on these results as well as those from prior work to characterize the role of calculus for the whole introductory sequence.

The methodology involved three researchers coding instances of the concepts and skills from the CCF in a commonly used introductory physics textbook. The three researchers independently coded chapters 2 through 12 and 21 through 34 of the text (Resnick et al., 2014) for the presence of absence of each of the skills described in the CCF. We neglected chapter 1 in our analysis, which introduces quantities and units, and chapters 13-20 which cover topics that are skipped in the introductory sequence at our university (fluids, thermal physics).

After each section was coded, the researchers met to discuss codes and coding principles and re-negotiated how codes were to be interpreted. As we proceeded, we recorded inter-coder agreement after independent coding. By the end of the first-semester analysis, the agreement between coders was very high, e.g., 97% agreement for Chapter 11.

Three examples of portions of the text that were coded can be found below in the Results. Each example states what concepts or skills were coded from that passage. Note that the coding was based on the entire section, so the absence of a concept or skill in the included passage does not necessarily mean that the concept or skill is not present in the section that includes it.

There were several repeated instances of mathematical practices that were not present in the CCF. Some were instances of calculus concepts beyond the scope of the CCF, due to being covered in multivariable or vector calculus: dot and cross products, partial derivatives, and ordinary differential equations. We also added concepts and skills to our codebook for units, vectors and unit vector notation, and reasoning with graphs and diagrams. While one might argue that none of these are distinctly calculus, each of these were more common than many of the skills and concepts identified in the CCF and part of mathematical practice for the course.

While all three researchers identify themselves as physics faculty or students, one of the three is a supplemental instruction leader in calculus courses and so is regularly in contact with instructional practices in mathematics. This researcher initially coded some concepts based on practice in a calculus classroom. This made it clear that our coding is from the point of view of physics practice, and that coding might be different were mathematicians doing the coding. As the project has progressed, we have also sought to characterize the ways in which mathematical practice in introductory physics differs from what students encounter in their calculus courses, as well as a broader characterization of the mathematical skills beyond the CCF. These aspects are ongoing and will include interviews with instructors and students.

Results

We summarize the results in tabular form in Figure 1. The number in each cell refers to the number of codes for this concept or skill in each chapter. While we coded all the concepts and skills in the CCF (and a few others), the table displays only a few of the categories. The reason for this is twofold. First, many of the categories were coded very infrequently (or even not at all). CCF categories that were seldom / never coded included: (concepts) Riemann sums, Parametric and Polar Equations, Continuity, Optimization, Transformation; and (skills) Parametric Equations, Polar Coordinates, Epsilon Delta Proofs. Many of these were lower prevalence in the original CCF study. There were two additional skills that we omitted because their coding was so ubiquitous to be meaningless: Listening and Reading Comprehension and Facility with Definitions and Notation.

We added to our analysis several concepts and skills that were not included on the CCF, either because they are not considered calculus or are part of courses after the first year: vectors, units, graphs, diagrams, dot and cross products, and partial derivatives. In the diagram we show only the last three of these. Vectors, units, and diagrams were ubiquitous in the data set.

Some of the integrals that students encounter are no longer integrals of a single variable as students would encounter in first-year calculus. In this textbook, students encounter partial derivatives and functions of multiple variables, as well as path integrals, surface integrals, and volume integrals. These concepts are concentrated in the early sections of second semester physics, which at many institutions has Calculus 2 as co-requisite.

Concepts	Chp22	Chp23	Chp24	Chp25	Chp26	Chp27	Chp28	Chp29	Chp30	Chp31	Count	% of Total Codes
Derivative	0	0	5	0	3	6	0	0	20	9	43	8.62%
Integral	9	8	10	9	6	0	0	12	9	3	66	13.23%
Limit	0	0	0	0	0	2	2	2	0	0	6	1.20%
Approximation	4	0	3	0	0	0	0	2	0	0	9	1.80%
Skills												
Derivative Computations	0	0	0	0	0	3	0	0	2	0	5	1.00%
Techniques of Integration	6	0	0	0	0	0	0	0	0	0	6	1.20%
Area / Volume	0	15	0	10	6	0	0	0	6	0	37	7.41%
Trigonometric Manipulations	9	3	15	0	1	0	15	12	0	15	70	14.03%
Manipulating Algebraic Expressions	15	14	22	15	13	9	19	15	20	18	160	32.06%
Facility with Logarithms and Exponentials	0	0	2	2	0	3	0	0	3	3	13	2.61%
Advanced skills												
Dot Product	2	5	15	6	3	0	0	4	6	0	41	8.22%
Cross Product	2	6	0	0	0	0	20	9	3	0	40	8.02%
Partial Derivatives	0	0	3	0	0	0	0	0	0	0	3	0.60%
											499	100.00%

Figure 1. The prevalence of calculus concepts and skills in the chapters of introductory physics corresponding to electricity and magnetism. The corresponding data for optics and modern physics are not shown, as explained.

Algebra and trigonometry appeared more frequently than any other codes. Manipulating Algebraic Expressions was the single most common code, appearing in all chapters, and represented over 30% of all codes. Trigonometric Manipulations was second more frequent, representing approximately 14% of all codes, though it was more present in some chapters than others. Notably, several of the physical quantities and relationships in electricity and magnetism use vectors and the dot and cross product are both prevalent in this portion of the course.

Whereas derivatives were more prevalent than integrals in the mechanics portion of the class, this trend was reversed in electricity and magnetism, with integrals being coded approximately twice as frequently as derivatives. In addition, for integrals and derivatives, the concept was coded an order of magnitude more frequently than the corresponding skills of computing derivatives or using techniques of integration.

The three semesters of introductory physics have considerably different calculus usage. As described in prior work, the first semester of introductory physics (classical mechanics) features intermittent use of calculus. Derivatives and integrals appear most heavily in chapters on kinematics, work and energy, and rotational motion (2, 4, 7, and 10) but were completely absent from three of the ten chapters in that course. In electricity and magnetism, in contrast, each of the first five chapters, and seven of ten overall, include heavy emphasis on integration, frequently of vector quantities. Students encounter integrals of cross products (Biot-Savart law) and results from vector calculus, including the divergence theorem and Stokes' theorem. Derivatives are less common but are prominent in Chapters 24, 27, 30, and 31.

When the course turns to optics and modern physics in the third semester, calculus essentially vanishes entirely. While we coded through Chapter 37 in this portion of the project, the table in Figure 1 stops Chapter 31. This is because the chapters covering optics and modern

physics had essentially zero calculus concepts coded. These sections follow a common pattern; the only codes were for algebra and trigonometry (as well as diagrams and units, not included in the figure) but essentially zero codes for calculus-specific concepts and skills like derivatives, integrals, or limits. There were single instances of integrals in two sections and a derivative in one, but otherwise calculus is entirely absent for this portion of the course.

Discussion and Implications

Our findings have significant implications for instruction in physics as well as calculus. We make some observations and discuss these implications.

Several of the concepts and skills in the CCF have different interpretations in physics.

Two of the concepts and skills identified in the CCF are limits and approximations. These have specific and formal meaning in mathematics. In physics, students encounter limits and approximations as well, but the instantiation of these ideas looks quite different. The passage shown in Figure 2 came from the chapter on electric fields (22) and this section was coded as including both limit and approximation. This illustrates mathematical moves that are common in physics: a complicated expression is simplified by considering what physicists would call a limiting case. This allows the derivation of an approximate expression (equation 22-8 in the figure) that is conditionally valid. A mathematician might or might not code this differently.

> We are usually interested in the electrical effect of a dipole only at distances that are large compared with the dimensions of the dipole — that is, at distances such that $z \gg d$. At such large distances, we have $d/2z \ll 1$ in Eq. 22-7. Thus, in our approximation, we can neglect the $d/2z$ term in the denominator, which leaves us with
>
> $$E = \frac{1}{2\pi\varepsilon_0}\frac{qd}{z^3}. \qquad (22\text{-}8)$$

Figure 2. An example of limiting arguments and approximations that highlights differences in instructional practices between calculus and physics.

While the contexts are complex, the actual derivatives and integrals are often not.

While the contexts in second-semester physics involve integrals of vector quantities in multiple dimensions, as a practical matter the integrals that are evaluated are often quite simple. As an example, in Chapter 22 the text derives expressions for the electric field of various configurations of static charges, including continuous charge distributions. To derive the electric field of a line of charge, the text begins with a charged ring. To an observer (probably including many students), this might seem more complex, but the symmetry of the charged ring simplifies

> ***Integrating.*** Because we must sum a huge number of these components, each small, we set up an integral that moves along the ring, from element to element, from a starting point (call it $s = 0$) through the full circumference ($s = 2\pi R$). Only the quantity s varies as we go through the elements; the other symbols in Eq. 22-14 remain the same, so we move them outside the integral. We find
>
> $$E = \int dE \cos\theta = \frac{z\lambda}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}}\int_0^{2\pi R} ds$$
>
> $$= \frac{z\lambda(2\pi R)}{4\pi\varepsilon_0(z^2 + R^2)^{3/2}}. \qquad (22\text{-}15)$$

Figure 3: The conclusion of the textbook derivation of the electric field due to a charged ring. While the integral in principle is a vector integral in three dimensions, in practice the integral is of a single variable.

the task. Figure 3 shows the concluding steps. Despite the complexity of the expression for the electric field, the integral to be evaluated is a definite integral of a single variable: $\int_0^{2\pi} ds$. All the complexity of the situation is in setting up and simplifying the integral; evaluation is, in principle, trivial. (Though we note that there is evidence that this may not be so trivial to students.) White Brahmia [2023] has noted that the functions that students encounter with in introductory physics are simpler than their calculus courses might suggest.

In the following chapter, students encounter Gauss' law, which is a physics application of the divergence theorem. The presentation in the text sets up a surface integral over the surface of a sphere but then simplifies the calculation with physical reasoning and symmetry. Notably, the integral is now with respect to an apparent single variable A, but that simplified expression might conceal the complexity of a two-dimensional surface area of a three-dimensional object.

> The drill here is the same as previously. Pick a patch element on the surface and draw its area vector $d\vec{A}$ perpendicular to the patch and directed outward. From the symmetry of the situation, we know that the electric field $\vec{E}$ at the patch is also radially outward and thus at angle $\theta = 0$ with $d\vec{A}$. So, we rewrite Gauss' law as
>
> $$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 \oint E\, dA = q_{\text{enc}}. \qquad (23\text{-}8)$$
>
> Here $q_{\text{enc}} = q$. Because the field magnitude E is the same at every patch element, E can be pulled outside the integral:
>
> $$\varepsilon_0 E \oint dA = q. \qquad (23\text{-}9)$$

Figure 4. A surface integral that is simplified through symmetry and physical reasoning to a more straightforward (to physicists, at least) example of integration.

In summary, we note several implications and future thoughts. First, the sequence of calculus usage in introductory physics is a mismatch with co- and pre-requisite structures. In many institutions, including our own, first semester physics is a co-requisite for Calculus I. For students in first semester physics, the first integral might appear in week two; at this time in Calculus I, students might not have even encountered derivatives. In electricity and magnetism, students encounter integrals of vector quantities in three-dimensional situations by the second week of the semester, when students in the co-requisite Calculus II are learning single-variable integration. The third semester then has minimal calculus.

Second, and perhaps a result of the above, when calculus does appear, there are often steps taken to reduce the complexity of calculations. Situations with two or more dimensions are regularly reduced to integrals of a single variable. While these integrals are conceptually complex, as examples above indicate, structurally they appear as integrals of a single variable.

Third, the CCF is an imperfect match to the mathematics that students encounter in introductory physics. As we previously reported, many of the mathematical skills required in the course were not included in the CCF, and many of the skills and concepts identified in the CCF appeared rarely (or never appeared). For the second semester of the course, this reflects the need to attend to vector quantities and situations in multiple dimensions, each normally the province of multivariable calculus and outside the scope of the CCF.

While the textbook strongly guides instructional practice, we note that analysis of a textbook provides only an incomplete description of what instructors and students do in class. We have performed several interviews with instructors related to first semester content, and while preliminary analysis indicates that they perceive their practice to be closely aligned with our findings, that work is still ongoing.

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College Students' Perspectives on and Use of Exam Note Sheets in Intermediate Algebra

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Creating and bringing a sheet of notes to a math exam can help students perform better and feel less anxious, but not all note sheets are equally helpful. This study explores how undergraduate students prepare and use exam note sheets in an intermediate algebra course. We surveyed 119 students about how they created and used their exam note sheet and how they perceived the usefulness of their sheet. We also collected students' note sheets and exam scores. Preliminary analyses revealed that most students spent less than two hours preparing their note sheet. Most students felt the note sheet helped them study for and perform well on the exam and lessened their anxiety. Forthcoming analysis will explore relationships between students' exam performance and features and use of their sheet. Findings can be used to develop guidance for students on how to more effectively prepare and use exam note sheets.

Keywords: Note sheets, crib sheets, support sheets, exam studying, college algebra

Sometimes college instructors allow students to bring an index card or sheet of notes to use on exams (sometimes referred to as a "cheat sheet"). These note sheets can usually include anything students think might be helpful on the exam, such as formulas, definitions, or descriptions of procedures. Instructors often allow this resource because they believe it helps alleviate students' anxiety and because they want students to focus more on conceptualization and problem solving and less on memorizing and procedures (Danielian et al., 2020; de Raat, 2012; Edwards & Loch, 2016). Some studies have shown that students report feeling less anxious or stressed about an exam when they are allowed to bring a note sheet (Danielian et al., 2020; Drake et al., 1998; Erbe, 2007). While some research has found that using note sheets is associated with better exam performance (Larwin et al., 2013), other studies have shown no significant relationship to performance (Hamouda & Shaffer, 2016).

The relationship between a students' exam score and their use of a note sheet may depend on the types of information they include and other characteristics of their note sheets (Song & Thuente, 2015). For example, studies analyzing students' exam note sheets have found that including more example problems (Edwards & Loch, 2015) and creating a more densely packed sheet (Ludorf & Clark, 2014) are associated with lower exam scores, while including a lower number of abstract concepts is associated with higher scores (Danielian & Buswell, 2019).

The benefit of a note sheet may not come simply from having it as a source of information during the exam; the process of creating the note sheet can be a valuable part of the study process. While preparing their note sheet, a student may consider which course concepts are most important, strategize about how to organize the information, and determine which ideas they know by heart and what reminders they might need. A high-quality note sheet that reflects a student's understanding and is tailored to their needs is a product of a thorough study process. For example, de Raadt (2012) suggests that the connection they found between including fewer abstract concepts and performing better on the exam may reflect that students who internalize key concepts develop a deeper understanding of the material and are thus likely to perform well.

In this study, we explore how college students in an intermediate algebra course create and use exam note sheets, as well as their perspectives on the utility of the note sheet. We also consider how various features of students' use of note sheets relate to their exam performance.

Insights gleaned from this research can indicate if and how preparing and using an exam note sheet is helpful to students. Instructors can use this information to develop guidance for students on how to prepare and use an exam note sheet in a way that helps them learn material deeply and perform better on exams.

Background Literature and Research Questions

Much of the existing research on the use of exam note sheets has focused on coding students' note sheets for particular features, such as number of example problems and formulas included, organization, and density (e.g., de Raat, 2012; Ludorf & Clark, 2014; Song & Thuente, 2015). Many of these studies have attempted to identify which features are associated with better performance, by comparing students' note sheets to their exam scores (e.g., Capaldi, 2019; Edwards & Loch, 2015). Most of these studies have used small samples, and many analyses failed to reach statistical significance.

Another approach has been to survey students to explore their perspectives on how they create and use exam note sheets (e.g., Danielian et al., 2020; McCaskey, 2013; Peters, 2024). These surveys have revealed insights into how the note sheet fits into students' larger study process and how they see the benefit of the resource. For example, when asked to rate the extent to which the process of making a note sheet helps them study for exams, undergraduate students in an engineering class gave an average rating of 4.11 out of 5, with 5 indicating that making a note sheet helps a great deal in studying (Peters, 2024). Most students felt that the note sheet was effective in helping them perform on the exam as well (Peters, 2024). In another study conducted in an undergraduate engineering class, students reported that preparing a note sheet helped them summarize, retain, and organize content and allowed them to avoid memorization (Danielian et al., 2020). Students also noted that having a note sheet alleviated their stress about the exam (Danielian et al., 2020). Most survey studies involved small samples, and many did not consider exam performance in their analysis.

More research is needed to investigate the relationships among students' perspectives on and use of note sheets, individual features of the note sheets themselves, and scores from the exam on which they used the note sheet. Exploring these three data sources and relationships among them should provide a deeper understanding of how preparation of a note sheet fits into students' overall study process, and how their use of the resource relates to their exam performance. Additionally, only a few studies have explored the use of note sheets in mathematics courses specifically (Capaldi, 2019; Edwards & Loch, 2015). To build on the existing literature on students' use of exam note sheets, this study investigates the following research questions:

1. In what ways do college students enrolled in an intermediate algebra course create an exam note sheet and use it on an exam?
2. How do students perceive the usefulness of their note sheet as an exam resource?
3. What are common features and properties of the exam note sheets created by the intermediate algebra students?
4. What relationships exist between how a student creates and uses their note sheet, characteristics of the note sheet itself, and their exam performance?

Method

Participants and Context

Participants were 119 undergraduate students enrolled in three sections of an intermediate algebra course at a large research institution located in the mid-Atlantic region of the United

States. Students in this course are allowed to prepare and bring one 8.5" x 11" sheet of notes to use on each exam. This course was chosen for study because instructors have noticed wide variability in what students include on their note sheets and want to better support future students in preparing and using their note sheets.

Ninety-one percent of participants were in their freshmen or sophomore year, and 35% were repeating the course. At least 90% of the participants elected to provided demographic information. Of the students who did, 68% identified as women, 30% as men, and 2% as non-binary. Seventy-three percent of respondents identified as white; 19% as Black/African American; 14% as Latine/Hispanic; 9% as Asian, Pacific Islander, or Desi; and 2% as indigenous/Native American. (Percentages add to more than 100 because participants were allowed to select multiple racial/ethnic identities.) Twenty-nine percent of respondents reported being a first-generation college student.

Data Collection

After the third exam of the Spring 2025 semester, we invited all 187 students enrolled in the course to complete a survey about how they created their note sheet, how they used it on the exam, and how they perceived the usefulness of their sheet. The exam focused on systems of linear equations, understanding and manipulating exponents, simplifying and factoring polynomials and rational expressions, and using the zero product property to solve equations.

Survey questions included a mix of multiple-choice, Likert, and open-response questions. Multiple choice questions asked students about their note sheet creation process (e.g., how long they spent creating their sheet and whether they worked with others), as well as what types of information they included and how they used their note sheet on the exam. Likert questions asked students how useful their note sheet was in helping them study for and perform on the exam and whether having the sheet lessened their anxiety/nervousness about the exam. The open-response questions, several of which were inspired by McCaskey (2013) and Peters (2024), asked students to describe their process for creating their note sheet, how they used it on the exam, and any changes they plan to make when preparing their sheet for the next exam.

Students were offered a small number of course points to incentivize participation. We received 128 responses, but some were not included in the analysis because students either did not complete enough of the survey or did not take the exam. There were 119 usable survey responses, a response rate of 64%. We asked the 119 students for consent to analyze their note sheets and exam scores. Seventy-three students allowed access to their note sheets and 72 to their exam scores. In total, 71 students allowed access to all three data sources.

Data Analysis

Preliminary analyses have been conducted. Specifically, we have analyzed the responses to the 18 multiple-choice and 10 Likert items on the survey. We created frequency tables for responses to each question. To analyze responses to the open-ended survey items, we will use inductive thematic analysis (Braun and Clarke, 2006).

We plan to code students' note sheets according to various features and characteristics of each sheet, inspired by strategies used by Capaldi (2019) and Edwards and Loch (2015). For example, we will count how many example problems and descriptions of procedures are included. In addition, we may categorize students' strategies for organizing the information. We will decide precisely which features to attend to after analyzing the survey responses and exam scores in more depth. Insights gathered from these first two data sources could illuminate which

features of the note sheets may be most beneficial to attend to, which will help target the coding process.

We will use chi-square analysis, ANOVA, and multiple regression to investigate correlations between and among the survey responses, exam scores, and quantifiable note sheet features.

Preliminary Findings

We present preliminary findings from analysis of the multiple-choice and Likert survey items. Of the 119 participants, 105 created a note sheet for the third exam. Of these students, 91% reported spending less than two hours making their sheet, and 54% reported spending less than one hour. About a quarter of students (28%) started making their note sheet on the day of the exam, with another quarter starting the day before, and 48% starting at least two days in advance. Only 13% of students reported working with others when creating their sheet.

Regarding the kinds of information students included on their note sheet, a large majority said they included formulas (89%), example problems (84%), and instructions or steps for solving certain types of problems (78%). Students also reported including definitions (43%), graphs (11%), and supportive notes to themselves, e.g., "You can do it!" (9%).

Participants who created a note sheet were asked how often they used it during the exam. Survey responses indicate that 41% of students used their note sheet three to five times, 29% used their sheet one to two times, and 27% used it more than five times. Only 4% of students reported not using their note sheet at all. Most students reported using their note sheet to look for examples of similar problems (74%), steps for solving certain types of problems (70%), or a formula (66%). Students were less likely to use their sheet to look up a definition or fact (29%) or a graph (6%).

Most students reported perceiving the note sheet as a positive resource. Fifty-two percent strongly agreed and 36% somewhat agreed that the process of creating their note sheet helped them study for the exam. Forty-five percent strongly agreed and 33% somewhat agreed that having a note sheet helped them perform better on the exam. Fifty-two percent strongly agreed and 31% somewhat agreed that having a note sheet helped lessen their anxiety or nervousness about the exam.

A small number of students (n=14, 12% of survey respondents) reported that they did not make a note sheet for the exam. When asked why they chose not to make one, half of these students reported that they felt confident about the material and did not feel the need for a note sheet. Other reasons included forgetting to make a note sheet (29%), not thinking a note sheet would be helpful for the exam content (21%), not having time to make one (21%), being unsure what to put on the note sheet (21%), and finding a note sheet they made for a previous exam unhelpful (14%). (Percentages add to more than 100 because participants were allowed to select multiple reasons.) Of the students who did not create a note sheet, 50% strongly agreed and 36% somewhat agreed that having one would have helped them study for the exam. Fifty-seven percent strongly agreed and 21% somewhat agreed it would have helped them perform better.

Discussion

Survey responses revealed that students spent a minimal amount of time creating their note sheets, with more than half of respondents spending less than an hour, and almost all (91%) spending less than two hours. Most students started making their note sheets close to the exam day, with slightly over half starting the day of or day before the exam. As our analysis proceeds, we plan to investigate if and how these characteristics of students' note sheet creation process relate to their exam performance.

Results indicated that formulas, example problems, and instructions or steps for solving certain types of problems were the top three pieces of information included on note sheets as well as the top three used on the exam. Interestingly, while formulas were the most common type of information students reported including on the note sheets, with 89% of students including them, they were only the third most common type of information used, with only 66% of students looking up a formula on their sheet during the exam. When we analyze the note sheets themselves, we will be able to see which formulas students included, which could reveal insights into why they were not as relevant on the exam as students expected. Students' expectations regarding examples and instructions/steps were more in line with their use: 84% of students reported including example problems, and 74% ended up looking for examples during the exam. Similarly, 78% of students included instructions/steps, and 70% looked for them during the exam. The low percentages of graphs and definitions/facts included likely reflects the fact that the content of the exam focused on simplifying and factoring expressions and solving equations and thus did not call for many definitions, facts, or graphs.

The finding that a majority of students (83%) felt that having a note sheet decreased their anxiety or nervousness about the exam resonates with others who have found that note sheets help students feel less anxious or stressed (Danielian et al., 2020; Drake et al., 1998; Erbe, 2007).

Next Steps

In the next phase of analysis, we will investigate how students' exam performance relates to various aspects of how they created and used their note sheets. We will look for correlations between students' survey responses and their exam scores and plan to report on our findings at the conference. For example, we will analyze whether the amount of time a student spent preparing their note sheet is associated with their exam score. We will also analyze students' responses to the five open-ended survey questions and plan to present salient themes at the conference. For example, we hope to describe students' various strategies for deciding what information to include on their note sheet and how to organize the information. The final phase of analysis will consist of developing a coding scheme for the note sheets themselves, based on salient features identified in earlier phases of analysis. After coding the sheets, we will investigate relationships between features of the note sheets and students' exam scores.

Previous studies have suggested that students need guidance in how to use the note sheet resource effectively (Capaldi, 2019; Edwards and Loch, 2015; Ludorf and Clark, 2015). We hope this analysis will reveal insights that can help researchers and practitioners develop guidance for students on how to create a note sheet that helps them learn the material and succeed on exams.

Questions for the Audience

1. What features would be fruitful to attend to when analyzing students' note sheets?
2. What conceptual or theoretical framings would be useful to consider when interpreting our findings?
3. Based on your teaching experience, what advice would you give students on how to prepare and use their note sheet?

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Approaches to College Mathematics: Reimagining Remediation and Improving Student Outcomes in Precalculus

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This proposal reports on the impact of a course designed to prepare first-year students for success in Precalculus. The course, Approaches to College Mathematics (ACM), was developed as a prerequisite course in response to one university's high failure rate in Precalculus. In addition to a carefully designed yet flexible curriculum that aims to have students explore familiar topics in fresh ways, the course focuses on growth objectives meant to help students transition to their first semester as a college student. Institutional data and survey data from both students and instructors were collected. Preliminary results show that the overall failure rate in Precalculus decreased the semester after ACM was implemented. Additionally, a high percentage of students in ACM reported feeling that ACM was the right placement for them. Finally, ACM grades and the ACM instructors' sense of their students' mathematical confidence significantly predicted grades in Precalculus.

Keywords: Precalculus preparation, mindset, retention

We report on a study that involved the development and implementation of a course, Approaches to College Mathematics (ACM), for students who require an additional college course to prepare for Precalculus. The study is based on 15 sections of the course from Fall 2024. Going in, we were acutely aware of the unique circumstances that were a reality for these students who, as first-year undergraduates in 2024, had been in eighth grade at the onset of the COVID pandemic. This meant that they had spent most of their freshman year in high school learning mathematics online, even though their teachers had not been supported in online pedagogies given the pandemic's abrupt and urgent demands (Engelbrecht et al., 2020). In other words, it is likely that many of the students entering college in 2024 had significant gaps in their mathematical skills and knowledge, particularly with respect to content covered in Pre-Algebra and Algebra I. In addition to these mathematical obstacles, we knew that ACM students would be navigating additional challenges commonly experienced by students transitioning from high school to college learning.

Because the primary goal of the ACM course was to prepare students for Precalculus, we chose to focus on content and skills that would be foundational to that course (e.g., proportional reasoning, graphing, and order of operations). However, we kept this list relatively short to allow for flexibility and responsiveness in our practice, providing space for instructors to address student misconceptions as they emerged in class. Further, we designed several inquiry-based group work activities for ACM that would promote the development of rich, conceptual understanding. Finally, to prepare ACM students more broadly for learning mathematics at the college level, we incorporated a number of *growth objectives* intended to encourage productive habits of mind and actions. These objectives focused on seeking support from professors, managing time, taking advantage of college resources (such as the tutoring center), and managing stress. Thus, we took on the charge to reconceptualize what it means to prepare

students for college level mathematics, and to investigate the efficacy of our new course. The following research question framed our study: *What is the impact of the Approaches to College Mathematics course on students' preparedness for Precalculus, as assessed by their grades in Precalculus and their own sense of the usefulness of ACM?*

Conceptual Framing

The early history of remedial education began with the passage of the GI Bill in 1944, which sparked a proliferation of postsecondary institutions (Adams, 2000). This meant that thousands of high school graduates, many of whom were not prepared to study mathematics at a college level, suddenly had access to a tertiary education. In this environment, mathematics departments began to offer remedial mathematics courses to help improve students' mathematical knowledge and skills. By the end of the 20th century, remedial mathematics had become "a formal discipline of study, as institutions of higher education recognized that the need to provide remedial education to their students was now a continuing one" (Shaddai, 2018, pp. 389-390).

Whether or not remedial courses in mathematics ultimately result in the intended benefits for students is a matter of debate. While successfully completing a remedial mathematics course may improve the likelihood a student will earn a college degree (Sanabria et al., 2020), the high failure rates in these courses position them as "a systemic black hole from which students are unlikely to emerge" (Jimenez et al., 2016, p. 1). In recent years, this perception has led several institutions to reconsider how best to support their undergraduate students who are not yet ready to study college level mathematics. For example, some institutions have reported improved student outcomes in introductory mathematics courses after experimenting with a co-requisite model (Bertrand et al., 2021; Ran & Lin, 2022). In this model, students that would traditionally be placed in remedial courses are mainstreamed into college-level math courses and provided with additional concurrent supports, such as tutoring services or extra lessons that review prerequisite skills just before they are required in the college-level course.

This need to reconsider 'remedial' or 'developmental' mathematics education for first-semester college students motivates our study. Having tried the co-requisite model, a computer-based self-paced emporium-style model, and other support systems with little success, our department proposed the development of a new course that acknowledges the unique (and varied) needs of this student population. The research team agreed that the overarching focus driving all of our efforts was to prepare students for Precalculus, and that this preparation must necessarily encompass more than just mathematical knowledge and skills. Given that, we eventually landed on two major goals: to enhance students' understanding of mathematics content that is considered prerequisite for Precalculus (math objectives) and to help them acclimate to being a college student (growth objectives). The team met weekly over the summer to develop these objectives and to design tasks that aimed to have students explore familiar mathematics using novel approaches.

Methods

The study took place at a mid-sized state university in the northeastern United States that has experienced substantial growth in the last two decades. Most of the students at this university come through local school districts or community colleges, and the current acceptance rate is roughly 87%. Precalculus is a required course for all STEM (science, technology, engineering, and mathematics) majors. Students take an internally developed placement test before their first semester, and are placed in ACM if their score is below 60%. In Fall 2024, approximately 30% of students were placed into ACM, with the remainder taking either Precalculus or Calculus. The

average class size of each ACM section was 23 students. Of the 343 students enrolled in the course in Fall 2024, 118 consented to participate in the study. Data was collected from both students and instructors. Students completed a survey at the beginning of the Fall 2024 semester in ACM and at the beginning of the next semester, Spring 2025, in Precalculus. This survey included several short-answer questions asking them, for example, why they were studying mathematics, their perception about what it means to be a 'math person,' and whether or not they thought ACM was the right course for them. The survey also included 14 Likert scale questions taken from the Attitudes Towards Mathematics Inventory Short Form (Lin & Huang, 2016).

At the end of the Fall 2024 semester, the seven ACM instructors were asked to report each student's grade (as a percentage), attendance (as a percentage), as well as their subjective assessment of each student with respect to seven categories: content knowledge; readiness for college; academic integrity; classroom presence; mathematical dispositions; and mathematical confidence. These assessments of each student were reported as scores of 1 (low), 2 (moderate), or 3 (high). Additionally, instructors were asked to predict each student's letter grade for Precalculus. Final letter grades for consenting students in ACM in Fall 2024 and Precalculus in Spring 2025 were collected from our institution.

Results

We report on three key findings: the percentage of students receiving either a D, F, or W grade (the DFW rate) in Precalculus, students' perceptions of their placement in ACM (both before and after having taken ACM), and a regression model predicting students' letter grades in Precalculus. In the semester prior to the implementation of ACM, the DFW rate in Precalculus was 34.7% (a rate that has historically been between 30% and 50% at our institution). This rate is significant because STEM students who receive a D grade or lower are unable to take the next required mathematics course. The Precalculus DFW rate in the first semester we held ACM (Fall 2024) was 18.4%, notably (and predictably) lower than the previous semester, due to placing the less-prepared students into ACM. When the ACM students took Precalculus in Spring 2025 (accounting for 61.3% of Precalculus students), the overall DFW rate increased slightly to 20.3%, but was well below the historical average of 30% to 50%. The DFW rate in Precalculus for students who had taken ACM was also comparable to that of those who had placed directly into Precalculus (20.5% and 19.5%, respectively). This last result indicates that students who took ACM were able to earn their required Precalculus credit at rates roughly similar to their peers.

Another notable result had to do with students' sense of whether or not they felt that taking ACM before Precalculus was the right course of action for them. Students were asked to answer the following open-ended question at the beginning of their ACM semester and at the beginning of Precalculus: *Do you believe ACM is/was the right course for you?* Students' responses to this question included straightforward 'yes' or 'no' responses, with some responses being more nuanced (e.g., "I feel like I should've been placed in [Precalculus] instead of [ACM], but it was a nice refresher."). However, we categorized all responses as either 'yes' or 'no,' based on our understanding of the intent behind the more nuanced responses. In the pre-ACM survey, 91.5% were coded as a 'yes' response; by the start of their Precalculus semester, this percentage had dropped somewhat to 80.5%. Despite this decrease, it was clear that the overwhelming majority of students felt as though their placement in ACM had been appropriate.

Finally, in order to investigate whether some of the data we collected about the ACM students might have predicted their subsequent success in Precalculus, we ran a multiple linear

regression model using SPSS. Students' Spring 2025 Precalculus grades[1] were used as the dependent variable; predictor variables included students' ACM final letter grades[1], ACM attendance, the ACM instructors' seven subjective assessments, and the instructors' predicted Precalculus letter grade[1] for each student. The model of best fit ($p < .001$) included only two statistically significant predictors for students' Precalculus letter grades: their final letter grade in ACM ($p = .024$) and their mathematical confidence as assessed by their instructor ($p = .001$). However, this model left much of the variance in students' Precalculus grades unexplained ($R^2 = .187$; see Figure 1).

Model Summary[c]

					Change Statistics					Selection Criteria			
Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	R Square Change	F Change	df1	df2	Sig. F Change	Akaike Information Criterion	Amemiya Prediction Criterion	Mallows' Prediction Criterion	Schwarz Bayesian Criterion
1	.405[a]	.164	.157	1.1516	.164	22.592	1	115	<.001	35.008	.865	7.106	40.532
2	.448[b]	.201	.187	1.1310	.037	5.218	1	114	.024	31.771	.841	3.849	40.058

a. Predictors: (Constant), InstrConf
b. Predictors: (Constant), InstrConf, ACMLetter
c. Dependent Variable: PrecalcLetter

Figure 1. SPSS output for the multiple linear regression showing students' ACM letter grade and instructors' assessments of students' mathematical confidence as significant predictors of their Precalculus letter grades.

Discussion

In our preliminary investigation of the impact of the ACM course on students' preparedness for Precalculus, we found some encouraging evidence that this new course was associated with improved outcomes for these students. There was a marked decrease in the DFW rate in Precalculus after the ACM course was implemented, from 34.7% in Spring 2024 to 20.3% in Spring 2025. We also found that the vast majority of students placed in ACM in Fall 2024 believed that this placement was appropriate, both before and after they had taken the course. Finally, our multiple linear regression found that students' ACM course grades were significant predictors of their success in Precalculus. Taken together, these results suggest that our implementation of a reimagined prerequisite course for Precalculus had a positive impact on students' success.

Perhaps the most surprising finding in our preliminary analysis was the emergence of only one of the ACM instructors' subjective student ratings as a significant predictor of their Precalculus grade, namely their ratings of students' mathematical confidence. This finding merits further investigation because the term *mathematical confidence* likely meant something different to each instructor. In future iterations of the course and data collection, we intend to clarify the meaning of this term to help determine whether this rating is truly a significant predictor of students' eventual success in Precalculus. We are also interested in investigating the relationship between instructors' ratings of students' mathematical confidence and students' assessment of their own self-confidence in mathematics.

Other institutions have found success in supporting students who are not yet ready for college-level mathematics with, for example, just-in-time teaching (e.g., Bertrand et al., 2021) and co-requisite course enrollment (e.g., Ran & Lin, 2022) models; what seems to have worked at our institution has been a return to a standalone prerequisite course for Precalculus. We

[1] All letter grades (ACM/Precalculus letter grades and instructors' letter grade predictions) were converted to numerical GPA equivalents as follows: A=4, A-=3.7, B+=3.3, B=3, B-=2.7, C+=2.3, C=2, C-=1.7, D=1.

reimagined a traditional 'remedial' mathematics course by focusing on both presenting familiar mathematics content in novel ways and by focusing on growth objectives. The mathematics content was often explored through integrated STEM problems, gamified lessons, student collaboration in different forms (e.g., pairs, small groups, whole group) and an emphasis on creativity and connection-making. Our inclusion of growth objectives recognized the transitional nature of the students' lives and addressed skills that are essential to learning mathematics at the college level that go beyond understanding content, such as seeking out resources, study skills, and stress management. Not only are these unique aspects of our course design likely responsible for the improvements observed in students' performance in Precalculus, but we believe that they also serve as contributions to the research on supporting mathematically-underprepared STEM majors.

We have only reported on three major findings of our study in this report, but anticipate the emergence of additional, notable results as our analysis continues. As mentioned previously, we collected survey data with respect to students' attitudes towards the ACM course and mathematics in general, as well as several open-ended questions addressing their views of mathematics as a discipline. We intend to analyze this data to identify any attitudinal shifts students experienced from ACM to Precalculus, as well as to determine whether the ACM students' attitudes were significant predictors of their eventual Precalculus performance. Additionally, because the ACM course was developed in response to the high DFW rates for Precalculus at our institution, and because we saw a marked decrease in this rate following the implementation of ACM, we will investigate whether this trend of lower DFW rates persists for the cohorts of ACM students as they progress further through the calculus sequence. Furthermore, we continue to collect data from ACM courses. We have begun our Fall 2025 academic year with a new round of data collection for our incoming (approximately 900) ACM students.

There is a continuing need to find new ways to support STEM majors that arrive on college campuses in need of additional preparation to study mathematics at the college level. We look forward to adding our preliminary results to ongoing discussions in the RUME community about addressing this need. Furthermore, we hope that by describing the development of ACM, sharing its design, and presenting these preliminary results, we will inspire others to reimagine 'remedial' mathematics and consider implementing a similar course at their institutions.

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Identifying Changes for Student-Centered Instruction in Graduate Student Teacher Professional Development Across Mathematics Departments

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Providers of teaching-focused professional development (TPD) for novice college mathematics instructors (e.g. graduate students) are a critical part of undergraduate education. Yet, little is known about the work done by Providers, including their local efforts to change TPD. We seek to capture details, summarizing a departmental change journey in a succinct advice packet, so other departments might leverage that advice in their own TPD. This preliminary study used a four frames model (people, power, symbols, structure) with two dimensions (process and product) to document TPD change in 10 mathematics departments. Through qualitative analysis of semi-structured interviews and member-checking with Providers, we identified interactions among key change elements. Preliminary findings suggest attention to certain interactions (e.g. people and power) was more common in successful change than others. Examination of the products and processes of change at the granularity reported here shows promise for sharing across departments for program improvement.

Keywords: Novice College Mathematics Instructor, Teaching-Focused Professional Development, Provider, Systemic Change

For over 30 years, evidence has continued to build the case that student-centered curriculum and instructional practices are effective for students' success in undergraduate mathematics education (Friedlander et al., 2014; Gehrtz et al., 2022; Smith et al., 2021). Productive methods include active inquiry-oriented approaches where collaboration among students is common (Laursen & Rasmussen, 2019). A large proportion of undergraduates' first (and sometimes only) experience with post-secondary mathematics is in classes taught by graduate students (Blair et al., 2018). Graduate students who teach, as assistants for sections adjacent to lectures and as instructor-of-record, are essential in spreading the use of student-centered instructional practices. Recent national initiatives have taken a closer look at teaching-focused professional development (TPD) and the work among faculty who provide that TPD (CoMInDS, 2020; Hauk et al., 2024; National Academies, 2023; Smith et al., 2021). Providers of TPD play a crucial role in shaping the teaching of college mathematics instructors (CMIs), particularly graduate student novice instructors. A *novice* instructor is anyone with fewer than 1,000 hours of classroom experience (e.g., five or fewer years)**.**

The most common forms of TPD include some combination of at least three of the following: pre-semester orientation, seminars, pedagogy courses, course coordination meetings (Yee et al., 2023a; Braley et al., 2023). Providers are change agents – they are on the front lines of their department's changes in undergraduate mathematics education because they are responsible for helping novice CMIs implement those changes. Research on how Providers initiate and/or navigate systemic changes is limited. Each mathematics department has its own profile of characteristics of TPD (e.g. seminar, course, coordination meetings) and institutional context (e.g. master's or doctoral granting, private or public). Here we report on our efforts to identify and disseminate findings about systemic change that is rooted in departments' efforts in novice instructor TPD.

Context and Theoretical Framework

This study is part of a larger federally funded project that initially implemented a national survey to identify information about Providers and the TPD they offer (Yee et al., 2023a, 2023b). The aim of the larger project is the development of an online "Design Tool" (currently in beta-testing) where one can go to see what other departments with similar characteristics are doing with/about TPD for novice instructors. In addition, the tool offers advice from the field from Providers that goes beyond *what* a department has done to succinctly describe *how* it was accomplished. "Advice Packets" include how Providers have designed and implemented opportunities for novices to learn about student-centered instruction, and describe the changes needed to arrive at their current state of TPD, whatever it may be. Each three-page Advice Packet includes a two-page description of the department's TPD change journey as well as a final table about the components of change. Creating these tables has allowed us to explore similarities and differences across TPD programs. Developing the table required a theoretical framework flexible enough to recognize and document the critical factors of systemic change.

Reinholz and Apkarian's (2018) revision to Bolman and Deal's (2008) four frames for systemic change provided a theoretical framework that was both flexible enough and structured enough to meet the needs of the study. Originally constructed to model organizational change, Reinholz and Apkarian tailored the names and descriptions of the four frames to support seeing and describing change in STEM fields in higher education (structures, symbols, power, people):

> *Structures* are the roles, routines, and practices of a department; their enactment and meaning are dependent on *symbols*, which are the norms, values, and ways of thinking in a department; changes are ultimately enacted by *people* whose individuality impacts their intentions and perceptions; and the distribution of *power* determines who makes certain decisions and influences interactions (p. 5).

Bolman and Deal (2008) argued, "Like maps, frames are both windows on a territory and tools for navigation" (p. 13). The purpose of the four frames in Reinholz and Apkarian's framework is to be a department's "analytic tool for understanding change" (p. 2). We have combined the metaphor of frames as windows on a territory and tools for navigation with Reinholz and Apkarian's focus on undergraduate STEM education. This allows us to describe departments' systemic change as well as providing a means to attend to and compare aspects of local departmental culture to support others in navigating in similar landscapes.

Research Questions

Reinholz and Apkarian (2018) describe a framework with both *process* and *product* components for each framing concept. Both the actors and actions must be taken into account. Reinholz and Apkarian say, "The four frames speak to both *what* aspects one may try to change (i.e., the product) and *how* they go about changing it (i.e., the process)" (p. 3). Our research questions parallel this structure:

> **RQ1:** What aspects and frame(s) (if any) are prevalent in Provider reporting about their department's TPD change experience?
>
> **RQ2:** How do certain aspects and frame(s) appear to interact or influence each other in Provider reports of navigating TPD change?

In RQ1, we use prevalence to focus on which frames and aspects within the frames frequently show up as key elements for a Provider of process and product. In RQ2, we are looking to see how the frames and aspects within those frames interact, seeking to see patterns across different Provider/department characteristics and ways of navigating change.

Methods

We report here on analysis of data from 10 Providers (of 95 for whom we have some data). Data sources include responses to our national survey, video-recorded semi-structured interviews, and member-checking feedback on Advice Packet drafts. Provider interviewees were chosen to ensure diversity of national survey responses, instructional characteristics, and geographical location. Of the 10 Providers included here, 2 were at private and 8 at public universities with 7 doctoral- and 3 masters-granting.

Data Collection

Data from the national survey provided information prior to the interview about the size of the department, as well as formats, topics, and activities of local TPD (Yee et al., 2023a). The interview included targeted discussion of (1) department goals for TPD, (2) Provider objectives, (3) drivers of change/stasis in local TPD, (4) specific instructional practices focused on in TPD, (5) description of the current TPD state or completed change, (6) departmental history of change, (7) critical factors for completed changes, (8) interactions of critical factors necessary for change, (9) MAA instructional practices (MAA, 2018) regularly addressed in the TPD.

Data Analysis

As each online semi-structured interview was completed, members of the research team watched the recording of the interview and, in consultation with the interviewer, composed a first draft of the Advice Packet. Using narrative analysis (Atkinson & Delamont, 2010), the research team categorized and summarized the details across Providers' interview and survey responses. The summary aimed for a granularity of description that satisfied the goal of differentiating among Providers and for succinct communication in the Advice Packet for the target audience (other Providers). Thematic analysis across the 9-part structure of the interview used inductive coding based in the four frames, broken into processes and products. Analysis for each Provider was synthesized in a table of process and products of change. This table (referred to as the *four frames table*) summarizes the change journey in the Advice Packet using the Reinholz and Apkarian (2018) framework. Subsequently, we did a constant comparative analysis across all 10 four frame tables to look for patterns across TPD programs.

Preliminary Findings

A critical aspect to the validity of our Advice Packets' trustworthiness is Provider feedback on their Advice Packet. This member-checking is currently ongoing, thus the preliminary nature of this report and a limitation (n=10) on the aggregate analysis of four frames tables. Next, we share details in a vignette and a four frames table that has been member-checked.

Vignette of Big State University (BSU)

Big State University (BSU) is a doctoral-granting institution in the midwestern United States with 60 to 75 graduate students working in instructional positions (as graduate teaching assistant, GTA, or instructor-of-record, GSI). Over a decade of work resulted in the current state of TPD at the institution. Like nearly half of programs in the national survey, the format for TPD included pre-semester orientation, a course on teaching, and regular course coordination meetings. At the same time, this university's program had uncommon characteristics, including the development and sharing of curriculum and instruction materials that generated revenue to support the costs of offering TPD. Some aspects, like shared values development, came from the Provider using the people lens, meeting the needs, skills, and motivations of individuals within the TPD program.

Also through the people lens, and that of power, the interviewed Provider talked about an integration of TPD and course coordination that was led by mathematics education professors within the department.

Table 1. BSU Table of Process and Products of Sustainable Cultural Change Using the Four Frames.

	Change Products	Change Processes
Structures	● TPD includes multi-day pre-term orientation, 1 Pedagogy Course, and weekly Course Coordination meetings. ● Online library of tested resources for Providers, Coordinators, and GTA/GSIs.	● First-Year director initiated a structure where coordination was supported by GTA/GSI creation of workbooks. ● Active-learning centered room changes due to larger classes with GTA/GSI teaching nine credits per year for stipend.
Symbols	● Workbooks from GTA worksheets created royalties to help run GTA TPD. ● Value of modifying classrooms to be student-centered supported by data justifying utility of active learning through curricular changes	● Collected data to refine workbooks/texts for improving instructional practices. ● Textbook adoption moved to open-source textbooks for all math classes. ● Graduate students not unionized; report feeling valued by course release.
People	● Administration and teaching faculty develop a shared vision about curricular coherence with graduate students ● GTAs/GSIs get "course buyouts" when taking the pedagogy course.	● Instructors retain relative pedagogical autonomy; agency for initiatives remains with department members. ● People in leadership valued efforts and time of teaching faculty and GTA/GSIs.
Power	● Director of First-Year teaching faculty elevated to department leadership. ● TPD of graduate students is considered not only important pedagogically, but also administratively.	● Leadership in administration was committed to these efforts by having changes in leadership. ● Change in classroom size, structure, and workbooks initiated by external funding.
*	*GTA=Graduate Teaching Assistant, GSI=Graduate Student Instructor-of-record*	

Table 1 illustrates that the people and power dynamics of change were strongly connected. Administrative **power** advocated for resources to improve TPD that supported the Provider's role. The **people** placed in positions of power made this possible because they valued the Provider's use of data-driven education research and valued novice instructors' ability to improve curricular resources for teaching. The respect of the administrators within the department was the strongest **symbol** when the Provider's role impacted decision-making, which allowed for changes in TPD **structure** with the provider taking a leadership position. Thus, for RQ1, the frames of people and power were more prevalent for BSU, and the aspects of textbooks, worksheets, and administrative support were critical for the systemic change.

Preliminary Connections

Two other departments out of the 10 had similar changes, with Providers indicating a view of connection between people and power. Other department Advice Packets illustrate different connections. Analysis of the four frames tables across all 10 Providers is ongoing (currently member-checking for connections). However, so far, all Providers explicitly spoke to at least two connections among ways of navigating change (i.e., frames). A preliminary analysis across all 10 Advice Packets resulted in Table 2, summarizing which pairs of frames were connected.

Table 2. Number of Connections between Four Frames among All 10 Advice Packets

Connections	Power	People	Symbols	Structures
Power	-	7	0	6
People	7	-	4	2
Symbols	0	4	-	1
Structures	6	2	1	-

In Table 2, notice that people and power was the most frequent connection between two frames (N=7) with power and structure being the second (N=6). Also noteworthy is the absence of discussion of how symbols and power might be connected. Table 2 suggests pathways for answering RQ2. Once member-checking is complete we will explore the stories behind how Providers indicated the connections happened (e.g., the very short vignette of BSU, above, can be expanded into a case story that illustrates the mechanisms of connections and change at BSU).

Discussion and Questions

These preliminary findings illustrate the viability of using the four frames (Reinholz & Apkarian, 2018) not only for self-analysis of STEM departmental change, but as a means to compare and discuss multiple department changes within and across "window frames" (Bolman & Dean, 2008). Specifically, the summary table from BSU illustrated that the four frames can provide means to identify critical change levers within frames (RQ1), while giving a means to compare connections among ways of navigating change (frames) and across departments (RQ2).

The basic questions related to "What works" have been answered in many ways, including in post-secondary TPD for novice CMIs. The question now is "How to make what works, work for us?" Inherent to addressing this question is that it is an iterative process rather than a destination. What holds promise is *scaling across*: "releasing knowledge, practices, and resources, and allowing them to circulate freely so that others may adapt them to their local community" (Wheatley & Frieze, 2011, p. 34). This is in distinction to the idea of reproductive scaling-up that dominates many western educational reforms. This is not to imply that the structures, values, and resources found in one department are not like those found in others. But similarity is not identity. Singular "best practices" do not exist across the different environments. As Wheatley and Frieze note, "scaling across invites communities to learn from one another and solve their own problems in their own particular way" (p. 36). They would also use the term "trans-local" to describe what happens when people carry an idea from one place (e.g., an Advice Packet) to let it loose in a new environment, often nurturing its growth into something quite different. Understanding the products and processes of change can support preparation for scaling across.

Questions for attendees at the session will focus on this idea and include: *How else might we look for connections between the four frames that would better align with how you would naturally investigate a Provider-described change to TPD? What infographics might be useful?*

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Students' Initial Goals, Beliefs and Conceptions of Proof in an Introduction to Proof Course

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The shift to proof-based mathematics in university programs often takes place in an introduction to proof (ITP) course. Literature exists describing the nature of these courses and some of the difficulties students experience in the course, but we know little about students' goal, beliefs, and conceptions of proof at the beginning of an ITP course (David & Zazkis, 2020; Bleiler-Baxter & Pair, 2017). Seven students were interviewed during the first two weeks of their ITP course. These same students are participating in a longitudinal study examining how ITP students' goals, beliefs, and engagement patterns evolve during an ITP course. Students were uniformly capable of providing a description of proof compatible with those commonly used in ITP courses, though students' ability to construct proof was varied. Students could identify components of proof writing that were different from other writing (e.g. the use of the word "or"). Students' goal orientations were seen to be malleable at the beginning of the semester, with most students adopting a learning goal orientation (Dweck & Leggett, 1988).

Keywords: introduction to proof, goal orientation, conception of proof

Mathematical proof is sometimes introduced in secondary school in the US, but frequently the first thorough introduction to proof that US students in some STEM programs receive occurs at the university level (Herbst, 2002). To combat the difficulty many students experience adjusting to proof-centered mathematics, many universities designate an introduction to proof (ITP) course dedicated to the comprehension and construction of mathematical proofs (Knuth, 2002; Moore, 1994). While research has been done examining the nature of ITP courses (David & Zazkis, 2020) and examining how students experience particular components of proof (e.g. Bleiler-Baxter & Pair, 2017; Hoffman, et al., 2024), there is not research that addresses students' goals and beliefs at the beginning of an ITP course. This study aims to shed light on students' goals, beliefs about mathematical learning, and conceptions of proof before extensive formal instruction in an ITP course. This study is situated within a larger project studying how students' goals, beliefs, and engagement habits evolve during an ITP semester and an intermediate proofs course immediately thereafter. The data presented in this report provides insights into the students' initial goals and beliefs that will allow us to see how these goals and beliefs change (or not) and affect engagement patterns. The following research question is addressed here: What are ITP students' initial goals, beliefs about mathematics, conceptions of mathematical proof?

Literature Review & Theoretical Framework

Mathematics education researchers have struggled to produce a definition of proof that draws an indisputable line between proof and non-proof (Stylianides, et al., 2017). This problem is sometimes called the "hard boundary problem" (Czocher & Weber, 2020). For example, there is debate about whether non-syntactic proof, such as computer-generated proofs, picture proofs, or probabilistic proofs qualify as proof (Aberdein, 2009). A definition of proof should provide justification for the inclusion or exclusion of such arguments. Experts lack of an explicit definition for proof and, consequently, there is a lack of clarity about proof for new participants in the community.

While the norms of proof writing, and consequently the definition of proof, held by distinct communities may differ, the values that underly those norms are likely similar (Dawkins & Weber, 2017). For example, most communities considering mathematical proof would likely agree that proof should increase mathematical understanding. A proof submitted for publication may be held to the norm that routine calculations or obvious justifications be omitted from the proof; however, in an ITP course a student may be required to show such calculations and justifications. It is critical to understand the norms established within the community from which a proof originates. Instructors of the same course at the same university may even have differing goals and beliefs about norms that should be adopted by students. We suspect that while practical norms differ between communities, the underlying values of what it means to prove something are relatively consistent across mathematical communities. For example, a value held by most communities considering mathematical proof is that a proof should have explanatory power. What constitutes the best way to explain differs between communities. When presented with arguments presented during a classroom discussion about why the sum of two odd numbers is even, a group of pre-service teachers valued a well-explained empirical example over a valid deductive argument (Morris, 2007). Many mathematicians would likely find a deductive argument more helpful in explaining that mathematical claim. The two communities adopted different norms to uphold the same underlying value that proofs should explain. Even subtle changes to setting such as having a different instructor or sitting next to a different classmate can influence the norms a student chooses to adopt in proof construction.

The purpose of ITP courses is to teach logic rules, proof structures, and set theory (David & Zazkis, 2020). ITP courses in the US have been classified into three categories: standard only, standard + topic, and topic-based. The data for this study was gathered in a standard + topic. Standard + topic courses (40% of ITP courses surveyed) teach skills necessary for proof comprehension and production such as understanding symbolic logic, quantifiers, set theory, and truth tables. They also include an exploration of a new mathematical topic, in this case the topic was abstract algebra.

Zone of Proximal Development

Two theoretical frameworks underpin this study: the zone of proximal development (Vygotsky, 1987) and motivation theory (Dweck & Leggett, 1988). The zone of proximal development (ZPD) describes the interplay between intra-psychological and inter-psychological domains that enables the emergence of more advanced cognitive activity (Vygotsky, 1987). The traditional interpretation of ZPD is the interplay between what an individual can do on their own and what they can do with the help of a more knowledgeable other (p. 208-209). Vygotsky also described ZPD as the dynamic interplay between spontaneous concepts and scientific concepts (p. 214-222).

A *spontaneous concept*, often referred to as an everyday concept, develops through everyday experience, usually without the aid of formal instruction (Vygotsky, 1987). Ideas within a spontaneous concept are specific and often lack generalization. For example, a child learning a native language is developing a spontaneous concept because the child knows many specific words but may not be able to generalize about categories such as “noun” or “fruit.” Ideas within a spontaneous concept are specific and lack conscious awareness, volitional control, and generalization. A student entering an ITP course likely has a spontaneous conception of what it means to prove a statement, even if the student has never been introduced to formal proof. His spontaneous conception of proof may rely on the principles of proving things in familiar settings such as justifying an answer to a classmate, explaining a solution to a teacher, or convincing a

friend. While these spontaneous conceptions may proport the same purpose of proof, they are likely inadequate to equip students to comprehend or construct valid mathematical proofs. A student's spontaneous conception of proof provides important grounding to their emerging scientific conception of proof. That is, as students learn the nuanced rhetoric of mathematical proof, their spontaneous conception of proof grounds them in the purpose of the practice, which is to explain, justify, convince, etc. A *scientific concept* is one learned through mediated interaction with their formal meanings. Students develop a scientific conception of proof as they engage in formal instruction in an ITP course. This study will focus on students' spontaneous conceptions of proof and how the first 2-4 days of engagement in the ITP course support the emergence of a scientific conception.

Goal Orientation

The goals students set during a mathematics course influence the types of activities students engage in and, consequently, the conceptions of proof they develop. Thus, the proximal and distal goals students set at the beginning of an ITP course are of interest to mathematics education researchers. Dweck and Leggett (1988) proposed the nature of goals can be sorted into two categories: performance goals and learning goals. Individuals who adopt performance goals in a particular context are concerned with maintaining or gaining favorable judgements of ability from external sources. This may lead them to avoid challenging problems, which is typically seen as a maladaptive behavior since many worthwhile pursuits in both academic and non-academic life come only after confronting a challenge. Individuals who adopt learning goals in a particular context are concerned with increasing their ability. Individuals may adopt different types of goals when they are in different settings or learning different subjects. Selection of performance and learning goals are sensitive to outside factors such as task difficulty and task consistency (Steele-Johnson, et al, 2000). In this study we consider students goal orientation in the context of an ITP courses with task difficulty and content consistent with that required by the course. A student's goal orientation is influenced by motivation, personality, willingness to persist through challenges, and personal beliefs about intelligence (Dweck & Leggett, 1988). Students who believe intelligence to be a fixed entity (sometimes referred to as having a fixed mindset) are prone to adopt performance goals; students who view intelligence as malleable (sometimes referred to as having a growth mindset) are prone to adopt learning goals.

Methodology

The university where this data was gathered ran one section of their ITP course in Fall 2025 with 19 students enrolled. Seven students enrolled in the course participated in individual clinical interviews during the first week and a half of the course. Students had attended 2-5 class session at the time of the interview. Interview questions were clustered into three categories. First, students discussed their goals and expectations for the course. Second, students discussed their experience learning mathematics in prior mathematics classes. They described what has helped them be successful in past classes and what they planned to do in the ITP course to achieve success. Third, students were given five number theory claims and asked to determine whether they were true or false and explain what they though a math proof might look like for each claim. Follow up questions about these claims did not include words such as prove, justify, convince, or explain as these words may encourage the student to adopt a particular purpose of proof for the interview (Hanna, 2000). The purpose of this section of the interview was to make inferences about students' spontaneous conception of proof after minimal formal instruction. This section was concluded by asking students to share their current understanding of mathematical proof.

Upon completion of the interviews, a transcription was made and used to make preliminary inferences about student goals, beliefs, and conceptions of mathematical proof. The inferences were used to create narrative accounts for each student. Formal coding of the data will occur at the conclusion of the ITP course when most of the data for the larger project has been gathered.

Results

Conceptions of Proof

Most of the students were able to verbalize a description of mathematical proof compatible with proof descriptions frequently used in ITP courses. Their descriptions generally contained two parts: (1) a proof should show a statement is true or infallible and (2) a proof is made by piecing together correct mathematical statements to reach the conclusion. When asked to give a proof for number theory statements, some students were able to give arguments that were closely aligned with what may be acceptable in their course. For example, consider Max's written argument that the product of two perfect square integers is a perfect square.

"Given two perfect square integers m and n.
Let $m = k^2$ and let $n = c^2$.
Then $m \times n = k^2 \cdot c^2 = (kc)^2$.
Let $(kc)^2 = I$, integer. "

Note that Max has some issues with introducing variables (i.e. he does not give descriptors for k and l). Max felt the argument produced here was more akin to proof scratchwork than a proof. One thing he said was missing in his argument is stating the definition of perfect square. Max felt that the use of definitions was critical to a proof and even more preferrable if the definition could be given in symbols that could be manipulated. Other students had a hard time constructing an argument that resembled a proof but were able to recognize that what their arguments were far from being considered mathematical proof.

Two of the three statements given to the students were false statements. One statement was false for the entire domain of interest ("The product of two primes numbers is prime.") and one was false for part of the domain of interest ("A multiple of 3 is a multiple of 6."). With the exception of Luke, all students felt confident giving a singular counterexample to prove the falsity of a statement. Chase rewrote the false statements with explicit quantification (e.g. "All multiples of 3 are multiples of 6.") and then showed an example that showed the statement not true for all the domain. He said he was inclined to make the quantification explicit because he had heard statements of the form "for all, there is something." Rewording the statement helped Luke justify why a counterexample is sufficient evidence to prove a statement false.

Goal Orientations

While most of the students expressed a desire to earn a particular grade or pass the course, they each indicated their primary goal was the understand the complexities of proof at a deeper level by the end of the course. Max showed the most consistent evidence of have learning goal orientations. The first goal Max verbalized was to earn a high A in the course, but he rarely mentioned grades or course outcomes based on performance afterwards. He showed his eagerness to understand at a deeper level when he stated, "I just guess like a feeling of like satisfaction when it like makes sense, finally." He shared his plans to achieve his goal, including reading his textbook, attending office hours, and engaging in homework assignments and other practice opportunities. Many of these plans were based off practices he had established for success in previous mathematics class. He felt he would likely not be able to watch YouTube

videos as he had in the past because he felt there would be fewer videos available to support his learning.

Luke changed his goal orientation somewhat consciously based on his interest in the material and how important he esteemed the material to be. He expressed that he began the semester focused on getting a good grade, which is indicative of a performance goal orientation. After a few days in the class he seemed to be shifting to a learning goal orientation. He said, "I think approaching it initially it was more so just get an A because it's not relevant to my major, but I find it really interesting so far. So I think my my opinion on that's going to change, I'd rather understand the subject at the—at its level." In another part of the interview, Luke said that "however it goes, it goes," which, based on the surrounding text, we infer to mean that he is not concerned with achieving a certain letter grade in the course overall. This perceived shift in goal orientation is also supported by his statement that he hopes to do better communicating with his instructor via in-person interactions to deepen his understanding, something he has not prioritized in the past.

Discussion

After less than two weeks of instruction, students' proof construction abilities were varied. Max was fairly adept constructing proofs after only 2 class sessions while Luke was not confident producing an argument resembling a proof after 3 class sessions. Since this class serves as an introduction to proof, it may be tempting for instructors to assume that all students are starting at the same base level. The contrast among student abilities at the beginning of the class is evidence that students begin the course with very different skill sets. We suggest that these differences in skill sets can be attributed, at least in part, to students' differing spontaneous conceptions of proof. Students may use their spontaneous conceptions of proof as the foundation upon which to build a scientific conception of proof. Thus, if the spontaneous foundation differs among students, the resulting scientific conceptions and skill sets will differ.

At the time of the interview, students had seen one formal proof in class (a proof that the sum of two even integers is even). The rest of the class sessions were spent discussing logical principles and the nuances of using formal logic. For example, students spoke of the difference between their spontaneous understanding of the word "or" and the new, scientific meaning of the same word. These discussions spawned in students a belief that when writing proofs, careful attention must be given to grammar, definitions, and language. Some students became so wrapped up in the scientific conception of proof that they ignored their spontaneous conceptions. For example, when deciding if an argument was a proof, they focused on grammar, language, and introduction of variables rather than the convincing or explaining component of proof.

Luke's encounters with the material during the first week of class changed his goals for the course. He began the course with a desire to earn and A and after three class sessions he was prioritizing learning the material over earning a good grade. He attributes this shift to the material being interesting. Literature on goal orientation tells us that a learning goal orientation is preferrable (Dweck & Leggett, 1988). We note that students' goal orientations may be malleable at the beginning of a course. Future research could investigate what activities led to Luke's shift to a more productive goal orientation.

The results presented here provide insights into students' initial goals, beliefs, and conceptions of mathematical proof. As we continue to interview these same seven students throughout their ITP course we will monitor how their goals and beliefs evolve from this initial starting point. This data provides a snapshot of the students at the beginning of the semester, which will compare to other data gathered throughout the semester.

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Values in Tension: Mathematics Teacher Educator's Decisions on Generative AI

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This analytic autoethnography preliminary report examines how early-career mathematics teacher educators (MTEs) make decisions about if and how to use generative AI (GenAI) in alignment with their values. Using reflective journals and group discussions, with Value-Sensitive Design, our analysis highlights three themes in GenAI use: preserving autonomy and creativity, protecting human elements of our job while viewing GenAI as a supplemental tool, and managing institutional pressures. This work highlights values-based perspectives informing the broader understanding of MTE GenAI decision making.

Keywords: GenAI, Values, Professional Development, Ownership, MTEs

Emerging research focuses on how generative AI (GenAI) impacts student thinking and learning, ethical considerations surrounding the infrastructure needed for large language models (LLMs), and classroom practices. Critics in education have raised concerns over GenAI's impact on students' perception on mathematics, critical thinking skills, autonomy, and problem solving (Kosmyna, 2025; Lee et al., 2025; Schorcht et al., 2024; Wang et al., 2024). Further, researchers found that students need support in determining how to critically engage with these tools and not passively accept their outputs (Yoon et al., 2024), and teacher educators specifically support preservice teachers' developing critical dispositions (Sawyer & Aga, 2024). However, far less is known about how teacher educators navigate GenAI, especially for those early in their career who are still finding their professional identity. Their choices not only inform their practice but set norms for preservice teachers about how to integrate, or resist, GenAI in their classrooms.

Understanding how teacher educators make choices about GenAI humanizes the broader conversation about its integration in education. During this period of sociocultural shifts in response to GenAI, we as educators should be transparent about when and if we use GenAI. This is particularly important for mathematics teacher educators (MTEs) as mathematics emphasizes reasoning, justification, and sensemaking which can be undermined if GenAI produces answers without engaging in mathematical practices. Given the lack of professional guidelines, early career MTEs are facing nebulous choices about GenAI in their teaching and professional development while navigating their induction into the profession. As a group of young MTEs, we ask the question: *How do we as mathematics teacher educators make decisions to align with our basic values about using GenAI in our teaching, scholarship, and service?* To answer this research question, we use analytic autoethnography methodology coupled with Value-Sensitive Design (VSD) to understand the different tensions that emerge when early career MTEs navigate GenAI use. By examining our own experiences and reflections, this preliminary study aims to uncover the core values MTEs draw on when making choices around the use of GenAI.

Positionality Statement

We are four early career mathematics teacher educators who share a commitment to preparing ambitious mathematics teachers, supporting in-service teachers, and contributing to

research-based interventions. We identify across a range of racial and gendered identities as we are composed of one Middle Eastern man, one Asian woman, one Black man, and one White woman. Professionally, we represent R1, R2, and Research Colleges and Universities. We are located across the US in education and mathematics departments. Collectively, we bring our different lived experiences to the research as we have nuanced dialogue that explores the similarities and differences in our approaches to GenAI in mathematics teacher education.

Theoretical Framing

We draw on Value-Sensitive Design (VSD), a theory and methodology used to provide a systematic process to account for human values in technology development within a multi-stakeholder system (Friendman et al., 2002). VSD positions technology as non-neutral since developed technology will always be intertwined with human values. These values manifest in layers, and within education technology we can understand values as basic values, learning values, or technology use values (Prieto et al., 2025) which should be iteratively coded. This allows us a lens for understanding the interaction between GenAI and our values, as well as to look across multiple layers of stakeholders in education, both direct and indirect (Shen et al., 2025) because as MTEs, we are in within a department that serves students within a university within academia. We can trace both how and why we make decisions about using GenAI by uncovering the personal and professional values that guide those choices. These choices are not simply pragmatic but are intertwined with our professional identities, institutional pressures, and underlying commitments as MTEs. For value categorization we used Schwartz's (2006) core-values. Under this theory of basic values, values are desirable goals that motivate action that often serve as standards or criteria. People's values dictate an ordered system of priorities and the relative importance of multiple values guide action. There are ten core values given: self-direction, stimulation, hedonism, achievement, power, security, conformity, tradition, benevolence, and universalism.

Methodology

We used analytic ethnography to investigate the personal and professional values that guide how we, as MTEs, chose to use (or not use) GenAI. This type of ethnography is where the researcher is "(1) a full member in the research group or setting, (2) visible as such a member in published texts, and (3) committed to developing theoretical understandings of broader social phenomena" (Anderson, 2006). This methodology focused on our roles as both researcher and participant, thereby allowing for reflexivity in the analysis process and situating our narratives in a broader theoretical and institutional context. Data collection consisted of iterative individual reflective journals as well as group discussions over a three-month period. The journal prompts focused on our feelings and choices surrounding GenAI in our own teaching, service, and research. After the initial journaling, we read each other's narratives and rejoined for a group discussion where we could ask follow-up questions and focus on what was shared among members and where our approaches diverged. All meetings were transcribed and then we engaged in an iterative coding process to engage in individual sense-making and collective meaning-making. We finally coded all data against Schwartz (2006) core-values for our preliminary VSD basic values and proceeded to group codes and look for themes.

Preliminary Results

Our analysis revealed three key themes, each reflecting core values, namely self-direction, security, and ownership, that shaped our decision about GenAI use: (1) Preserving autonomy and

creativity when deciding whether to use GenAI, (2) recognizing that GenAI cannot replace the fundamentally human parts of being an MTE, but could serve as a supplemental tool, and (3) GenAI is a way to negotiate institutional demands that we do not view at the core of teaching.

Self-Direction

A core value that we drew on when deciding whether or not to use GenAI was self-direction, which focuses on independent thought and action, as well as choosing, creating, and exploring. This value manifested within our data as our commitments to autonomy, critical thinking, and creativity in our teaching and research. Within the classroom, we positioned GenAI as a tool for exploration and problem-solving if used, rather than shortcutting mathematical reasoning. We chose to allow for the use of GenAI in this manner because we wanted to have students make choices about their learning and have the opportunity to critically engage with GenAI and discourage simply copying and pasting answers. This reflected our commitment to autonomy as students are encouraged to direct their own learning and think both critically and independently. Within our research, we chose not to have GenAI independently generate references or citations for our work as this was in violation of autonomy, and we value self-direction. By choosing our resources or engaging in a systematic review process that we designed, we maintained our self-directed inquiry and focused on how we actively shaped our research rather than outsourcing intellectual labor to a machine. This focus on research and intellectual labor centered a larger conversation on the humanization of research and what does knowledge and ownership look like when human and machine contributions overlap?

Ownership and ideas. Ownership emerged as a central theme and value that shaped how we chose to use, or not use, GenAI. As ownership of ideas focused on making decisions about what to create, explore, or think critically about, we viewed it as a basic value under self-direction as it fundamentally centers on our autonomy over thinking and contributions. There was concern about not only what was produced but who was responsible for creating knowledge. This value informed decisions across multiple aspects of our work. In teaching, many of us described clear boundaries around students' work. Entering student assignments and work into AI tools felt like a breach of ownership and trust, since students shared their work with us, but never consented to AI, and this work belongs to the student. This demonstrates that ownership is not simply an abstract principle, but a value that is operationalized in a lived ethical boundary embedded in daily pedagogical practice.

Within our professional writing and communication, the value of ownership was often in tension with the need for efficiency within the academic system. Many expressed concern that over relying on GenAI would weaken their professional voice, thereby hiding identity and authorship from their writing. This led to us heavily editing AI outputs when used, and to treat them as starting points, rather than the final product. This allowed us to make decisions aligned with our desire for self-direction while maintaining productivity. However, many expressed the concern on homogenization of ideas and writing tone when using GenAI and wanting to promote individual thought and diverse ideas and writing styles.

One of the clearest choices made around GenAI as MTEs was protecting the humanization of ideas and roles. We consistently drew lines around activities that we considered core to our professional role and identity. Teaching and research, in particular, were framed as fundamentally human endeavors. As one reflection noted, education must remain a “human endeavor” where feedback, instruction, and inquiry flow from authentic human engagement, not automated outputs. This manifested within research as many of us expressed reluctance to use AI tools within qualitative analyses, as having a weighted algorithm categorizing, coding, and

grouping thereby undermines the reflexive research process and may cause harm. However, for elements of research that were not necessarily viewed as essential or relational, like debugging code, checking grammar, or organizing references, we were much more inclined to use AI tools, so long as the central intellectual labor was still-human driven. These instances highlighted how our effort to preserve ownership and autonomy in thinking forces GenAI tools to be viewed as a supportive tool rather than replacement of work.

Our choice of whether to use GenAI tools was not solely confined to our own work, but also for students. We raised concerns that if students turned to GenAI too quickly for answers, they would "cheat themselves" out of the process of thinking. It was important to all of us to emphasize the importance of cultivating critical evaluation skills and perseverance, so students could use GenAI as a dialogue partner rather than a replacement for their own reasoning. Allowing students to co-construct classroom AI policies was seen as one way of reinforcing ownership, making explicit the terms under which GenAI would or would not be used.

Ultimately many of us discussed feelings of guilt surrounding the use of GenAI when tasks were within our professional capacity because there was a feeling of inauthenticity as well as lack of ownership. Further, we were deeply troubled with current research highlighting the psychological impact of ownership and GenAI (Kosmyna, 2025; Lee et al., 2025). The more secure individuals felt in their intellectual capacities, the less likely they were to outsource critical thinking to machines. On the other hand, lack of confidence could lead to over-reliance, undermining ownership of ideas and even memory of the work produced. We had greater concerns of our GenAI use as early career MTEs, as this stage of academic work is often associated with imposter syndrome and developing our own professional identity.

Conformity and Security

Our values were not in isolation, rather our values of conformity and security interacted with the universities' institutional values, such as achievement and power. As early career MTEs thinking about their professional identity, we often focused on our belonging within professional communities and stability within our job and research; however, this was in competition with the expectations that are placed upon 'measurable' achievements and productivity. Within our context, we operationalized conformity as making decisions to not violate expectations of us as MTEs. These social expectations came from our universities, professional communities, and students. As mentioned, this was often tied with security, which focused on our safety within our job and the research community. Simultaneously, the university values of power and achievement impacted our decisions around GenAI as universities wanted to maintain a rigorous professional image, in alignment with power, and many times used physical measures that are social standards to determine achievement.

Due to the focus on measurable outcomes and the social norms imparted on us, and out of a sense of wanting stability in our profession, this created a context in which GenAI was a pragmatic strategy that was used to reconcile competing value demands. Often GenAI is what allowed us to meet the needs of the university and conform to expectations while being able to preserve what we viewed as the heart of teaching. When our commitment to our students, our research, or other job elements did not align with the universities' measures of achievement, this led to a greater likelihood of using AI. AI was not simply a technological tool but rather was a way to try and protect our core teaching and research values while meeting social norms and keeping our jobs. Even if using AI was not consistent with personal values like self-direction or universalism, including environmental concerns, some resulted to AI to meet these needs. For example, consider one of our experiences navigating AI use:

> "I struggle because it feels like I should be able to do all these things on my own and yet trying to do it all is not physically possible… I am trying to prioritize students… Many times, for generic things that I have to do as a faculty member that no one really reads, I will have ChatGPT do a first draft because if no one else is really caring about it, then why should I put my time and effort into it, especially if it is basically adult busywork. However, why should I make my life easier at the expense of others?"

Within this quote, it is evident that this MTE wants to focus on students; however, the university values of achievement and their desire for conformity created a situation where they felt they had to use GenAI. This choice was inconsistent with their value of universalism, but the value of security outweighed this concern.

The time pressure related to academic work and the change in communication expectations were expressed across participants and in these situations, our values combined with university values, created a space where GenAI allowed us to maintain our commitment to teaching while meeting the social expectations. A common situation was with regards to email and electronic communications. This practice was important to the university, but we did not feel was at the core of teaching. One member noted that "the need to communicate consistently, and keep the right tone, and the fact that I am not confident in this skill is the reason I have used AI to help me rephrase my email drafts." Another stated that "one of the things that I feel like that has changed, especially since COVID is the expectation of speed and time to reply to things. And so the only way for me sometimes to meet those deadlines with emails [which] to me it isn't at the core of teaching [is to use AI]". These are specifically tied to being "[measured] in number of articles or something like that or number of emails sent" which are norms set by the universities and professional communities. Further, despite the desire of universities to have achievement and power, one member, with agreement of others, expressed frustration in the university not willing to provide resources. Our reflections highlight that the use of GenAI in our work is more of a values-mediated coping tool as we respond to additional institutional and societal pressures related to tasks we do not highly value or view as innately human. It is not that we wanted to actively ignore GenAI, rather, to maintain the critical thinking desired in the classroom and to have time for mentoring students and conducting research, there are areas of our job that we were willing to compromise and use GenAI to complete. These areas are what the institution may deem necessary but are not at the core of our professional values.

Moving Forward

This preliminary report focuses on the first layer of basic value coding with our initial data. While we have identified some personal values and how they interact with institutional values over time, the application of VSD is still being worked on, particularly when we think of the interaction and creating a framing of the implications on GenAI use across all values. Further, we view this preliminary report and subsequent discussion with the larger community and stakeholders as part of the analytic autoethnography process. Questions that we plan to pose to the audience include (1) How could VSD be adapted to account for evolving professional expectations when core values are fixed? (2) Are there different values or tensions in the VSD framework that attendees feel we overlooked either in our analysis or based on their experience? (3) How can this analytic autoethnography serve a broader MTE audience for GenAI decision making? Feedback from attendees will further refine our framing and analysis while ensuring that this autoethnography is linking our experiences to the broader phenomena of MTE GenAI decision making. This feedback will help us explore the subsequent layers of VSD with learning and technology values.

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HaLLMos: Constitutional AI for Feedback on Mathematical Proofs

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Learning to construct and critique proofs is central to advanced mathematics but challenging for students. This paper reports on HaLLMos*, an AI model that helps students learn to write mathematical proofs. HaLLMos uses a constitutional approach along with expert-curated feedback exemplars to generate responses to students' proof efforts. Guided by Schoenfeld's Goals–Orientations–Resources framework and perspectives on proof as discourse, we investigate the tool's alignment with its constitutional principles and student perspectives on HaLLMos' value. Preliminary data from interviews and system logs suggest that HaLLMos supports students' independence while also preparing them to participate in mathematical conversations about proof. Preliminary findings suggest HaLLMos can expand resources for students and faculty while raising new questions about how AI-generated feedback can support students' mathematical interactions and their sense of belonging in mathematics.*

Keywords: Mathematical Proof, Feedback, Constitutional AI

Learning to construct proofs is one of the most significant challenges students face in the transition to postsecondary mathematics (Selden, 2012; Moore, 1994; Clark & Lovric, 2009). Proofs are not only logical structures, but also discourse practices, shaped by the norms of justification, explanation, and persuasion defined by the mathematical community (Stylianides, 2007, 2018; Weber, 2014). Many undergraduates struggle with both the technical and discursive demands of proof (e.g., Selden & Selden, 2008; Dreyfus, 1999) and how to participate in mathematical discourse (Sword et al., in press; Herbst & Chazan, 2010).

Mathematics and mathematics education faculty who seek to support students in learning to write proofs face the ongoing challenge of providing personalized (Pedemonte, 2018), immediate (Bokhove & Drijvers, 2012), iterative (Barker & Pinard, 2014), and well-framed (Alvidrez, Louie, & Tchoshanov, 2022) feedback on students' mathematical proofs. There remains a gap between what research suggests students need and what professors can provide, given the constraints of time and resources (Stylianides & Stylianides, 2018), a gap that widens where faculty are under-resourced, and for students who may be coming to college with less experience with proof methods (Fyfe & Rittle-Johnson, 2017).

To close this gap, HaLLMos, an AI tool, takes a *Constitutional AI* approach (Askell et al., 2021; Bai et al., 2022). To build HaLLMos, we convened a group of experts to create a *constitution* guiding the development and training of a Large Language Model specifically trained to give students feedback in line with constitutional principles. This tool is not intended to replace faculty as teachers and mentors but can offload some of the work of feedback on proofs so that faculty and students can focus on human aspects of mathematics (Jones & Herbst, 2012; Selden & Selden, 2008). HaLLMos.com is a prototype website for students to receive feedback on their proofs and a setting for research on how HaLLMos serves students and faculty.

Drawing on Schoenfeld's Goals–Orientations–Resources framework and perspectives on proof as discourse, we investigate the following questions: How does feedback from HaLLMos

align with its constitutional principles? How do students use HaLLMos, and how do they perceive its value in building their capacity to read, critique, and construct proofs? How does use of the tool affect students' preparation for mathematical discourse?

How HaLLMos Works

On HaLLMos.com, students enter a proof for a pre-loaded or custom-written exercise and receive feedback from the model. Unlike general-purpose chatbots, HaLLMos does not provide complete proofs or engage in open-ended conversation; it responds only to student proof attempts, encouraging revision rather than replacement.

To generate feedback, the model acts in two stages. In the first "critic" stage, the model analyzes each step of the proof for correctness, relevance, and support. This critique is not shared with the student but is fed into a second "coach" stage, in which the model, explicitly guided by constitutional principles and fine-tuned on over 350 expert-curated feedback exemplars, generates the feedback students receive. Sample constitutional principles include: the feedback should begin by identifying one or more correct elements of the proof, if any; feedback should always be connected to the content of the work, not about the person who wrote it; and feedback should not be overly prescriptive in how issues should be fixed, preferring instead to give the student enough information to fix problems on their own.

HaLLMos does not have "conversations" the way ChatGPT does. Students get iterative feedback by revising their proofs and resubmitting. Notably, unlike existing LLMs, HaLLMos does *not* provide correct proofs after giving feedback. Figure 1 illustrates this process, showing five iterations of a student proof with changes between iterations highlighted.

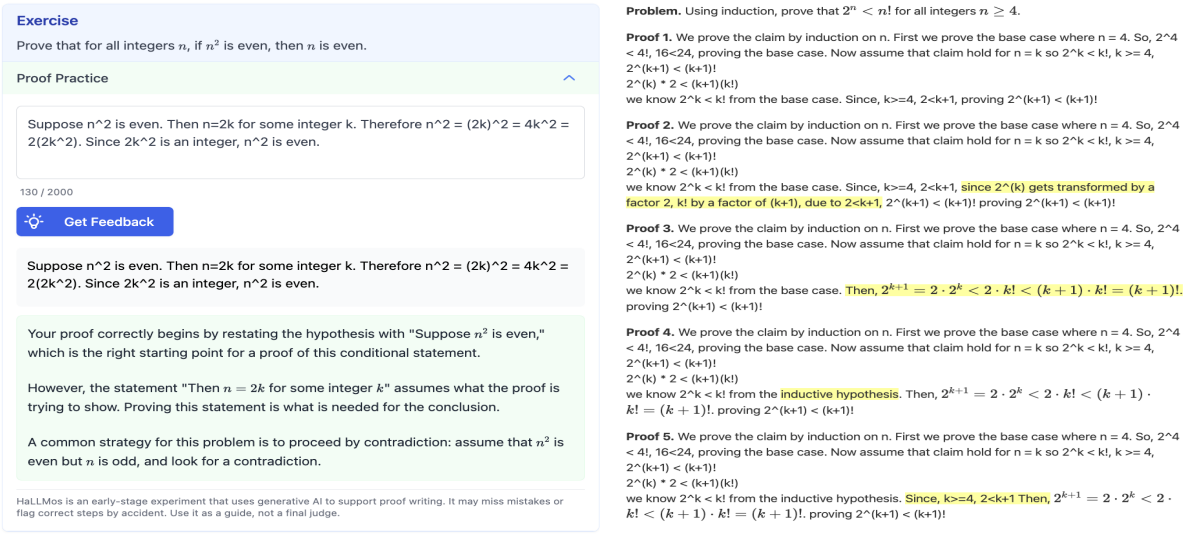

Figure 1. A screenshot from HaLLMos.com (left) and an example of a student iterating on a proof (right).

What Makes This Paper Preliminary

We are writing this paper at the end of a year of work largely dedicated to developing HaLLMos. At the time of writing, we have collected over 14,000 user-input proofs, surveyed students at four pilot sites, interviewed three self-selected students and one faculty member. In addition, we have had extensive informal conversations with pilot faculty, all of which have informed our interview protocol design. Multiple pilots of the tool are scheduled for spring 2026. We will conduct faculty interviews, as well as additional student surveys and interviews. While the

current size of the study limits the conclusions we can draw, we anticipate that the emerging themes will interest the mathematics education community as the field considers how to incorporate AI tools responsibly and productively into college mathematics.

As we continue this work, we pose the following questions for discussion with the RUME community: (1) What additional frameworks might illuminate the relationship between AI-mediated and human-mediated mathematical discourse? (2) How might we better assess alignment between AI feedback and constitutional principles at scale? (3) What safeguards should guide implementation of AI feedback tools in proof-based courses?

Conceptual Frameworks

This study is framed by Schoenfeld's theory of decision-making (2010, 2011) and the discourse perspective on proof (e.g., Stylianides, 2007, 2018; Weber, 2014; Herbst & Chazan, 2010). Together, these frameworks help us conceptualize HaLLMos as a pedagogical actor whose effectiveness lies not only in correctness but also in its capacity to realign decision-making resources and support students' proof discourse.

Schoenfeld (2010, 2011) describes teachers' instructional choices arising from the interaction of goals, orientations, and resources. We extend this framework to HaLLMos itself. HaLLMos' *goals* are to provide immediate, useful feedback and reduce faculty feedback load. Its *orientations* are encoded in the constitution, which emphasizes humanizing feedback that centers students' ideas. Its *resources* include extensive training exemplars. This framework allows us to evaluate HaLLMos' alignment with its constitution through its outputs to students.

Stylianides (2007, 2018) argues that learning proof requires appropriation of the sociomathematical norms governing acceptable justification. Herbst and Chazan (2010) demonstrate that classroom discourse norms determine who can participate and how, with students often excluded less by logical incapacity than by unfamiliarity with discourse expectations. Feedback thus functions as more than correction: it apprentices students into proof discourse by modeling the language, structure, and norms of mathematical argumentation.

Methods

HaLLMos was piloted in multiple Introduction to Proofs courses at R1 universities as an optional feedback tool for homework assignments. Data sources included website logs from student submissions, interviews with three self-selected students (majoring in engineering, statistics, and mathematics), and course materials. Interviews lasted approximately 30–45 minutes, were audio-recorded, and transcribed for analysis. Website logs captured student proof submissions, model feedback, and revision patterns. Interview protocols focused on students' experiences with HaLLMos, perceptions of feedback quality, and changes in mathematical interactions.

We employed template analysis (Brooks et al., 2015) informed by the GOR and proof-as-discourse frameworks. Initial codes were drawn deductively from both frameworks, with additional emergent codes identified through iterative analysis. Analysis focused on instances where students described their interactions with HaLLMos and how those interactions shaped subsequent mathematical activity.

Preliminary Findings

Our analysis of three focal student cases reveals themes illuminating how HaLLMos aligns with its constitutional principles, how students perceive its value, and how it reshapes their preparation for mathematical discourse.

Theme 1: Alignment with Constitutional Principles

Students consistently reported that HaLLMos feedback was useful in identifying errors and suggesting improvements: “My biggest complaint was that inputting my proofs took a while... But once I got all the way in there, the feedback was, for the most part—I never got any, like, *bad* feedback.” Another student compared HaLLMos to other AI tools: “The tone voice was always good... HaLLMos definitely gave me better answers compared to ChatGPT.”

These responses reflect alignment with key constitutional principles. Students did not report feeling personally criticized, consistent with the principle that feedback should address the work rather than the person. Descriptions of HaLLMos helping students “further develop the proof” rather than simply providing answers suggest feedback avoids being overly prescriptive.

However, students also identified areas where alignment could be strengthened: “It would help diagnose those problems. But when it had just one minor thing that really didn’t need to be changed, it would still tell me to change it... if it could differentiate between the major fundamental problems and the minor... that would be helpful.” Another noted, “Sometimes I would submit my proof... and maybe submit the same thing again, and I would get different feedback.” These observations point to opportunities for refining the model and highlight where human instructor judgment remains essential.

Theme 2: Building Confidence for Independent Work

Students described HaLLMos as an accessible “extra set of eyes” available outside office hours. One noted, “It definitely helped me with my homework, and the homework is where I learned.” Another began to think “more like a logician,” explaining, “As a mathematician, we have to basically explain and justify why our solutions work. Using HaLLMos helped me reinforce that understanding.”

One student pointed to the use of HaLLMos for getting started: “the biggest difference is when starting a proof by myself can be a little challenging... even if you start with a bare bones introduction to your proof in HaLLMos, it can provide feedback on how you can further develop the proof from there.” However, students consistently cautioned against over-reliance, emphasizing the importance of struggling through proofs. As one put it, “being stuck matters.”

Theme 3: Preparation for Human Interactions and Mathematical Discourse

One student pointed to the value of using HaLLMos before talking to a professor: “I didn’t want to just say, ‘I don’t know, I’m doing *anything!* Help me!’ So I wanted to have some understanding, and HaLLMos helped me with the first steps.” Another described the value of having a “peek behind the curtain” of mathematics that HaLLMos afforded him. Together, these findings point to HaLLMos as both a decision-making resource (in the Schoenfeld framework) and an apprenticeship tool for discourse, with potential to humanize students’ experience of learning to write proofs.

Faculty Perspectives

Faculty emphasized the pedagogical value of immediacy. As one instructor noted, "Anything with feedback that comes much later is not going to motivate students. But if the feedback is immediate, that helps students a lot." Another elaborated on HaLLMos' role in the iterative process of proof-writing: "Writing proofs—there's many ways you can go. A few right ways and many more wrong. Getting real-time feedback like 'don't go that way, try this'... that's absolutely beneficial." These observations align with the literature on feedback timing (Bokhove &

Drijvers, 2012) while suggesting that faculty see HaLLMos as addressing a gap they experience but cannot fill with existing resources.

Although our faculty data collection is ongoing, early interviews and informal communications from pilot instructors echo the student themes. One email exchange, while not part of our formal research data, has shaped our thinking on what kinds of things we can learn in a study like this one. The faculty member shared that a student "came to him after class and said she is so thankful for him sharing HaLLMos with her, because it doesn't judge her ideas." This unsolicited comment points to an affective dimension we had not fully anticipated: for some students, the non-evaluative nature of AI feedback may create space to take risks that feel too costly in human interactions.

We want to emphasize two important points about what the faculty member shared: he valued this student's experience of *feeling* not judged, not because he himself was judging her, but because she saw that aspect of HaLLMos as valuable. Second, as a team, we deeply value the human relationships that spring up around mathematics. We are not sharing that quotation because we believe HaLLMos could or should replace those human interactions. Our hope for HaLLMos is that it offer students and faculty more common ground for mathematical interactions, not less, lowering the barriers that keep some students from showing up to mathematical conversations at all. When routine feedback is offloaded to a non-judgmental machine, faculty and students can spend their time together on the aspects of mathematics that most require human presence: intuition, meaning, and community.

Implications

HaLLMos can relieve some of the burden on faculty by providing timely, principled feedback on proofs. This enables faculty to reallocate time towards activities often underemphasized in transition-to-proof courses, such as conjecturing, developing mathematical intuition, and cultivating mathematical habits of mind (Lin et al., 2012; Cuoco, Goldenberg, & Mark, 1996; Thurston, 1994). Furthermore, by modeling proof language and structure, HaLLMos lowers the threshold for students to participate in office hours and class discussions. Student reports that HaLLMos helped them know "what to ask" highlight the tool's role in apprenticing learners into proof discourse.

AI tools are proliferating quickly across education. There is a real need for research that informs the field about how to build and implement those tools responsibly, and for oversight about tools' effects on students. Such research should be situated in current literature rather than treated as "AI education research" independent of existing work.

Beyond studying the use of AI tools in undergraduate mathematics, this research offers a lens on proof learning. The constitutional process required us to work with our advisors to articulate criteria for high-quality feedback that often remain tacit in instructional practice. Website logs reveal the iterative, nonlinear nature of student proof development in ways that submitted homework cannot. And student responses, especially around students' perspectives on non-judgment and freedom to make mistakes, surface affective dimensions of proof learning that may be difficult to access through direct questioning about human instructors. AI tools, when designed and studied carefully, can function not only as interventions but also as instruments that make visible aspects of mathematical learning that are otherwise hard to see.

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Investigating the Role of Various Conceptions of Subtraction in Calculus: A Textbook Analysis

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The concept of difference, and its representation as distance within a graphical representation pervades Calculus. Indeed, difference expressions represent change, the very foundation of the subject. Yet, students may often default to a takeaway conception of subtraction, which may be incompatible with viewing subtraction as finding change. Other ways of conceiving of subtraction may support additional ideas in Calculus, related to translating functions, finding areas and volumes, or indexing terms of a sequence. In this preliminary report, we investigate the prevalence of various conceptions of subtraction in the study of Calculus through a textbook analysis of Stewart's Calculus: Early Transcendentals. We report our findings from our analysis and future directions for research.

Keywords: subtraction, difference, Calculus, representations, textbook analysis

The notion of difference, and its graphical representation as a distance between two points, is foundational in the study of Calculus, as it finds the change in values of a variable. Some instances include the limit definition of the derivative ($f(x+h)–f(x)$), the Mean Value Theorem ($f(b)–f(a)$), and the Fundamental Theorem of Calculus ($F(b)–F(a)$). To use a difference expression to model change in a quantity, one must conceive of subtraction as comparison, rather than as takeaway. Yet, the takeaway conception may be at the forefront of students' minds instead, possibly reinforced by concrete activities such as physically taking away objects or "going back" used to introduce subtraction (Selter et al., 2012). Though first teaching subtraction as takeaway at the elementary level is recommended, research has suggested that students need to hold multiple conceptions of subtraction for deep comprehension across the mathematics curricula (Bofferding & Wessman-Enzinger, 2017; Sievert, 2021). However, many undergraduate students may not readily use such a conception of subtraction to represent a horizontal distance between two functions in the Cartesian plane (Sinojia et al., 2023). Other ways of conceiving of subtraction may also be beneficial in Calculus, such as interpreting subtraction as translating a function down some number of units. To better understand the ways in which subtraction conceptions are embedded in the study of Calculus, we decided to conduct a textbook analysis. We chose Stewart's (2021) *Calculus: Early Transcendentals, 9th edition* because of its ubiquity in undergraduate courses. We follow Weinberg (2010) in adopting the lens of the *implied reader of a Calculus textbook* to consider the various conceptions of subtraction needed to meaningfully respond to the text. With this framing, we ask the following questions:

1. How do readers of *Calculus: Early Transcendentals* need to conceive of subtraction to meaningfully and accurately engage with the text?
2. What is the prevalence of various conceptions of subtraction in the text and by topic?

Theoretical Perspective

We adopt the framing of reader-oriented theory for mathematics textbooks, as explicated by Weinberg and Wiesner (2011), in which the text can be viewed as a site of interaction between the author and reader. The authors of a textbook may envision the reader of their text and address them as their target audience, referred to as the *intended reader*. In contrast, the *implied reader*

describes the capacities required of the reader to "respond to the text in a way that is both meaningful and accurate" (Weinberg & Wisener, 2011, p. 52). We take up the idea of the implied reader to examine the ways in which subtraction is required to be conceived of in order to meaningfully and accurately engage with Stewart's *Calculus: Early Transcendentals*.

To consider various conceptions of subtraction, we adopt the four models of subtraction identified by Parr et al. (2025) which they observed with undergraduate students on difference tasks designed for Calculus. These four models are: (1) magnitude, (2) takeaway values, (3) takeaway segments, and (4) translation. Table 1 contains the description of each model of subtraction as observed on a number line task or single axis.

Table 1. Four Models of Subtraction adapted from Parr et al. (2025)

	Models $b–a$ as
Magnitude	The distance between positions a and b
Takeaway values	The resulting position starting at position b and moving left a units
Translation	The distance between $b–a$ and 0
Takeaway segments	The resulting segment from removing a segment of length a from a segment of length b

Methods

We performed an initial textbook analysis of James Stewart's *Calculus: Early Transcendentals* in search of each representation of subtraction, led by the first author. We used each representation's description from Table 1. In order to begin to investigate our research questions, the first author identified whether various conceptions of subtraction occurred in each of the 116 subsection of the text (e.g., 4.7). The decision to document the presence of a particular conception, rather than code each individual expression or figure allowed us to avoid double counting repeated properties, definitions, or equations within each chapter. Preliminary coding was only done by the first author, with input as needed from the co-authors. Thus, the current findings remain exclusive to one perspective until others also code more extensively for reliability.

A subchapter would be coded with an "M" if it included any indication of a magnitude representation of subtraction. Mentions of "distance" or "length" were identified as such, as well as any graphical representation containing the length between two points. These concepts require a magnitude representation because this particular form conceives of subtraction as a representation of distance and change between points.

A subchapter would be coded with a "TV" if it contained any instances of a takeaway values form of subtraction. Language including "go back" or "shift down" were considered. The typical visual cue was any subtraction where the result was an independent value similar to the initial point before subtraction. Such instances demonstrate subtraction as a means of producing a new value rather than moving between two existing positions.

Subchapters would be coded with a "T" if they demonstrated a translation form of subtraction. The primary requirement for this representation to be identified was the translation of a segment representing length. If the endpoints of the visual subtraction on display did not match with the minuend and subtrahend, then some form of translation was involved. Otherwise, the subtraction was just considered an instance of the magnitude representation.

Subchapters would be coded with a "TS" if they involved the takeaway segments form at least once. Similar to translation, takeaway segments uses the same minuend and subtrahend as the magnitude form. This fourth model was considered when language involving "pieces" or "segments" was involved, which appeared in visual models as the comparison of smaller and larger lines that overlapped.

As these are visual representations of subtraction, we considered excluding coding to the explicitly graphical applications within the textbook. However, it became apparent early on that not all graphical concepts, such as the representation of the difference between two functions, were given a visual aid (Stewart et al., 2021, p. 40). Because of the space constraint of the textbook, we decided to code all applications of subtraction rather than exclusively graphical representations. Even if subtraction was not written out as an expression in a subchapter, we would still code for representations of subtraction if concepts applying the subtraction forms were referenced to get to these higher-level concepts. The most notable example of this was the expression Δx, which is used to represent the magnitude of the change in x over a particular interval. This symbol is used in later subchapters to represent this change in x while introducing higher concepts like rates of change (p. 146), double integrals in polar coordinates (p. 1063), and surface area (p. 1177). As a subject that builds upon its smaller concepts, we chose to code for references to earlier subchapters requiring subtraction.

Results

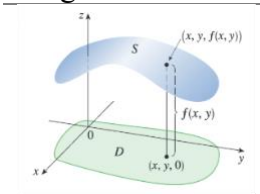
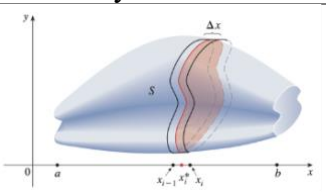
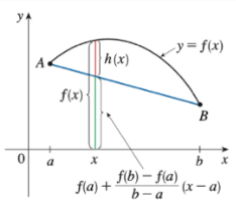
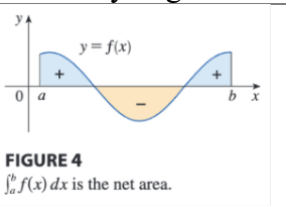
All four representations of subtraction were present within Stewart et al.'s *Calculus* textbook. The magnitude form of subtraction was the most prominent, occurring in 100 of the 116 subchapters (86.96%). Most observed instances can be divided between representations of distance/length and value comparison. Subtraction as length was applied in subchapters exploring rates of change, the precise definition of a limit, polar coordinates, series limits, and vectors. Each of these concepts build off of the basic concept of distance between two points. The distance between an initial and final x value is used in both the creation of derivatives as well as the defining of an integral's interval. These two central Calculus concepts were much of the reason for the magnitude form of subtraction being so prominent across the textbook. Value comparison appeared in additional concepts like the difference of limits, function value differences ($f - g$), rate comparisons, and the difference between two series. Real world examples using this representation of subtraction often involved physical positioning or distance.

Graphing general functions demonstrates how frequently concepts of distance are applied in Calculus. Rather than label the output of a function itself, the textbook would frequently label the length between the x-axis and the output with the function (Table 2). The expression of function outputs as distances from zero puts an emphasis on magnitude and the space between an initial and final value. The epsilon-delta definition of a limit is another interesting representation of distance through the magnitude form of subtraction. The distance between the limit $f(x)$ and $f(x) - \varepsilon$, for some $\varepsilon > 0$, is used as a reference for output value proximity to $f(x)$.

The takeaway values representation was the second most frequent model, and it appeared in 44 of the subchapters (38.26%). There was a greater variety in the applications of the model here, including function transformations, the creation of new functions, sequence notation, value reduction, and repeated subtraction. Function transformations were indicated by the "shifting" of a function around a graph. This became more prominent in later chapters when adding different values of c to an antiderivative. Value reduction occurred in scattered chapters, including the placing of epsilon ball boundaries, subtracting angles to get a third angle, and modeling

radioactive decay. Repeated subtraction demonstrated an instance of the operation unique to the takeaway values model, and it was best demonstrated with the concept of an alternating series.

Table 2. Four conceptions of subtraction identified in Stewart's Calculus: Early Transcendentals

Magnitude	Takeaway Values	Translation	Takeaway Segments
			
Mapping a function output. The distance from 0 to f(x,y) is labeled f(x,y). Emphasis on distance.	*The (i-1)th term is one term before the ith term. Moving between values performed by counting down.*	*h(x) is the difference between f(x) and the bottom equation. It is then translated onto the x-axis so that Rolle's Theorem can be used to prove the MVT.*	*Subtraction of areas used to find "net area." Introduces integrals with subtraction.*

Sequence notation was the most frequent application of the takeaway values representation of subtraction, even if it is in a more abstract form. Working with the $(n-1)$th term requires students to "count back" one term from the nth value. The $(n-1)$th term was frequently depicted in graphs as an independent point directly preceding to the nth term. Table 2 presents a visual instance of this more abstract subtraction. When explaining the division of a shape into volume cross-sections, the textbook used the subscript of sequence notation to generalize the visual partitioning beyond any fixed number of segments. Across the chapters, more complicated concepts often employed index notation like this to generalize processes and demonstrate concept application beyond any limiting context. Integral subintervals used subscript subtraction most frequently.

The translation representation of subtraction was the least present within the textbook. There were only 13 subchapters, making up 11.3% of the textbook's sections. The most common occurrence of this representation of subtraction was within settings where the initial point was constant. The greatest example of this is the subtraction of two functions to create a new function. No matter what the result of subtracting two outputs is, the new function's output must be a magnitude relative to zero. The best visual depiction of this was in the proof of the Mean Value Theorem (Table 2). Function subtraction was used to produce a new function $h(x)$, which would have root values for the use of Rolle's Theorem. In a similar manner, subtracting vectors creates a new vector that can share the same initial value as the first two vectors. Problems involving centers of mass also applied this model, visualizing all points with a distance relative to the center. Other real-world problems such as the concept of displacement from a starting point can use this form of subtraction.

The takeaway segments representation of subtraction appeared in 29 of the subchapters (25.22%). Instances of a takeaway segments form being used include the comparison of lengths or quantities as well as area or volume subtraction. Length comparison is helpful when trying to visualize concepts like change over time, differences in rate, or radius subtraction. Displacement and position functions demonstrate subtraction as a tool for finding the remaining or added

length when comparing two different points or positions. Area and volume subtraction use the takeaway segments model more consistently because they represent the comparison of established "spaces," which are closer in concept to a "segment" than an individual point on a graph. Subtracting inner volumes from outer volumes, calculating the area of a portion of a shape, depicting change in area, and representing segments of polar rectangles are all examples of Calculus concepts that applied the takeaway segments form of subtraction in this way. Even integrals were introduced with help from this representation of subtraction, referring to an integral as "net area" (Table 2). Though the positive and negative areas are not overlapping in the same way that the position function's segments did, their subtraction still represents the comparison of spaces instead of points.

In this study's current state, the subtraction of two functions to produce a third function was the only Calculus concept where three or more representations of subtraction are applied. If $h(x) = f(x) - g(x)$, then $h(x)$ is the magnitude between the points $f(x)$ and $g(x)$. As a function of its own, all values of h are graphically represented as a distance from zero. This means that the magnitude of $f(x) - g(x)$ is translated to be a distance in relation to the x-axis. This distance can be understood as the difference between the distance from zero to $f(x)$ and the distance from zero to $g(x)$, alluding to the takeaway segments model. The best instance of this is in the proof of the Mean Value Theorem.

After determining the function for the secant line between the two endpoints of the designated interval, function subtraction is used to determine the magnitude of the distance between the secant line and the original function. This new function has two zeros at the endpoints of the interval, as the outputs on the secant line and in the initial function are equivalent. This allows for the use of the previously proven Rolle's Theorem to complete the remainder of the proof.

Discussion & Future Development

These findings assert that a magnitude representation of subtraction is the most essential for students to understand in Calculus. Graphical interpretations of both derivatives and integrals rely on expressing or representing the distance between points, making the magnitude model of subtraction a necessary resource for Calculus. Yet, this does not mean that the other representations of subtraction should be considered as nonessential. Each of the representations appeared within their own contexts, and thus, students can benefit from each of them to varying degrees. Rather than teach only a magnitude representation of subtraction, it would be beneficial for students to learn how to identify the situations where each form can be helpful.

We pose the following questions for discussion with the audience as we consider our next round of coding of the data:

1. Multiple representations of subtraction can be considered for our data. What is the audience's perception of the primary (and secondary) representations for the "net area" example?
2. Are there any other examples from the Stewart text that you do not think can be described by one of the models?
3. Are there any limitations of this framework as we consider examples in higher dimensions?

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Community College Students' Definitions of and Experiences With Equity in College Algebra Classrooms

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In mathematics classrooms, equity has been researched, defined, and reported by those who have the ability and authority (professional organizations, instructors, state and federal educational agencies, local administrators, universities, and teaching faculty). However, the students who are susceptible to these definitions have not been asked for their perspectives of what equity would look like or what would remove the inequities they experience in their mathematics classes. Our goal is to center community college students' experiences with and definitions of equity as they reflect on their mathematics' learning journey, significant moments that shaped their engagement in mathematics, how they thought about equity, and whether and how they had experienced equity or inequities specifically in College Algebra. We found that participants' understandings and definitions of equity varied significantly, as did their experiences in mathematics classes. Additionally, we found that participants situated their understanding of equity in features of their identities.

Keywords: Equity, college algebra, student experiences, classroom practices

Introduction

In 2000, the National Council for Teachers of Mathematics published the *Principles and Standards for Teaching Mathematics*. One of the five key principles was equity. Since that time, researchers have argued for more robust understandings of equity in the context of mathematics teaching and learning (e.g., Cobb & Hodge, 2002; Gutiérrez, 2002) and many studies have been published about equity in the context of mathematics education. In conjunction with this movement, more professional organizations also began to focus on equity as a critical component of mathematics instruction and mathematics communities (e.g., American Mathematical Association of Two-Year Colleges, the Mathematical Association of America *Instructional Practices Guide*, 2018), noting that equity and equitable teaching practices have the ability to diversify the field of mathematics, create more inclusive spaces, and, as a result, enrich the field of mathematics.

For our work, we aimed to understand how equity was conceptualized in mathematics education research. In a review of 177 papers (Boeck et al., 2024) that focused on equity in mathematics teaching and learning, we found that only 21 explicitly defined and/or operationalized equity. Some of the earliest definitions of equity included Gutiérrez: "develop proficiency in dominant mathematics, contribute towards and develop critical stance and new perspectives on relationship between mathematics and society" (2002, p. 174) and Esmonde: "Fair distribution of opportunities to learn or opportunities to participate" (2009, p. 249). We observed in our literature review that many researchers relied heavily on Esmonde (2009) and Gutiérrez (2002) as the basis for their studies. However, we found that the definitions in the 21 articles were mostly created out of the researchers' understandings and interpretations of educational contexts and that student voices appeared to be absent in the literature.

The goal of our research is to add student voices to the mathematics education research literature focused on equity. Our focus is on community college students because they play an important role in the preparation of the next generation of STEM majors, the rise in the

percentage of students taking courses at community college to complete requirements for their undergraduate majors, and community colleges are well positioned to contribute to the diversification of the STEM workforce. In this preliminary research report, we share findings from the first of four interviews exploring community college students' experiences with mathematics generally and College Algebra specifically. Our inquiry focuses on students' definitions of and experiences of equity in College Algebra classes. We aim to answer the questions: (1) What are community college students' definitions and understandings of equity? (2) What do students' voices add to our understanding of equity?

Methods

We recruited participants from community colleges across four diverse states by announcing the study to classes of students, posting flyers in public spaces on community college campuses, and asking community college mathematics faculty to share the recruitment flyer through an announcement on the learning management system for the course. If students expressed interest in participating, we asked them to first answer survey questions about: the amount of time and the number of mathematics classes they had taken in community college; their work and family obligations; their past experiences with mathematics courses; and their definitions and experiences of equity. We chose students based on the survey responses, using a few criteria: age 18+, experiences with College Algebra, prioritizing those who intended to continue their academic studies after community college – looking for diverse backgrounds across the participants. This process resulted in a total of 37 participants across the four states. We then conducted interviews with the participants on zoom, asking questions about their current mathematics classes, including: How do the students in your class interact with each other?; Please describe a moment when you feel supported or motivated (or not supported or motivated); Have you had an experience in a college math class in which you or a classmate were either highlighted/singled out/ignored based on race, ethnicity, religion, gender, accent, or other identity? Based on their answers to the intake survey, we asked more questions about equity.

Questions based on intake survey about equity (question a, b, c and/or d were chosen depending on participants' survey responses)

(a) In the intake survey you said that you have **heard** the term equity used. How and where have you heard it being used?

(b) In the intake survey you said that you have **used** the term equity. How have you used it?

(c) In the intake survey you said that you have **defined** the word equity before. What is your definition for equity?

(d) In the intake survey you said that you have **explained** to others what the term equity means. Can you please share the explanation you used?

Questions about equity in mathematics classrooms

What would equity look like in a college math class?

- Can you describe specific examples of how your math instructors supported equity (e.g., representation, topics, perspectives)?
- Can you describe specific examples of how your math instructors do not support equity (e.g., representation, topics, perspectives)?
- What specific actions or approaches would support equity in your math classroom? (either by instructors or peers)

Based on our literature review (Boeck et al., 2024), we created an analytic codebook using the definitions and operationalizations described in the 21 articles, which we then used for deductive coding of the interviews. For example, one of our codes relied on Esmonde's definition of equity as the "fair distribution of opportunities to learn" (2009, p. 249). Another code was based on Gutiérrez's (2002) notion that students should develop proficiency in dominant mathematics. Other codes included: relational equity among students (Sengupta-Irving, 2014), agency (Gutiérrez, 2002), and opportunity to learn quality mathematics (Gregson, 2013), among others. For many of the codes, we created more nuanced categories; for example, for "fair distribution of opportunities to learn" we added the categories *unfair* and *occurred*, for "students' sense of agency," we added *lack of*, *exhibits*, and *struggling with agency*. Cleaned transcripts were uploaded to Dedoose qualitative software program and we independently applied the codes to the transcripts. We then reconciled the students' statements that we had captured with the different codes, refined our descriptions of each code, found examples for each of the codes to assist with future coding, and coded additional transcripts. For the purpose of this preliminary report, we focus on our analysis of students' definitions of equity and how they described what equity would look like in mathematics classrooms, providing examples from a diverse group of five participants.

Findings

From the codes that centered around equity, we analyzed students' statements to further understand the contexts in which students use the word equity in a college mathematics classroom and their functional definitions. Findings are derived from analyzing interviews with a team of three researchers. We first share different ways the participants defined equity, confounding equity and equality, and how they described equity in a mathematics class. Finally, we explore how the participants situated equity in their identities and their experiences.

Defining Equity Was Difficult for Some and Students' Definitions of Equity Varied

There were significant differences across participants' definition of equity. Below we provide examples for the different types of thinking and definitions.

- *Defining equity was difficult for some*: Will stated "I struggle with it, I think. Maybe more than either diversity or inclusion. And I struggle with all of them" while Elena was very unsure and asked if she could look up the definition of equity during the interview.
- *Some students thought of equity in a financial context*: George said equity is "the amount of money you possibly gain like an income like for a house", and Althea said "I think I would define it as, like, equal or, like, money wise, like a part of a business, but, or like the value you have issued in a business."
- *Some confounded equity and equality*: Will acknowledged that he "struggles to parse out equity, equality, and fairness," and Elena stated "equality, I mean is, we're all treated the same. We're all treated with the same, you know, we don't, you know, look at anyone different, treat in that sense as far as equality," while Althea remarked "sometimes equity and equality kind of, like, combine in my head." Althea suggested that teachers should give each student the same attention and further implied the teacher was unfair as they "sectioned us out, like there was, like, like, the smart people that, like, don't really need help. I don't really need to focus on them too much. I have to focus on the people who don't understand that much or like it.

- *Some participants provided more concrete definitions of equity*: As one example, Ayami said "I think equity is having the tools there to make it possible for all of us to work together [...] deep equity looks like, is that those tools aren't uh… aren't either forced on us or like and uh… an intrusive presence. Like, they're just part of the way things are."

Students' Descriptions of What Equity Looks Like in a Mathematics Classroom

We also heard a variety of responses to the question "what would equity look like in a mathematics classroom?" Below are some examples of their responses:

- *Varied understanding of equity in the classroom*: George answered "In the math classroom. I don't really know. I don't really know the word equity yet. I'm equity enough to say that," while Will stated "equity in a college classroom focuses on student needs and accommodations for academic success, over comparative treatment for all students." Althea defined equity as "everyone gets the same opportunity to understand the coursework and the way, like the teacher is teaching."
- *Personal and social identities are taken into consideration when considering what equity in the classroom*: Ayami provided a very in-depth conception of equity in the classroom that incorporated how both students and teachers could promote equity: "I think for classmates it'd just be more so, kinda being willing to help versus like being very closed off. Not really like, you know, just being willing to help someone if they ask, and then, in the sense and like kind of kind of fostering a non-judgmental zone… because, you know, when it comes to like things like neurodivergency, or like struggles it can be embarrassing to kind of like, talk about it, or like reach out for help, so kind of providing and creating a nonjudgmental zone."
- *Students identified meaningful interactions with their peers surrounding mathematics as a form of equity*: When asking for specific actions or approaches that would support equity in each participant's mathematics classrooms, there was a high frequency of students desiring meaningful interactions with their peers surrounding mathematics. For example, Ayami described her experiences in study groups as negative: "It wasn't really like the study group conversation of trying to learn math together." This implies that social interactions with teachers and students help define whether or not a space is equitable. Other examples of specific actions or approaches that would support equity that were provided included giving extra time to students, providing multiple resources to learn from, and adjusting for different learning styles. Ayami explained that the role of a teacher is to "guide the way. Not really dictating [...] I feel like also kind of exploring different kinds of learning styles, cause I feel like it's very rigid."

Situating Equity in Identities, Experiences, and Broader Viewpoints

Lastly, using the participants' examples of equity in a mathematics classroom, we observed participants provided some insight into their personal and social identities when considering equitable outcomes. Compiling a list from all participants, they expressed that race, neurodivergence, social classes, immigration status, physical health, and mental health are factors that equity should address in a college mathematics classroom. In addition, Ayami stated that "home life" itself may be enough to create inequities. It is important to note that no two participants had the same set of identities and this leads to a difference in beliefs as to what

impacts a student's experience in a mathematics classroom. Here we provide two different situations for understanding and describing equity through their identities:

- Will noted his identity as a white, middle-aged man and said: "And obviously, that's a certain amount of privilege, right? Because places aren't built to make me uncomfortable. It's very easy for me to make myself comfortable in them."
- Ayami noted her identity as a young, West Asian woman with caregiving responsibilities and part-time employment, who had recently been diagnosed as neurodivergent. Her definition of equity was rooted in her positionality: **"**I usually describe it. I'm trying to think of, like, a specific time that we use the word equity. I usually describe it more in a racial sense or neuro, like, in a sense that uses neurodivergency or social classes. I typically use examples to describe equity versus, like, giving like a definite, like a strict definition, because equity in general is kind of, because in order to use equity, it's like a social. It's a social construct. So, we have to like, you know, kind of give it context in order for it to seem. So, yeah, I kind of use more examples when I talk with them."

Discussion

We focused our *first research question* on hearing how students defined equity. Unsurprisingly, we found significant variation across the participants in this preliminary analysis, that equity can be conflated with equality or sameness, and a definition of equity that might suit all students does not, and likely never will, exist. That said, we found that student definitions centered around the three main ideas identified above and that these ideas varied by their ability to conceptualize equity in education vs. finance and their ability to concretize that conceptualization based on their classroom experience or their identity. With our *second research question*, our goal was to answer the question "What do students' voices add to our understanding of equity?" Our preliminary data from students highlights a number of important facets of the definitions of equity reported in the literature. Some of the terms used more frequently by the students, such as descriptions of what they would like to see in their mathematics classroom that would demonstrate equity, were related to *relational equity* as described by Sengupta-Irving (2014), which means learning to effectively communicate with one another and appreciate other perspectives. However, there were more instances of students not experiencing relational equity than reporting its occurrence, and the participants reported that their college algebra classrooms should focus more on collaborative work rather than "just a relationship between one student and the teacher." The second most coded definition of equity was Esmonde's fair distribution of opportunities to learn or opportunities to participate (2009). The occurrence of this element was almost evenly distributed between noticing that it had occurred in the classroom and noticing that it had not occurred, but would be an important element to promote equity in the classroom.

Discussion Prompts for the Audience at RUME

We have two prompts for the audience at RUME: (1) We have coded primarily looking for instances of students' reflections and utterances that resonate with definitions of equity that we found in the literature as well as what they say directly when we ask them what is equity and how would you see it in a mathematics classroom. Is there another approach for coding that comes to mind that we might consider as we move forward? (2) How might our findings inform equity frameworks as well as research studies that center student voices in mathematics?

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The Nature of a Community of Practice: Social Expectations at an Informal Summer Mathematics Program

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We present results from a pilot study conducted to inform possible directions for further investigation focused on a specific community of practice: an informal summer mathematics program for mathematically talented students. While the program is attended by high school students, they take undergraduate level proof-based courses and engage in research at a level comparable to a research experience for undergraduates. Journals and interviews were used to investigate aspects of the program using Wenger's Communities of Practice theoretical framework, which describes social characteristics of a community of practice in terms of mutual engagement, joint enterprise, and shared repertoire. Suggestions for feedback from the reader and audience are included.

Keywords: Communities of Practice, Informal Educational Context, Undergraduate Research

A plethora of communities exist around mathematics, both educational and professional, and it is worthwhile as researchers to describe the diversity of these communities. Wenger (1999) describes the importance of Communities of Practice (CoP): "Communities of practice are the basic building blocks of a social learning system because they are the social "containers" of the competences that make up such a system." (p. 4). The CoP framework is a useful analytic tool to study how knowledge is created and competency is assigned within a community. From a situated learning perspective, Lave and Wenger (1991) describe legitimate peripheral participation as a tool to study learning within CoPs. The purpose of this work is to explore the way knowledge is generated within an informal summer mathematics program (ISMP) by describing essential aspects of the community and observing the legitimate peripheral participation of its members. The purpose of this paper specifically is to overview the early stages of the project including the initial round of data collection and preliminary observations about the community of practice.

Components of a Community of Practice

Wenger (1998) establishes that CoPs combine three elements: mutual engagement, a joint enterprise, and a shared repertoire. The first component, mutual engagement, refers to when members interact and establish norms for how they will engage in their practice. Mutual engagement creates relationships among members of a CoP. However, Wenger notes that engagement may not always be positive. It is possible for conflict or tension to characterize the engagement of a community. The second component, a joint enterprise, refers to agreement among members of the community about what they're doing, what their goals are, and they hold each other accountable to those expectations. Wenger describes how the joint enterprise is established, "It is defined by the participants in the very process of pursing it." (Wenger, 1998, p. 77). Finally, shared repertoire refers to resources that are shared with all members such as specific vocabulary, artifacts, and experiences. Wenger points out that a shared repertoire reflects a history of mutual engagement and remains inherently ambiguous meaning it can be reused or applied in novel ways. Biza et al. (2014) describe transforming the CoP framework into *Communities of Inquiry* (CoI) (Goodchild et al., 2013), which they explain to be, "… a CoP in

which inquiry is a fundamental way of being in practice." (p. 164). In a CoI, inquiry manifests itself in all three components of a CoP. Members engage in inquiry-based processes, their goals are aligned with better understanding of concepts, and they share inquiry-based tasks and approaches (Biza et al., 2014).

A range of empirical work has been done using the CoP theoretical framework to study undergraduate participation in mathematics and related fields. Solomon (2007) reports on a study examining undergraduate students' relationship to mathematics within the context of specific communities. Solomon observed a lack of students who identify as Legitimate Peripheral Participants and attributed this scarcity to the dominant discourses present in the community. The continuation of the project in this paper may provide further insight into barriers to legitimate peripheral participation undergraduate students face.

The following sections describe an ISMP using the CoP theoretical framework by giving examples of a joint enterprise, mutual engagement, and shared repertoire within the program. Specifically, we address the following research questions:

1. What social values or expectations do the students hold for themselves and their classmates regarding the way in which the ISMP community engages with mathematics?
2. What is the nature of mutual engagement, joint enterprise, and shared repertoire in the context of informal mathematics communities?

Methods

Context

The data used in this analysis was collected from students participating in a six-week ISMP in the southwestern United States. The program invites students who demonstrate excellence in mathematics to participate for up to three summers during their high school years. Students take the following undergraduate level mathematics courses taught by faculty at the host institution: number theory and Python programming (first year), abstract algebra and real analysis (second year), and topology and combinatorics (third year). Second- and third-year students also undertake research projects supervised by faculty at the host institution. Students spend their afternoons and weekends engaging in a variety of recreational activities and field trips.

After graduating from high school, students may return to join a group of undergraduate and graduate mentors. The mentors are responsible for holding daily problem sessions with eight students to review material from class as well as nightly study groups with four students to work on problem sets. In addition to their academic responsibilities, the mentors, particularly those who were previously students in the program, also uphold the camp culture by organizing activities and honoring traditions.

Data and Participants

Students submit two journal entries reflecting on their ongoing experiences. The first journal entry is after the second week, and the second journal entry is after the fourth week. The journal prompts were written by the program faculty with the purpose of ensuring all students are supported mathematically and socially throughout the summer. For example, some questions inquire about the pacing of the courses, some about the dynamics of their study group, and some about recreational activities. While the prompts were not written specifically for this project, the questions the students are asked to reflect upon are still aligned with the goals of the research.

Journals written by fifteen first-year students, thirteen second-year students, and four third-year students were collected for this project.

In addition to the journals, nine students were chosen to participate in semi-structured interviews. The students were chosen based on availability out of the group of second- and third-year students. This allowed for questions about research, which we know from prior experience to be a treasured component of the experience for returning students. A common interview protocol was used for all participating students; however, for several students, questions were added based on their journal entries to give the students the opportunity to elaborate on their written responses. The interviews were conducted during the fifth and sixth weeks of the program. Each interview lasted approximately 30 minutes. Zoom was used to record the audio and generate transcripts. The interviews were conducted by the first author, who was familiar to the participants as a mentor and staff member at the program. The study was supervised by the second author, who is on the program faculty. It is important to note that the students were aware that their journal entries would be read by the two authors and by other faculty in the program.

Results

In this section we provide examples of themes in the data regarding the students' values and expectations, which point to possible directions for future analysis. The themes reported below were selected when several students shared the same or similar sentiments in response to the journal or interview prompts. The first theme is related to mutual engagement, the second is related to the students' joint enterprise, and the third is related to their shared repertoire.

Mutual Engagement: During the Program, Students Expect to Work Collaboratively.

During the interviews, a conversation contrasting the way students do math at the program versus the way they do math at school often came up. We did not include an interview question specifically aimed at this; this theme emerged in response to different prompts depending on the participant. The majority of participants noted the collaborative nature of mathematics at the summer program compared to school. The following are two illustrative examples.

One student, Eugene, described why he returned for a second summer:

> "I thought like being in a study group with other people and like working on problems together, instead of just like being at home and doing all the problems myself, it's a lot more interesting. And I think it's like a lot more enjoyable, so that's why I came back."

Eugene returned to camp with the expectation that his study group would be collaborative. Later in the interview, Eugene compared his first-year study group to his second-year study group. He described how the first-year group would work together to solve all the problems, use the chalkboards, and bounce ideas off each other. Sounding somewhat disappointed, he shared that his second-year study group was often quiet, and students in the group worked on their own. Eugene's disappointment in his second-year group indicated that his expectation to collaborate with his study group was not met, and their dynamic was abnormal.

Another student, Miles, shared that the summer program allowed him to find friends who shared his interest in math. When asked why it is nice to have friends with the common interest, he shared,

> "…math can definitely get bland at some point if you're working on the same problem for like hours on end. So, like just having another person, even if they're not like helping

doing the same problem as you, just having another person doing something similar and like able to help or talk to about the problem is nice."

Miles came to camp with the expectation that others there would share his interest in math and provide collaborative support that he did not have at home.

The students' shared expectation that they will work together with their peers suggests that collaborating mathematically is a social norm at the summer program. They come to the program expecting that their classmates will engage with them in problem solving. Further investigation into and analysis of the types of socially expected collaboration will help inform the characterization of mutual engagement at the program.

Joint Enterprise: CoP Members are Committed to Supporting Each Other's Learning.

Many of the students wrote in their journals or shared in their interviews about the techniques they used to support their own learning and the learning of others. As above, this was not an explicit question; it came up in a variety of contexts depending on the student.

One expectation set by the faculty at the program is that mentors provide students with written feedback on their proofs (often referred to as "grading" even though no numerical grades are assigned) and assign several proofs per problem set for the student to rewrite and resubmit. One student, James, created his own strategy for responding to feedback. When prompted in the second journal about the feedback he received from his mentor, James described how, rather than just rewriting the proofs on his own, he preferred to think about the feedback he received and have a discussion with his mentor about it during study group. He wrote, "This turns out to be a very productive strategy for me, as rather than 'doing corrections,' we can instead have constructive conversations in person during study group." James' reflection is notable for two reasons. First, his motivation for adopting this new strategy of responding to feedback was his own desire to be more productive and engage with the ideas in a more meaningful way. Second, his mentor willingly adopted the new way to engage James with feedback, going against the faculty's expectations, but honoring the fact that it supported James' learning.

Another example of the joint enterprise is from the perspectives of two students, Kaitlyn and Aaron, who were in a study group together. Kaitlyn wrote in her second journal that their study group was very collaborative, often solving all the problems together on the board. However, she claimed that Aaron was the most "advanced" out of all of them, so his contributions were often diagrams or hints to help the others arrive at the solution on their own. Aaron also wrote about this dynamic in his first journal. He says, "I usually tend to guide the discussion, but when I realize I am being too overbearing, I often decide to let the others figure out the problem without me." Both Kaitlyn and Aaron were asked follow-up questions about this dynamic. Kaitlyn expressed her appreciation of Aaron's approach, and Aaron shared that he used it because he thought it created a better learning opportunity for his group mates. These examples suggest that the students in the program have a shared purpose to engage deeply with mathematical ideas and support one another's endeavors to do so. More investigation into this shared value and its manifestation across contexts will be useful in describing the joint enterprise of the students in the program.

Shared Repertoire: The Program Provides Students With Shared Artifacts and Experiences.

During this program, students participate in intense mathematical exploration together as well as build memories through recreational activities and free time. During their courses and

research, they receive and create shared artifacts including their problem sets, proofs, and past research projects. An illustrative example of a shared artifact comes from the interviews with Eugene and Miles. When asked how their research groups chose their respective topics, they both explained that their groups built upon the research of groups from previous summers. Both referenced a shared document listing past projects, and future directions students could go. Eugene explained that his group was working on extending a result by a previous group, who showed a property is true for primes, to now be true for all natural numbers. The shared artifact in this case is the previous result and proof, which was available to Eugene's research group. Further investigation is warranted into other shared artifacts of the community, as well as the shared stories which live on in the community members' memories.

Discussion and Future Work

The examples above are the result of the early stages of an investigation into the community of practice at the summer mathematics program. From the journals and interviews, there are clear examples of the three aspects of a community of practice: mutual engagement, joint enterprise, and shared repertoire. This investigation is also practically useful for future study design. For example, the summer 2025 data collection provided an estimate of how many participants to expect in the future and what logistic challenges may arise. Guiding feedback on the direction for future data analysis would be appreciated from the readers and audience, as well as feedback on the theoretical framing of CoPs.

Future work will continue to build descriptions of the three aspects of the community of practice for this ISMP, examine how they influence each other, and ultimately lead to a useful way of understanding the way learning takes place in the context of the program. The results from future work will inform disciplinary knowledge about summer programs as well as other non-traditional communities within undergraduate mathematics such as REUs, clubs, and cohorts of mathematics majors. Intended future steps of this work include collecting more journal and interview data as well as observational data during the summer 2026 program and develop a coherent analytic approach to triangulate themes in the journals, interviews, and observations.

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Students' Embodied Reinvention of the Bijectivity of Invertible Functions

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We present a preliminary local instructional theory of students' embodied guided reinvention of the bijectivity theorem of invertible functions. We leveraged the design principles of Realistic Mathematics Education (Freudenthal, 1991) and Embodied Design (Abrahamson et al., 2020). We present our task sequence, describe implementation in a classroom, and provide an analysis of student reasoning. We discuss our plans for reimplementation and implications for teaching.

Keywords: Realistic Mathematics Education, Embodied Cognition, Invertible Functions

Introduction and Literature Background

It is essential for students to understand properties of functions and inverse functions. The extensive literature on functions documents various student difficulties such as inattention to defining properties (e.g., Markovits et al., 1986) and issues with domain and codomain (e.g., Even, 1990; Hitt, 1998). Recent work has identified productive techniques for approaching defining properties of functions (e.g., Uscanga et al., 2024). Even when students attend to defining properties, they may conflate well-defined and one-to-one (e.g., Clement, 2001; Even 1993), and everywhere-defined and onto (e.g., Melhuish et al., 2020; Vinner & Dreyfus, 1989).

Reasoning about inverse functions is multifaceted. It may entail conceptualizing an inverse function as an "undoing" (Cook et al., 2023, p. 6), "manipulated element" (p. 7), or "coordination of the binary operation, identity, and set" (p. 8). This may involve recognizing the composition of a function and its inverse as the identity function, which acts as a "do-nothing function" (Bagley et al., 2015; Serbin, 2023). Reasoning about inverse functions may involve using analytic, graphical, or covariational approaches (e.g., Oehrtman et al., 2008; Paoletti et al., 2018; Paoletti, 2020). Students must also make sense of why a function needs to be bijective (i.e., one-to-one and onto) for its inverse relation to be a function (i.e., well-and everywhere-defined) (Bansilal et al., 2017), which is challenging for them.

Here, we explicate our local instructional theory (LIT; Gravemeijer, 2004) to engage students in reasoning about the properties of well- and everywhere-defined, one-to-one, and onto to reinvent a direction of this target theorem: *Let* $f: A \to B$ *be a function. An inverse relation* $f^{-1}: B \to A$ *is everywhere- and well-defined iff* f *is onto and one-to-one.* An LIT is "the description of and rationale for the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). We provide the rationale for our preliminary instructional task sequence and evidence of students' progress. We merge the design principles from Realistic Mathematics Education (RME; Gravemeijer, 1999) and embodied cognition (Abrahamson et al., 2020) to inform the design of our task sequence. We address the research question: *What learning route can students follow to reinvent the target theorem?*

Theoretical Background: RME × Embodied Design Principles

Design research (Gravemeijer et al., 2003) involves a cycle of designing instructional task sequences, implementing them with students, analyzing students' reasoning, and revising the task sequences accordingly. We leveraged RME (Freudenthal, 1973; 1991), an instructional design theory in which, "students are expected to develop formal mathematics by way of mathematizing their own informal activities" (Gravemeijer, 1999, p. 160). RME has three

heuristics that guide researchers' design of an LIT. The first is using problems with "contexts that are *experientially real* for the students and can be used as starting points for progressive mathematization" (p. 158). The second is *guided reinvention*, by which instructors leverage students' ways of reasoning to guide them to reinvent formal mathematics. The third is the opportunity for students to develop an *emergent model*, an object or representation that students project meaning onto as they model the mathematics in play. We also use embodied design principles (Abrahamson et al., 2020; Nathan, 2021) to frame students' emergent models as *multiparty* enactments (Kelton & Ma, 2018) in which students use their bodies, physical materials/props, and space to carry their personal meanings for the mathematics (e.g., Soto, 2022; Lajos & Katz, 2025). Embodied design principles suggest the instructor should leverage a variety of students' embodiments (e.g., Soto et al., 2024) to support their guided reinvention.

As students reinvent mathematical concepts and transition from using their emergent model as a "model of" to a "model for," they engage in *horizontal* and *vertical mathematizing* (Rasmussen et al., 2005; Treffers, 1987). *Horizontal mathematizing* refers to "formulating a problem situation in such a way that it is amenable to further mathematical analysis" (p. 54) and entails a shift towards symbolizing via inscriptions and taking stock of what is noticed across a variety of emergent models. *Vertical mathematizing* is defined as performing mathematical activities that are built on and are grounded in one's own horizontal mathematical activities. For our context, this entails the act of moving from an embodied model to a formal definition, conjecturing, and arriving at classroom generated theorems using refined language.

Methods

We conducted a teaching experiment (Cobb, 2012) in a Fundamentals of Algebraic Structures course at a public university in the southern US. The course was taught by a teacher researcher (TR) who is a coauthor. The students were junior and senior prospective secondary math teachers. The task sequence implementation took place in one 75-minute class. Before this lesson, the course had addressed the definitions of relations, well- and everywhere-definedness of functions, and inverse relations, but not one-to-one and onto, which were focal properties in this LIT. We note that one group of students used "one-to-one" language as they worked on the tasks, but they did not recall the term or meaning of onto. We video-recorded two groups of four students each to capture their multiparty embodied models and class discussions. Each author watched the class videos and wrote content logs (Lesh & Lehrer, 2012) on students' reasoning, evidence of mathematization, embodied models used, and how the TR leveraged the students' embodiments to guide their reinvention. The authors compared and discussed content logs.

Results

In Activity 1 of our LIT, students embodied the functions in Figure 1 and their inverse relations. Students leveraged their embodiments (e.g., multi-party enactments, inscriptions) to make sense of everywhere- and well-definedness issues of the inverse relations. In Activity 2, students used their embodied models to formulate definitions for one-to-one and onto, make conjectures relating properties of functions and inverse relations, and arrive at the target theorem.

Activity 1: Enacting Puddle Diagrams of Functions and Their Inverse Relations

Activity 1 prompts students to physically act out puddle diagrams for functions and their inverse relations and determine whether each inverse relation is a function. Students played the role of elements in the domain and codomain for various functions (Figure 1). TR provided element name tags labeled 1, 2, 3, 4, 5, and 6, strings, and domain and codomain labels. We

present Group 2's (G2) embodied models for inverse relations. Next, students symbolized their embodied models using sets of ordered pairs and puddle diagram inscriptions. They leveraged their body formations to discuss whether their inverse relations were functions. This task was designed to give students an opportunity to model functions' inverse relations, which they could later use to make conjectures about the relationships between function properties.

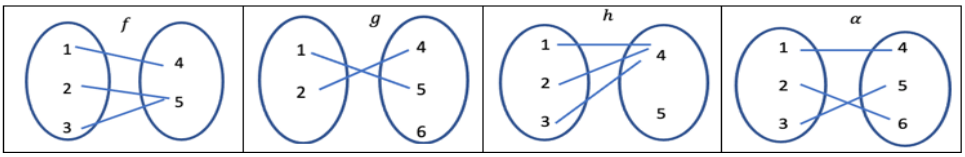
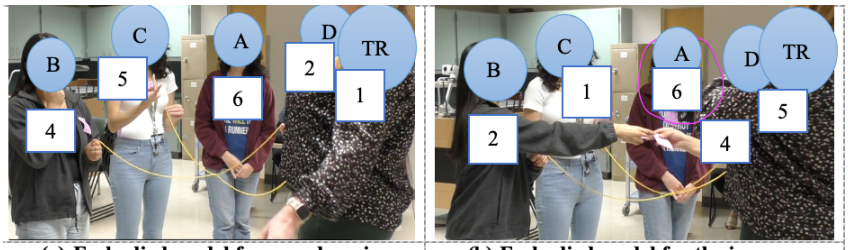


Figure 1. Parent functions given to students in Activity 1.

G2's embodied model for the inverse relation of g: "Switching (element) name tags." G2 made an embodied formation for the puddle diagram of g, and student C playing element 5 gestured a swap with her fingers (Figure 2a). To embody the inverse relation of g, the students playing elements 2 and 4 switched element name tags, and the students playing elements 1 and 5 switched tags (Figure 2). They were perturbed when the student playing 6 had no one to switch tags with, and their "swapping x and y" procedure was abrupted. The instructor remarked, "That's weird [glanced at 6 who was still standing in codomain area], should 6 be in the domain or codomain?" (Figure 2b). B questioned, "if 6 doesn't have a partner, can it still switch? No, right? because there is no one there." TR probed further: "What do we do with 6?" The students' strategy of switching name tags among paired elements in the domain and codomain did not work. They recognized that this embodied breakdown occurred because student A (6) did not have someone to switch tags with. The group next began to mix features of their embodied model (e.g., no one to swap with, kick her out, no partner) with formal property language.



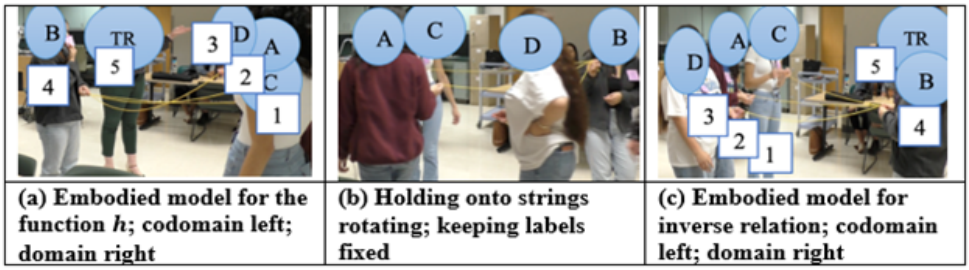
(a) Embodied model for g; codomain left; domain right

(b) Embodied model for the inverse relation; codomain left; domain right

Figure 2. Embodied model for function g and its inverse relation "switching element name tags."

G2's adjustment and symbolization of their embodied model: "Kicking out" a peer in the codomain of g^{-1}. The group continued to focus on A (6) questioning where A belonged, the codomain or domain side. B (1) and C (2) suggested, "kick her [6] out." TR leveraged A's eviction and body location to invite her to stand on the domain side. She asked, "Why do we not want her in our domain set?" B replied, "Because she didn't have a connection in the first place." TR asked what function property was violated. B replied "Every x has to have a y," and D said, "Everywhere-defined." The group concluded g^{-1} is not a function, unless they restrict its domain. G2 confirmed g^{-1} was well-defined and concluded it was not a function because it was not everywhere-defined. Following their multi-body enaction, B symbolized g and g^{-1} as the sets, $\{(1,5), (2,4)\}$ and $\{(5,1), (4,2)\}$. Student B claimed to not know how to symbolize that "6 did not have a partner," so she wrote "Either remove or switch to domain without partner."

G2's embodied model for the inverse relation of h: "Holding onto strings and rotating." G2 embodied h as B (4 in the codomain) held three strings, TR (5 in the codomain) held no strings, and students C (1), A (2), and D (3) (in the domain) each held one string emanating from B (4) (Figure 3a). After looking at her peers across from her, student B exclaimed, "I only have one heart guys. I can't split it." G2 then decided their prior tag switching model would not work. At best, B (4) could only switch tags with one peer on the other side. This foreshadowed an issue that h^{-1} is not well-defined because h is not one-to-one. G1 suggested that they "just switch the [domain and codomain] papers on the table." G2 seemed puzzled and fell silent when TR asked, "is this [h^{-1}] a function?" TR offered a change in embodied perspective and motioned for them to rotate their entire embodied model for h (Figure 3b). After arriving at their new locations, C (1), A (2), and D (3) looked down at the codomain label on their side (Figure 3c). D stated, "It would be the same situation as [g] where it's well-defined but not everywhere-defined" and pointed to TR (5) in the domain who was not holding a string. TR noted D's partial meaning of "not everywhere-defined" and revisited issues of "well-definedness" during the class discussion.



(a) Embodied model for the function h; codomain left; domain right | (b) Holding onto strings rotating; keeping labels fixed | (c) Embodied model for inverse relation; codomain left; domain right

Figure 3. Group 2's embodied model for h and inverse relation "rotating multi-body string formation"

Activity 2: Mathematizing Embodied Models to Inform Definition and Conjecture Drafts

Activity 2 was designed for students take stock of their embodied models. The first part prompted students to "make conjectures about what properties f must have for f^{-1} to be a function." This task was intended to engage students in horizontal mathematizing by explicating their informal ways of reasoning about one-to-one and onto grounded in their embodied models. TR could then leverage their embodied reasoning and introduce formal mathematical language to refer to their ideas. Next, students had to "fill in the blanks" in these definitions with either one-to-one or onto: (1) Let $f: X \to Y$ be a function. We say f is ___ when we have if $f(x_1) = f(x_2)$, then $x_1 = x_2$; (2) Let $f: X \to Y$ be a function. We say f is ___ if and only if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. This encouraged students to further mathematize their embodied models as they identified mathematical symbols to represent their enacted properties. The last prompt asked students to formalize their conjectures as theorems. This engaged students in vertical mathematizing as they explicated relationships between a function's properties of one-to-one and onto and the inverse relation's properties of well- and everywhere-definedness. The class's negotiation of the theorem drafts concluded the guided reinvention of the target theorem.

Horizontal mathematizing: Taking stock of prior embodied models. During class discussion, TR asked about issues with the inverse relations. G1 explained that f^{-1} is not a function because "5 had two outputs" which violates well-definedness. G2 stated "our 6 doesn't have a partner [for g]," so g^{-1} is not a function. For h, G2 explained, "When we did the inverse to it, we got it was not well- or everywhere-defined, which meant the inverse relation was not a function." For α, TR mentioned that α^{-1} was a function, and G2 described this as "everyone has a partner and only one partner." TR encouraged more explicit language, "that's what well-

defined and everywhere-defined means, but now it's more special than that." Students drew from their embodied formations to formalize their language. They had a spatial sense for one-to-one and onto but were not familiar with the formal definitions. As they attempted to formalize their language, TR revoiced students' contributions: "We cannot have repeated partners." She posed the question, "What about the issue of everybody having a partner?" Revoicing again TR stated, "Everyone has a partner, and it's a unique partner" and introduced the term, one-to-one.

Vertical mathematizing: Defining, conjecturing, and reinventing target theorem. TR shared G1's conjecture that a function being one-to-one is sufficient for the inverse relation to be a function. Student E in G1 shared that g is a counterexample because it is one-to-one, but g^{-1} is not everywhere-defined. Drawing on their embodied model [gestured three fingers on each hand], E revised the conjecture and added that there "has to be the same amount of inputs as there are outputs." TR rephrased their contribution, "everything in the codomain needs to have a buddy." TR then introduced the term onto. To encourage symbolizing and formalizing, TR provided a fill-in-the-blank definition box. Students matched their notions of one-to-one and onto to the formal symbolic definition to fill in the boxes. Student F in G1 explained, "if it's one-to-one, there would be one input for an output and if it's onto, everything in the codomain has to have a partner, so that would be everywhere-defined." TR asked students what the "it" referred to. She wrote templates of students' conjectures and asked them to fill in the blanks: *If _ is one-to-one, then _ is well-defined. If _ is onto, then _ is everywhere-defined. If _ is one-to-one and onto, then _ is a function.* Student B said the first blank "has to be the function." TR followed with, "if f is one-to-one, then what is well-defined?" Student D replied "f inverse." TR then transitioned to the onto conjecture. She wrote student G's conjecture that *If f^{-1} is onto, then f is everywhere-defined*, and asked for input. Student B said, "I think he's okay, but the last one, would it be *If f is one-to-one and onto, then the f inverse is a function.*" TR validated this theorem draft and asked about the puddle diagram of f, "is that onto? [Students: "yes"] and is f inverse everywhere-defined?" [Students: "yes"]. She validated G's conjecture, "We could have f inverse or f, but from what we noticed in our little diagrams if f is onto, f inverse is everywhere-defined." TR wrote the class's target theorem: *If f is one-to-one and onto, then f^{-1} is a function.* This concluded the class's reinvention of the target theorem.

Discussion of Preliminary Findings and Conclusion

We shared our preliminary LIT on students' guided reinvention of the target theorem (refer to intro). Students' progressive mathematization of their embodied models supported them to reinvent the target theorem. Thus, we recommend the use of embodied enactments of puddle mapping diagrams to reason about function properties. Transitioning from horizontal to vertical mathematizing seemed difficult for students in this study. We suggest that more research should identify how tasks and instructors can scaffold that transition for students. A contribution is crossing design principles (RME and embodied cognition) to arrive at an instructional design framework for embodied reinvention. By leveraging this hybrid framework, our LIT contributes to the literature on instructional design and the teaching and learning of (invertible) functions. Future research could test the efficacy of this intersection in other design research studies. This is a preliminary task sequence which we plan to implement again as a new iteration of the design cycle to refine the tasks. Our revisions involve incorporating examples of functions with infinite domains/codomains and having students refine their conjectures to capture the bidirectional implication the target theorem could have. We seek advice from the audience on the task sequence and our merging of RME and embodied cognition.

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Constructive or Procedural? Student Engagement With an AI Tutor in Multivariable Calculus

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Large language models (LLMs) such as ChatGPT are increasingly used in higher education, raising questions about how they influence student learning. While unstructured use often leads to shallow engagement, carefully designed AI tutors may foster deeper reasoning. This study examines student engagement with an AI tutor in a multivariable calculus course, drawing on the ICAP framework. The focal task asked students to use dot products and vector reasoning to estimate storm velocity from radar data. Preliminary results suggest that while the AI tutor frequently posed prompts aligned with constructive engagement, student responses were most often active, reflecting a "completionist" approach to problem solving. Constructive reasoning was difficult to sustain, and tutor follow-up was sometimes limited. These findings highlight both the promise and the complexity of designing AI tutors to elicit and maintain constructive mathematical engagement.

Keywords: Large Language Models, AI Tutoring, Student Engagement, ICAP Framework, Multivariable Calculus

Literature Review

Large language models (LLMs) such as ChatGPT have quickly become part of higher education, with students and instructors experimenting with their use for studying, writing, and problem solving. Stakeholders describe LLMs as convenient, confidence-building, and accessible, but also express concerns about over-reliance, inequity of access, and diminished opportunities for peer or instructor interaction (Govender, 2023; Hasanein & Sobaih, 2023; Kasneci et al., 2023). Systematic reviews in mathematics education confirm that AI tools, especially chatbots, are increasingly common across secondary and higher education, though most work is exploratory or tool-focused rather than analyzing learning processes in detail (Awang et al., 2025; Almarashdi et al., 2024; Walkington, 2025).

Empirical studies present mixed evidence on the effectiveness of LLMs. On one hand, research suggests that unstructured use of ChatGPT may reduce cognitive engagement and lead to shallow learning. For example, MIT researchers found that students relying on ChatGPT for essay writing showed reduced neural engagement and poorer recall compared to those using search engines or working unaided (Kosmyna et al., 2024). Similar results emerged in inquiry-based tasks: students using ChatGPT reported lower cognitive load but also produced weaker justifications (Stadler et al., 2024). In physics education, students frequently accepted incorrect ChatGPT solutions uncritically and performed worse than peers using a search engine (Krupp et al., 2023). These findings underscore the risk that LLMs may encourage passive or procedural engagement unless structured to promote deeper reasoning.

On the other hand, carefully designed AI tutors show promise. In a recent randomized controlled trial, Kestin and colleagues (2025) demonstrated that students working with a custom AI tutor, TeachGPT, achieved significantly greater learning gains in less time than students in active-learning classrooms. Similarly, Bastani et al. (2024) found that while unstructured ChatGPT use could hinder later exam performance, a safeguarded "GPT Tutor" version, engineered to provide incremental hints and anticipate misconceptions, avoided these negative effects. A meta-analysis of 51 experimental studies similarly concluded that ChatGPT can

substantially improve student performance and moderately support higher-order thinking, particularly when embedded in problem-based learning (Wang & Fan, 2025). In mathematics, ChatGPT-generated hints have produced learning gains comparable to those authored by human tutors, although accuracy concerns remain (Pardos & Bhandari, 2024). These findings suggest that the context and design of AI use are critical determinants of effectiveness.

One useful lens for understanding this contrast is tutoring research. Decades of work on human tutoring demonstrates that learning gains are often attributable not to tutor expertise but to the ways tutors elicit student thinking. Graesser et al. (1999) found that even untrained tutors often engaged students through feedback, hints, pumps, and elaborations, and that these discourse moves, not tutor expertise, were key features of effective tutoring. Chi et al. (2001) likewise found that tutoring effectiveness was driven by students' constructive responses, not tutor explanations. More recent innovations with AI tutors build on these insights, leveraging scaffolding strategies such as flipped dialogic approaches (Pavlova, 2024) or structured prompting frameworks (Tassoti, 2024) to push students beyond copying answers toward reflection and justification. Together, this body of work positions AI tutors not as replacements for human instructors but as systems that can be designed to elicit particular kinds of engagement.

Theoretical Framework

To analyze engagement systematically, this study draws on the ICAP framework (Chi, 2009; Chi & Wylie, 2014). ICAP proposes a hierarchy of engagement modes: Passive (receiving information), Active (manipulating or applying given information), Constructive (generating new inferences or explanations), and Interactive (co-constructing knowledge with peers). Research consistently shows that constructive and interactive activities lead to deeper learning than active or passive ones. In tutoring contexts, the constructive level is particularly salient: Chi et al. (2001) found that students' constructive responses predicted gains regardless of tutor expertise. Applications of ICAP to AI use likewise suggest that generative tools can promote deeper learning when designed to elicit constructive contributions (Gao et al., 2024).

We determined that in our context, interactive engagement is not feasible, since AI-tutor sessions involve a single student. We therefore adapt ICAP into three categories, Passive, Active, and Constructive, to code both AI prompts and student responses. Passive responses include minimal or off-task engagement (e.g., "idk," blank entries, or repeating AI text). Active responses recall facts, carry out straightforward operations, or follow procedures without interpretation. Constructive responses involve inference, justification, or proposing new ideas, such as setting up equations or reasoning about vector direction. We applied the same categories to AI prompts, distinguishing between explanatory (Passive), recall/procedural (Active), and generative/reflective (Constructive) tutor moves. This adaptation allows us to investigate how the design of AI prompts relates to the depth of student engagement.

Finally, while there is growing research on AI in mathematics education, most studies remain focused on procedural domains. ChatGPT explanations of multiplication, for instance, often reduce to repeated addition rather than deeper conceptual models (Hankeln, 2024). Studies on proof assistance emphasize correctness and formal revision rather than conceptual reasoning (Park & Manley, 2024). In calculus, student surveys report benefits of ChatGPT as a supplementary explainer but also concerns about accuracy and over-reliance (Serhan & Welcome, 2024). Reviews of AI in mathematics education confirm that most studies emphasize procedural problem solving, tool adoption, or broad perceptions (Almarashdi et al., 2024; Awang

et al., 2025; Walkington, 2025). In contrast, very few investigate how AI tutors can foster conceptual reasoning in advanced mathematics.

This study fits in that gap by analyzing AI tutor logs from a multivariable calculus course to investigate how the phrasing of AI prompts shapes the depth of student engagement on a conceptual vector problem. Using an ICAP-based coding scheme, we classified both AI prompts and student responses as Passive, Active, or Constructive, allowing us to examine the relationship between tutor design and student engagement. In doing so, our work extends prior research that highlights both the promise and pitfalls of LLMs: when carefully designed, they can foster individualized learning and constructive reasoning, but in unstructured use they often default to procedural explanations and promote shallow engagement. At the same time, most studies of AI in mathematics education emphasize procedural tasks such as algebraic manipulation or proof revision, leaving open the question of how AI tutors might support conceptual reasoning in advanced courses.

Guided by this framework, we investigate two research questions:

1. What types of AI prompts lead to the most constructive student-generated responses?
2. To what extent do students demonstrate active or constructive engagement in AI-tutor sessions?

Methods

In this section, we describe the course context, participants, focal task, data collection procedures, and analytic approach.

Course Context

Data were collected during a multivariable calculus course offered at a selective private university in the US. The course enrolled approximately 400 students across multiple coordinated sections, each meeting three times per week with 25–30 students per section. In this course, "workshops" are weekly applied problem sets designed to highlight conceptual applications of multivariable calculus and real-world data. Traditionally, these workshops were completed in small groups with support from undergraduate course assistants. In Spring 2025, the course offered an AI tutor as an optional support, revised from an earlier Fall pilot to encourage shorter, more guiding prompts, delayed solution suggestions, and more frequent checks for understanding.

AI Tutor

The AI tutor platform used in this study, TeachGPT, was created by Dr. Greg Kestin at Harvard University. Prior to the course, the tutor was configured to be supportive and encouraging but not to provide direct answers. Instead, it was designed to elicit students' thinking and ideas by offering prompts, hints, and checks for understanding.

Participants

This study was approved by the Institutional Research Board (IRB). Students were invited to participate via a survey distributed at the end of class, which asked about their experiences with the AI tutor and whether they consented to have their chat logs included in research. Of the total course enrollment, 123 students consented (approximately 31%).

Mathematical Task

The focal problem required students to use dot product calculations to build a system of equations for estimating the velocity of a storm using radar data from two different stations. Students worked with graphics and real data from a published research paper. Rather than providing exact numerical values, the task asked students to approximate storm speed and direction by interpreting vector arrows and color-coded visualizations. Evaluation emphasized conceptual application and reasoning processes, not exact numeric answers.

Data Collection

TeachGPT logs all student interactions, which can be downloaded by the instructor. The 123 consenting students produced over 1,500 turns of dialogue with the AI tutor. Each interaction turn (one uninterrupted chat entry) was recorded in the log. For this analysis, we focus on Spring 2025 logs, though the platform was also used in Fall 2024.

Data Analysis

Using the ICAP framework (Chi, 2009; Chi & Wylie, 2014), each turn of dialogue was coded as Passive, Active, or Constructive. Both AI prompts and student responses were coded, and turns could receive multiple codes. Two authors independently coded all responses and then met to reconcile disagreements. Agreement was generally high; precise interrater reliability will be reported in future analyses. At the time of this preliminary report, approximately 50% of the data has been coded. Codes are described in Table 1 below.

Table 1. Adapted ICAP-based coding scheme for student and AI tutor utterances.

Code	Student Responses	AI Prompts
P (Passive)	Minimal or off-task response; no meaningful engagement	Provides explanation without asking for a response
A (Active)	Recalls a fact, performs a basic operation, or follows a known procedure without interpretation	Prompts for recall or straightforward use of given information
C (Constructive)	Makes inferences, proposes ideas, sets up equations, estimates, or justifies choices	Prompts student to generate, infer, reflect, or justify

In what follows, we report descriptive patterns from the coded sample, focusing on the distribution of student and AI codes and illustrating how different types of AI prompts were associated with varying levels of student engagement.

Results and Discussion

Of the 564 turns coded so far roughly equally split between the student (n = 275) and the AI-tutor (n = 289) we found that the Active code occurred relatively equally between the students (n = 144) and the AI-tutor (n = 136). The passive code was actually more commonly applied to the AI tutor (n = 148) than the students (n = 98). However the form this code took was distinctly different. In the case of the students this code most often represented some amount of disengagement from the work as they said they just “didn’t understand” or asked the AI to give

them the answer. The AI tutor got a passive tag when it provided substantial information to the student such as laying out a procedure for how to solve a portion of the problem. The passive code was often applied alone for the student but for the AI it often was coded alongside Active or Constructive. The largest difference was seen in the constructive code with the AI tutor (n = 146) receiving the code much more frequently than the student interactions (n = 35).

The preliminary analysis provides insight into the depth and character of student engagement in AI-tutor sessions. Constructive responses were relatively rare, with most turns coded as active and oriented toward carrying out procedures or recalling information. When constructive responses did occur, they were often followed by active turns rather than sustained elaboration, suggesting that constructive reasoning was more difficult to maintain over the course of an interaction. Many students appeared to approach the task with a "completionist mindset," focusing on moving efficiently through the problem rather than pausing to reflect or justify their ideas.

At the same time, the AI tutor frequently generated prompts that were well targeted to the mathematical content and designed to elicit reasoning. These prompts often invited inference, justification, or reflection, aligning with the constructive category of ICAP. However, constructive prompts did not always yield constructive responses. More often, students replied at the active level, providing correct but concise procedural steps rather than extended explanations.

Some challenges also emerged in the tutor's interactional pattern. The AI occasionally asked several questions in succession without fully engaging with student input, which may have limited opportunities for extended dialogue. In certain cases, it appeared to misinterpret student responses, resulting in exchanges that did not progress as productively as intended. These patterns would then constrain the tutor's ability to sustain constructive engagement, even when its questions were well designed.

Taken together, these findings suggest both the promise and the complexity of fostering constructive engagement through AI tutoring. The AI posed strong and targeted questions, yet students' responses did not consistently rise to the constructive level, and the tutor's responsiveness at times made it difficult to deepen reasoning. As a result, interactions often emphasized procedural progress more than conceptual exploration, highlighting opportunities to refine tutor design in ways that better support sustained constructive engagement.

Preliminary Nature of Report

Because this analysis is drawn from an ongoing project, the findings should be considered preliminary. We focus here on a single conceptual vector problem from one workshop, which generated over 1,500 turns of dialogue, of which 564 have been coded to date. This problem was selected because it emphasizes the kind of conceptual reasoning central to the course, provides a strong context for examining constructive engagement and we have film of students engaging with the problem during in-person workshops. As coding continues, our next steps include completing analysis of this problem and extending the comparison to video data from in-person workshop sessions, allowing us to contrast AI-facilitated and human-facilitated interactions. We invite feedback from the audience and engagement with the following question: what additional coding categories or analytic lenses could help capture engagement not fully addressed by ICAP?

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"But You Gotta Put In Work": The Difference of Groupwork

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We report on an extension of previous findings from a project aimed at improving equitable groupwork in the context of undergraduate proof-based courses through designing and implementing group-worthy tasks. This quantitative study reports on three group-worthy tasks implemented in two topology courses. We investigated how student interactions in small groups reflect participatory equity when relating to perceived academic status along with differences in participation patterns between the two classes. Results are indicative of the tasks having the effect of balancing talk turns regardless of status, disrupting status hierarchy, or neither; similar to what has been reported previously. These findings suggest that implementation of group-worthy tasks is not sufficient to ensure equitable participation in small groups and that normalized groupwork has no impact on the effectiveness of the utilized tasks.

Keywords: Status, Equity, Group-Worthy Tasks, Proof, Groupwork

There is a growing consensus that students in undergraduate math courses can benefit from active learning approaches (Freeman et al., 2014). Evidence shows that active learning techniques can lead to more equitable outcomes (see Laursen et al., 2014; Theobald et al., 2020), but these outcomes are not guaranteed. Research shows that inequities can persist in settings including proof-based courses where groupwork is central to the classroom culture (Brown, 2018; Ernest et al., 2019; Johnson et al., 2020). One possible source for these inequities may be how students are perceiving each other's status, that is, student assumptions about who is "smarter" or "better" at math influences participation during groupwork (Esmonde, 2009b).

Research has shown that the nature of groupwork tasks themselves can magnify or depress inequitable participation (Shah & Lewis, 2019). The structure and content of the task can create or hinder access to students contributing and engaging mathematically. A well-designed group-worthy task can help manage status-related dynamics that would otherwise worsen participation for some students (Cohen et al., 1999; Esmonde, 2009a). We are far from understanding the complex interactions between status dynamics, group-worthy tasks, and equity in undergraduate courses that incorporate active learning strategies (Adiredja & Andrews-Larson, 2017); this lack of understanding is exacerbated in upper-level math courses. This particular setting shifts students' center of attention to formal proof, which may consequently trigger status issues (Weber & Melhuish, 2022).

In this report we share an analysis of a subset of data from the STructuring Equitable Participation in Undergraduate Proof (STEP UP) project whose aim is to assist instructors in the design and implementation of group-worthy tasks in proof-based courses. Previously, we shared results from a single class leading to the development of conjectures about how different types of tasks structures may disrupt or amplify status issues (Ellis et al., 2025). In this preliminary report, we provide a comparison between the original results and an analysis of a different topology course implementing the same structures. We sought to answer the following research question: How does groupwork, participation, and status interact in group-worthy tasks in two

topology courses? Do we find differences in these participation patterns in a course that regularly engaged students in groupwork and one that was typically lecture?

Background and Framing

We conceptualize equity as when an individual in a group has the requisite resources to support their learning (Shah & Lewis, 2019). On the other hand, inequity is when power dynamics are structured in such a way that halts students' access to these requisite resources. These resources can come in many different mediums such as physical access to a shared workspace like a whiteboard or worksheet, the opportunity to contribute to groupwork whether written or verbal, or accessing another group member's ideas. While equity is multifaceted, for this study we are particularly interested in participatory equity. Participatory equity refers to the fair distribution of opportunities to participate, and the realized participation shared amongst all individuals in a learning situation (Shah & Lewis, 2019).

These inequities in group settings may emerge because of imbalanced power dynamics that are rooted in the difference in status among group members. Status characteristics are defined as a "social ranking that everyone agrees is preferable to have a higher rank rather than a lower one" (Cohen et al., 1999, p. 84). Academic status characteristics, like perceived mathematical ability, have been found to be particularly influential inside the classroom (Cohen & Lotan, 2014). These academic status characteristics can be influenced by much broader status characteristics such as gender, race, and ethnicity (Cohen et al., 1999; Ridgeway, 2018). Evidence points towards students with higher status having more opportunities to contribute and participate (Cohen & Lotan, 2014). Increased opportunities to participate leads to greater learning opportunities (Cohen & Lotan, 2014; Esmonde, 2009a).

In addition to the dynamics between students, the nature of the task can influence the extent that participation can be inequitable in a group setting (Chizhik, 1999, 2001). Borrowing from Complex Instruction, the use of group-worthy tasks has been shown as one possible strategy to balance status and participation (Cohen et al., 1999; Featherstone et al., 2011). A task is considered "group-worthy" if it meets the following criteria: mechanisms are put into place that ensure every individual in the group is learning and engaged, the task affords multiple approaches, and the task endorses a sense of interdependence between group members in a positive fashion (Cohen & Lotan, 2014). These tasks may potentially disrupt imbalanced participation brought on by status differences by affording more opportunities for students to engage in mathematically meaningful ways.

Methods

Participants and Procedures

This study draws data from a larger NSF-supported partnership between mathematics educators and mathematicians aimed at developing and testing group-worthy tasks in proof-based settings. The three task types focused on proof comprehension, proof comparison, and proof construction, respectively. For this paper, we are focused on the set of tasks developed for Topology. The comprehension task (Universal Property of Quotient Spaces Theorem) involved students exploring examples with specific responsibilities and then engaging in discussion of the proof. In the discussion, each group member took turns leading the discussion by asking questions to probe for meaning in the provided proof. The proof construction task involved groups of four students. First, they examined two examples and then constructed a proof of the claim that "The diagonal is closed if and only if X is Hausdorff." Every group member was given

a distinct role[1] and the details they needed to finish the assignment. The proof comparison task involved students working in pairs. Each student was given the opportunity to become an expert on a different version of a proof for the focal theorem (homeomorphism preserves connectedness). They then engaged in asking questions, comparing and contrasting, and analyzing the proofs and theorem.

The set of tasks were implemented in an undergraduate introductory topology course in Fall 2023 (Instructor A) and in Fall 2024 (Instructor B). The tasks were spaced throughout the courses. Both were taught at the same large research university in the United States. Instructor A used groupwork throughout their course. Instructor B typically lectured. We note that the proof comparison task was taught by a mathematician on the research team in Fall 2024, as they were substituting for Instructor B. As a result, this task may be influenced by confounding variables related to the change of the instructor.

Data collected included video and audio recordings of the students working in groups on the tasks. The audio was transcribed and cleaned to ensure accuracy and all statements were attributed to the correct student. We also collected data regarding demographics and students' perceived status towards the end of the course. Replicated from Cohen and Lotan (1995), students were provided with a list of students in their class and asked to circle any students perceived as "best at math." An overview of student level data can be found in Table 1.

Table 1. Participant's Student Demographics and Perceived Status

Fall 2023			Fall 2024		
Name (Status)	Gender	Race/Ethnicity	Name (Status)	Gender	Race/Ethnicity
Ariella (low)	Woman*	Hispanic*	Ashley (low)	Woman	White
Lily (low)	Woman	Hispanic	Ben (low)	Man	White
Samantha (low)	Woman*	White*	Carson (low)	Man	White
Zahir (low)	Man*	Hispanic*	Kassandra (low)	Woman	White
Brandon (mid)	Man	White	Olivia (low)	Woman	White
Joe (mid)	Man	Spanish	Aaron (mid)	Man	White
Kieran (mid)	N/A	South Asian	Adrian (mid)	Man*	Hispanic*
Luke (mid)	Man	White	Arturo (mid)	Man	Hispanic
N (mid)	Woman	Mexican	Joseph (mid)	Man	Asian
Pink (mid)	Woman	White, Hispanic	Pepe (mid)	Man*	Hispanic*
Alexis (high)	Woman	White	Alan (high)	Man	White
Julie (high)	Woman	White	Elaine (high)	Nonbinary	White
Kendra (high)	Woman*	Black*	James (high)	Man	White
Xander (high)	Man*	White*	Jane (high)	Nonbinary	White

*Indicates researcher assumption; all other data is student self-identified.

[1] (1) 'Linguist' assisted group with provided key definitions of Hausdorff, diagonal, and closed; (2) 'Zoologist' assisted group's understanding of provided "strange and exotic" topologies: trivial, discrete, finite-complement, and poset topologies; (3) 'Product Manager' assisted group with provided definition of the product topology; (4) 'Summoner' assisted group by recruiting additional resources for the group.

Data Analysis

The transcript data were analyzed to create two ratios related to participatory equity for each student, adapted from Reinholz and Shah (2018). The first ratio is the "Words Spoken Ratio." This ratio compares the total words spoken by a student compared to how many words they would have spoken if each member of the group spoke exactly equally. The "Talk Turn Ratio" was computed similarly, but this time was based on how many times a student spoke. One talk turn reflects a contribution until a new speaker contributes. A ratio greater than one indicates a student is participating more than expected if everyone contributed equally (e.g., a student is expected to contribute a quarter of the time in a group of four) and less than one indicates a student is contributing less than expected. The status survey data was analyzed by grouping students into quartiles based on how many peers identified them as "best at math." Due to little differences in the first two quartiles, they were combined into a single group. This resulted in three categories in which the lower third is classified as "low" status, the middle third as "mid", and the upper third as "high."

We then created a series of graphs with Words Spoken Ratio on the vertical axis and Talk Turn Ratio on the horizontal axis and data points colored by status (see Figure 1 below). We provide a descriptive analysis of these graphs to consider the degree that participation patterns from the Fall 23 implementation were consistent (or not) in the repeated Fall 24 implementation.

Results

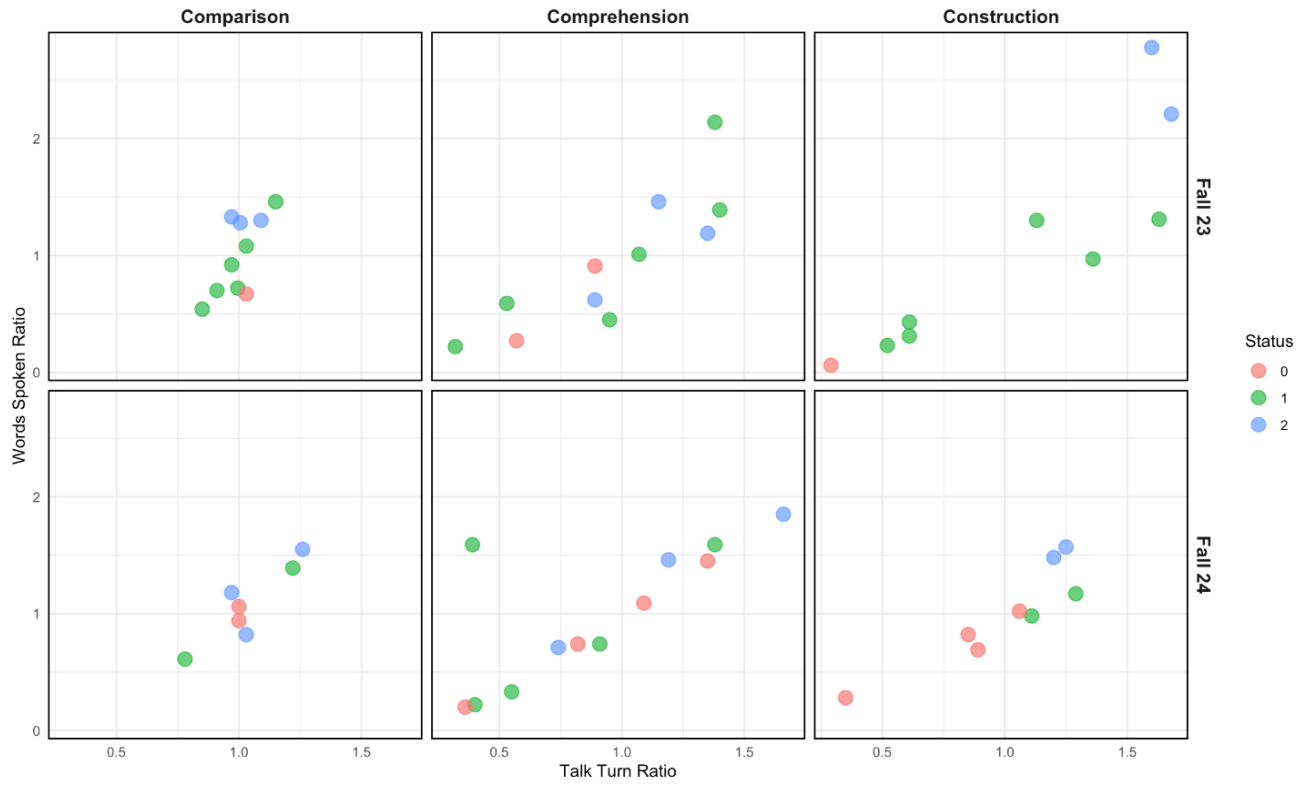

Figure 1. Correlations between Words Spoken Ratios and Talk Turn Ratios parsed by perceived status for each task for both classes.

When looking at these graphs, we want to attend to two features related to participatory equity. First, the closer all the ratios are to 1, the less differences there are in contribution ratios by student. Second, we want to see if there are clear orderings of status vertically and horizontally. There should be no pattern in a task that disrupts status hierarchy's relationship to status. In Figure 1, a task is said to not disrupt status if there is a hierarchical order from low to high status students along the vertical and horizontal coordinates. For example, the construction task in Fall 23 shows no disruption of status since the graph has high status students participating more and trickles down to the low status students.

In Ellis et al. (2025), we discussed the fact that the construction tasks seemed to maintain status hierarchy in participation both in terms of words spoken and number of contributions with the high-status students having the highest ratio of both. For the comprehension task, we noted that the status hierarchy was disrupted across both dimensions. Surprisingly, despite the substantial class differences for Instructor A and B, we see roughly the same outcome in both cases. The comprehension task with discussion prompts seems to be operating as intended. The construction task with group roles does not.

The comparison task was not as clear cut. This was a task with partner exchanges and comparisons across proof approaches. We claimed the implementation in Fall 23 did a good job balancing turns in talk, but not number of words. That is, the higher status students appeared to talk more than lower status students. In the Fall 24 implementation, we see the low status students are clumped right at 1. At this point, we can say that the structure of the task led to even turn taking in both classes, but the relationship with words spoken and status is less clear.

Discussion and Preliminary Nature of This Report

For this preliminary report, we expand upon our findings in the prior report by keeping the tasks and subject area constant with the instructor differing. We wanted to explore the degree to which we would see the same type of pattern across classes despite substantial deviation in the standard pedagogical approach to the class. We were surprised to find the degree to which the patterns in participation seemed to be replicated. This finding leads us to conjecture that the task features themselves may be the dominant variable for status disruption in a task. It seems that exploration and proof discussion prompts might disrupt status more than group roles in this context. This implies that a productive avenue of inquiry could be to test whether these patterns are consistent in other contexts. Additionally, we are interested in which task features contribute to these participation patterns. At this point, we have additional data gathered from abstract algebra, discrete mathematics, and real analysis. If we can demonstrate that it is simple changes to the task features that are disrupting status, then this may lighten the instructor's cognitive load when planning and creating classrooms that are more inclusive for students. Overall, we see promising implications for the teaching and learning of undergraduate proof-based courses.

We welcome feedback on ways we might account for the mathematical nature of student's contributions within our measures for participatory equity. Additionally, the current report was focused solely on participatory equity, but our team would benefit from feedback on how we can measure and quantify relational equity.

Acknowledgments

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Mapping the Demands of Thinking: Spatial Reasoning and Cognitive Demand in an Undergraduate Discrete Mathematics Textbook

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Spatial reasoning and cognitive demand are critical for mathematical learning, yet they are rarely examined together. This study analyzes 178 tasks from the graph theory chapter of Discrete Mathematics: An Open Introduction to explore how these dimensions present themselves and how they interact. Most tasks were of high cognitive demand and required spatial reasoning, especially spatial relations and visualization. Notably, spatial reasoning did not consistently appear as cognitive demand increased in tasks. These findings suggest no clear correlation between the cognitive demand of tasks and the spatial reasoning required.

Keywords: spatial reasoning, cognitive demand, task analysis

Introduction

Strong spatial skills demonstrate advantages for future STEM performance, and notably, there is extensive correlation linking spatial skill to mathematics performance (Gilligan-Lee, Hawes, and Mix, 2022). Spatial reasoning is often overlooked in mathematics curricula at all levels (Gilligan-Lee, Hawes, and Mix, 2022), despite the high demand for STEM skills and the proven malleability of spatial reasoning (Uttal et al., 2013).

Equally important are problem-solving and reasoning skills for STEM success (NRC, 2012). When tasks actively engage students at a high level, they help develop these essential skills. This study examines exercises in the widely used open-source textbook, *Discrete Mathematics: An Open Introduction,* to explore how spatial reasoning and cognitive demand are represented in tasks, aiming to identify patterns in how graph theory problems encourage different types of thinking. This research seeks to determine which spatial skills occur most often and whether their presence aligns with patterns of cognitive demand. This analysis contributes to ongoing discussions about the importance of task design in undergraduate mathematics education and highlights spatial reasoning as a valuable component of mathematical learning.

Literature Review

Spatial Reasoning and STEM Outcomes

Spatial reasoning is the ability to understand, visualize, and manipulate objects and their relationships in space. Spatial reasoning is often thought of in terms of three connected parts: spatial visualization, spatial relations, and spatial orientation (Lohman, 1979). *Spatial visualization* is the ability to mentally construct or interpret a visual representation from a diagram or description. *Spatial relations* refer to understanding how elements are arranged and connected within a visual structure. *Spatial orientation* requires mentally changing your point of view to see an object from a different angle. Research has shown that spatial skills are highly malleable, and small amounts of training can improve spatial reasoning in children and adults (Uttal et al., 2013). Despite its importance in STEM fields (Wheatley & Abshire, 2002; Newcombe, 2010), spatial reasoning remains underemphasized in classroom practice.

Cognitive Demand and Task Complexity

Mathematics education researchers often examine how task design affects the depth of students' thinking (Arbaugh & Brown, 2005). Stein and Lane (1996) emphasize the importance of "starting with high-level, cognitively complex tasks if the ultimate goal is to have students develop the capacity to think, reason, and problem solve" (p.344). Building on this, Stein and Smith (1998) introduced a four-level taxonomy for classifying mathematical tasks based on cognitive demand. In this taxonomy, tasks are categorized as either lower-level demands or higher-level demands. Stein and Smith argue that although choosing a high-level task does not guarantee student engagement at a high level, it is a necessary prerequisite for it (1998).

Intersecting Spatial and Cognitive Demands

Although both cognitive demand and spatial reasoning are recognized as critical, few studies examine their overlap. Cognitive demand of math tasks has often been categorized using Smith and Stein's framework (1998) (Jones & Tarr, 2007; Sapkota and Bondurant, 2024) while spatial reasoning research has focused primarily in K–12 settings (Battista, 2007; Younger, 2018; Hawes, Gilligan-Lee, & Mix, 2023). Few, if any, studies have examined their overlap, and none have systematically addressed this in undergraduate mathematics. Given this gap, undergraduate discrete mathematics provides a useful setting to explore how spatial reasoning and cognitive demand intersect.

The Case of Undergraduate Discrete Mathematics

Discrete mathematics courses often ask students to construct and navigate graphs, networks, and combinatorial structures, activities that are inherently spatial. Graph theory is a core component in discrete math curricula and is a spatially rich domain. Unlike geometric contexts, spatial reasoning in graph theory often involves abstract structures that require students to reason about connectivity, adjacency, and relational structure. An analysis of the graph theory chapter of Levin's *Discrete Mathematics: An Open Introduction* explores which visual-spatial skills appear most frequently, how they align with levels of cognitive challenge.

Research Questions

This study of spatial reasoning and cognitive demand of tasks in *Discrete Mathematics: An Open Introduction* was guided by the following research question: *How are levels of cognitive demand distributed across tasks requiring different types of spatial reasoning and task types in Discrete Mathematics: An Open Introduction?* To address this, the study focused on three sub-questions:

1. How are levels of cognitive demand distributed across task types (e.g., preview activities, in-text exercises, end-of-section problems, etc.)?
2. How does the nature of spatial reasoning vary across these task types?
3. Are there patterns that link levels of cognitive demand with specific types of spatial reasoning required in these tasks?

By investigating these questions, the aim is to understand how spatial reasoning, task type, and cognitive demand interact in this text. These goals shape the study design and analysis outlined in the following methods section.

Methods

This study focused on a chapter introducing one core discrete topic, graph theory, to identify opportunities for students to engage in high cognitive demand tasks and spatial reasoning. *Discrete Mathematics: An Open Introduction* was chosen because it's a free, open-source

resource that's "especially well-suited for courses that incorporate inquiry-based learning" (Levin, n.d.) and is widely used as a primary or supplemental text in colleges and universities. The graph theory chapter was selected for this analysis because it appears early in the textbook (Chapter 2), making it easier to determine what topics students know when approaching each task. Additionally, given that spatial processing plays a key role in understanding complex graphs (Trickett & Trafton, 2007), this section provided a fitting context for this study.

All tasks in Chapter 2, except the summary, were analyzed. After examining results holistically, results were disaggregated by the type of problem: investigate, preview activities, in-text examples, reading questions, end-of-text exercises, and additional exercises. Initial coding assigned each task to the type of problem, and tasks were left out if they did not contain a question or ask students to do something. Tasks were the unit of analysis, and overall, n=178 tasks were examined. Grouping by task type, two frameworks were adapted into deductive coding schemes to measure the level of cognitive demand and type of spatial reasoning required for each task.

To identify cognitive demand, each task was coded based on Smith and Stein's Levels of Demand framework (1998). Tasks were identified as lower-level demands or higher-level demands. Lower-level demand tasks were broken into the categories of *memorization* and *procedures without connections*, while higher-level demand tasks were broken into the categories of *procedures with connections* and *doing mathematics*.

Next, a coding scheme was developed to identify the spatial reasoning required for each task. Each task was coded for the type(s) of spatial reasoning required, if any, using Lohman's (1979) framework, resulting in the following codes: *spatial visualization, spatial relations*, *spatial orientation,* and *no spatial reasoning*.

As many tasks contained more than one type of spatial reasoning, codes were treated as mutually exclusive combinations. For example, 'SV' refers to tasks requiring only spatial visualization, while 'SV & SR' includes tasks requiring both spatial visualization and relations. This led to eight mutually exclusive codes, clarifying both which types of spatial reasoning were most frequently used and how often students would need to integrate multiple forms of spatial thinking to complete a task. Figure 1 shows the classic Königsberg bridge problem. This task was coded as *doing mathematics*, since students must explore and reason to solve this task. This task was also coded as *spatial relations and visualization*, as students must understand how the islands and riverbanks (nodes) are connected by bridges (edges) while also mentally tracing and manipulating paths.

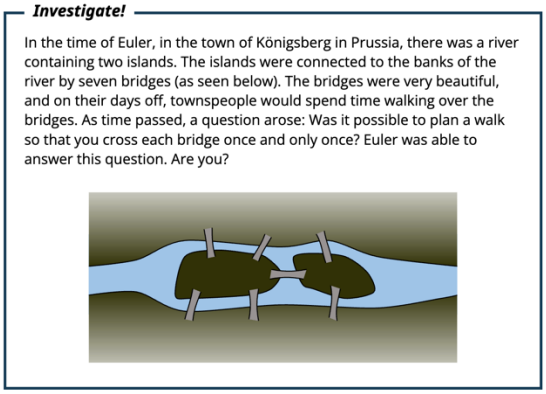

Figure 1. Task coded as doing mathematics and requiring spatial relations and orientation

Findings

Research Question 1: Cognitive Demand of Tasks

This study examined the cognitive demand of 178 tasks in Chapter 2. About 85% were of high cognitive demand (54% *procedures with connections*, 32% *doing mathematics*). The remaining 15% were *procedures without connections*; none were *memorization.*

Disaggregated by task type, most tasks were additional exercises (47%), and nearly all of these were of high cognitive demand, with a roughly even split *of doing mathematics* and *procedures with connections* tasks. End-of-text and reading questions leaned toward *procedures with connections,* while investigate tasks were exclusively high-level, mostly coded as *doing mathematics*.

Research Question 2: Spatial Reasoning Required of Tasks

Of the tasks examined, 67.4% required spatial relations (SR), 64% required spatial visualization (SV), 15.2% required spatial orientation (SO), and 28.7% required no spatial reasoning (NS). Note, that this does not add up to 100%, as some tasks were coded as requiring more than one type of spatial reasoning. Most tasks required some type of spatial reasoning (71.3%), with SR and SV showing up most frequently. Almost half of the tasks that required no spatial reasoning occurred in Section 2.6: Relations and Graphs. Tasks requiring SV and SR together occurred most frequently, with this pair present in 46.6% of all tasks (Table 1). Across task types, spatial reasoning appeared most in investigate and preview tasks (over 85%), somewhat less in additional exercises (76%), and least in end-of-text problems (63%).

Table 1. Mutually exclusive spatial reasoning codes with counts

Spatial Reasoning Only (SR)	Spatial Orientation Only (SO)	Spatial Visualization Only (SV)	SR & SV Only	SR & SO Only	SV & SO Only	SR, SV, & SO	No Spatial Reasoning (NS)
n=13	n=0	n=4	n=83	n=0	n=3	n=24	n=51

Research Question 3: Relationship Between Cognitive Demand and Spatial Reasoning

Spatial reasoning appeared across all cognitive demand levels but was most frequent in *procedures with connections* tasks (77%). *Doing mathematics* tasks required it less (63.2%), and *procedures without connections* the least (46.2%). Again, SV & SR were the most common combination across levels.

By task type, investigate tasks were all high-level, with almost all of these high-level tasks requiring spatial reasoning. Preview tasks also required spatial reasoning, while in-text examples and reading questions had many tasks without spatial reasoning. End-of-text exercises mixed cognitive demand and spatial reasoning. Interestingly, additional exercises showed higher spatial reasoning in *procedures with connections* tasks than in *doing mathematics* tasks.

Discussion and Future Work

Rather than evaluating the quality of tasks or making claims about student learning, this analysis documents how spatial reasoning and cognitive demand co-occur in graph theory tasks, positioning the textbook as a resource for understanding how mathematicians structure opportunities for spatial thinking. Across all task types, a clear pattern emerges in the data: **s**patial reasoning and high cognitive demand are both emphasized, but they are not always

coupled. Further, moving from *procedures with connections* to *doing mathematics* does not indicate that there will be more spatial reasoning required.

Overall, most tasks analyzed were of high cognitive demand, which was expected as this book is inquiry-based. Notably, tasks that appeared at the beginning of each section, investigate and preview activities, were almost all high cognitive demand, aligning with Smith and Stein's recommendation of "beginning with a task that has the potential to engage students at a high level if your goal is to increase students' ability to think and reason" (1998, p. 347). There was a difference in the breakdown of cognitive demand for end-of-text and additional exercises, both of which appear at the end of each section. Additional exercises included a significantly higher proportion of *doing mathematics* tasks, suggesting that assignments drawn from either the initial end-of-text exercises or the additional exercises could vary substantially in cognitive demand.

Spatial reasoning was present in many tasks (71.3% overall), but its distribution among tasks was uneven. Overall, spatial visualization and relations appeared most frequently, and were often grouped, as tasks that required students to think about how aspects of graphs related also required them to visualize the graph. Notably, almost all tasks that appeared at the beginning of each section, investigate and preview activities, required spatial reasoning, and additional exercises showed a slightly higher percentage of tasks that required spatial reasoning than end-of-text exercises.

Regarding the relationship between cognitive demand and spatial reasoning, spatial reasoning did not consistently increase with higher levels of cognitive demand. While many tasks coded as *procedures with connections* involved spatial thinking, spatial reasoning was less frequently seen in tasks categorized as *doing mathematics*. Many of these were focused on proof-based reasoning, where the cognitive demand was of the highest level, but spatial reasoning may not have been necessary. Though there was no clear relationship between cognitive demand and spatial reasoning, the data suggest that there may be a relationship between spatial reasoning and the type of task.

The data show two important findings: (1) spatial reasoning is not automatically embedded in mathematically rich textbook tasks, and (2) spatial relations and visualization are especially prominent forms of spatial reasoning in graph theory tasks. As spatial reasoning is highly malleable (Uttal et al., 2013), these findings can inform pre-college preparation or in-class scaffolding to equip students with the spatial skills necessary for success. These patterns also raise questions about whether selecting high-level tasks alone is sufficient to ensure consistent opportunities for students to engage in spatial reasoning, a hypothesis that warrants further study with student data.

Preliminary Nature of This Report

This report represents an early stage of analysis and is limited to a single textbook chapter with coding conducted by one researcher. Because this study analyzes textbook tasks rather than student work or classroom enactment, claims about how students engage in or develop spatial reasoning are necessarily inferential. Future directions could include expanding the analysis to additional chapters, additional textbooks, additional math content areas, and incorporating interrater reliability. I am especially seeking feedback on two areas still in development: (1) how the coding frameworks for spatial reasoning might be refined to capture nuances in undergraduate mathematics tasks, and (2) how the implications could be framed to best support both pre-college preparation and in-college scaffolding.

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Mathematical Content in Physics for Life Sciences Homework

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This study presents a task-based content analysis of homework problems from a year-long, calculus-based physics sequence designed for life science students. Using a codebook created by adapting ideas from the Calculus Content Framework (CCF), we coded each homework task for the types of mathematical reasoning required, including derivatives, integrals, manipulating algebraic expressions, geometric reasoning, arithmetic calculation, and graph interpretation. Across the year, we found that manipulating algebraic expressions, arithmetic calculation, and geometric concepts appeared more frequently than calculus procedures such as derivatives or integrals. These findings align with prior work suggesting that algebraic skills and modeling are more central than calculus topics like derivatives or integrals in many physics tasks. As a preliminary study, this work invites feedback on how to build toward broader analyses, such as alignment with modeling-first calculus courses, with the long-term goal of better supporting students in navigating mathematics across disciplinary boundaries.

Keywords: Calculus; Physics for Life Sciences; Task-Based Content Analysis

The overwhelming majority of students who take a calculus class in college are doing so as a prerequisite to other courses they are required to take to pursue their major. These chemistry, biological science, physics or engineering courses require the knowledge of calculus as a foundation of their discipline and courses. However, students' ability to transfer knowledge learned in their math courses to other subjects has long been a challenge.

We also see challenges in the mismatch of requirements for certain majors and career trajectories, such as toward medical school which generally encourage students to take a full year of physics, but only one semester of calculus (Ceder & Harrington, 2024; Columbia University – School of General Studies, n.d.; Rutgers University Health Professions Office, n.d.; University of Connecticut Pre-Medical and Pre-Dental Advising, n.d.). While these students do not usually continue taking calculus past the required courses, many of their other courses do require knowledge of calculus. To prepare students for medical school, universities might create versions of calculus or physics courses to specifically serve a population of students such as biology or pre-med majors which are tailored to that student population. These classes are especially important because research has shown that students' comfort with and interest in using mathematics to understand biology varies widely and is lower for women and first-generation students (Andrews & Airens, 2018). This project was motivated by a desire to understand the math present in a physics series specifically targeting biology or premed majors to assess how well prerequisite math courses prepare life science students for their subsequent coursework.

We hypothesize that misalignment between different courses might impede students' understanding and success (Ayene et al., 2010; Redish & Kuo, 2015). In particular, Bowers et al. (2023) emphasize the distinction between mathematical and scientific mastery, noting that students may successfully learn the necessary mathematics without achieving conceptual understanding in physics. Building on Redish and Kuo's (2015) work, we draw on their argument that success in physics depends not just on mastering mathematics, but on blending it meaningfully with physics reasoning. Their framework points to the need to consider how supporting students' use of math in a physics context may require reframing or extending what is

typically taught in mathematics courses. While this report shares our analysis of homework problems we intend to use it as a foundation to examine student understanding more directly in future work.

Generally the strength of students' mathematical background appeared to correlate with their success in these classes (Ayene, 2010; Morán-Soto, et al. 2023; Quimson, 2021). To begin the work of determining how to better prepare students for subsequent courses such as physics we conducted a systematic content analysis of the math skills required for the completion of homework problems in the physics classroom. By classifying each task by the type of math skills it required, we aimed to show the specific mathematical emphases in these physics courses and how they relate to the math classes that preceded them.

We see this work as contributing to the understanding of the alignment or misalignment of math content in follow-on courses. We also see the need to consider how to support students to succeed in both their math courses and the courses they will go on to take later that make use of the math they learned. At the 27th RUME conference Robles and colleagues (2025) presented what they called the first study of math content in introductory physics using the calculus concept framework. We desire to build on their work with this examination of math content in an introductory physics course specifically targeting biological science majors. These desires led us to the research question for this preliminary proposal: What math content is present in a year long general physics class directed toward life sciences majors?

Literature Review

While there have been research studies targeting the different ways math is taught in physics courses, not many have targeted the physics courses for life sciences. Much of the research has provided insights into how math in physics courses might differ from math in math courses. González-Martín (2021) and Hitier and González-Martín (2022) found that there is a difference between the way integrals and derivatives are taught and used in physics and math courses. Some authors even go as far as to say that the math taught in math courses is inherently different from the math in physics courses but can be bridged if math courses emphasized the ways to apply the tools math provides (Rabin, et al. 2025; Redish, 2006; Redish & Kuo, 2015).

There have also been studies that compare students' performance in STEM classes and how it correlates with their performance in math courses. In fact, three in particular concluded that it seems that students who have higher performance in math courses also perform better in their subsequent STEM courses (Ayene, 2010; Morán-Soto, et al. 2023; Quimson, 2021). In a different study of how math background correlated with results in Physics 1 and 2 Burkholder and colleagues (2021) found that while math background was not a predictor of Physics 1 final exam scores, having completed vector calculus was a predictor of success in Physics 2. They found this appeared to be primarily due to better reasoning about vectors and geometry, not specifically vector calculus skills.

Projects that analyze the math content in STEM courses find few uses of the math learned in prerequisite calculus courses (Faulkner, et al. 2020, Robles, et al. 2025). Ayene (2010) puts this in a greater context by comparing the math content in physics courses throughout a physics program and whether they align with content from each prerequisite math course and determines that there is in fact misalignment in math content within this sequence of courses. There appear to be few studies that look at the broad use of math in introductory physics courses specifically. Our project continues the research seen in these papers such as that completed by Robles and colleagues (2025) through our analysis of the mathematical content in a physics sequence designed for life sciences.

Analytic Framework

To analyze the mathematical content in subsequent physics courses we adopted a task-based content-analysis approach. We wanted to evaluate the math content to see how well calculus courses are preparing life science students for these calculus-based physics courses. We treated each subpart of a homework problem as a task that a student had to accomplish and used instructor solutions to determine which techniques were used in each task.

To determine the math content present in each problem we considered the definition of calculus from the Calculus Content Framework (CCF) (Sofronas et a., 2011). The framework breaks out a number of concepts and skills that calculus experts deemed part of first year calculus. Perhaps unsurprisingly the first few skills that showed up in their study were derivative computations, techniques of integration and limit calculations. After this is a longer list of items identified by 1-4 participants, these included: sequences/series, manipulating algebraic expressions, area/volume, parametric equations, polar coordinates, trigonometric manipulations, facility with logs and exponentials, epsilon delta proofs, listening/reading comprehension and facility with definitions and notations. After consideration of this list and initial review of the problems we were coding we decided to keep the first couple of categories but to bucket together, adjust or leave out several of the categories that came up among fewer of their participants and that did not appear relevant to the calculus problems. We also wanted to include a code specifically for arithmetic calculation which we saw show up a number of times in problems. In the next paragraph we define each of the codes we used to understand the math present in the physics homework.

For the concepts of *derivatives* and *integrals* we used Sofronas' and colleagues' (2011) discussion to develop short descriptions. We also included a code for *manipulating algebraic expressions* which encompassed solving for unknowns, rearranging equations, and symbolic manipulation (Puig & Rojano, 2004) and included the expression of physical relationships. This code also included two sub codes, *summation notation* and *delta notation*, which appeared frequently in the problems although generally used differently than is typical in calculus classrooms. The final bucket we considered was *geometry*, which included area/volume, spatial reasoning, diagrams, coordinate systems, and shape-based interpretation (Jablonski & Ludwig, 2023). This bucket also included *vectors* and *trigonometry* as specific child codes.

Finally, we added an emergent code, *graph interpretation*, for problems requiring students to extract quantitative relationships from visual representations, a skill recognized as distinct in both mathematics and science education research (Roth & Bowen, 2001). This skill also came up in the CCF in connection with derivatives, but many of the problems that appeared to involve graphical interpretation in this set did not necessarily involve derivatives.

Methods

To analyze the mathematics present in the physics sequence, we obtained the complete written homework problem sets and instructor-provided solutions for each semester. These solutions offered insight into the instructor's expectations and represented the problem-solving techniques students were likely expected to use. We conducted a task-based content analysis of the homework sets to identify the mathematical skills required to complete each problem. We also analyzed online homework sets that did not include instructor solutions. These questions were multiple choice or short numerical responses. While these tasks still provided some information, the absence of solutions and limited response options led us to question exactly what mathematical skills were required for the problems and to what degree they were required.

Each homework problem and its subparts were categorized as either requiring mathematics or not. In cases where mathematical work appeared in the instructor's solution but was not necessary to solve the problem, we applied an additional tag, “math in soln.” To code the math content in each problem we devised a codebook in line with the calculus concept framework, but adjusted as described in the theoretical framework above.

The codebook was implemented in Microsoft Excel, with problem numbers, brief descriptions, and math tags documented to ensure consistency in analysis. Each problem was independently coded by both authors, and disagreements were discussed and resolved through consensus. Initial agreement rates were high, and once coding was complete, we conducted an informal member check by sharing our coding with the physics instructor, who confirmed that our characterizations aligned with the intentions for the problems.

Findings

In classifying the homework problems, we found that the majority of problems required math skills with 63% of problems across the sequence coded as involving math content in some way (n=240 of 379). Similarly, in the online homework 74% of the problems were coded involving math content (n=221 of 297). We found that the majority of math content required of students as represented by the homework problems involved manipulating algebraic expressions but only a minority included calculus specific skills such as calculating derivatives or integrating.

The skill most commonly appearing in the written problems was manipulating algebraic expressions with it showing up in 77% of the problems that involved math skills (n=185). The second most commonly appearing code was arithmetic calculation which appeared in 50% of the math problems (n=121). Geometry was the third most common code occurring in 43% of problems (n=103) with its subcodes of vectors (n=54) and trigonometry (n=54) each appearing in 22% of problems. Derivatives came up in 8% of problems (n=19) and integrals in 4% of problems (n=9). The most common skill in the online homework problems was arithmetic calculation showing up in 76% of problems (n=167) and manipulating algebraic expressions was the second most common code showing up in 25% of the problems (n=62). Considering the format and intent of the online problems, these skills being most common makes sense; these questions are intended to check basic, often conceptual, physics skills newly introduced in the class sessions. Derivatives were only coded in 2% of the online problems (n=4).

Beyond the frequency of the problems we also noted in coding that summation notation (n=11) and delta notation (n=32) were used in ways not generally seen in calculus curriculum. In the physics curriculum summation notation was often used as a stand-in for writing “sum” when a problem asked for summing up a number of different forces for example. This use is distinct from how students usually see summation notation in calculus courses in the context of series. Although we only did a formal review for the homework assignments, we performed an informal review for the in-class session activity handouts and found more calculus there. The calculus involved derivatives, chain rule, integration, differential equations, and calculus in 3 dimensions. No analysis was done of section activities, exam study materials, or supplementary resources.

Discussion

Different Emphases

Our analysis of these homework problems demonstrates that the math skills students use most frequently in an introductory physics series like this one differ from what calculus experts deem the emphasis of their classes. This difference is not necessarily surprising as the focus of a

calculus class is the understanding and implementation of the mathematics content itself. Math teachers need to support math understanding for students with broad aspirations while in physics the math provides tools for understanding the content of that discipline. While not surprising, we think this finding is important because when a course is a prerequisite then students would expect that completing that course means they are prepared for subsequent courses that require it.

This work reveals however that for success with the mathematics present in a course like this a solid foundation in algebraic and geometric knowledge and skills may be more important than skills related to manipulating derivatives and integrals. Many students likely enter their calculus courses already with strong algebra and geometry skills. However, for the subset of students that do not have a strong background in algebraic manipulation there is no guarantee that their calculus course will address these skills. As the CCF shows, many calculus educators emphasize new content like manipulating derivatives and integrals, but not the previously mentioned skills.

This disconnect between the emphasis of calculus topics and possibility that students get through without addressing gaps in their algebraic manipulation skills is especially salient in courses targeting life sciences majors. Students in these majors, especially those selecting these targeted courses, may have a more negative relationship to mathematics than other STEM majors (Wachsmuth et al., 2017). With this population that may be interested in being veterinarians or doctors, we need to consider the best options to serve them mathematically. This work shows that within the context of a physics class specifically designed for this population, the most commonly used skills aligned more to the modeling aspect of calculus rather than advanced differentiation and integration techniques.

Directions for Future Research

Our findings align with prior research and show that a lot of mathematics work is required of students in physics classes and confirm why prior studies have shown alignment between students' math skills and success in physics coursework, even if content does not necessarily align (Ayene, 2010; Morán-Soto, et al. 2023; Quimson, 2021). We consider this work preliminary because our true interest is in how to better support students' transition and transfer from calculus to physics, a well known and complex challenge.

This study provided insight into the skills that are most commonly required for students to succeed in a physics course targeting life sciences majors. These results align with Robles and colleagues' prior study that showed that in introductory physics the most prevalent skill required is algebraic manipulation with trigonometry, vectors, and graphs coming up more frequently than integrals or derivatives even in a calculus based physics course.

Others have recognized this disconnect and created alternative forms of calculus such as modeling-first approaches to calculus for life sciences students which have shown some success (Garfinkel et al., 2022; O'Leary et al., 2021). In future work we would like to analyze problem sets from these courses and more traditional calculus courses to see how they compare. We also want to analyze more physics for life sciences classes to see trends in mathematical content. Furthermore, we would like to look at student outcomes or consider conducting cognitive interviews with students to look at what they know and how it translates.

Questions

As a preliminary report, audience engagement is much appreciated, and we offer questions. Would further content analysis likely bring new insight into knowledge requirements, or should additional methods, like interviews, be used? What suggestions would you have for us to deepen this study further and make it more relevant to how we can better support students?

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Women's Experiences Within the Communities of Practice of a Doctoral Mathematics Program: Insights Into Persistence

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For decades, women have consistently earned fewer than 30% of PhDs in mathematics in the United States each year. Advancement through a doctoral mathematics program can be viewed as a process of integration into the communities of practice within the department, and a student's persistence (or attrition) is influenced by experiences that enhance or inhibit their participation in the activities of each community. In this preliminary report, I discuss the experiences of six women in a doctoral program and their participation in the coursework, research, and graduate teaching communities of practice in their departments. In contrast to past studies that propose that these communities of practice are mutually exclusive, my participants' accounts illustrate some overlap between them.

Keywords: women, communities of practice, doctoral mathematics

Women have been underrepresented in the discipline of mathematics for decades, particularly in higher education settings. Data from the Survey of Earned Doctorates, conducted by the National Center for Science and Engineering Statistics (NCSES), shows that women consistently earned between 27%-29% of doctorates in mathematics and statistics each year between 2010 and 2020 (NCSES, 2024). Further, in the year 2020, non-white women accounted for only 8% of doctorate recipients in mathematics and statistics (NCSES, 2024). Since at least the 1970s, research dedicated to understanding the underrepresentation of women in mathematics and other STEM fields has revealed several cultural, institutional, and structural barriers to obtaining a graduate degree (Ceci et al., 2011; Roby, 1973). Research that focused specifically on the experiences of non-white women has highlighted additional layers of inequity that further pushes them to the margins (Malcom et al., 1976).

Background

Though much of the conversation has been focused on identifying barriers and highlighting inequities in higher education, some studies have also investigated factors that contribute to the persistence and success of women and women of color in graduate mathematics programs. For example, Borum and Walker (2012) found that among Black women who held PhDs in mathematics, early exposure to mathematics and participation in REUs prior to graduate school contributed to their interest and confidence, which helped them succeed. Among women who earned their PhDs in mathematics and secured positions at research institutions as tenure or tenure-track faculty, Miller (2015) found that two of the most important factors related to their success were strong undergraduate preparation in mathematics, and positive relationships with their graduate advisors.

Research addressing the persistence of women, and particularly women of color, in doctoral mathematics has several limitations. One is that many of these studies focus on women who have already completed their doctorates (e.g., Borum & Walker, 2012; Miller, 2015) which provides only retroactive accounts of their experiences. These women also attended different graduate institutions at different times, which makes it difficult to identify any political, structural, or institutional factors influencing their persistence. Additionally, differences in identity have

generally not been incorporated into the analysis of findings, which may unintentionally produce results that center the majority experience (in this case, that of cisgender white women) and erase or invalidate the experiences of minorities. In fact, the incorporation of multiple intersecting identities has often been limited to just race and gender. My study aims to fill this gap by focusing on womxn[1] currently enrolled in a doctoral program in the United States which graduates women at a rate higher than the national average, and by investigating their experiences through an intersectional lens (Crenshaw, 1989).

Theoretical Framework

Herzig's (2002, 2004) persistence framework for students in doctoral mathematics combines Lave and Wenger's (1991) theory of learning through participation in a community of practice with Tinto's (1993) model for doctoral persistence to describe the process through which mathematics doctoral students become academically and socially integrated into their departments and ultimately persist through completion of their graduate program (and thus become full participants in the community of practice of mathematicians). Herzig (2002, 2004) proposed that mathematics doctoral students encounter two sequential but disjoint communities of practice within their departments which they must participate in and become integrated into. These are the *course-taking community of practice* and the *research community of practice*, respectively. As these students become full participants in each, their likelihood of persisting through their doctoral program should increase. However, Herzig (2004) argued that this may not be as simple as moving through one community of practice and into the next.

> If students are to become full participants in the community of practice of mathematicians, they need access to all of the means of membership. … Instead, they are participating in a distinct course-taking community of practice, and their persistence at this stage depends on the degree to which they become integrated into this community. It is not clear to what extent this participation prepares them to participate in the community of practice of mathematics research they will encounter later (Herzig, 2004, p. 181).

In developing this framework, Herzig (2004) was particularly concerned with identifying factors that enhance or inhibit diverse students' access to the community of practice of mathematicians in order to understand how they become integrated into that community. She found that "while there are obstacles to participation and integration that all graduate students face, women and students of color face additional obstacles that limit their opportunities to participate fully and effectively in mathematics practice" (Herzig, 2004, p. 182).

At this juncture, it is worth noting that Herzig (2004) briefly acknowledged the existence of a third community of practice, that of graduate teaching assistants, though discussion of its relationship within the persistence model is absent. Perhaps this is because teaching assistantships aren't a requirement for obtaining a doctorate. However, many mathematics doctoral students, including all of the participants in my study, hold teaching assistantships for at least some portion of their program. Further, mathematics faculty are a large part of the community of mathematicians that doctoral students interact with during graduate school, so teaching is integral to the graduate students' overall practice as mathematicians. Therefore, I will incorporate the *graduate teaching community of practice* into my analysis.

[1] For the purposes of my study, I am choosing to adopt the definition of womxn from the Empowering Womxn of Color Conference, which "uses womxn to describe all who experience life through the lens of womxn in body, spirit, identity past, present, future and fluid." For more information, visit https://ewocc-berkeley.com/history/

While my study design and analysis incorporate intersectionality theory (Crenshaw, 1989), I want to note that this preliminary report focuses the discussion on communities of practice in doctoral mathematics. Intersectional analysis of the findings is in progress but lies outside the scope of the current paper.

Methods

I am conducting a thematic narrative inquiry (Riessman, 2008) to investigate the experiences of womxn currently enrolled in the same doctoral program in mathematics in the United States. The program was chosen because it shows a relatively high rate of women persisting through the doctorate, with over 30% of PhDs in mathematics awarded to women each year for several years (NCSES, 2024). Six womxn initially joined my study, but one discontinued her participation after completing the first set of tasks. Data collected from each individual includes three separate one-on-one interviews (conducted via Zoom), two personal artifacts, five reflective journal entries, and one focus group interview (to be conducted in person). Preliminary findings and discussion will draw from the first interview, the first artifact, and the first two journal entries.

The purpose of the first interview was to understand the participants' mathematical histories up to their decision to pursue a doctorate, and to get a glimpse into what their world as a doctoral student currently looks like. To accomplish the latter, each participant created a map of their doctoral world (their first artifact) that could include anything in their environment that has been meaningful to them during their time in the graduate program. The map was shared prior to the first interview so that questions about the map could be addressed. The interview protocol was largely open ended, beginning with the prompt, "Tell me the story of how you decided to pursue a doctorate in mathematics." Follow up questions inquired about influential people and inspiration, and their engagement with mathematics through time (coursework, research, conference participation, etc.). After obtaining the participant's mathematical history, we reviewed their map artifact together guided by the prompt, "Tell me about the places you included in your map." Follow up questions inquired more specifically about where their mathematical activity happens and where they choose to spend time. Each of the six interviews lasted between 60 and 90 minutes.

The five remaining participants completed their first journal entries within two weeks of their first interviews, and their second journal entries within two to four weeks of the first. Journal entries were structured, guided by prompts which asked about how they engaged in mathematics over the last couple of weeks, including questions such as "How did you share your own mathematical ideas? How did you feel they were received by others? Did your contributions become part of the mathematical conversation?" These journals are intended to capture the continuity of participants' experiences (Clandinin & Connelley, 2000; Riessman, 2008) with respect to their engagement with mathematics in the program over the course of a few months.

To analyze the data, I used a deductive coding scheme using the three communities of practice as broad category codes. Subcodes for each community of practice included "activities" which identified instances of activity or participation in the community, "factors enhance" which identified instances where a participant's experience facilitated integration into the community, and "factors inhibit" which identified instances where a participant's experience disrupted integration into the community.

Preliminary Findings

I found evidence of each of the communities of practice, described by Herzig (2002, 2004), that mathematics doctoral students are expected to encounter during their graduate program.

Each participant provided accounts in varying degrees of detail of their engagement with graduate coursework at their institution, research-related activities, and teaching activities.

Within the *course-taking community of practice*, important activities discussed by all participants, regardless of their current stage in the program, included taking a variety of courses, doing homework, and solving problems. One major factor that enhanced their participation in these activities was collaborating with their peers. Professors encouraged students to work together on their assignments, and the mathematics building provided ample space for doing so. Many participants spoke about their use of shared graduate student offices and openly accessible spaces in the hallways which had seating and whiteboards. During our discussion of her map, Sofya described one such space.

> *Sofya:* Our cohort stays very close to each other, so that – we do this on purpose – so we can like, work a lot together, and mine was the leftmost [office] in some sense. That's [the common table outside of the offices] where we sometimes just go out to all sit there and work, or they like to come to my office a lot. I realized that, like, we usually just end up with sharing one office. Because that's how - we were encouraged to have that. … Every professor will put like, 'You're encouraged to work together.' … I honestly think maybe we spend time together more in 1st year cause we were all in intro classes, and you know how frequent those homework came out. You need to work with each other in order to actually like, survive, sort of.

Having space to collaborate on homework, being encouraged to do so by her professors, and having easy access to others in her cohort all enhanced Sofya's experience and facilitated her integration into the course-taking community of practice in the department.

Though Herzig (2002, 2004) described the *course-taking* and *research communities of practice* as sequential and disjoint, I found that participants in their second and third years have all been able to access the research community of practice despite not having been fully integrated into the course-taking community of practice. For example, Anne was given her first research opportunity the summer before the start of her second year. In her first journal entry, she described working with a group of collaborators on specific tasks with the goal of writing a paper. In her second journal entry, she followed up by saying that she authored one paper and co-authored another by the end of the summer. This indicates that she was able to begin developing several skills needed to become integrated into the research community of practice of mathematicians.

Other forms of participation and activity in the *research community of practice* within the department are through seminars, colloquiums, and reading courses. Many of my participants mentioned attending seminars early on in their program, and the positive impacts this had on their trajectory.

> *Moose:* So I actually joined the seminar for my research group in my 1st year. It's the groups and topology seminar, and I found that to be a really helpful group. The professor, well, the professors behind that, two of which are my co-advisors, really wonderful at accepting people.

Moose was able to gain access to the research community in the area that she was interested in pursuing despite having just entered the program. Feeling accepted in that group further enhanced her experience and facilitated her integration into the research community of practice.

Interestingly, I also found some interaction between the *graduate teaching community of practice* and the *course-taking community of practice*. For example, when Maxine struggled in her first year with coursework despite having a collaborative relationship with peers.

Maxine: I started grad school during the pandemic, so I had, I think, like 2 or 3 friends that we had a little bubble with, and one of them was super high achieving – like, super duper fast, high achieving - and then the other one was also high achieving but did things a different way. And so, I was often just stuck in the middle, like 'I don't know what to do,' so definitely learning to ask for help. And, you know, you say this to our students so much, being like, 'Go to office hours.' *Going* to office hours [laughs].

Maxine described an isolating experience where she felt "stuck" between her two peers, who she positioned above herself as "high achieving" and who approached things differently, but the negative effects of this experience were mitigated by "learning to ask for help" during professors' office hours. In this case, her experience as a teacher who urges her students to attend office hours was reflected onto her own experience as a student struggling in a course and highlighted a path forward.

Discussion

Herzig (2004) argued that "the persistence of a doctoral student in mathematics, and the quality of her experience in graduate school, is a function of how well integrated she is in the academic communities of her department and discipline" (p. 201). My preliminary analysis investigated the experiences of womxn in different stages of their doctoral program to identify opportunities they had to participate in the mathematics community of the department, and factors that enhanced their engagement and integration into the *course-taking* and *research communities of practice*.

Previous research has found that competitive and individualistic environments isolate women from the communities of their departments (Herzig, 2002; Hollenshead et al., 1994; Stage & Maple, 1996). My analysis showed that peer collaboration was both encouraged and facilitated by aspects of the program environment, which promoted integration into the course-taking community of practice. One potential implication of this is that mathematics graduate programs which make concerted and intentional efforts to promote collaborative relationships among its graduate students will increase the likelihood of graduating their diverse students

During the first few years of graduate study, mathematics doctoral students are sequestered away from authentic practices of mathematicians while they focus on completing coursework and preparing for qualifying exams (Herzig, 2002). Herzig (2004) found that this structure denies students access to the *research community of practice* while also forcing them to spend significant time and energy developing skills which may not necessarily enhance their ability to perform mathematics research. However, my preliminary analysis suggests that the course-taking and research communities of practice overlap in this department. Students are offered summer research opportunities at the early stages of their program (even if they haven't yet passed qualifying exams), and they are also encouraged to attend research seminars and take reading courses in the department. I conjecture that this feature of the program plays a key role in the persistence of womxn pursuing PhDs.

Finally, my preliminary analysis suggests that participation in the *graduate teaching community of practice* may influence a student's ability to become integrated into the coursework community of practice. Attending to their students who are struggling may provide a mirror through which graduate students find ways to alleviate their own challenges and persist through their coursework. Further investigation is needed to determine whether and to what extent this is the case. I have the following questions for the audience: (a) Have you encountered graduate programs that offer students early research opportunities? (b) What other ways can graduate students be invited to the research community in their departments?

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The Alignment of Student Conceptions With Course Learning Outcomes: A Case Study on the Definite Integral

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Students pursuing science, technology, engineering, and mathematics (STEM) degrees require some proficiency in Calculus. The Fundamental Theorem of Calculus is arguably the most important theorem taught in a Calculus I course, yet it is not covered until the last weeks of the semester in a traditional three-part Calculus sequence. As a result, students tend not to hold deep conceptual understandings around integration, and the conceptions that they do hold are procedurally based. This paper illustrates the disconnection between student conceptions around integration (primarily area-based) and the general emphasis (from mathematics education literature) on accumulation. In addition, our work highlights how course learning objectives may drive a primarily area-based interpretation of integration. Our preliminary findings from interview data with Calculus III students confirm prior research, suggesting that students hold strong procedural and area-based conceptions, rather than accumulation conceptions.

Keywords: accumulation, definite integral, course learning objectives

Background

Extensive research has examined the diversity of student conceptions and challenges in understanding integration in introductory calculus (Jones et al., 2017; Serhan, 2015; Thompson & Silverman, 2008). As Thompson (1994) highlighted, "the realization that the accumulation of a quantity and the rate of change of its accumulation are tightly related - is one of the intellectual hallmarks in the development of the calculus." Some researchers have pushed for teaching integration with a pedagogical focus on "adding up pieces" to support productive conceptual development among learners (Jones & Ely, 2023). Even with this emphasis on accumulation, students often leave Calculus courses with a conception of integration as an antiderivative or a process for computing areas under or between curves (Jones, 2015; Nguyen & Rebello, 2011; Serhan, 2015; Wagner, 2017). In fact, students often learn about the Riemann sum as a means for calculating the definite integral, and later forget this procedure once they have learned the Evaluation Theorem (Jones et al., 2017).

It is not until the Calculus II course, at our institution, that students are exposed to applications of integral calculus. It is in these applications where having a conception about accumulation is more productive than simply holding an area conception. In Calculus III, students have opportunities to explore accumulation across multiple dimensions, though they may not always recognize it explicitly. Building on prior literature that documents this phenomenon, our research team aims to explore whether similar patterns emerge among students at our institution. We pose the following question:

1. What conceptions do Calculus III students hold on the definite integral?

Preliminary Work

The first research question was the primary subject of an undergraduate's (third author) summer research project. Upon completion of the data analysis for this work, the research team wondered about the relationship between the findings and the actual content that students learned

in their previous Calculus I and II courses at our institution. Thus, we pose the following question:

2. Based on the stated course learning objectives (CLOs) in Calculus I and II, what conceptions do we expect students to develop around integration after completing the first two courses in the three-part Calculus sequence?

In this preliminary report, we share findings from both research questions and pose further questions to explore. We intend to continue this work by systematically reviewing CLOs from various Calculus programs across the United States to observe how CLOs align with established perspectives around accumulation.

Methods

Author 1 and Author 3 participated in an independent study project where we set out to explore student conceptions around integration. During the first phase of the project, the research team performed a literature review and constructed an a priori codebook. The codebook was based on the work of Grundmeier et al. (2006) and Jones (2013), highlighting student conceptions that we anticipated students to hold around integration (see Table 1). Afterwards, the team developed a semi-structured interview protocol to engage with Calculus III students.

Table 1. A Priori Codebook of Student Conceptions of the Definite Integral.

Codes	Elaboration
Opposite	The student conceives the integral as the opposite of the derivative.
Area	The student conceives the integral as the area underneath a curve.
Infinite Sum	The student conceives the integral as an infinite sum of quantities.
Accumulation	The students discuss the idea of adding up, totaling, or accumulating areas. Without explicit reference to Riemann sums.
Procedure	The student only discusses procedures for evaluating a definite integral.

During the second phase of the project, Author 2 joined the research team. With IRB approval, the research team recruited ten students from our institution who were currently enrolled in Calculus III. Each participant participated in an hour-long video-recorded semi-structured interview, for which they were compensated with a $25 gift card to the campus bookstore. Each participant was asked eight Calculus I questions, where five were explicitly about integrals. Two questions were informal questions, posed as a means of capturing students' general ideas about integration (see Figure 1). Three traditional Calculus I questions that involved integrals were then posed to the participants (see Figure 2).

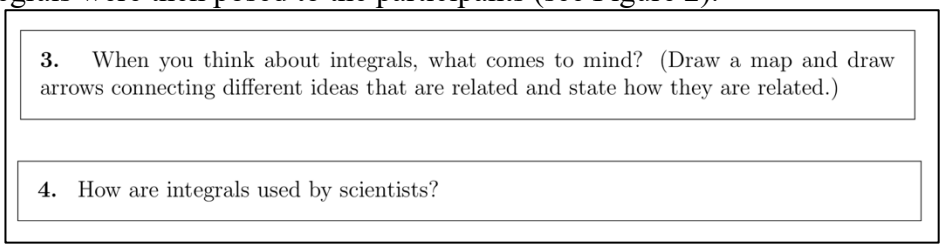

Figure 1. Informal questions about integrals posed to the participants.

The data from the semi-structured interviews were transcribed and coded using the a priori codebook. Each member of the research team coded the ten transcripts independently and met to discuss codes. Any disagreements around coding were discussed and resolved during research meetings.

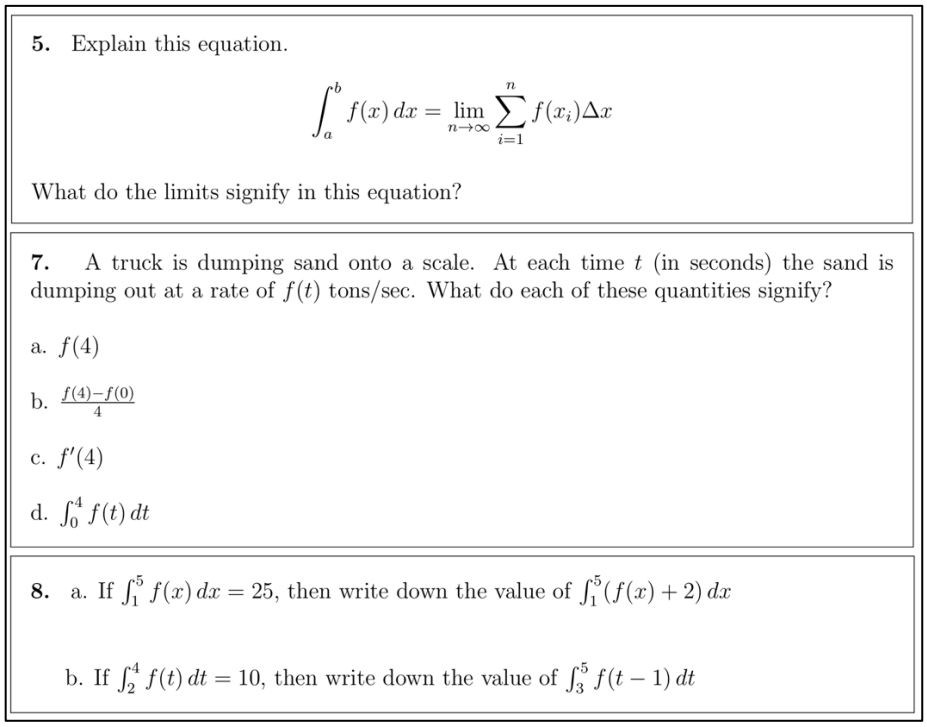

Figure 2. Formal questions involving integrals posed to the participants.

Considering preliminary findings from the student interviews, the research team decided to then explore and code the CLOs for the Calculus I and II courses that the student participants had taken in prior semesters. The CLOs were coded using the same codebook in order to make comparisons between the findings.

Preliminary Findings

In this section, we present findings in response to each research question.

Student Conceptions on the Definite Integral

We noticed several key observations from the student data. First, and consistent with the literature, all participants referenced the concept of area at least once during the interview. For instance, one participant sketched a picture of an arbitrary function $f(x)$ as they interpreted question 7d, "So because we have infinitely many rectangles, because that n approaches infinity… essentially it's like this whole area filled out." Second, all students demonstrated procedural understanding in solving at least one of the given problems (e.g., question 8).

Third, a majority of students articulated that integration is related to the notion of being the opposite of a derivative (7/10). One participant stated, "I think integrals are like antiderivatives…same thing as derivative just opposite." In addition, 7/10 students associated the integral with accumulation. Many students discussed an idea about totals. One student mentioned "total electric charge," and another mentioned "total work done" over some interval. In discussing question 7d, another student stated the following:

> If you're looking at the line, that's the output of the function. That's how much … sand got dumped. But, for an integral … it's kind of like saying, it's this much sand. It's this much sand. Like it's-it makes me think of saying like, that's all the sand ever between 0 seconds and 4 seconds.

This participant used gestures and repetition to emphasize the addition to a pile of sand as a representation of the integral in 7d.

Lastly, only 6/10 of the participants accurately discussed the idea of an infinite sum either within the definition of an integral or while explaining the process of integration. This was surprising given that each participant was presented with the Riemann sum in question 5.

Expected Learning as Outlined in CLOs

The Calculus I course at our institution covers the foundations of differential and integral calculus, with 12 CLOs. Six of these learning goals, as displayed in Table 2, pertain to student learning about integration. We noticed that the predominant emphasis in this Calculus I course was to facilitate the development of an area conception around integration. The Calculus II course covers techniques and applications of integration, sequences and series, and polar and parametric equations. There are 15 Calculus II CLOs, and only four were explicitly related to integration. All four of these CLOs were coded as Procedural. For instance, one CLO was, "to learn how to apply the definite integral and integration techniques to the problems of finding the arc length of a curve, the area of a surface of revolution, moments and centroids, and hydrostatic pressure and force." While it may occur in Calculus II, we do not speculate on whether an emphasis on an accumulation interpretation occurs during instruction.

Table 2. Six Relevant Calculus I learning goals Pertaining to Integration

CLO	Code
7. To study and understand the concept of an antiderivative of a function, and to find families of antiderivatives for some simple functions.	-
8. To grasp the meaning of the "area under the curve" problem, to be able to approximate such areas with left and right hand Riemann sums, and to evaluate the limit of these sums using summation formulas provided.	Area, Infinite Sum
9. To understand the meaning of the definite integral, to learn to use the notation correctly, and to be able to interpret its meaning in terms of areas.	Area
10. To learn how to use the Fundamental Theorem of Calculus to evaluate definite integrals and to find derivatives of definite integrals as a function of the upper limit.	Procedural
11. To learn how and when to apply the method of substitution to evaluate a definite integral, and to be able to interpret a definite integral using the Net Change Theorem.	Procedural
12. To understand how to apply the definite integral to find the area between two curves.	Area

Discussion and Conclusion

Though the sample size was small, when compared to the CLOs, these findings provide insight into potential conceptions students may develop about integration following their coursework in Calculus I and II at our institution. Findings indicated that students' conceptions of integration encompassed expected procedural methods, as well as connections to area under a curve and, in some cases, volume. Notably, students referenced accumulation (including "totaling" and "adding up") more frequently than anticipated. Overall, their conceptions were consistent with the stated CLOs.

To foster an understanding of accumulation, it is essential to identify concrete ways to *integrate* it into the course learning objectives. With that said, we pose the following questions for the audience to consider:

1. What CLOs would need to change if we want students to develop a robust conception of accumulation?
2. What barriers exist to shifting from a primarily area and procedural focus to an accumulation focus on integration? And, in what ways could we better leverage the introduction of the Riemann sum to cultivate an accumulation conception?

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How Proofs Are Organized: A Systemic Functional Linguistics Theme Analysis

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We present a preliminary linguistic analysis of how proofs are organized. In particular, we performed a Theme analysis—an analysis from systemic functional linguistics (SFL) that identifies the starting points of messages at the level of independent clauses—on proofs from undergraduate textbooks. Theme analysis is one of the ways in which SFL studies the textual metafunction, that is, how texts are organized to relate to their context. Our preliminary results show that the main marked (i.e., unusual) Theme constructions in our dataset are: (a) beginning clauses with Adjuncts, and (b) beginning clause complexes with dependent clauses. Further, we found that textual elements (e.g., thus, so, since, if) were common as part of multiple Theme constructions.

Keywords: discourse, proof, systemic functional linguistics, textual metafunction

We expect that the reader will find the following proof (of the theorem that every finite integral domain R is a field) to be written in a frustratingly opaque manner:

> We need only show that for each $a \neq 0_R$, the equation $ax = 1_R$ has a solution since R is a commutative ring with identity. You let a_1, a_2, …, a_n be the distinct elements of R. Also, suppose $a_t \neq 0_R$. Consider the products $a_t a_1$, $a_t a_2$, $a_t a_3$, …, $a_t a_n$ to show that $a_t x = 1_R$ has a solution. We must have $a_t a_i \neq a_t a_j$ if $a_i \neq a_j$ (because by cancelation $a_t a_i = a_t a_j$ would imply that $a_i = a_j$). $a_t a_1$, $a_t a_2$, …, $a_t a_n$ are, therefore, n distinct elements of R. Although, R has exactly n elements all together, and so all the elements of R in some order these must be. In particular, $a_t a_j = 1_R$ for some j. A solution, consequently, the equation $a_t x = 1_R$ has, and a field R is.

This proof is a rewritten version of a proof by Hungerford (2014). We created it by changing the order of clauses within sentences, the order of clause constituents (e.g., subjects, objects), and conjunctions. With this rewritten proof, we wish to emphasize that authors of proofs make choices to ensure that they "relate[] [their] messages to the prior and following text and context" (Schleppegrell, 2012, p. 21). This relating process is what systemic functional linguists refer to as the textual metafunction, and this preliminary report's goal is to study how this textual metafunction is realized in proof texts. Although we eventually intend to perform several of the textual metafunction analyses SFL offers, this report focuses on one textual metafunction analysis: a Theme analysis. We capitalize Theme, which is roughly the first constituent of a clause (more on this below), to distinguish it from its everyday meaning. Thus, we seek to address: How are Themes realized in proofs from undergraduate mathematics textbooks?

Literature Review

Systemic functional linguistics (SFL) originates with linguist Michael Halliday. It studies the grammatical choices humans make within linguistic systems and the functionality of these choices (Schleppegrell, 2012), and for over 50 years, SFL has crossed paths with mathematics education. In 1974, Halliday attended the "Symposium on Interactions Between Linguistics and

Mathematical Education," after which he published a chapter about the mathematics register (Herbel-Eisenmann et al., 2017)—a register being "the constellation of lexical and grammatical features that realizes a particular situational context" (Schleppegrell, 2004, p. 18). Key to understanding registers and SFL are the three metafunctions—ideational, interpersonal, and textual—which are three types of meanings realized in every clause "to talk about the world [...] to interact with other people [...] [and to] organiz[e] language to fit in its context" (Thompson, 2014, p. 30). Below, we describe the metafunction most relevant to our work: the textual.

The Textual Metafunction (in Mathematics)

Two types of features realize the textual metafunction, that is, how messages are organized: structural features (i.e., thematic structure, information structure and focus) and cohesive features (i.e., reference, ellipsis and substitution, conjunction, lexical cohesion) (Halliday, 1994). Whereas the structural features are internal to a clause (or clause complex), the cohesive features focus on how clauses (or clause complexes) relate to one another (Halliday, 1994). (A *clause* is "(potentially) any stretch of language centred around a verbal group," Thompson, 2014, p. 17; a *clause complex* is "a combination of two or more clauses into a larger unit," Thompson, 2014, p. 186). Important for this report is the thematic structure, particularly the concept of *Theme*. Theme, which we discuss in greater depth in our Conceptual Framework, refers to "the element that serves as the point of departure of the message" (Halliday & Matthiessen, 2014, p. 89).

In mathematics education, several scholars have studied aspects of the textual metafunction. For example, O'Halloran (2005) observed that in mathematics texts: (a) marked (i.e., nontypical, not the default) Theme constructions are used to emphasize important content (e.g., "From Equation 3", "When the derivative is small"); (b) dependent clauses as marked Themes are common (e.g., "If $x = 4$"), and (c) conjunctions and conjunctive Adjuncts as textual elements of (multiple) Themes are common (e.g., if, then, therefore). Morgan (1998) postulated that in deductive texts, it is likely that textual elements (e.g., hence, therefore) and reason-providing Adjuncts (e.g., "By Theorem 1") appear in the Theme. (*Adjuncts* are those clause constituents that provide information about the circumstances of a situation, for example, why something happens, Thompson, 2014.) Further, Morgan (1998) suspected that unlike scientific explanations, mathematical deductive texts may favor leading with the cause and then providing the result. On the other end of the textual metafunction spectrum, González and Herbst (2013) and González (2015) have focused on cohesive features by studying conjunctions, and conjunctions and lexical cohesion, respectively.

Last, we wish to acknowledge that there are also studies in mathematics education that focus on the other two metafunctions or combinations of the three metafunctions. For instance, there is work focused on the ideational metafunction (e.g., DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011), the interpersonal metafunction (e.g., DeJarnette, 2018; Herbel-Eisenmann & Wagner, 2010; Mesa & Chang, 2010; Zolkower & Shreyar, 2007), both of these (e.g., Burton & Morgan, 2000), or all three metafunctions (e.g., Herbel-Eisenmann, 2007; Morgan, 1998; O'Halloran, 2005). Finally, not all SFL-related work foregrounds the metafunctions. For example, there is also work that draws on SFL's writing on genre (e.g., Marks & Mousley, 1990; Solomon & O'Neill, 1998).

Conceptual Framework

This report focuses on the realization of the textual metafunction in proofs, and SFL offers multiple analytic tools for analyzing the textual metafunction (e.g., Theme analysis, conjunction analysis). The guiding concept for this report is the concept of *Theme*: "the element that serves as

the point of departure of the message; it is that which locates and orients the clause within its context" (Halliday & Matthiessen, 2014, p. 89). In English, placing something at the start signals that it is the Theme. The Theme typically contains given (i.e., already present) information, and the remainder of the clause, the *Rheme*, typically contains new information (Eggins, 2004; Fries, 1994). For each of the three basic grammatical moods in English—which SFL terms *declarative* (e.g., "Danielle watches TV."), *imperative* ("Danielle, watch TV!"), and *interrogative* ("Is Danielle watching TV?")—SFL proposes that there are *unmarked* (i.e., usual) and *marked* (i.e., less usual) Themes. For example, it is natural to start a declarative clause with a Subject (e.g., "I can do without grading") but less common to start with a Complement (e.g., "Grading I can do without"). (A *Complement* is a "nominal or adjectival group that refers to the same entity as the Subject, or describes the Subject," Thompson, 2014, p. 20). So, for declarative clauses, a Subject as Theme is referred to as an unmarked Theme, whereas other thematic choices are referred to as marked Themes. Note that it is SFL tradition to capitalize functional labels like Subject.

In our analysis of Theme, we follow Thompson's (2014) approach, and there are a few key points about his conceptualization of Theme that merit mention:

1. When a clause complex starts with a dependent clause, Thompson (2014) suggests treating the entire dependent clause as the Theme (e.g., "**If it rains**, I'll stay home").
2. Textual and interpersonal elements that precede the first experiential (or topical) element are part of the Theme, which is then called a *multiple Theme* (e.g., "**Thus, surely, you** agree"). Note that a multiple Theme is not multiple Themes, but one Theme consisting of multiple elements that correspond to different metafunctions.
3. Clauses with "there" plus the existential process (usually realized by "be") include the existential process in the Theme (e.g., "**There is/was** a bird outside").

Methods

Data Collection

Thus far, our dataset consists of ten proofs from five popular undergraduate textbooks (i.e., Abbott, 2015; Axler, 2024; Dummit & Foote, 2004; Fraleigh, 2003; Rudin, 1976) that cover typical undergraduate mathematics content: real analysis, abstract algebra, and linear algebra. Within each textbook, two proofs were randomly selected with a random number generator.

Data Analysis

First, the proof texts were separated into T-units: "independent clause[s] together with all the clauses that are dependent on it" (Thompson, 2014, p. 161). T-unit boundaries that occurred within sentences were indicated with a double slash (//). Next, the Theme (and Rheme) of each T-unit were identified, following Thompson (2014). Third, the Theme was studied to: (a) determine whether it was marked, and (b) identify any textual and interpersonal elements part of a multiple Theme. (Textual elements were highlighted in orange and underlined. Interpersonal elements were highlighted in blue and underlined with dots.) For an example of the T-unit separation and coding of a sentence from a proof by Abbott (2015), please see Table 1.

Table 1. Example Theme Analysis of a Sentence

Theme	Rheme	Marked?
If $a = b$,	then $\|a–b\| = 0$ //	marked (dependent clause)
and so certainly $\|a–b\|$	$<\varepsilon$ no matter what $\varepsilon>0$ is chosen.	unmarked (declarative)

The first author coded all ten proofs (i.e., 151 T-units), and, at the time of writing, we are in the process of having each proof coded by at least two authors. We are also currently studying the marked Themes to identify patterns. Our preliminary results are described below.

Preliminary Results

Theme Status

Theme status describes whether a Theme is marked or unmarked, and in our coding, marked constructions appeared roughly a third of the time (49 out of 151 T-units). In particular, we found two marked constructions: (a) beginning a clause complex with a dependent clause instead of the independent clause (33 times), and (b) beginning a declarative or imperative clause with an Adjunct (16 times). We note that among the possible marked options in declarative clauses, Adjuncts fall "somewhere in the middle on the scale of markedness" (Thompson, 2014, p. 150). As this breakdown shows, other marked options, like beginning a declarative clause with a Complement, did not appear at all. Artificial examples thereof (e.g., "To show that x is even is our goal", "A subset of Y the set X is") suggest that such Complement-as-Theme constructions sound strange and may be less functional for the flow and organization of proofs than their unmarked counterparts (i.e., "Our goal is to show that x is even", "X is a subset of Y").

Dependent clauses. For this subsection and the next, let P and Q refer to mathematical statements (e.g., "f is differentiable"). Among clause complexes beginning with dependent clauses, if–then constructions were most common, with "then" sometimes omitted (e.g., "If P, [then] Q"). There were also implicit conditional statements, such as "Assuming that P, [we get] Q", and "pseudo-conditional" statements that described depersonalized mathematical actions in the antecedent (e.g., "Renumbering [...], [we get] Q"). (We use the term "pseudo-conditional" to describe that although these clause complexes can be phrased in if–then terms [e.g., "If we renumber […], then we get Q"], they do not introduce a mathematical assumption.)

Another construction we observed was Thematizing mathematical warrants in dependent clauses (e.g., "Because P, [we get] Q"; "Since P, [we get] Q"). Lastly, we observed constructions that directed the reader toward a goal or direction (e.g., "To prove P, [...]", "Turning to multiplication, [...]"). We note that these uses of dependent clauses as Theme allow authors to adhere to the flow of logic—with data and warrants preceding claims—and, in the last case, orient the reader by framing the argument.

Adjuncts. As aforementioned, Adjuncts provide information about the circumstances of a situation (e.g., when, how, why) (Thompson, 2014). In our dataset, a majority of the Adjuncts appearing as Themes provided justification and typically took the form "By [*warrant*], P" (11 out of 16 times). Initial Adjuncts were also used (5 out of 16 times) to orient the reader and frame the Rheme (e.g., "In the forward direction", "For the [proof of the] first statement", "As regards (a')"). Initial Adjuncts almost exclusively appeared in declarative clauses (15 out of 16 times). The only example of an imperative clause beginning with a (framing-)Adjunct was: "In the general case, let [...]").

Theme Composition

Theme composition describes whether a Theme is simple (i.e., contains only an experiential element) or multiple (i.e., also contains textual and/or interpersonal elements).

Textual elements. There were many textual elements that initiated multiple Themes. To truly do these textual elements justice, a conjunction analysis (Martin & Rose, 2007) would be required. But even without such an analysis, there are several takeaways. Many textual elements

introduced (warranted) claims (e.g., "Hence", "So", "Therefore", "Then, "Thus") or antecedents (e.g., "If"). There were also textual elements that set a (new) stage (e.g., "Now", "First"), created contrast (e.g., "But", "However"), and were used to add (e.g., "and", "also", "again"). At times, multiple textual elements were combined, mostly to organize the text *and* do mathematical work (e.g., "Now if", "Again by", "And hence", "But then").

Interpersonal elements. There was only one interpersonal element in a multiple Theme: "certainly" in "**and so certainly** $|\mathbf{a}-\mathbf{b}|<\varepsilon$", which adjusts the force of the message (s. SFL's system of Graduation) (Derewianka, 2011). Since statements in mathematics are either true or false, there may be little need for resources like Graduation that intensify or weaken claims.

Experiential elements. The experiential elements of Themes were: Subjects in unmarked declarative clauses (e.g., "**The *i,j* entry of AB** is [SUM]") (80 times), dependent clauses in marked clause complex constructions (s. the "Dependent clauses" section) (33 times), Predicators (i.e., the verb that the Subject "does") in unmarked imperative clauses (e.g., "**Let** n = dim V") (22 times), and Adjuncts in marked declarative (and, rarely, imperative) clauses (s. the "Adjuncts" section) (16 times). Other than the markedness patterns, we noted that unmarked Subject-Themes often included Pre- and Postmodifiers. For example: "**The very definition of φ_a applied to a constant polynomial $a \in F[x]$, where $a \in F$** gives $\varphi_a(a)=a$", where the Pre- and Postmodifier are underlined with dashes. Thompson (2014) calls such Subjects *'heavy' Subjects*.

Other observations. Linguists disagree about the best way to perform a Theme analysis, particularly around where to draw the Theme boundaries (Thompson, 2014). We experienced this challenge and wish to highlight three structures we observed that were not captured by our Theme analysis but could have been captured if we had drawn different Theme boundaries: (a) leading with projected clauses of the form "We (+ *modal verb*) + *verb* + that + *mathematical statement*" (e.g., "We (must) show that […]", "We observe that […]", "We assume that […]"); (b) leading with projected imperative structures (e.g., "Assume that […]", "Note that […]", "Observe that"); and (c) leading with depersonalized versions of the previous two structures beginning with anticipatory "it" (e.g., "It follows that […]", "It needs to be shown that […]", "It may be noted that […]"). These constructions Thematize comments *on* the subsequent mathematical message instead of the message itself. We posit that this is a functional choice that allows authors to manage the flow of the argument and the links between claims.

Discussion

Although preliminary, our results align with O'Halloran's (2005) observations and Morgan's (1998) conjectures about Themes in (deductive) mathematics texts. Our next steps are to increase the number of proofs in our dataset and have all proofs coded by at least two authors. Since we are ultimately interested in what makes a proof flow for (student) readers, we want to also conduct a cohesion analysis. We believe that attending to cohesive features and making observations about how clauses (and clause complexes) are related to one another will complement our current results about how clauses (or clause complexes) are (internally) organized. We have several questions for the audience. First, we have thus far drawn on well-regarded mathematical texts, which appear to be making similar grammatical choices. How could we increase or diversify our dataset to better emphasize that choices are being made and to understand which choices are possibly more functional for communicating proofs in which contexts? Further, since our goal is to eventually understand the effects of these choices on the reading of proof, what methods could we draw on to study how Theme and cohesion choices influence the reading (and comprehending) of proof? Relatedly, how might our current clause-level observations inform the writing of proofs?

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"There's Nothing Abstract About This Math": Abstract Algebra Students' Interpretations of Abstraction

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Abstract Algebra is often cited as being a difficult course for undergraduate students because it is more abstract than computationally based courses such as Calculus. But what does it mean for something to be "abstract"? I conducted a grounded theory analysis of student survey data from an Introduction to Abstract Algebra course to explore how students define abstraction. The data show that students' personal definitions of abstraction go beyond conceptions of abstraction discussed in the literature, and the concept of abstraction acts as a mathematical artifact that influences the way students view abstract algebra concepts. Preliminary analysis also suggest that students' definitions of abstraction differed depending on their major area of study, indicating that abstraction is a boundary object whose definitions and uses vary depending on context.

Keywords: abstraction, abstract algebra, artifact, boundary object

As one of the first proof-based courses in the undergraduate math sequence, abstract algebra is often cited as being a difficult course for undergraduate students because of its abstraction level. The purpose of this study is to investigate how students conceptualize mathematical abstraction in an introductory abstract algebra class.

"Abstract" is used as both an adjective and a verb in various contexts in math education. The most well-known treatment of "abstract" as an adjective in Abstract Algebra is found in Hazzan's (1999) framing of Reducing Abstraction, which identified three interpretations of abstraction. Hazzan writes of "levels" of abstraction of abstract algebra concepts, defining this not as distinct enumerated stages but relative comparisons of how abstract a concept seems to a student.

In the first interpretation, Hazzan states that the level of abstraction of a concept or topic is not a property inherent to that concept, but a "reflection of the relationship between the object of thought and the thinking person" (1999, p. 76). Thus, a person's opinion of the abstraction level of a concept is influenced by their level of familiarity and prior experience with it.

According to Hazzan, the level of abstraction of a concept can also be described as a reflection of the process-object duality. Students working at a low level of abstraction are often focused on a process involving carrying out an algorithm or procedure, while students at a higher level of abstraction can view a computational process as a generalized object without referring to a specific example.

Hazzan's third interpretation of abstract is of complexity; "the more compound an entity is, the more abstract it is" (1999, p. 82). For example, a set is more abstract than one element in the set. Finally, Melhuish et al. (2018) expanded the framework and defined a fourth interpretation of abstraction level as the degree of formality or precision of mathematical objects. For example, a formal definition is more abstract than a metaphor or an informal definition.

The interpretations of abstraction identified by both Hazzan (1999) and Melhuish et al. (2018) were based on observations of student work and the strategies students use to make sense of abstract concepts. To date, no studies have directly addressed how students define what it means for a concept to be "abstract" in abstract algebra.

A limitation of Hazzan's study is the student population, which consisted of computer science students taking an abstract algebra class, after having previously taken a linear algebra class. However, some students take abstract algebra as their first introduction to abstract, proof-based mathematics, including students majoring in a variety of subjects. Thus, in this study I investigate how students with a variety of majors (including math, computer science, physics, chemistry, engineering, and economics) and differing levels of mathematical experience interpret abstraction in abstract algebra.

Theoretical Perspective

My theoretical perspective draws from sociocultural theory and the idea of mathematical artifacts as boundary objects. In the sociocultural view, learning occurs through social interaction and is mediated by artifacts, which includes language and symbols as well as physical or conceptual objects, tools, and concepts (Lerman, 2001). Mathematical artifacts are products of history and culture, and their meanings are "long-lasting but potentially changeable" (Lerman, 1996, p. 146).

An often-overlooked artifact encountered in abstract algebra is in the name, "abstract." While the definition of "abstract" is often taken to be self-evident, the concept of abstraction contains multiple meanings, each with their own culture and history. Students from differing backgrounds may have vastly different assumptions of what abstraction is, and these assumptions may influence the way they approach and interpret the subject.

Abstraction as an artifact is a boundary object, an artifact at the boundary between different communities of practice that are used differently by each community (Hoyles & Noss, 2004). Abstraction has long been at the boundary of mathematicians, math educators, and laypeople, who each have different conceptions and uses of abstraction (e.g., Thorndike et al., 1923; Corry, 2003). Students majoring in disciplines such as physics and engineering have different experiences with abstraction compared to math majors. As Hoyles and Noss observe when it comes to boundary objects, "communication of meaning and 'transfer' can be problematic unless and until the different conceptualizations—and the language in which they are expressed—are brought into alignment" (p. 3).

Methodology

Data were collected from students in an Introduction to Abstract Algebra course taught using a rings-first curriculum sequence. The course had an enrollment of 44 students with varied academic majors, including math, computer science, economics, physics, and others.

Halfway through the semester, I administered a written survey to students in the course, to which 33 students responded. For this paper, I analyzed student responses to the following questions:

1. This course is called Abstract Algebra. What does "abstract" mean to you?
2. What concepts in this course so far have seemed the most abstract to you?
3. What concepts in this course so far have seemed the least abstract to you?

I conducted a Grounded Theory analysis (Charmaz, 2014) of all 33 student responses to the survey question. Initial line-by-line coding resulted in 135 codes related to students' experiences with abstraction in this course. After several rounds of coding, I grouped these codes into three major categories representing students' definitions of abstraction, which will be described in the Results section. Following the survey, I conducted one-on-one, semi-structured interviews with 14 students. I used my coding results and memos to select six of the 14 interviewed students

representing all major codes. I coded portions of these interview transcripts and selected excerpts to represent each category.

Results

I identified three main categories that students consider in their definitions of abstraction: Structure, Purpose, and Beyond Sensory Experience. Each category was interpreted in two different ways by students. Three of these categories are further split into subcategories (Table 1).

Table 1. Student Interpretations of Abstraction

Category	Interpretation	Subcategory	Examples
Structure	Presence of Structure	Underlying reasons	Abstract algebra "breaks down" math to its fundamentals and explains the underlying reasons behind familiar procedures
		Simplicity	Abstract algebra concepts have simple definitions and axioms which require less justification than contextualized concepts
	Absence of Structure	Arbitrary	Abstract algebra changes familiar algebra rules and definitions without justification
		Indirect	Abstract algebra uses words and "math tricks" instead of numbers
Purpose	Widely Applicable		Abstract begins in applied concepts but moves away from applications Generalizing across dissimilar contexts Able to represent multiple things
	No Applications		Not relevant to student's major or future career

Beyond Sensory Experience	Basis of reality		Abstract math is a "sixth sense" that helps us learn about the world beyond the five senses
	Not Real	Outside the Real World	Things that are beyond perception by the senses are difficult to grasp
		Too strange to believe	Students have difficulty believing concepts are valid, even with proof

Structure

In the first category, Structure, some students interpreted abstraction as the presence of structure. One student, James (all participant names used are pseudonyms), described Underlying Reasons as a "broader view" of algebra:

> We are taking a look at algebra in a broader view. In simple algebra classes you learn the mechanics of algebra and how to apply it. In this class we are looking at things one level higher and proving why the mechanisms we previously used work.

Other students described abstraction as a lack of structure, marked by chaos and uncertainty. Hanna, a math and art double major, described frustration with the seemingly arbitrary nature of rules in abstract algebra, and couldn't see any underlying reasoning behind it. Hanna defined abstraction as "unstructured, random, chaotic… [groups and rings] seem like random 'things' with very specific conditions that someone decided randomly one day."

Purpose

A common theme involved how abstraction related to the purpose or applications of math. Some students said that abstract math is widely applicable to many different areas, seeing it as a generalization of more specific math. Travis described abstraction in terms of the number of applications: "Abstract means abstracting away from applications… groups feel the most abstract to me because they can represent so many things because of how simple the axioms for them are."

On the other hand, Rye, an electrical engineering major, expressed frustration with the lack of practical applications to her major:

> I am a very practical person. And my degree is very practical stuff. And so abstract just feels like, I don't feel like I'm ever going to use this material…it feels like we just chose different conditions, and we are like, 'this is now a ring' and 'this is now a group'. But what is it doing for us? …I actually don't see any importance. Like it just feels like someone's made it up.

The fact that Rye didn't "see any importance" to rings and groups indicates how necessary it is for her to have practical applications when learning a new concept. However, some students enjoyed the fact that there were no practical applications for the course material. Abraham, a biochemistry major, said about the course, "It all seems not very applicable to the real world, which I love. I love that. It's just like exercising the mind for the sake of knowing more things about numbers rather than for, you know, building a skyscraper or something."

Outside the Five Senses

Some students described abstraction in terms of the senses, contrasting abstract math with math that is visual or tangible. Chuck, a math major, suggested that abstraction goes beyond the five senses to learn more than we can normally perceive:

> Maybe math is another human sense. I think that the abstract can go beyond observations and make predictions… actually figuring out what's going on with the universe, sometimes we just can't see or hear or feel it. And I think that's where the abstract and mathematics comes into play and is super useful.

To other students, math that is not tangible and perceivable by the five senses is "not real math," which is how Taylor, a chemistry major, described abstraction. Ian, a physics major, described abstract math as "math that will never be used," "intangible," and "not actually proven." However, he found that a lot of concepts he learned in the course were connected to concepts he had learned in previous courses, leading him to conclude that "there's nothing abstract" about abstract algebra.

Discussion

Much of the literature on abstract algebra assumes that abstraction is inherently difficult for students. However, the students in my study did not consider "abstract" to mean the same as "difficult." There were several students who responded that nothing in the course was difficult for them, but everything in the course was abstract.

The categories of abstraction I found are different from Hazzan (1999) and Melhuish et al. (2018), which were the only previous literature that broadly addressed student perceptions of abstraction in abstract algebra. One possible reason for this is the student population. All of the students in Hazzan's study were computer science majors in an abstract algebra class, while my study included students majoring in math, computer science, and a variety of other STEM fields.

There were notable differences in how students with different academic majors viewed abstraction. A student's major constitutes a community of practice to which they belong, with each community holding different perspectives on and uses for abstraction. The majority of math and computer science majors in this study shared a similar view of abstraction as the presence of structure and purpose. Other majors, the majority holding opposing views, were likely not familiar with the conventions of abstraction in the mathematics community, and instead viewed abstraction through the lens of their primary area of study. In their interviews, several students described how abstraction is used in their major fields like chemistry and economics, and how that influenced their views of abstraction in abstract algebra. Future analyses will explore in depth the relationship between students' major and their understanding of abstraction.

I am in the process of analyzing further data from this same group of students, including a subsequent survey administered at the end of the semester, in which I ask students to rate their agreement with each interpretation of abstraction identified here. These data will reveal whether students' definitions of abstraction changed over the course of the semester. In addition, future analysis of task-based data from this same group of students will shed light on whether students' views on abstraction influenced the way they solved abstract algebra problems and constructed proofs.

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From Noticing to Reflection: Graduate Teaching Assistants' Experiences With Written Classroom Accounts

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Noticing is the first step to reflective teaching. Mathematics graduate teachers used 'brief-but-vivid' classroom accounts to reflect on teaching, documenting weekly moments and discussing them in facilitated groups. Two major themes emerged: (1) Capturing Fleeting Moments, emphasizing subtle classroom dynamics, student behavior, and practical strategies, and (2) Camaraderie and Learning Through Shared Accounts, where peer interaction fostered camaraderie, accountability, and co-construction of professional knowledge. Participants focused more on pedagogy than content, highlighting early instructors' priorities. Findings demonstrate that socially mediated reflection shapes teaching identity, informs instructional decisions, and links individual noticing to collective learning, with implications for designing more effective teaching professional development.

Keywords: Reflective practice, professional development, collaborative learning, teaching assistants, mathematics graduate instructor

Introduction

Overburdened with attending classes, reading, research, teaching, preparing, and grading, Mathematics Graduate Teaching Assistants (MGTAs) seldom have time to reflect on their teaching practices. The result? Many MGTAs exit their doctoral programs having missed opportunities to reflect on and improve their teaching.

Math graduate students juggle multiple roles in graduate school—student, researcher, and teacher. In research universities across the U.S., mathematics classes are often large, frequently exceeding 50 to 100 students (Cuseo, 2007). MGTAs are typically appointed to lead recitation or discussion sessions that support these large classes. Despite their minimal prior teaching experience, MGTAs often begin teaching early in their programs, sometimes on the very first day (Nyquist et al., 1991). Although a significant number may later pursue teaching faculty positions, teaching might be viewed as secondary to research in research-intensive universities (Fairweather, 2002). Especially when time is scarce and more emphasis is placed on research than on teaching, the instructional habits they form during this period can have a lasting impact on their careers.

The purpose of this pilot study is to explore how MGTAs, particularly Mathematics Graduate Teachers (MGTs), experience using "brief-but-vivid" accounts of classroom incidents to reflect on their teaching experience with peers and faculty. These accounts are intentionally descriptive rather than emotive, asking participants to focus on observable events and actions while omitting explanations, justifications, or personal interpretations. In doing so, the accounts functioned as entry points into discussions, enabling peers to engage with the teaching moments without being overly influenced by the writer's subjective stance. Also, respecting and valuing their time as they navigate the many roles they have. The research questions guiding this study are:

1. How do novice instructors experience and evaluate the process of using writing accounts as a tool for reflection within teaching professional development?

2. What do these accounts focus on? Specifically, what aspects of teaching do novices notice and choose to capture?

Literature Review

Teaching professional development (TPD) trajectories that begin early and continue throughout the teaching experiences of graduate teaching assistants (GTAs) are essential for fostering their growth as educators (Deshler et al., 2015). Continuous PD equips GTAs with the skills, knowledge, and experiences needed to succeed in their teaching roles. Lortie (1975) emphasizes that early teaching experiences play a pivotal role in shaping an individual's instructional identity. For GTAs, these initial teaching roles act as a bridge between their immediate responsibilities as graduate students and their potential long-term academic careers. Consequently, early TPD, combined with opportunities to reflect on teaching practice, can significantly impact not only the quality of MGTAs' instruction but also the broader departmental educational ecosystem and undergraduate learning outcomes (Åkerlind, 2008).

Recent scholarship has highlighted the importance of sustainable TPD programs that support long-term retention and growth for MGTAs as educators (Sadera et al., 2024). While many TPD initiatives provide opportunities for professional growth, these programs often feel short, sporadic, and disconnected from the realities of classroom teaching (Herrmann & Grossman, 2021). Without sufficient support and contextually grounded reflection, GTAs may default to replicating their prior educational experiences or rely on limited personal beliefs about teaching—patterns that can persist into their future faculty roles (Lee, 2019). One way to address this gap is through communities of practice, where novice instructors can share experiences and co-construct professional knowledge with peers and mentors (Wenger, 1998).

Reflection in teacher education is a complex process that involves negotiating values, attitudes, and beliefs, alongside developing cognitive skills (Calderhead & Gates, 1993). For MGTAs and novice MGTs, reflection could allow them to cognitively and introspectively transform their teaching experiences, consistent with Dewey's (1933) view that reflective thought emerges from actively examining one's experiences. Similarly, Schön's (1983) concept of the reflective practitioner highlights the importance of learning through critical reflection on action, both during and after practice. Engaging in reflective practice helps MGTAs and MGTs make sense of challenges in discussion sessions and classrooms, refine instructional strategies, and develop a stronger professional teaching identity, underscoring reflection as a central component of professional growth (Brookfield, 1995; Korthagen, 2017).

Schön (1983) describes the expert practitioner as someone who is awake to and aware of their practice, rather than simply immersed in it (Mason, 2002). Mason (2002) extends this idea, emphasizing that reflection requires *noticing* incidents before one can act upon them. To facilitate this noticing, Mason advocates the use of written 'brief-but-vivid' accounts, which capture key classroom events and provide a starting point for reflection. He explains that brevity is achieved by omitting distracting details, while vividness focuses on observable behaviors that others could independently verify. Furthermore, he distinguishes between giving an "account-of" an event versus an "account-for," highlighting that the former encourages an inner split between actor and observer, fostering reflective awareness (Breen et al., 2014). These accounts can also function as entry points, connecting individual noticing to collaborative reflection in group discussions, which is consistent with the idea of boundary objects in professional learning (Akkerman & Bakker, 2011).

Building on this work, our research team seeks to create conditions in which MGTAs bring their own expertise into reflection, discussion, and critique. By crafting stories that are

meaningful and valuable to them, MGTAs can engage in reflective practices that not only deepen their understanding of teaching but also contribute to the collective professional knowledge of the departmental community.

Theoretical Frameworks & Methodology

This study is framed through a multi-faceted lens of reflection and sociocultural learning theory, emphasizing the lived experiences of MGTs. Schön (1983) explained that *reflection-in-action* occurs during teaching, as instructors respond in real-time to students' needs, confusion, or questions, while *reflection-on-action* takes place afterward, when instructors review classroom events and analyze what happened. Writing 'brief-but-vivid' accounts facilitates this reflective process, providing a concrete means for MGTs to capture and examine their teaching experiences.

Thompson and Thompson (2018) emphasize that reflection should not only help practitioners make sense of past experiences but also inform their future actions, a process they refer to as *reflection-for-action*. This type of reflection serves as a complementary perspective, highlighting how insights gained from writing and discussion inform subsequent teaching and guide future classroom decisions.

Sociocultural learning theory situates these reflective practices within a hands-off facilitated community of practice, emphasizing that professional development is socially constructed through interaction with peers and faculty. In the context of this study, the 'brief-but-vivid' accounts function as objects to enter the discussion, connecting individual noticing to collaborative reflection. Through discussion, MGTs compare experiences, propose strategies, generalize from specific incidents, and co-construct professional knowledge, thereby integrating individual reflection with collective learning in a hands-off facilitated community of practice.

Context and Methods

This pilot study was conducted at a PhD-granting HSI in the southwest. The structure of the mathematics graduate program is that graduate students dedicate their first and second years to coursework, preparing for two written qualifying exams. MGTA starts teaching four discussion sessions on the first day of the program and is expected to teach every quarter if they are offered a TAship as a source of funding. After meeting specific departmental requirements, these MGTAs are encouraged to serve as the primary teachers of undergraduate courses over the summer, when classes are typically smaller, with approximately 30 students per class. We then refer to them as mathematics graduate teachers (MGT).

Study Participants

Study participants included four MGTs who are the primary teachers for summer undergraduate math classes ranging from pre-calculus to calculus II for life science majors and business calculus. Participants were all in the 4th or 5th year of the PhD program. Their teaching experience varied from their second to third time being the primary teacher at that institution. All study participants had been teaching assistants during their time in this program.

Methods

The data collected for this qualitative study included written 'brief-but-vivid' accounts, semi-structured group meetings, and an individual exit interview. Participants were asked to write three 'brief-but-vivid' accounts of their teaching each week and share those accounts via Google Docs, which all participants could access. These accounts were prompted as follows: Think of a

moment or experience from your class that stood out to you. Describe it as simply and directly as possible—focus on the facts, omitting unnecessary details and emotions. Don't explain why it happened or try to justify the actions involved. Instead, aim to give an account of what happened as if you were an outside observer, just as you would if you were telling someone about the moment without adding your interpretation.

The group meetings served as a space we thought of as a hands-off, facilitated community of practice. Participants met once each week for an hour and a half over a period of five weeks, during which they were teaching. The first author, a peer math graduate student, and the second author, a faculty member who serves as one of the TPD providers in the department, facilitated the group discussions in order to maintain a fruitful, continuous conversation. These group meetings were semi-structured and audio-recorded throughout the 5-week summer term. There was one participant who could not make the group meetings but wrote the 'brief-but-vivid' accounts.

All participants participated in an exit interview, which helped us capture the meta-reflections on the whole process of writing the accounts and the subsequent group discussions.

Preliminary Findings and Discussion

In examining how MGTs described their experiences using the 'brief-but-vivid' accounts in this study, two major themes emerged. These themes highlight both the social and cognitive dimensions of reflective practice, illustrating how the accounts functioned as tools for noticing, processing, and sharing classroom experiences. The first theme, Capturing Fleeting Moments, underscores how documenting subtle, often overlooked classroom events enabled MGTs to engage in reflection-in-action, reflection-on-action, and reflection-for-action, ultimately informing future teaching practices. The second theme, Camaraderie and Learning Through Shared Accounts, emphasizes the ways in which peer interaction, discussion, and mutual support fostered professional growth and a sense of camaraderie. Together, these findings illustrate that reflective practice among MGTs is both individually meaningful and socially mediated, aligning with sociocultural learning theory and concepts of communities of practice.

Capturing Fleeting Moments

A key theme was the accounts' ability to capture fleeting, often-overlooked moments in teaching. Participants noted that documenting small but significant classroom interactions—such as unexpected questions or subtle changes in student behavior—enabled them to reflect on real-time classroom dynamics and their responses. By writing these incidents shortly after they occurred, MGTAs engaged in reflection-in-action (Schön, 1983) while also generating material for reflection-on-action. One participant reflected on how a small observation sparked practical reflection:

> *But some other things. Like, another interesting one that I had written about was the fact that, like, that one time I left to get water, yeah, like, one minute, you know, and I come back and everyone's just much more bubbly, much more social. This sort of prompted something within me to at least consider maybe I don't have a great answer yet, but like, How can I simulate this without me having to physically leave the classroom for XYZ reason, whatever. So it affected me, I suppose, in these ways, most mostly practical.*

This quote illustrates that fleeting classroom moments, when captured and reflected upon, can lead to practical insights and potential strategies for future teaching. The act of writing accounts not only promotes noticing but also supports reflection-for-action (Thompson & Thompson, 2018), helping MGTs anticipate and plan responses to similar situations in the future.

Camaraderie and Learning Through Shared Accounts

A significant theme that emerged was the social dimension of reflection—participants emphasized learning from their peers and experiencing a sense of camaraderie. One participant remarked, "It doesn't only happen to me," highlighting that challenges and successes in teaching were shared across the group rather than experienced in isolation. Another participant reflected on the supportive yet accountable nature of the group:

> *Like more intention about, like, how we're teaching, or like how we're dealing with situations…in a group that's like an accountability group as well…we do reflect on it, and I share what we did…Sure, it's like a judge-free environment, but like, to some extent…*

Participants described how reading and discussing peers' accounts provided opportunities to compare experiences, share strategies, and learn from one another. One participant noted:

> *Yeah, I definitely learned a lot from other people's accounts. I also felt like I got to share some experience…you're not alone in those experiences…you hear somebody else has something similar going on in their classroom…And then there were some new ones for me…so now I feel like I would think about them now, like, how am I going to address that if it ever happened to me?*

Therefore, engaging with peers positions reflection as a collective rather than individual practice, allowing participants to recognize their teaching experiences as shared and socially situated. Through this process, the group functioned not only as a supportive, judgment-free space but also as an informal accountability structure in which participants exchanged strategies, surfaced common challenges, and jointly made sense of their instructional decisions.

What Aspects of Teaching Do MGTs Capture & Discuss?

Analysis of the accounts revealed that MGTs primarily focused on pedagogical and classroom management aspects rather than on mathematical content itself. Common observations included students not answering questions, minimal student interaction, writing more visibly on the board, use of technology, and other practical classroom strategies. These findings suggest that novice instructors are especially attuned to the flow and dynamics of teaching—how students engage, respond, and interact—rather than the technical content being delivered.

Interestingly, the research team noted that there was minimal attention to mathematics itself in the accounts, which was surprising given that the participants were teaching math courses. This highlights that early-career instructors may be more focused on how to teach effectively, rather than on content mastery, particularly when balancing multiple teaching, research, and student roles.

Through capturing these observations, MGTs engaged in reflection-in-action and reflection-on-action, attending to observable behaviors and classroom interactions that directly impacted the teaching and learning environment. The accounts provided a space for noticing both challenges and small successes, creating opportunities to improve instructional strategies and develop pedagogical awareness.

These findings raise several questions for future research. One particularly compelling question is: How might TPD programs better integrate content-focused reflection with pedagogical reflection, so that novice instructors attend both to what they teach and how they teach it? Addressing this question could provide insights into designing professional development that more fully supports the dual goals of content mastery and effective pedagogy.

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Poster Reports

Designing a Study to Explore the Quantitative Literacy Practices of U.S. Undergraduates

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Keywords: numeracy, quantitative literacy, social practices

In this document and accompanying poster, I share motivating literature and the outline of a study that started in Fall 2025 and that will continue through Spring 2026.

Over the last three decades, coursework centered on quantitative literacy (QL) has become increasingly common at postsecondary institutions across the U.S. (Madison, 2019). A common definition of QL accompanying such courses is that QL is a skill set and disposition to engage with quantitative information in various arenas of life (e.g., work, school, daily life) (Boersma & Klyve, 2013). Given this framework for QL, the logic of such courses is that there are certain skills students need to thrive in the modern world, and that if we can teach students those skills, as well as instill in them the habit of mind to engage with quantitative information, students will be prepared for a modern world "awash in numbers" (Steen et al., 2001, p. 1). That logic is flawed, as research in anthropology (Lave et al., 1984; Street et al., 2005), psychology (Carraher et al., 1985), and numeracy studies (Yasukawa et al., 2018) suggests that the ways in which individuals reason and engage with quantitative information in daily life is not well aligned with how related mathematics might have been presented in formal schooling. An alternative to the ability/disposition framework for QL is to view QL as a social practice (Graig & Guzmán, 2018). A social practices approach to QL focuses on what people do with quantitative information—their QL practices—rather than what people can or cannot do, which is a common focus that follows from viewing QL as an ability.

Understanding the QL practices of a group is a critical part of designing a QL curriculum for such a group (Baker, 1998). While there is work in the area of QL practices among children (Anderson & Gold, 2006; Street et al. 2005), adults in India (Rampal, 2018), vocation-school students (Boistrup et al., 2018), and among workers in select professions (e.g., Kalman & Solares, 2018), to date there are no studies in relation to college students (of any age) in the U.S. This year-long multiple case study aims to address that gap by providing insights into the numeracy practices of eight undergraduate students at a small liberal arts college in the Southeastern U.S. The overarching research question that drives the project is: What are the numeracy practices of the students I am working with? The theoretical framework of numeracy as a social practice necessitates the use of ethnographic methods (Yasukawa et al., 2018). In this study, I engage in in-depth interviews with the eight participants several times per semester over the course of an academic year; while interviews from month to month have a similar structure, they evolve based on the student and their interests. During both the interviews and the focus groups, students share artifacts (e.g., course slides, homework, photos) that relate to the topics discussed. The purpose of working with students over the course of a full years is not to examine change over time—though that is likely to occur—but instead to capture information that a one-time interview simply cannot (Barton & Hamilton, 1998).

Analysis of data from the project is ongoing and informs the questions used in the interviews and focus groups. By the time of the conference, I will have early results to share.

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The Impact of Near-Peer Mentors' Interactions Within Lower-Division Mathematics Classrooms

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Keywords: Near-Peer Mentors, Positioning Theory, Bloom's Taxonomy, Developmental Math

A near-peer mentor (NPM) is a general term for an undergraduate student who supports other undergraduate students' learning, typically in STEM courses. NPMs are associated with increased student performance and retention in STEM courses (Alzen et al., 2018; Barrasso & Spilios, 2021) and additionally help promote active learning techniques, reduce barriers to student help-seeking, and foster community within the classroom and throughout campus (Bronstein, 2008; Knight et al., 2015; Sellami et al., 2017) . However, there is a gap in the literature regarding exactly *how* NPMs have these impacts in classes. This study aims to address that gap – using positioning theory (Harré et al., 2009) as a theoretical framework – by examining the *positions*, *acts*, and *storylines* that structure NPM-student interactions. Additionally, the framework of Bloom's taxonomy (Bloom, 1956), which categorizes a hierarchy of cognitive learning (ranging from recalling to innovating), will inform the ways that NPMs encourage student learning through their interactions. This study is guided by the research question: "**How do NPMs impact both students and themselves through their in-class practices in lower-division mathematics classrooms?**"

This study will take place at Sageview University (SU), which is a mid-sized public university in the Mountain West region of the United States that serves roughly 27,000 undergraduate and graduate students. This study focuses on two developmental math course sections supported by NPMs: MATH 101 - Elementary Algebra and MATH 110 - Intermediate Algebra. MATH 101 focuses on academic success and self-efficacy in mathematics and is designed to prepare students for all other future math courses at SU. MATH 110 is intended to support students in their transition to college algebra as well as fulfill general education requirements. Both of these courses incorporate small-group activities and individualized work time during class, which provide opportunities for NPMs to interact with students. Data sources for this study include classroom observations of NPM-supported and non-supported sections, focus group interviews with NPMs and students, and student success metrics (grades, test scores, and pass rates). All titles used are pseudonyms to maintain anonymity.

In-class interactions between NPMs and students will be analyzed by identifying how questions that NPMs ask align with the six categories in Bloom's taxonomy, which will help identify how the NPMs are guiding students through their learning processes. Focus group data will be analyzed using positioning theory to identify themes that arise from student and NPM perspectives about their interactions in class. Their shared perspectives in conjunction with the classroom observation data will reveal patterns (*storylines*) that emerge from the *acts* and *positions* observed in class and discussed in focus group interviews.

Data collection for this study is ongoing during the Fall 2025 semester, but we anticipate that findings will highlight how NPM roles shape classroom culture and engagement. This research will additionally contribute to understanding how peer mentorship structures can strengthen student success in introductory mathematics and inform training for NPM programs.

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The Impact of AI Tools on Metacognition and Study Skills

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Keywords: Artificial Intelligence, Metacognition, Study Skills, Introductory Math

The growing use of Artificial Intelligence (AI) tools by students in mathematics education presents both opportunities and concerns related to student learning. AI tools provide students with personalized learning experiences and instant feedback, but overreliance on these tools can undermine critical thinking skills and create opportunities for academic misconduct (Alasadi & Baiz, 2023; Lo, 2023; Yoon et al., 2024). Essential for addressing the concerns posed by AI tools are students' metacognitive skills and their adherence to academic integrity, both of which play critical roles in effective and ethical learning. Metacognition refers to one's knowledge of and ability to monitor, control, and regulate one's own thinking and learning processes and allows students to understand and regulate their thinking processes, leading to improved problem-solving skills and deeper learning (Desoete & De Craene, 2019; Flavell, 1979; Velzen & H, 2016). Academic integrity entails acting honestly, transparently, and responsibly in all academic activities, which is essential for fostering genuine learning (Riedesel et al., 2012; Seaton, 2019). The central question guiding this study is: **How do AI tools impact students' metacognitive awareness and study skills in introductory mathematics courses?**

Methods

This study will take place at Timberline University (TU), a mid-sized public university located in an urban area in the Mountain West region of the United States, serving approximately 27,000 undergraduate and graduate students. It focuses on students and faculty from six introductory mathematics courses, ranging from developmental math to precalculus. These courses are part of an award-winning program within TU's Mathematics Department, known for its innovative practices and strong student success rates. As part of the initial intervention efforts, instructors will have weekly discussions with their students about how to effectively and responsibly use AI tools.

Data sources for this research include pre- and post-semester student surveys assessing AI tool use and learning behaviors and student focus groups exploring study habits and AI usage. A qualitative survey will be sent to math instructors teaching these courses to explore their perspectives on AI's impact on student learning and academic practices. The data will be analyzed using a mixed methods approach to explore trends in AI use, metacognitive development, learning behaviors, and academic integrity. The pre- and post-semester surveys will be compared to assess changes in students' AI tool use, metacognitive awareness, and study behaviors following the initial interventions. The qualitative survey and focus group data will undergo thematic analysis to identify recurring patterns in students' AI use, learning strategies, metacognitive reflections, and perceptions of academic responsibility. Additionally institutional data will be used to contextualize the self-reported behaviors within broader academic trends.

Data collection for this study is ongoing during the Fall 2025 semester. We anticipate findings will reveal trends in how students engage with AI tools in relation to their metacognitive development and adherence to academic integrity. In addition, this study will contribute to the development of practices and guidelines regarding AI tools that enable both faculty and students to foster metacognitive awareness and support academic integrity.

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Unpacking Students' Intuitive Limit Views Using Graphical Representational Approach

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Keywords: Graphical, Intuitive, Limits, Object, Tool

Students' intuitive reasonings of the limit of a function at a point play a role in their overall conceptual understanding of the limit concept. Some researchers have defined intuitive limit views as students' understanding of the limit concept based on their everyday experiences (e.g., Adiredja, 2014; Adiredja et al., 2020; Rathnayake & Jayakody, 2023). In this study, I characterize limit views that are based primarily on reasonings with less emphasis on *mathematizing* those ideas as *intuitive.* Research shows that intuitive views are closely linked to students' graphical reasoning (Adamoah & Strayer, 2025). Building on the findings of Adamoah and Strayer (2025), this study explored how students' intuitive understandings of limits are connected with graphical representations. The results of this study highlight an instructional approach to help overcome what past research identified as students' misconceptions about the difference between $lim_{x\to a} f(x)$ and $f(a)$ (Bezuidenhout, 2001; Denbel, 2014; Fernández, 2004). Specifically, I addressed the research question: *How do students' intuitive limit views influence their **reasoning with** and **use of** graphical representations in calculus?*

This poster presentation shares a portion of the results of a larger study that framed students' limit views as *intuitive*, *informal*, and *formal* and their representational approaches as *graphical*, *numerical*, and *algebraic* to understanding the limit of a function at a point. This study was conducted at a large public university in the southeastern United States in Fall 2024 with 63 Calculus II students. First, all the 63 participants completed a pre-assessment survey that focused on the three limit views and three representational approaches. Afterwards, I chose four participants for each limit view type (intuitive, informal, and formal), and that resulted in 12 participants purposively selected for the structured task-based interviews where they explained more fully how they reason about limits with each of the three representational approaches.

In this study, I found that students' *intuitive* limit views were linked with the graphical representational approach in two ways: (1) viewing limits with graphical representation as an *object* and (2) viewing limits with graphical representation as a *tool.* Graphical representation as an object implies students' belief that "*it shows me*" and graphical representation as a tool implies students' belief that "*I'm using it to show*". Below was Jake's explanation of how the graph with removable discontinuity showed and helped him to distinguish $lim_{x\to 2} f(x)$ and $f(2)$.

> *The graph helps me understand that the limit isn't going to just be like a point* [referring to the (2, 5) on the graph]*. We're going to get to, so it's not going to be the value of F of 2 is 5. This helps me understand* [referring to the graph] *we're getting closer and closer to a number, just as like we're evaluating a function as it gets closer and closer to a point.*

Jake's explanation above characterizes students' reasoning with graphical representation as an *object*. On the other hand, students' view of graphical representation as a *tool* was seen as they used it intuitively to prove the formal conventional $\varepsilon - \delta$ definition as well as using graphs to visualize and validate analytic properties of limits. The results of this study offer calculus instructors ways to integrate and effectively use graphical representations either as an *object* or as a *tool* to help deepen students' understanding of limits including the formal limit definition.

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"I'm Making This Math, Not Only Taking This Math": Poetic Transcription of a Trans Student's IO Calculus Experience

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Keywords: Poetic transcription, Inquiry-oriented instruction, Student experience

Inquiry-oriented (IO) mathematics instruction holds promise for engaging undergraduate students in meaningful mathematics. Yet, recent research shows that this promise may not be realized equitably without careful attention to classroom norms and participation structures. IO instruction can reproduce existing hierarchies, privileging dominant voices and marginalizing others (Ellis & Alzaga Elizondo, 2023; Ernest et al., 2019; Reinholz et al., 2022). Our project explores what IO instruction can feel like when it creates a more affirming space, especially for students from historically excluded groups.

We use poetic transcription to share the experience of Mal (first author), a trans undergraduate student in an IO Calculus 2 course. Mal was a student in Kristen's class, which used research-based IO materials on sequences and series (Vroom et al., 2025; Vroom & Davis, 2024; Vroom & Alzaga Elizondo, 2025). To address potential inequities, Kristen drew on Universal Design for Learning (Lambert, 2021, 2024; Vroom & Alzaga Elizondo, this issue). Mal participated in a reflective interview with Saúl, where they discussed both classroom experiences and research findings. Mal's reflections felt especially resonant, offering insight into the IO experiences that engage students in meaningful mathematics *and* affirm their identities.

We chose poetic transcription to center Mal's voice and emotional experience (Prendergast, 2009). Our work aligns with trans* methodology, attempting to "disrupt, disorder, and rupture" narratives of collegiate mathematics classrooms and cishegemonic research through a mix-gender research team including gender-nonconforming members (Jourian, 2017, p. 431). Like Rios et al. (2024), each team member created an individual poem by re-listening to the interview, selecting emotionally significant quotes, and juxtaposing them (Tremaine, 2022). Since each author created a poem, Mal's voice transcended from transcript data into personal connections for each of us (Lather, 1991; Richardson & St. Pierre, 2005). We shared drafts, provided feedback, and created a line bank from excerpts. Prioritizing Mal's preferences, we grouped lines by emergent themes and iteratively rearranged them to shape a cohesive emotional arc.

The final poem tells a hopeful story. It begins with Mal's previous view of math: "Math is a more straightforward thing // It's either right or wrong / No / You're wrong // It just ends there / People shutting down." It traces how their questions were dismissed and how identity shaped their experiences: "I would ask a question and I wouldn't really get an answer // I am a visibly queer person / Is this why this is happening...?" But in the IO course, something shifted: "The environment very much set the tone that it is absolutely okay to ask questions // I have never doubted that I am not being heard." Mal reflects on collaboration ("Working together, Not forced together"), how instruction recognized their whole self ("I am more than just the question I asked"), and growing mathematical agency ("I'm making this math / Not only taking this math"). The full poem illustrates that IO instruction can support belonging and joy in mathematics.

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An Analytic Perspective Expanding the Construct of Practice-Linked Identities for Undergraduate Mathematics Students

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Keywords: mathematical identity, analytic procedures, theoretical perspective, calculus

Mathematical identity is broadly defined as the dynamically negotiated self-understandings students develop in the context of mathematics (Martin, 2006). With growing interest in mathematical identity among researchers and emerging conceptualizations continuing to appear, Darragh (2016) critiqued that mathematical identity has been thought of inconsistently, in terms of studies not clearly aligning choices such as the definition employed, theories, or methods. As distinct studies have different purposes that will inform their choices, authors using varying conceptualizations would be less problematic if there was more "theoretical coherence" in individual studies, avoiding "defining identity in a contradictory manner to the methods used and conclusions drawn" (pp. 24, 28). In this poster, we will propose a clarified analytic expansion of Nasir's (2011) "practice-linked identities", which has been used in cultural community settings such as high school basketball, mathematics, and track (Nasir & Hand, 2008; Nasir & Cooks, 2009) but not yet explored further in terms of analytic considerations or undergraduate settings. We offer ways to explicitly analyze practice-linked identities of undergraduate students as they develop a personal perspective towards themselves linked to practices they engage with in mathematics classrooms. We will also outline empirical examples from our work with Calculus I students to illustrate the analytic perspective. We are guided by the research question, "How can analyzing undergraduate students' development of practice-linked identities be clarified?"

Expansion of Practice-Linked Identities

Our perspective meets Nasir's practice-linked identities that individuals develop as they build a connection to local practices with Ernest's (2009) social constructivist view towards learning in that there is a reflexive relationship between individuals' understandings and the social world they encounter. This reflexive relationship includes students' understandings towards who they see themselves as within contextual classroom practices. In that realm, we emphasize the weight of the student perspective in developing their sense of self and how that perspective impacts how they define and take on practices. We expand Nasir and Hand's (2008) insights that practices become salient when students achieve access to the domain (the discipline of the community, i.e., undergraduate mathematics), access to self-expression (the ability to enact personal meanings), or integral roles (making valued contributions) for the purposes of analytic considerations in two ways. First, we clarify that this access occurs from the student point of view, and next, we introduce that there also must be evidence of personal importance. That is, it is not enough to consider a student to have developed a practice-linked identity by one of the three access criteria alone; students should express personal connections by either citing why the practice personally matters, such as aligning with their values (Seah & Andersson, 2015), using first-person active voice (Morgan, 2006), or expressing emotion (Kaasila, 2007; Tannen, 1979).

In the poster presentation, we will detail our analysis from a study of six Calculus I students' accounts via interview data to illustrate considerations for what evidence should exist to classify units of analysis as part of a student's developing practice-linked identities. We will provide several examples and non-examples while connecting to the perspective discussed above.

Acknowledgments

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That's My Job!: A Preliminary Analysis of How Mathematicians View Teaching

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Keywords: Pedagogy, Abstract Algebra

As a person engages in the act of teaching, they make numerous decisions before, during, and after the instructional period. In making these decisions, teachers (mathematicians in our case) justify their decisions based on their perceptions of their obligations. For example, Fukawa-Connelly, et al. (2016) found that most mathematicians choose to lecture. Others, such as Woods and Weber (2020) corroborated these findings and suggested that lecture best served various goals such as the goal to cover certain content. While mathematicians engage in large scale decision making (such as lecturing), they also engage in smaller scale curricular decisions such as content ordering (e.g., Mirin and Rupnow, 2022). While research has started addressing why mathematicians make specific decisions, we began to ask the question, how do mathematicians conceptualize teaching and their role in the process– specifically which aspects of teaching and learning do mathematicians view as their responsibility and which aspects of teaching and learning are put on the student?

In this poster, we report on our initial steps towards addressing this question. Specifically, we report on our preliminary analysis of 14 mathematicians' answer to the questions: *What do you view as your role in teaching [abstract algebra]? What do you view as the students' role in learning [abstract algebra]?* The data originated from a larger dissertation study focused on pedagogical decision making in Abstract Algebra. Of these mathematicians, 4 self-identified as taking an inquiry-oriented approach, while 10 self-identified as taking a lecture-based approach. Each author independently conducted thematic analysis on the instructor role and student role questions. We then compared our themes and came to a consensus through discussion. We noticed six themes across how mathematicians conceptualized their role and four themes in how they viewed the role of students. From the instructors' view, the role of the instructor is: to spend time outside of class preparing for class and grading, to ensure that content moves forward within the class time frame, to inform overall trajectory of course content, to steward the discipline of mathematics - that entails providing correct information and fostering disciplinary norms, to spark students' interest in mathematics in order to motivate them to learn the math, to adhere to institutional guidelines. From the instructors' view, the role of the student is: to engage with the materials that I provide, engage with materials that I provide outside of class, to engage in self-exploration of the content, to positively interact with their peers.

We noticed that given information about how the instructor viewed their role, we could see who did or who did not identify as inquiry-oriented, with some exception. However, we noticed that there was no clear pattern in the relationship between instructional choice and how the instructor viewed the role of the student. This result is intriguing considering students tend to engage in different activities in IO classes compared to lecture based classes. This leads us to wonder if those patterns would hold true with more data. For this poster, we will report the results of our pilot analysis and present a study design which seeks to further address our research question.

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Empowering Undergraduates Through Community-Engaged Mathematics Outreach: A Model for Undergraduate-Led AMC Workshops

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Keywords: community engagement, STEM outreach, experiential learning

This study investigates the impact of undergraduate-led mathematics workshops on both undergraduate instructors and middle school participants. The workshops are built around topics from the American Mathematics Competition (AMC), providing a rich context for exploring problem-solving and conceptual understanding beyond the standard K-12 mathematics curriculum. Specifically, the study examines how designing and teaching these AMC-based workshops influence undergraduate students' communication, leadership, and research skills. It explores whether participation in these teaching experiences enhances undergraduates' confidence in explaining mathematical concepts, strengthens their collaboration and pedagogical abilities, and deepens their understanding of the broader societal impact of mathematics education. In parallel, the study investigates the extent to which middle school students demonstrate measurable knowledge gains after participating in the workshops. It assesses improvements in their problem-solving abilities and conceptual understanding through pre- and post-assessments, providing insight into how undergraduate-led initiatives can effectively support K–12 learning in mathematics.

This project builds on existing research highlighting the dual benefits of STEM outreach for both undergraduate instructors and K–12 learners. Studies consistently show that undergraduates who engage in teaching or outreach develop stronger communication, leadership, and content knowledge skills (Felege et al., 2022; Schalk et al., 2009). Feldon et al. (2011) found that graduate students with teaching experience demonstrated stronger research design and hypothesis development than peers focused solely on research.

STEM outreach further promotes confidence, collaboration, and public engagement skills essential for future educators and researchers (Curtis, 2017; Pickering et al., 2004; Philipp et al., 2016). Beyond undergraduate benefits, outreach programs have been shown to enhance younger students' engagement and conceptual understanding in science and mathematics (Volbrecht et al., 2019). Embedding such experiences within the undergraduate curriculum strengthens students' preparedness for professional STEM careers while fostering community connections and mentoring relationships (Caldwell et al., 2007; Pfund et al., 2012).

Overall, the research demonstrates that structured outreach creates reciprocal learning benefits—advancing STEM education through meaningful engagement between college and pre-college students.

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Addressing the Need for an Undergraduate Statistics Teaching Observation Protocol

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Keywords: introductory statistics, teacher practices, observation protocol, statistical knowledge for teaching

Historically, teacher knowledge has been categorized by the work of Shulman (1987): general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, content knowledge, curriculum knowledge, and pedagogical content knowledge (Shulman, 1987). Ball, Thames, and Phelps (2008) expanded on these ideas by identifying mathematical tasks of teaching and tying them to Ball et al.'s (2008) adapted domains for mathematical knowledge for teaching (MKT).

The work of Gleason, Livers, and Zelkowski (2017) led to the Mathematical Classroom Observation Protocol for Practices ($MCOP^2$). The instrument "examines teaching mathematics for conceptual understanding through lenses examining teacher facilitation and student engagement" (p. 113). This research has served as the basis for continued observation protocol work, such as that conducted by the CoMInDS program (Lauren & Haberler, 2017). Another instrument that has been developed around the concept of observing teacher practices is the Mathematical Quality of Instruction (MQI) (Hill, Schilling, and Ball, 2004) which was used to measure Mathematical Knowledge for Teaching (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008).

However, there are few resources that pertain to observing practices in statistics education. Cobb and Moore (1997) argue that statistics' inherent focus on variability lends itself to a need for contextualization. They point out that "design, exploration, and interpretation [which] are core elements of statistical thinking" (p. 805) do not fall under traditional mathematical skills. Likewise, content in statistics differs from typical mathematics content. Thus, it is necessary to teach statistics differently than mathematics.

Groth's (2013) statistical knowledge for teaching (SKT) framework adds to Hill et al.'s (2004) domains with key developmental understandings (Simon, 2006, as cited in Groth, 2013), and pedagogically powerful ideas (Silverman and Thompson, 2008, as cited in Groth, 2013). Callingham and Watson (2011) published research on levels of SKT: aware, emerging, competent, and accomplished. In a hypothesized model of SKT, Batur (2021) embedded Callingham and Watson's (2011) levels with Groth's (2013) framework to develop a five domain framework that can be assessed at four levels, all with the aim of centralizing statistical literacy.

It is using this model that I believe I could create items for the development of a new instrument to observe a teachers' current abilities in statistical teaching, and potentially assess or evaluate their SKT. Therefore, my current questions for this study are 1) How are teachers currently enacting SKT in the classroom? And 2) How can we observe and evaluate SKT in the classroom? My plan to answer these questions would be to 1) observe for qualitative data collection what is currently happening in statistics classrooms, and 2) use the extant literature and my observations to find or write items for a new observation protocol. The target population for this instrument would be AP Statistics high school teachers or introductory statistics college course instructors. Further development of the instrument would require a validation study.

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Understanding Strategies of Multiplication and Division Through Worked-Out Examples Using an Inquiry-Based Approach

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Keywords: worked-out examples, multiplication, division, inquiry-based approach

Aim and Literature

The study aimed to enhance prospective elementary teachers' conceptual understanding of multiplication and division strategies. This project examined whether combining worked-out examples with an inquiry-based approach could improve comprehension, confidence, and instructional readiness.

Cognitive science shows that worked-out examples with step-by-step reasoning build procedural and conceptual understanding when paired with self-explanation and peer discussion (Atkinson et al., 2000; Renkl, 2002; Chiu & Chi, 2014). Inquiry-based learning fosters exploration but can overwhelm learners without scaffolding (Kirschner et al., 2006). Integrating both provides structure, guided inquiry, and collaborative meaning-making in mathematics.

Methods

Pre-service elementary teachers were randomly placed in groups of four and given worksheets with fully solved examples. Each group studied two different multiplication or division strategies (e.g., area model, lattice, box method, repeated subtraction, grid, and traditional algorithms). For 15–20 minutes, groups discussed the correctness of each solution and compared methods, leading to rich inquiry about how multiplication strategies relate to identifying divisors in division. After the initial discussions, a carousel activity began: two members from each group rotated to teach their strategies to another group while learning two new ones from the remaining pair. Each rotation lasted 20 minutes, and after four rotations all teachers had learned all eight strategies before returning to their original groups. Throughout the activity, students engaged actively, communicated clearly, identified patterns, and explained their reasoning, showing deeper inquiry than in previous semesters.

Discussion

Students exhibited greater mastery and confidence in explaining and applying all multiplication and division strategies. Students expressed satisfaction and reported that teaching peers enhanced their confidence and understanding. Combining worked-out examples with inquiry-based instruction created an engaging and supportive learning environment where all students participated meaningfully. The approach promoted mathematical discourse, peer collaboration, and self-explanation, key components of conceptual understanding. Integrating worked-out examples within an inquiry-based framework effectively supported preservice teachers' conceptual and procedural fluency in multiplication and division. This approach empowers future teachers to confidently teach diverse mathematical strategies and foster conceptual understanding among their own students.

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Mathematics Self-Efficacy in Collegiate Calculus

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Keywords: beliefs, calculus, mathematics self-efficacy

Mathematics self-efficacy (MSE) refers to students' beliefs in their ability to successfully perform mathematics (Bandura, 1997). MSE is an important construct because it has been shown to impact students' perseverance, college major choice, likelihood of remaining in college, and success or failure in mathematics (Gill, 2019; Grigg et al., 2018; Hackett, 1985; Street et al., 2017). Generally high MSE has been linked to more beneficial outcomes, such as higher grades or choosing a STEM major. MSE can be measured globally, for an overall subject like mathematics, or locally, for specific topics or problems (Bandura, 1997).

A review of self-efficacy in higher education found that students' self-efficacy can change in higher education and that interventions designed around social cognitive theory or practical experiences that allow students to feel successful can improve self-efficacy (van Dinther et al., 2011). For MSE and MSE in calculus specifically, less research has been conducted though Nicolescu (2015) found that calculus at a community college generally raised students' MSE.

Research Design

This study examined the global and local MSE of students enrolled in collegiate calculus, with the goal of identifying and qualitatively describing cases of participants who reported changing MSE. Participants were 14 students enrolled in a first collegiate calculus class. A semi-structured interview was used to explore participants' mathematics experiences prior to calculus, their experiences in calculus, and their global self-efficacy beliefs before calculus and in calculus. Participants were also asked to rate their local MSE for three calculus problems (one on evaluating a limit graphically, one on finding a derivative that required the product rule and knowledge of the derivative of trigonometric functions, and one on using the first derivative test to find local extrema). Participants were asked to describe why they felt that confidence level and what solution strategy they would use to solve the problem.

Findings and Implications

Results show that some participants reported changed global MSE because of calculus, with 7 reporting lower MSE in calculus than previous mathematics courses, 4 reporting higher MSE in calculus than held previously, and 3 reporting the same MSE in calculus as other mathematics classes. Participants reported reasons for changes to their MSE included difficulty of the exams, pace of the courses, and differences in teaching style. For local MSE, participants were generally confident on the problems, reporting the highest confidence in finding the procedural derivative using the product rule (mean confidence 4.5 out of 5) and the lowest confidence on using the first derivative test (mean confidence 3.71 out of 5). Results have implications for our knowledge of MSE, a relatively stable belief that can still change under certain conditions (Bandura, 1997; Byars-Winston, et al., 2017; Friedel et al., 2010). Results also have implications for collegiate calculus instructors who wish to better support their students.

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Relationships Between Pre-Service Teachers' Decentering Actions and Quantitative Reasoning

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Keywords: Quantitative Reasoning, Decentering, Pre-Service Teachers

The emergence of quantitative reasoning—conceiving situations in terms of measurable attributes and relationships between those attributes (Carlson et al., 2002; Karagöz Akar et al., 2022; Thompson, 2011; Thompson & Carlson, 2017)—as critical for K-16 students' mathematical development has generated a need for exploring its role in preparing future teachers. Also related to supporting K-16 students' mathematical development is the act of decentering, which involves a teacher putting aside their own ways of thinking for the purpose of discerning and building a model of their students' thinking (Baş-Ader & Carlson, 2022; Carlson et al., 2022; Ellis, 2022; Hackenberg et al., 2024; Silverman & Thompson, 2008; Teuscher et al., 2016). Notably, researchers have recently illustrated an important link between teachers' decentering and quantitative reasoning capacities (Tallman & Frank, 2020; Tallman et al., 2024).

In this poster, we explore secondary mathematic pre-service teachers' (PSTs') decentering actions and their quantitative reasoning. Specifically, we draw on clinical interview data to illustrate relationships between the PSTs' enacted meanings and the ways in which they appraise secondary students' solutions and reasoning. The PSTs' enacted two primary types of meanings. As one type, they drew on meanings that consisted of declarative facts and indexical associations. For example, some PSTs drew on a learned fact that involved a type of function (e.g., a quadratic function) necessarily being associated with a shape (e.g., curving up). As a second type, they drew on meanings that involved enacting quantitative operations to make sense of the posed solutions and reasoning. These PSTs made sense of a graph by considering how it might have emerged via the covariation of the two quantities along the axes (e.g., y changes by three times the amount x changes). We noted that in some cases, PSTs drew on both types of meanings to make sense of the posed student solutions and reasoning.

With respect to the PSTs' decentering actions, their dispositions varied depending on the enacted meanings. When PSTs enacted meanings foregrounding declarative facts and indexical associations, they typically focused on the extent the posed solutions and reasoning either violated or maintained those learned facts and associations. For instance, if a given graph had a learned shape they associated with a function the secondary student intended to graph, they deemed it correct. If it did not have the learned shape they associated with the function, they deemed the student incorrect regardless of the student reasoning presented. When PSTs enacted meanings foregrounding quantitative reasoning, they predominantly searched for various ways posed student solutions and reasoning can be considered a viable and correct solution to the posed task. For instance, regardless of the familiarity of a given graph's shape, these PSTs considered conventional and unconventional ways in which the graph might convey the quantitative relationships the graph was to convey. This enabled them to account for unfamiliar student reasoning that was viable. In general, we illustrate that PSTs' quantitative reasoning provided a flexible grounding for considering students' diverse ways of reasoning as compared to instances in which PSTs enacted declarative facts or indexical associations.

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Readiness for the Next Disruption: Lessons From Instructors' Struggles During COVID-19

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Keywords: Crisis-readiness; Pandemic pedagogy, Emergency remote teaching, STEM instruction, Resilience theory

When higher education rapidly shifted online in 2020, instructors across STEM disciplines faced fundamental challenges. They include academic integrity concerns, inequitable access to technology, insufficient institutional training, unclear communication, and significant emotional strain (Daniel, 2020; Pokhrel & Chhetri, 2021). Using a national dataset of 934 chemistry, mathematics, and physics instructors' open-ended survey responses, this study investigates which pandemic-era challenges instructors faced during pandemic response teaching.

This poster is theoretically grounded in resilience theory (Hoiling, 1973; Norris et al., 2008). While it can be argued that undergraduate education has survived the pandemic, the disruption it ushered suggests that a strategy of survival is insufficient. To better prepare for the potential disruptions in the future, it is helpful to adopt a strategy of "community resilience" that integrates robustness, redundancy, resource mobilization, rapidity, communication, and social capital into a web of "adaptive capacities" (Norris et al., 2008, p. 127). Applying these frameworks to instructional systems, resilience is not simply the readiness to make the in-person-online transition on the outset of the next disruption. Rather, it is the extent to which such a move does not compromise academic integrity, dismantle professional communities, widen inequity, or encroach on student learning.

This study draws on text responses in a large-scale survey of STEM instructors across U.S. institutions. We conduct a qualitative thematic analysis focused on two research questions:

1. What pandemic-era instructional challenges did instructors face?
2. What capacities or structures did they identify as lacking?

Preliminary review reveals recurring themes: (a) insufficient technological preparation and emergency training; (b) academic integrity and assessment validity concerns; (c) lack of communication and unclear expectations; (d) inequitable access to devices, software, or stable internet; (e) emotional exhaustion and social isolation for both faculty and students; and (f) the need for departmental and institutional infrastructure that is proactive rather than reactive. These findings indicate that in addition to asking whether instructors can teach wholly online again, more focus should be on how they define readiness—technologically, pedagogically, institutionally, and psychologically. Findings can inform a framework that explicates fundamental criteria for emergency response teaching preparedness needed in STEM departments: rapid development of online assessments resistant to cheating, proactive instructor training, better communication systems, equitable access to digital tools, and structures that preserve community and belonging.

This poster contributes to the growing scholarship on lessons learned from COVID-19 by: (1) shifting the conversation toward future-focused resilience; (2) documenting instructors' own reflections on challenges; and (3) identifying the kinds of instructional, technological, and institutional capacities STEM that we can be investing in now.

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Leveraging Open Educational Resources and Interactive Technologies to Enhance Learning in Undergraduate Mathematics: A SoTL Approach

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This study investigates the impact of Open Educational Resource (OER) modules on student learning and perceptions in a mathematics content course for elementary education majors. Rising textbook costs have created significant barriers to student access and learning (Perry, 2022; Osterman, 2021). Research demonstrates that interactive technologies, such as H5P and GeoGebra, can increase student engagement, motivation, conceptual clarity, and self-managed learning (Shadaan & Eu, 2013). Additionally, OER initiatives have been associated with both academic and economic benefits for students (Thurman, 2020).

Keywords: Open Educational Resources, Teacher Education, Elementary Content Courses

Methods

Fifteen weekly OER modules were developed using H5P and GeoGebra and embedded in the university Learning Management System, replacing a $45 publisher inclusive access homework platform. Student outcomes were assessed via final exams and surveys measuring engagement, perceptions of learning, self-paced learning, and feedback utilization. Using he Scholarship of Teaching and Learning (SoTL) provides a research-based framework for systematically examining how teaching practices affect student learning outcomes. Additionally, the study explores whether interactive OER activities enhance engagement, understanding, and self-directed learning of undergraduate education majors, and considers how this approach can be applied to other mathematics courses.

Results and Discussion

Initial analysis comparing final exam scores across semesters prior to the OER modules and those using the OER modules indicated no statistically significant difference ($P = 0.1151$), suggesting that students using the OER modules performed comparably to those using the publisher platform. Despite similar performance outcomes, student survey responses revealed strong positive perceptions of the OER activities. Approximately 79% of students reported that the interactive modules helped them learn the course content, 88% valued the ability to work at their own pace, and 94% indicated that immediate feedback encouraged continued engagement. Qualitative feedback reinforced these findings, with students noting that the weekly OER activities were more engaging and beneficial than the traditional platform, increasing motivation and enjoyment in learning.

While OER modules did not significantly alter final exam performance, they positively influenced student engagement, motivation, and perceptions of learning. The ability to work independently and receive immediate feedback fostered self-directed learning and a more meaningful learning experience. Framed through the SoTL perspective, this study illustrates how systematically investigating teaching innovations can provide actionable insights to improve instructional practices and support student success. This model can be applied to other undergraduate mathematics courses, promoting affordable, interactive, and evidence-based teaching approaches that enhance both learning outcomes and student confidence, particularly for prospective teachers.

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Examining the Nature of Participation in NREUPs: Narratives of Racially Minoritized Students

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Keywords: Mentoring, racially minoritized students, REUs, undergraduate research experience

The Mathematical Association of America's (MAA) National Research Experience for Undergraduates Program (NREUP) provides opportunities for racially minoritized students to collaborate with faculty in the mathematical sciences on summer research projects at a diverse range of institutions. Through a rigorous and accessible research experience, NREUP's goals are to increase undergraduate degree completion rates and ignite students' interest in obtaining graduate degrees and embarking upon careers in the mathematical sciences.

Since 2003, the NREUP has supported approximately 100 faculty members and 768 students, with about half of them identifying as Black/African American and about 35% as Latinx/Hispanic. In this poster presentation, we will share results of the analysis of follow-up interview data from students who participated in the NREUPs during the summers of 2019-2024. This work focused on this time frame as part of evaluative efforts examining the grant project's five-year period. Of the 151 former NREUP students, one third did not provide information about their post-undergraduate whereabouts, and the remaining students were placed into two distinct categories — (1) those who were attending graduate school in the mathematical sciences and (2) those who were pursuing careers in the mathematical sciences.

We randomly selected 10 students from each category to capture the longitudinal impact of the program. Ten students from category 1 participated in the study; however, only eight students from category 2 did so. The reason why we did not reach our target number of participants from category 2 during the given time frame (until August 31, 2025) was because we did not have access to the working professionals' email addresses. As a result, our total sample size consisted of 18 former NREUP students from racially minoritized groups. Our research question was: What is the nature of NREUP participants' experiences from diverse institutions?

This qualitative study was evaluative in nature (Patton, 2015). We conducted online, semi-structured interviews that lasted about 50 minutes. Sample interview questions included: In what ways did the program shape your decision about whether or not to attend graduate school? In what ways did the program shape your decision about your future career? Each researcher read through the individual transcripts to engage in preliminary analysis and highlight the positive aspects and areas of improvement for NREUPs. Then, we used open coding to perform data analysis and to extract overarching themes (Saldaña, 2021). Next, we explored math identity work and mapped our findings onto this theoretical frame (McGee, 2015; Voigt et al., 2023).

Across both categories, students consistently reported gains in knowledge about the research process, the benefits of having formal and informal mentors to learn more about discipline-based opportunities, and the advantages of collaborating with undergraduate and graduate student peers. Students also praised the structure of the program and expressed that it led to increased confidence as emerging scholars within the mathematical sciences. Given this, we highlight potential avenues for future RUME work that considers different programmatic structures and mentoring approaches associated with NREUPs. In all, our study extends prior work regarding undergraduate research experiences (Castillo-Garsow & Castillo-Chavez, 2016; Sturner et al., 2017) by following up with participants from racially minoritized groups who participated in NREUPs.

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Exploring the Relationship Between Language and two Problem Types

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Keywords: Problem types, Language, Mathematics register, Mathematical domains

Much of the study of mathematics is reducible to the problems students are asked to complete. We are thus motivated to explore the language within problems – which would be part of the mathematics register (Halliday, 1975; Pimm, 1987) or even its own genre (Bakken & Andersson-Bakken, 2021). Yet, whereas mathematical terms are often both well-defined and consistently used, the language used for posing mathematical problems – our interest – is neither.

In a previous RUME report, Wasserman and Dawkins (2025) elaborated 8 problem types. In that preliminary theoretical paper, two common problem types across K-16 mathematics were noted as distinct, yet at times difficult to differentiate. SOLVE problems were about producing "the complete solution set of mathematical objects that satisfies a given set of properties" and EVALUATE problems were "the outcome of a procedure or operation" (p. 1023). The first might be understood as a search for particular elements in set builder notation, e.g., $\{x \mid +(5,x)=12\}$; the latter as the output of a function, e.g., +(5,7). This proposal reports on work that further explored these two categories of problems in relation to the language used to pose them.

Methodology

Specifically, our work was guided by the following research question: "What, if any, linguistic features mark and distinguish prototypical SOLVE and EVALUATE problems across K-16 mathematics textbooks?" Five coders worked on the project. After discussing the two problem types, each selected a different elementary, middle, high school, and college textbook; and each added about 75 problems per book, from its breadth, that they felt could be unambiguously categorized as either SOLVE or EVALUATE. Coding then included the "problem stem" – often an imperative verb (e.g., Find), though sometimes an interrogative (What is) – and the problem type, along with some other basic information (e.g., grade level). Next, the research team coded each other's problems, with disagreements being adjudicated or problems being removed as ambiguous. The result was a collection of (n=915) problems across K-16 that the five coders could unambiguously code as either SOLVE or EVALUATE.

Findings and Implications

From this work, we briefly share three claims (data supporting them, and others, will be shared on the poster): i) Some verb stems were distinctly associated with a problem type, e.g., "Add" and "Compute" were EVALUATE whereas "List" and "What are" were SOLVE, and other common verb stems like "Find" and (interestingly) "Solve" may have been either type of problem; ii) Two-verb structures (e.g., Use… to determine…) were primarily EVALUATE; and iii) Grade level impacted the distribution of problem types as well as particular verb stems.

We see this work as valuable because it informs how we might clarify to students the nature of a mathematical problem – either as SOLVE or EVALUATE – by incorporating specific linguistic choices into how we pose it.

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Why are We in the RUME?

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Keywords: Conference, Motivation, Career Development

Academic disciplinary conferences are believed to be critical sites in academic trajectories (Henderson, 2015), yet the people who attend them likely have many different reasons for doing so. These roles and motivations remain under-researched. In this study, we examine both the intellectual and social community of the SIGMAA on RUME Annual Conference as part of people's decision to attend (or not), including barriers and issues of access to the community.

We developed a survey based on the work of (Fukawa-Connelly et al., 2025). The survey included demographic questions (about participants' conference attendance, home department, career stage, and research expectations), questions about their current and first reasons for attending the SIGMAA-RUME conference, and questions about the participants' experienced barriers to participation. Links to the survey were posted in various locations at the 2025 SIGMAA-RUME conference and via the RUME Google Group listserv. In total, 165 people completed all survey items, and five more completed at least two-thirds of the survey.

RUME participants reported a wide range of reasons as "most" or "very" important for attending the conference. The three reasons with the largest share rated as "most" important were "being part of a warm/welcoming community" (41%), feeling like the conference is their academic or intellectual home (38%) and networking (34%). When including ratings of "very important," the most commonly reported reason is learning about other people's work because it's interesting (72%), although many other reasons were close behind. The two reasons participants were most likely to rate as either "somewhat" or not important were participation as a job expectation (84%) and advancing their career (54%). These reasons differed slightly depending on the career stage of the participant. For example, graduate students and new faculty members were much more likely to assign increased importance to "advancing their career," while experienced faculty members were much less likely to do so. However, within each career stage, the relative importance of the various reasons stayed roughly consistent. Comparing participants' reasons for attending the conference to their reasons for *first* attending suggests modest shifts as participants become more experienced.

A total of 91 participants identified at least one barrier to their recent participation in the SIGMAA-RUME conference. These barriers included (in decreasing frequency) the feeling of a lack of belonging, financial barriers, the perceived relevance of conference topics, the feeling of "imposter syndrome," scheduling and workload conflicts, accessibility, and the political climate.

These findings suggest that community—both intellectual and social—matters when people decide whether to attend the conference. These aspects bring people to the conference initially and lead them to return. The disciplinary focus, tightly connected research groups, and personal relationships mean that people can feel "at home" both as scholars and as people, but these same aspects can also pose barriers to participation.

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From Belief to Practice: Investigating Growth Mindset as a Bridge Between Teacher Cognition and Student Experience in Mathematics

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Keywords: growth mindset, motivation, confidence, mathematics, beliefs

Teacher beliefs are integral in shaping classroom culture as they affect the teacher-student relationship, students' learning-related processes, and learning outcomes (Wolf & Brown, 2023). Yeager et al. (2019) believes that embracing a growth mindset can enhance students' academic outcomes and teachers can cultivate growth mindset cultures in mathematics classrooms through specific practices and interactions (Murphy et al., 2021). Many students experience negative attitudes in math-related contexts, and these responses result from lack of confidence in solving mathematical problems, which has an impact on the student's motivation and learning (Aguilar, 2021; Metu et al., 2023; 2025). These negative responses could be impacted by the teacher's belief found to be associated with an educational system's overall efficacy (Li & Ma, 2025). Teachers with growth mindset are able to influence their pedagogical decisions and their classroom conduct (Laine & Tirri, 2023; Zhang & He, 2025). However, previous studies focused primarily on student-centered beliefs, while giving little attention to the teacher's role in shaping classroom behavior, especially in mathematics classrooms. Despite growing literature on the importance of growth mindset in classroom settings, there is limited literature on how undergraduate mathematics teachers' growth mindset influences students' belief, motivation and confidence. This study examined: (i) the beliefs teachers hold about the nature of mathematical ability and learning; (ii) the extent the teachers' growth mindset predicts students' beliefs, motivation, and mathematics confidence.

This study adopted a descriptive correlational research design, which allowed for both understanding the teacher's perception of growth mindset and determining the strength of the relationships between the teacher variable and the students' variables. The sample size included Seven (7) teachers and 181 first year students from the mathematics education unit, University of Nigeria, Nsukka. The instruments "Implicit Theory of Intelligence Scale (Dwerk et al., 1995)" for teachers and students growth mindset; "Student Opinion Scale (Sundre, 2000)" for motivation; and "General Self-Efficacy Scale (Schwarzer & Jerusalem, 1995)" for students' confidence were all adapted. These surveys (with likert scale 1-5) were distributed and collected through the face-to-face approach. The collected data were analyzed using descriptive statistics and regression analysis.

Result showed that $Mean = 3.73; SD = 1.04$ was obtained about the teachers' belief. While the scores were moderately dispersed, the majority of the teachers leaned towards the growth mindset belief. This implies that the teachers support the principles of growth mindset that with practice and strategies, an individual's ability in mathematics can be developed. Regression analysis revealed that teachers' belief accounted for 14.4% of variation in students' growth mindset($r = 0.38$), 21.2% of motivation($r = 0.46$), and 2.9% of confidence($r = 0.17$). Therefore, students are encouraged to seek out faculty with growth mindset attributes as they are more inclined to creating a supportive learning environment. This result will also encourage students to form peer groups that share a growth mindset, thus increasing their motivation and participation.

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Integrated Math/Biology Lessons for Approaches to College Mathematics

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Keywords: Incoming Math Students, Graphing, Integrated STEM, Design Experiment

We report on our analysis of student reasoning during the first iteration of a design experiment (Cobb et al., 2003) in which we implemented a lesson that integrates mathematics and biology content through graph interpretation activities. Montclair State University (MSU) has seen an increase in the number of first-year STEM majors who fail college Precalculus, contributing to lower retention and graduation rates. In response, the MSU math department developed a course called Approaches to College Mathematics (ACM) to prepare these students for Precalculus. Research shows that integrating graphing with other STEM content can lead to improved learning across the integrated content areas (e.g., Basu & Panorkou, 2019; Gültepe, 2016; Panorkou et al., 2022). We conjectured that activities we designed for ACM with an integrated STEM context might support students in reasoning emergently about graphs and their covariational meanings. Moore and Thompson (2015) define emergent (in contrast to static) shape thinking as reasoning about a graph as a meaningful trace of the relationship between two covarying quantities. Therefore, we explored the following research question: What kinds of reasoning do ACM students engage in as they work with the integrated lesson we designed?

The lesson consisted of a whole-group graph comparison task, an in-depth analysis of graphs in student pairs, and a concluding presentation by students to showcase their learning. Our analysis was based upon multiple data sources: student handouts, group digital posters, and instructor (second author) field notes. Two researchers, neither of whom were the instructor, independently coded the student data, employing a qualitative coding process that began with deductive analysis and then moved into inductive analysis (Hammersley, 2004; Patton, 2015). After discussion and comparison of codes, there was 100% intercoder agreement.

We noted three major categories to classify students' static versus emergent thinking, as well as integrated versus unintegrated reasoning: unintegrated (biology only) with no shape thinking, unintegrated (math only) with static shape thinking, and integrated thinking (both biology and math) with emergent shape thinking. We found that these students engaged in reasoning about integrated math and biology content only when they were engaging in emergent shape thinking. When their shape thinking was static, their reasoning was unintegrated. We argue that this shows the potential for lessons with an integrated design to foster deeper mathematical understanding in ACM students. At the same time, we noted that students struggled with details of mathematical terminology and notation, such as the origin, domain and range, scale, and the labeling of points. We also noted a significant amount of interval-point confusion, a long-known graphing difficulty (Leinhardt et al., 1990). These difficulties are typical of the struggles ACM students face and may be exacerbated by the differences between graphs that appear in biology and math texts.

Our poster will include the research design, graphs and interpretation tasks, students' written responses, and a discussion of how we make sense of their integrated reasoning through a static-emergent shape thinking framework. Student data will be showcased, organized according to major themes from our findings.

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Reimagining Calculus Through Team-Based Course Redesign

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Keywords: Transformational change, Calculus redesign, Value-based vision, Team efforts

Change guided by a shared vision and aligned with a department's cultural norms can be transformative. With a charge from administrative leadership, Notre Dame (ND) Learning has established the Foundational Course Transformation Academy (FCTA) as a team-based process grounded in literature to facilitate sustainable change. Currently, ND Learning is partnering with the Department of Mathematics to redesign the Calculus I course with an eye toward building learning-centered environments that are better for students and instructors.

According to Henderson et al. (2011), effective change strategies must align with the beliefs of those involved, avoid uniform policies that deter individualized solutions, and be compatible with the complex ecosystems of the university. Reinholz & Apkarian (2018) emphasize that sustainable changes require attending to the structures, symbols, people, and powers that define a department. Quardokus Fisher & Henderson (2018) and Quardokus Fisher & Koretsky (2021) further argue that change is impactful when it grows from faculty's concrete visions, aligns with their aspirations, and provides them with autonomy and resources.

In Phase 1 of the process, informed by this literature, the FCTA team created community norms, reviewed literature on course redesign and effective mathematical practices, and created a shared value-based vision. They also engaged with stakeholders within and beyond the Department of Mathematics, benchmarked with peer institutions, and acquired data from the Office of Institutional Research. Aligning their individual proposals with the data, stakeholder feedback, and relevant literature, the team crafted the following transformational goals: a) implementing a learning cycle of exploration, sense-making, formalization, and application (Schoenfeld, 2016); b) redesigning content to include targeted precalculus review; c) redesigning assessments to align with the five strands of mathematical proficiency (National Research Council, 2001); d) embedding holistic students' support structures (Johnson & Hanson, 2015); e) restructuring course coordination and formalizing ongoing training and support for graduate student instructors and postdocs (Rasmussen & Ellis, 2015); and f) designing 1-credit support courses (one for pre-calculus review, and others connecting calculus to disciplinary contexts).

Currently, the team is in Phase 2, applying the backward design framework (Wiggins & McTighe, 2005) to define course competencies and learning outcomes aligned with the transformational goals crafted in Phase 1. Next, the team plans on creating instructional material (syllabus, modules, assessments) informed by the learning outcomes, while continuously sharing progress and receiving feedback from stakeholders. In Phase 3, the team will implement the redesigned course, meet regularly with instructors, revise and assess the course redesign, develop a sustainability process, and conduct further research. In this poster, we share our redesign process, along with how we plan to assess it for faculty, administrators, and students involved.

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Navigating Calculus: Challenges and Assets of First-Generation STEM Students

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Keywords: Assets, Calculus, First-generation students, Persistence

Students tend to encounter challenges when facing gateway courses, such as calculus. For example, calculus courses tend to have high failure rates (Bressoud et al., 2013), discourage students from persisting because of certain classroom experiences (Rasmussen & Ellis, 2013), and have an implicit mechanism that pushes students away from STEM (Olson & Riordan, 2012). There is little research on how first-generation college students navigate calculus. However, there is research that has shown these students often face challenges throughout their college journey, such as financial hardship, work and have caregiving responsibilities, gaps in academic preparation, and feelings of isolation (Chen & Carroll, 2005; Ardoin, 2017; Covarrubias et al., 2019).

Nevertheless, Gutiérrez (2008) warns that research on the achievement gap overshadows the advancement of marginalized groups and "suggest[s] a research agenda that focuses on advancement, on excellence, and on gains within marginalized communities" (p. 362). Therefore, this poster will also present findings on the assets first-generation college students activated while navigating through calculus. That is, I will present what specific challenges they encountered throughout their calculus journey and the assets that they employed that supported their persistence. To stay consistent with literature, I define first-generation college students as those whose parents or guardians have not earned a four-year degree. My specific research questions are:

1. What obstacles do first-generation STEM students encounter as they transition into and navigate their calculus journey?
2. What assets do first-generation STEM students draw on to support their persistence in their calculus journey?

I will note that when I refer to their calculus journey, I am including what they experienced within the university and outside the university. Sometimes these obstacles get intertwined with their experience taking their semester long course, which participants still need to navigate with challenges that happen outside the classroom. Moreover, all participants passed their calculus course and took the course within the same university.

This study uses interpretative phenomenological analysis (IPA) to explore how participants understand and reflect on their lived experiences. Rather than stopping at description, IPA seeks to highlight participants' perspectives while also interpreting the significance of their accounts (Larkin et al., 2006). The approach combines three elements: phenomenology, hermeneutics, and idiography. Phenomenology guides the focus on participant's shared experiences, in this case, how they navigate and persist through calculus. Hermeneutics introduces a double layer of interpretation. That is, participants work to make sense of their own experiences, and the researcher then interprets those reflections. Idiography centers on detailed analysis of individual cases before drawing connections to broader patterns across participants.

My poster will present initial findings from eleven STEM, first-generation students who are reflecting on their calculus journey from their semi-structured interviews. I look forward to sharing my study with the RUME community and gathering feedback on how to conceptualize their calculus journey in relation to challenges and assets.

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The Impact of Attending Mathematics Tutoring on STEM Retention

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Keywords: explanatory sequential design, mixed methods, STEM retention, peer tutoring

University students who choose STEM majors often do not complete their degrees in STEM. A 2009 study found that 55% of STEM majors leave STEM fields altogether (Chen, 2009). Many STEM majors require a sequence of mathematics courses. The PCAST 2012 report found that engineering majors who were enrolled in math courses that utilized active learning strategies were more likely to persist in engineering majors (PCAST, 2012). Students have been shown to be more actively engaged during a peer tutoring session than in lecture, as measured by the number of questions asked by the student (Graesser & Person, 1994). This study examines how attending peer tutoring for mathematics courses impacts STEM retention.

Multiple studies have found that attending peer tutoring correlates to higher grades in mathematics classes (Byerley et.al, 2018; Rickard & Mills, 2018; Xu, et al., 2001). In fact, a study of ten mathematics tutoring centers in the USA found that five centers had a statistically significant positive correlation between course grades and number of visits, three had no significant correlation, and two had a negative correlation (Byerley et al., 2022). Clearly, passing required mathematics classes is necessary for a student to persist in their major, but there may be other social benefits to attending tutoring such as students' sense of belonging and formation of a positive mathematical identity (Bjorkman, 2019).

Our research questions are: What is the relationship between math tutoring center usage and STEM persistence? What do students identify as factors contributing to their STEM retention?

For this study, we employed an explanatory sequential mixed methods design. Our first quantitative phase examined the statistical correlation between tutoring visits and STEM persistence. These results informed the second qualitative phase which consisted of interviews with seven STEM majors who attended tutoring.

We gathered quantitative data for all STEM majors every semester for a two-year period. These measures included their major at the beginning of each semester, math course grades each semester, and the number of visits to the math learning center each semester. We utilized a logistic regression model to determine the likelihood of STEM retention over the two-year period based on the students' passing rate for their math courses and average tutoring attendance per course. We found a statistically significant positive correlation between tutoring visits and persistence, and that the odds of persisting in STEM increases by 10.92% for each additional visit to the MLSC per math course.

We then recruited STEM majors who passed 4 or more math courses to participate in an interview. Seven students agreed to participate, and six of these students utilized the math learning center. We utilized the constant comparative method (Glaser & Strauss, 1967) to identify themes in the interview data. All seven said that they had mostly positive experiences in their mathematics courses. They viewed the peer tutors as helpful and several mentioned that they met up with friends at the math learning center. Several of the students discussed how the tutors helped him understand the process but didn't give away answers, which implies that they were actively engaged in the tutoring session. They also mentioned other spaces where they connected with classmates to study, both virtually and in person.

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The Role of Self-Efficacy in Student Personas for Redesigning a Calculus Recitation

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Keywords: Calculus, Self-Efficacy, Recitation, Personas

A common format for Calculus 1 at large universities across the United States is a large lecture with an accompanying recitation led by a graduate teaching assistant. A Merit recitation program, based on a model developed by Uri Treisman (1983) at the University of California, Berkeley, replaces the typical recitation with a Merit one that meets for twice as long. Merit recitations also have smaller class size and a specifically trained facilitator who guides students through tough problems and understanding the core concepts of calculus. Our design problem concerns building personas of students participating in Merit recitation to understand (1) their choice to enroll, (2) their in-recitation learning experience, and (3) self-efficacy. Personas, which are "life-like archetypes that stress users' diverse goals and motivations and embrace their corresponding needs and challenges" (Huynh et al., 2020) can help a Merit program director to predict how students will engage with the program and can guide program refinement to more fully support students in achieving their academic goals. Our Merit student personas address mathematical self-efficacy. Bandura argued that self-efficacy, an individual's belief that they can successfully execute a behavior to produce a desirable outcome, influences the effort and persistence a person puts into a task (Bandura, 1978). Further, he posits four sources of information that strengthen or hinder self-efficacy: performance accomplishments, vicarious experiences, verbal persuasion, and emotional arousal. Students can provide insight into how and why these sources of information change students' self-efficacy (White, 2025).

To generate our personas, we conducted two in-person focus groups with a total of 12 students. These were audio-recorded, transcribed by AI (Otter and Rev), and cleaned by the first author. Both authors independently coded the transcripts for sources of self-efficacy, then met and discussed codes to reach consensus on all codes. Code mapping was used with each source of information to form the basis for the development of each persona (Saldaña, 2016). From these results, we constructed 2 personas - Maddie and Jacob - who had distinct motivations for enrolling in a Merit-style recitation. Maddie had low mathematical self-efficacy coming in and just wanted to pass calculus. Maddie's parents strongly encouraged her to enroll, affirming that she needed the additional support outside of the large lecture. Jacob had at least modest mathematical self-efficacy in general, however had very low self-efficacy with respect to passing Calculus 2 at our institution, which they had heard was "apparently really hard." His parents were more neutral than Maddie's, telling their student to decide for himself if he wanted to take Merit. Being able to ask questions during their Merit recitation was extremely important to both student personas. Jacob just needed the time and space for questions, especially in contrast to lecture and regular-timed recitations. In addition to time and space, Maddie additionally needed to feel safe to ask "humbling" questions, particularly about prerequisite material.

With these newly developed personas, the Merit director better understands the role of parents in the enrollment process and can better market the program. Additionally, the director can stress particular aspects of the Merit graduate teaching assistant training, to better support students' comfort in asking questions during recitation. Thus, these personas guide structural adjustments that better support the wide variety of students who choose a Merit recitation.

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Mathematicians' Views on Intuitive Concepts in Abstract Algebra

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Keywords: Abstract Algebra, Intuitive Knowledge, Intuitive Concepts, Mathematicians' Beliefs

Developing an intuitive feel for mathematical objects (e.g., concepts, problems, principles, languages/representations, etc.) (Godino et al., 2022) and thinking intuitively is considered an important skill as students transition into more advanced university coursework (Burton, 1999; Davis et al., 2012). While developing an intuitive feel for the mathematics is important, many studies have documented that abstract algebra, a foundational undergraduate course for advancing math majors and pre-service teachers, continues to be difficult for students to grasp (e.g., Melhuish & Fasteen, 2016; Melhuish et al., 2020). Given these continued difficulties, a more clarified understanding of how to better support students in developing an intuitive feel for key abstract algebra concepts is needed. However, it seems that experts' views on what constitutes an intuitive or non-intuitive concept, the methods they used as learners to develop an intuitive feel for key abstract algebra concepts, and how they integrate such methods into their teaching practices has yet to be explored and compiled for specific undergraduate course contexts. Considering these gaps, we address the following research questions: 1) What criteria do mathematicians (who are also instructors of abstract algebra) use to characterize a concept as intuitive/non-intuitive? 2) Is the "intuitive" quality of an abstract algebra concept fixed, or can it change? If it can change, what methods or events do mathematicians credit with helping them shift an abstract algebra concept(s) from having a non-intuitive to intuitive quality?

While the goal of this work is to explicate what mathematicians mean by "intuitive", we are informed by prior work on intuition. Currently, there are various understandings of what it means for something to be intuitive. In light of this, we chose to draw on Semadeni's (2007) theorization of "deep intuition" (p. 3) as it relates to concepts and intuiting methods (e.g. Nathan et al., 2021).

The data comes from a completed pilot study aimed at investigating mathematicians' views of "intuitive'' versus "non-intuitive" qualifiers in connection with various abstract algebra concepts. These concepts were drawn from Melhuish's (2019) Delphi study that ranked abstract algebra concepts for difficulty and importance (e.g., first isomorphism theorem, quotient groups, group properties, etc.). We conducted hour long interviews with 7 mathematicians from various universities across the United States selected through convenience sampling. We note that all participants regularly teach abstract algebra at their respective universities. Mathematicians were initially asked a series of questions regarding their beliefs on what it means to them for a concept to be "intuitive". They then engaged in card sorting activities (Rugg & McGeorge, 2005) which further illuminated the criteria used to decide if a concept is intuitive. To analyze our data, we used Thematic Analysis (Braun & Clark, 2006).

Based on a compilation of our interview data, we found a multitude of themes or criteria that mathematicians used to characterize a concept as intuitive. One such criteria was being able to work with the concept in practice. We also found that they used varied approaches to develop an intuitive feel such as practice and teaching a course. We expand further on these results and provide future directions for research.

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Students' Perturbations and Accommodations in the Context of Graphing

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Keywords: Covariational Reasoning, Student Cognition, Graphing

Covariational reasoning can support students' thinking across mathematical domains including functions, graphing, calculus, and modeling. Researchers have developed several frameworks for classifying students' covariational reasoning. For example, Thompson and Carlson's (2017) levels of variational reasoning apply broadly across mathematical contexts and attend to the nature of the students' images of variation and the construction of a multiplicative object in students' reasoning. While researchers have documented the ways students' reason about and construct graphs of covarying quantities, it remains an open question how covariational reasoning develops in students' thinking and whether the levels articulated by Thompson and Carlson are developmental.

Our work seeks to provide empirical evidence for the extent to which Thompson and Carlson's levels of covariational reasoning might be developmental, and to understand the mechanisms and instructional decisions that might contribute to the development of students' covariational reasoning. To address these goals, we conducted one-on-one task-based clinical interviews (Hunting, 1997) with college Calculus I students to understand the extent to which task sequencing of non-normative graphing tasks might facilitate the development of students' covariational reasoning.

In this poster, we focus on one student, Meg, to explore how her engagement with a sequence of non-normative graphing tasks contributed to her demonstration of more propitious levels of covariational reasoning. Our analysis is grounded in Piaget's conceptualization of schemes where students' schemes to develop through processes of assimilation and accommodation (Piaget, 1985). We examine how Meg assimilated tasks into her existing graphing schemes, we document the perturbations Meg experienced when assimilating these tasks, and classify the nature of the accommodations she made to her graphing scheme to resolve her perturbations. Through this process, Meg's reasoning evolved from gross images of variation to chunky images of covariation in four distinct phases. We take this as preliminary evidence that there might be a developmental progression between the levels of covariational reasoning.

Additionally, our analysis revealed that the interplay between Meg's activity constructing graphs and then reasoning about her constructed graph resulted in Meg experiencing cognitive perturbations that ultimately led her to make accommodations to her graphing scheme. This finding builds on our previous work that students reason differently when constructing graphs and interpreting graphs and that students' graph interpretations more often align to their images of covarying quantities than their graph construction (Frank, 2018). Thus, as Meg interpreted her constructed graphs in ways that did not align with her images of covarying quantities, she experienced perturbations that challenged her graphing scheme.

This work provides empirical support that levels of covariational reasoning might be developmental and if a student has robust images of covarying quantities, then their work constructing and reasoning about their constructed graphs can serve as a productive mechanism for the development of their graphing scheme.

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One Prospective Secondary Mathematics Teacher's Responsive Task Design Through a Locally-grounded Experience

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Keywords: pre-service teacher education, culturally responsive mathematics teaching, curriculum

Critical mathematics (Skovsmose, 1994) can promote critical thinking about the world through mathematics, leading to a rising *critical consciousness* (e.g., Ladson-Billings, 1995). However, critical mathematics curriculum intending to be meaningful or relevant to students may lead to tensions with respect to the authenticity of the tasks and their connection to societally-valued mathematics content (Brantlinger, 2022). One proposal for teaching mathematics that supports students' deep understanding and learning of mathematics through meaningful activities reflective of students' lived experiences is culturally responsive mathematics teaching (CRMT; del Rosario Zavala & Aguirre, 2024). However, what CRMT looks like in prospective secondary teacher (PST) education remains an under explored area of research.

In this poster presentation, we share the results of an exploration of PSTs' emerging critical consciousness as they engaged in an activity based on Zavala & Aguirre's (2024) "Mathematizing Your World" (p.39). During a variation of the activity, 13 PSTs were instructed to "*Take 15 minutes to walk with your partner. Go outside, return in 20 minutes (build in any personal breaks you may need). When you return, be ready to share about this experience."* and reflect upon the following, *"(1) What do you notice about the places you visit? (2) What feelings do you have about the things you see? (3) What questions come to mind as you do this? (4) What mathematical problems might you pose about what you observe?*" For homework, PSTs were instructed to design a task suitable for teaching content to students in grades 6-12 based on their partnered walk.

This poster will share one PST's (Colin, first author) reflections about the activity and a task that Colin developed through engaging in the *Mathematizing Your World* activity. The poster will provide examples of how Colin's critical consciousness with respect to the socio-ecological (e.g., Coles et al., 2024) was activated through the activity and how this informed his task design which centered an invasive species that he encountered during the walk. For example, he noted how the observation of several spotted lantern flies (an invasive species in the region) informed his decision to construct an exploratory activity for students where they would engage with their local environment and collect and analyze data regarding the number of lantern flies they encounter. This work serves as an example of how engaging PSTs with CRMT activities can creatively inform their curriculum design, promoting a responsiveness to locally-relevant issues.

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Exploring Undergraduate Mathematics Instructors' Perspectives and Considerations About the Literature Related to the Teaching and Learning of Mathematics

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Keywords: Documentational Approach to Didactics, College instructors' engagement with research, Research dissemination, Generative Study

Cobb (2007) characterized mathematics education as a design science, "the collective mission of which involves developing, testing, and revising conjectured designs for supporting envisioned learning processes" (p.3). Thompson (2008) similarly defined the discipline of mathematics education: "students' mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics" (p. 45). Mathematics educators' investigations into a multitude of social, cognitive, affective, and cultural phenomena serve to inform the development of innovative designs—pedagogical, curricular, social, political, and otherwise—to improve the quality of students' mathematics learning. The extent to which mathematics educators' contributions accomplish this aim depends on the effective dissemination of their scholarship to those who might be positioned to leverage its insights to improve students' learning. We argue that the effective dissemination of mathematics education research must be informed by an understanding of practitioners' motives for accessing the literature and the facilitations and hinderances they experience to implementing its findings.

To address this need, the research reported in this poster examines how and for what purpose college mathematics faculty and instructors apply insights from the mathematics education literature. Our study was guided by the following research questions: (*RQ 1*) What are the perspectives and considerations of instructors regarding the utility of literature related to the teaching and learning of mathematics? (*RQ 2*) What facilitates or hinders instructors in their efforts to access and utilize literature related to the teaching and learning of mathematics?

We applied survey and clinical interview methods (Clement, 2000; Goldin, 2000) in a manner informed by our literature review to explore instructors' perspectives about the utility of the mathematics education literature, including the facilitations and hindrances they experienced in accessing and applying its insights. Informed by the elements of schemes of usage from the Documentational Approach to Didactics (Trouche, Gueudet, & Pepin, 2018), our thematic analysis (Braun & Clarke, 2006) revealed several convergent themes related to instructors' engagement with the literature. We present these themes and discuss their implications for improving the content and dissemination practices of scholarly products to address the barriers practitioners experience in accessing, applying, and learning from the mathematics education literature.

Based on our thematic analysis of the survey data, we invited a subset of respondents to participate in a one-hour virtual interview. Our analysis of these interviews enabled us to further explore provisional hypotheses that emerged from our thematic analysis of the survey data and to elaborate our emergent themes. Our poster will present and exemplify these themes with sample interview responses. Additionally, we reflect on the implications of our research for the dissemination practices of mathematics education scholarship and ways of supporting practitioners' engagement with this literature that reduces the barriers abstracted from the survey and interview data.

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Creation, Implementation, and Evaluation of a Training Program for Undergraduate Guided Study Group Leaders

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Keywords: Communities of Practice, Peer-Assisted Learning, Group tutoring, Tutor training

Duah, Croft, and Inglis (2014) define peer-assisted learning (PAL) as "a formal scheme of assistance that is provided by one or more students to other students to enable them to develop their understanding of course content, and also to assist them [in developing] the study skills required to be successful [in] the course" (p. 554). Inspired by other PAL models such as *Supplemental Instruction* (Fayowski & MacMillan, 2008) and *Learning Assistants* (Barrasso & Spilios, 2021), we discuss a PAL model called a Guided Study Group (GSG). A GSG is a small group tutoring session focusing on a singular mathematics course and guided by an undergraduate GSG leader who is employed as a mathematics tutor specializing in the mathematical topics within the course. The overarching goal of each GSG session is to help establish students within a learning community specifically catered to their course, engage them in a collaborative learning environment, equip them with study skills specifically pertaining to their course, and encourage them to rely on and connect with their peers.

The purpose of this research project is to create, implement, and evaluate a training program for undergraduate GSG leaders at a mathematics learning success center. Through colloquial conversations with past GSG leaders, I was able to determine and design training sessions that I believed would be beneficial for GSG leaders. Fifteen GSG leaders engaged in a training session at the beginning of the Fall 2025 semester. The training lasted approx. 1 hour and involved a review of GSG policies, procedures, and expectations as well as two training activities: *Exploring Mathematical Identities* and *Mock-Tutoring Scenarios*. The creation and implementation of the training was informed by the theoretical perspective of *Communities of Practice* (Wenger, 1998). Specifically, the training sessions required the GSG leaders to be active participants in the training in hopes that the training topics being conveyed would be accepted as a *reified object* in the community of practice we hope the GSG leaders will form. We will evaluate the training according to the following research questions: *(1) What impact does the training have on the practice of guided study group leaders? (2) In what ways does the training facilitate a community of practice among the guided study group leaders? (3) In what ways does the training equip the leaders to support the development of a community of practice among students?*

The data collection methods being employed in this study consist of: Session Reflections, Observations, and Interviews. This poster will present a case study of one GSG leader in particular. This participant was chosen based on preliminary data from session reflections and observation field notes which seem to indicate that the GSG leader displays behaviors that point to a positive impact from the training. In particular, this GSG leader showed that they were focused on making the students active members of the problem-solving process, repeatedly encouraged students to work together, and noticed common misconceptions which were then addressed as a whole group.

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Improvement in Undergraduate Student Survey Responses About Engaging and Belonging in ELITE PD MGTAs' Mathematics Courses

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Keywords: Graduate Teaching Assistants, Professional Development, Student Experiences, Survey

Research on teaching practices in post-secondary mathematics classrooms consistently show that students benefit from engaging, active learning environments (Harper & Quaye, 2009; Laursen, 2013; Laursen et al., 2014; Laursen & Rasmussen, 2019; PCAST, 2012; Wieman, 2017). However, such benefits are not necessarily possible for all students unless those active learning environments also foster a sense of belonging (Brown, 2018; Ernest et al., 2019; Johnson et al., 2020; Voigt et al., 2023). Given that mathematics graduate teaching assistants (MGTAs) teach a significant number of undergraduate students over the course of their graduate studies (Selinski & Milbourne, 2015), it is important for MGTAs to develop teaching practices that engage students and increase their sense of belonging.

Our research team implemented a three-year PD program (ELITE PD) for MGTAs that is focused on teaching practices that support each and every student to meaningfully engage in learning mathematics, especially through group work. The goal of this poster is to share results from our analysis of undergraduate student survey responses from the mathematics classes of MGTAs enrolled in ELITE PD. Our research question is: *Do MGTAs receive improved student survey responses over time in the ELITE PD program, specifically from Winter to Spring during the 2023-24 and 2024-25 academic years?*

We focused our analysis on one cohort of 12 MGTAs who were new to teaching in 2023-24 and who completed two years of ELITE PD during their first two years of the mathematics graduate program. The MGTAs experienced a three-day, pre-fall term orientation and graduate teaching seminar in their first year. During fall and winter of their second year, they participated in seminars that emphasized active learning and student engagement through exploration and discussion of numerous evidence-based teaching practices. As part of our study, we surveyed the MGTAs' students during each term to see whether we saw improvement in students' responses to the MGTAs' teaching. We collected over 1000 responses for this cohort over this time period.

We conducted two statistical analyses by comparing means and standard deviations of survey responses for our cohort of 12 MGTAs from winter to spring: (1) in 2024 (2) and in 2025. Wilcoxon Signed Ranks Tests revealed significant differences (at $p=.05$ and $p=.10$) in means and standard deviations for a number of Likert-scale items (ranked from Strongly Disagree (1) to Strongly Agree (5)) in 2024 and 2025. For example, in 2024 the mean scores across all 15 items increased from 4.04 in winter to 4.15 in spring, and statistically significant differences in means (at $p=.05$) were found for four survey items. Two such items included: *I felt welcomed, comfortable, and a sense of belonging in this course*; *This course was structured so that I could work effectively with others who were different from me.* We also found statistically significant differences in means (at $p=.05$) for two survey items in 2025: *I felt safe contributing my ideas in class discussions*; *My TA presented examples that resonated with me or related to my interests.*

For our poster, we present our statistical results in more detail, discuss their implications, and future analyses. Primarily, these results demonstrate the promise of professional development for MGTAs to improve all students' engagement and sense of belonging in mathematics classrooms.

Acknowledgments

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Prospective Teachers' Modeling of Student Thinking About Sampling Distributions

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Keywords: sampling distributions, prospective teachers, second-order modelling

Although prospective secondary teachers (PSTs) are expected to teach ideas involving sampling and sampling distributions, research shows there may be limitations in their understanding of these concepts (Begué et al., 2023; Valenzuela-Ruiz et al., 2023). Prior work suggests that analyzing student work can deepen knowledge for teaching (Crespo, 2000; Heritage, 2022), yet little is known about how PSTs' attention toward statistical ideas shapes their ways of engaging student thinking. This study addresses that gap by examining how PSTs make sense of and respond to hypothetical student statements about sampling distributions.

We adopt Steffe and Thompson's *first* and *second-order models* as the central theoretical construct guiding this work. A first-order model refers to a learner's own construction of knowledge, while a second-order model represents a teacher's conception of a learner's thinking based on observations of their reasoning and work. With this framework, we pose the following questions: *Which features of sampling distributions do prospective teachers attend to when interpreting hypothetical students' statements about sampling distribution? Also, how do PSTs construct second-order models in response to students' reasoning about sampling distributions, and how are these models shaped by the statistical features the PSTs attend to?*

This study was part of a teaching experiment (Brown, 1992) within a statistics unit in a curriculum course for PSTs at a large public university. After introductory discussions about sample and sampling distributions, 11 PSTs discussed two hypothetical student statements that reflected common misconceptions about sampling distributions (Chance et al., 2004): with one hypothetical student conflating the sampling and population distributions, and another overlooking how averaging reduces variability. Transcripts of recordings of group discussions were used for analysis. In our analysis, we used Wilson et al.'s (2011) and Didis et al.'s (2016) frameworks to deductively code PSTs' talk turns into increasingly elaborate categories of second-order model-building. We identified instances in the transcripts where PSTs showed evidence of *describing* the hypothetical students' work, *questioning* aspects of that work, *comparing* it to their own reasoning, *inferring* about the students' thinking, and *restructuring* their understanding or ideas of teaching the sampling distribution concept. We also inductively coded the transcripts to identify the features of the sampling distribution concept PSTs were attending to in their discussions. Through pattern coding (Miles et al., 2014), we examined how PSTs' focus on statistical ideas influenced their second-order models of the students' thinking.

The analysis revealed that the PSTs' discussions of the statements reflected productive engagement with statistical ideas and elaborate forms of second-order model-building, while also indicating opportunities to deepen their understanding of sampling distributions, particularly how averaging influences variability. Given that PSTs often receive limited preparation in statistics compared to mathematics (Lovett & Lee, 2017), their second-order models reflected this imbalance, showing emerging insights but somewhat limited attention to the full scope of sampling distribution concepts. In the poster session, we will share illustrative episodes from PSTs' discussions that exemplify these findings and hope to foster discussions about how teacher educators can help PSTs build models of student thinking while identifying and addressing overlooked aspects of the subject matter to deepen their own knowledge for teaching.

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Creativity in Proving Through Individual, Small Group, and Whole Class Contexts

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Keywords: group work, collaboration, proving, creativity

Calls to foster creativity and collaboration in undergraduate mathematics have grown in recent years (e.g., Conference Board of the Mathematical Sciences, 2016; Saxe & Braddy, 2015; Schumacher & Siegel, 2015). Yet, little is known about how students perceive collaboration or individual work as supporting their mathematical creativity. Prior studies have largely measured creativity before and after instructional interventions (e.g., Asahid & Lomibao, 2020; Gilat & Amit, 2014, Levav-Waynberg & Leikin, 2012, Yuan & Sriraman, 2011; Zazkis & Holton, 2009), with limited attention to students' lived experiences. Further, few have examined proof-based courses or explicitly explored how collaboration may shape mathematical creativity.

This study investigates students' perspectives on experiences of mathematical creativity in an introduction-to-proof (ITP) course. Specifically, after a semester-long introduction-to-proof course taught with collaborative, inquiry-based methods, how do students recall individual, small-group, and whole-class activities as fostering their mathematical creativity?

Theoretical Framing and Methods

I adopt a relative perspective on mathematical creativity, defining it as the process of generating or implementing mathematical ideas or strategies perceived by the individual as non-routine. Feeling creative is thus taken as evidence of being creative (Satyam et al., 2022).

Data were collected from undergraduate students (n = 7) enrolled in an ITP course at a large public university in the southeast United States. Primary data for this study were participant responses to an end-of course reflection asking, "Did you feel creative in your Introduction to Proof course? Provide as many details as possible in your response." Responses were first open coded (Saldaña, 2016) to highlight students' words describing the contexts in which they felt creative. Codes were refined through iterative analysis, and findings were triangulated with course artifacts and video data to contextualize students' perspectives.

Findings

Students described feeling mathematically creative across individual, small-group, and whole-class contexts. Students felt creative in **individual** work when they independently overcame proving impasses, often by revisiting class videos, applying learned strategies, or experimenting through scratch work. Participants reported working in **small groups** allowed them to be creative because collaboration fostered shared ownership of ideas, helped with moving past stuck points, and revealed multiple valid approaches to a proof. Lastly, **whole-class** work fostered long-term creativity by positioning students as responsible for gaining their own understanding and short-term creativity when the instructor provided specific guidance and questioning. Insights from this study can inform the design of instruction in proof-based courses to intentionally use collaboration to cultivate students' creativity in mathematics.

This poster presentation will communicate detailed findings, including participant quotation and secondary data such as course artifacts and invite feedback and conversation about how to continue this work for a more explicit comparison of individual, small group, and whole-class contexts in undergraduate mathematics courses.

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Examining How Students Leverage Code as a Representational Tool in Combinatorics

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Keywords: combinatorics, discrete mathematics, computing, programming.

In order to have students use computational tools in exploring mathematical ideas (e.g., Papert, 1980) and to explore the role of computing in students' mathematical reasoning (e.g., Geraniou et al., 2024; Lockwood & Morken, 2020), we wanted to explore how students engage with coding modules while solving combinatorics problems. While analyzing student work, we noticed that students pointed out different aspects of their code in their explanation of how they solved the problems. In this poster, we seek to answer the following research question: *What aspects of the code did students attend to as they used code as a representation in solving combinatorial problems, and how did they leverage these aspects of code*?

Literature and Theoretical Perspectives

This poster will focus on connecting counting process and sets of outcomes from Lockwood's (2013) model, whose components are counting processes (the procedures used when solving counting problems, formulas/expressions (the number, variables, and operations used to get an answer to a counting problem), and sets of outcomes (the sets of objects that are being counted). We build upon prior work that has explored students' use of programming while solving counting problems (e.g., De Chenne & Lockwood, 2022; Lockwood & De Chenne, 2020). Here, we will expand on such prior work on students' use of code and aspects of the code as a representation. In this poster, we will also focus on *machine-based computing*: "the practice of developing and precisely articulating algorithms that may be run on a machine" (Lockwood & Morken, 2021, p. 405), which in this poster is programming (both for writing and editing code) (in alignment with Weintrop et al., 2016). The code itself is the counting process, and the output of the code is the set of outcomes.

Methods

A set of modules were designed that focused on students engaging with coding in Python and counting problems, and they were implemented in an upper-level mathematics classroom for pre-service teachers. The whole class worked through the modules, and we video- and audio-recorded pairs of students during six class sessions while they worked; the instructor and another researcher checked in and asked questions about their thinking and work. For data analysis, we reviewed the paired sessions and identified instances in which students referred to particular aspects of the code (including initialized sets and lists, lines of code, and output), and we examined, compared, and contrasted episodes where students focused on different parts of code.

Results

In this poster, we will show different ways that the students used various aspects of the code to represent different aspects of their combinatorics process. We found that some of the ways that the code was used to represent their listing process, to represent their counting process, and using the output of the code as a representation. In all, the code aligned with counting processes, sets of outcomes, and the connection between the two from Lockwood's (2013) model.

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Telling Latina Stories

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Keywords: identity, racism, sexism, Latina students

Today in the United States the rights of people of color, specifically Hispanic people, and women and transgender individuals are under fire with these populations being told to or made to stay silent and disappear. The focus of this poster, and the larger project behind it, is to highlight and amplify the voices of Latina STEM majors and explore their navigations of their own identities in the context of their math courses. This work brings students' own self-identifications to the forefront of the conversation and examines student experiences in undergraduate mathematics. Specifically, I focus on Latina women of various Latin American backgrounds, their identity co-constructions, and how if at all it affects their experiences as doers of mathematics (Oppland-Cordell, 2013; Leyva, 2022). For students with intersectional identities, the classroom environment and peer-to-peer interactions or instructor-to-student interactions can be incredibly difficult to navigate; when individuals identify in ways other than cisgender, straight, white, and male, these racialized and gendered themes emerge (Leyva et al., 2021). Works like Adiredja and Zandieh (2020) highlight the counter-stories of women in mathematics and STEM in order to accurately portray what these women go through and how they persist in mathematical spaces where it can be easy to disappear. This project stems from my effort to not only contribute to a body of work that tells the stories of people like me, but also to join my participants in their desire to be role models for young women who want to go into STEM fields despite experiencing discouragement and silence. This project explores the question: *how do Latina women in STEM majors feel in Calculus I courses and do those feelings affect the negotiation of their identities as math doers, STEM majors, and Hispanic women?*

Over the course of this study, I interviewed six women ages 18 and 19 from a variety of STEM majors who are currently taking Calculus I. The initial interviews focused on forming initial student profiles about each woman's background and how they perceive three main aspects of their identity: (a) their identities as Hispanic or Latina, (b) their identities as women, and (c) their identities as doers of mathematics. After being asked directly about their experience with racism or sexism in the math classroom, the majority of the women interviewed were able to either explicitly describe an instance of racism or sexism they saw or experienced in their schooling or mentioned taking note of racialized or gendered experiences. Furthermore, at least two women mentioned refusing to stay silent or make themselves smaller in the face of these experiences. For the women who identified explicit racist or sexist encounters, they expressed finding comfort in their support systems of family or roommates and friends who would hear them out and let them vent their frustration in the face of these aggressions. Preliminary results provide insights into what participants' responses are to discrimination inside and outside of the classroom and evidence of how they view themselves and the different intersections of their identities. Potential implications of this work in the research world encourage more lived experience storytelling for Latina STEM majors, the public will hear more voices of women in STEM, and Latina women and girls can see and hear the stories of women like them to show what is possible.

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A Collaborative Rubic Development and Scoring Program

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Keywords: coordinated courses, rubric scoring, inter-rater reliability

Research Aims and Relationship to the Research Literature

The literature speaks to the benefits of course coordination (Rasmussen et al., 2014) and the importance of consistency in grading across multi-section courses. Achieving grading consistency can be challenging (Reddy & Andrade, 2010) but can be enhanced by using task-specific rubrics and rater training (Jonsson & Svingby, 2007; Crisp, 2017; Schoepp, et al., 2018). It is well-documented in the literature that rubric training can be a time-consuming process and that faculty time constraints can be a barrier to implementation of assessment practices in general and rubric scoring in particular (Charlton & Newsham-West, 2024; Holmes & Oakleaf, 2013; Williams, 2024). However, the research literature does not seem to adequately address how, after an initial time investment, rubric use can become routine. We seek to investigate the following research question: after instructors invest time in rubric development and training, how does the time needed for rubric implementation change?

Preliminary Data Collection and Results

Our investigation of this research question has two phases. Phase one is the initial time investment in rubric development and training. Phase two is rubric utilization. For the first phase, we collected student responses ($n = 147$) on assessment items involving exponential growth, cooling, and trigonometry. Faculty members (two associate professors and two assistant professors in their first year of teaching post Ph.D.) teaching a coordinated precalculus course participated in a collaborative rubric development and training program: after selecting a task aligned with learning outcomes (e.g., problem-solving, communication, modeling exponential growth), we collaboratively developed a draft task-specific rubric specifying partial credit based on the demonstrated level of understanding, then refined it by anticipating potential student solutions, errors, and omissions. We scored a small sample of blinded student responses as a whole group to discuss how to apply the rubric and make necessary revisions. Next, we independently scored a sample of student work and discussed results to identify discrepancies and come to a consensus rating. After the training session, we independently scored the remaining student solutions, then reviewed discrepancies to identify lessons learned. In round one, the percent agreement across all raters was 50.3% and the percent adjacent agreement was 73.1%. In round four, the percent agreement was 84.7% and the percent adjacent agreement was 91.5%. Implementation of the second phase is forthcoming. Using already collected student work on different assessment items, each faculty participant will write a task-specific scoring rubric without additional training beyond the initial time investment in phase one. For each rubric, all faculty participants will independently score the student work and percent agreement will be computed. Faculty perceptions of changes in time required for rubric development and implementation will be captured through a survey and a structured interview process with each faculty participant.

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A Corequisite College Algebra Support Program With Undergraduate Peer Facilitators

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Keywords: co-requisite support, college algebra, calculus sequence placement practices

Many undergraduate science, technology, engineering and mathematics (STEM) degrees require one or more calculus courses. For some, the path to those courses includes one or more prior-to-calculus courses. Not all students who begin this journey end up succeeding, and the higher the number of pre-requisite courses students need, the less likely they are to eventually succeed in the course needed for their program (Jaggars & Stacey, 2014). Many factors influence student success, however, pre-requisite skill fluency and understanding of foundational ideas can impact learning in calculus sequence courses and are thus often the foci of placement tests.

We report on a student support program in college algebra, designed by combining principles of co-requisite support (Zaback et al., 2016) and Peer Led Team Learning (PLTL) (Willson & Varma-Nelson, 2016). Co-requisite support typically takes the form of a supplementary course taken concurrently with the target course and provided as an alternative to having students take a term-long course on the prerequisite material.

Our program is a variation on this model. The university does not offer a course as preparation for college algebra. Historically, students who did not place into college algebra were unable to begin their path towards calculus. Now, students whose placement scores are below the requirement for college algebra enroll in the combination of mainstream 3-credit college algebra and a 1-credit support course. The course coordinator developed groupwork materials and supervised undergraduate facilitators, overseeing hiring and preparation meetings. This approach was modeled after PLTL which had been implemented previously on campus (for other purposes), including in mathematics. Experienced undergraduates were student group facilitators and time was spent strengthening prerequisite skills needed for up-coming topics, reducing potential impediments to student learning.

We share findings from the first year of implementation at a research-intensive university in the northeastern U.S. Data included grades from the 63 participants for both the semester when they were in college algebra and for subsequent semesters. Findings confirmed our hypothesis: students who were under-prepared for college algebra (as measured by placement scores) could succeed (earn grade needed to move to the next course) with additional support. A chi-square test revealed that success rates of those in the course and those in the course plus support were statistically indistinguishable (at the $p<.05$ level). In addition, the vast majority who enrolled in a next course in the calculus sequence were successful in that course.

Our findings suggest that what was found in community college settings (i.e., Zaback et al., 2016) can also be true in university settings. In particular, findings suggest that co-requisite support enabled student success in college algebra and in subsequent calculus sequence courses. In addition, findings suggest that undergraduate peers can staff such programs which could help with wider-spread implementation. The poster includes additional details and we seek feedback on evaluation design and ideas for additional research questions to pursue. Of particular interest are the undergraduate peer facilitators since their involvement is a variation from the typical co-requisite support program model. In addition, we seek discussion about further research questions that might be possible to investigate if this program were expanded to other courses in the calculus sequence (e.g., pre-calculus and first-semester calculus).

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Using Provided Guided Notes in Coordinated First-year Mathematics Courses

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Keywords: coordinated courses; guided notes; post-secondary mathematics

Course coordination has been successfully implemented at the mathematics departmental level at various universities (Regier et al., 2024; Jensen-Vallin, 2013; Golnabi et al., 2020; Rahman et al., 2021). Adeyemi et al. (2023) described "Course coordination implies coordination in the design, structure, and teaching of a course to the extent that multiple instructors within a multi-section course have come to a common agreement." Another path to improving student experience, explored by instructors, has been implementing guided notes in the classrooms. Feudal and Panse (2022) defined guided notes as "a modified version of the instructor's notes with certain blanks the students have to fill in during the lecture." The department in which this research study is being conducted has begun to use standardized guided notes as part of the provided curricular materials in course coordination. There has been little studied about the impact of course-wide standardized guided notes, created by someone other than the instructor, in post-secondary mathematics. The research question is: In what ways does the use of standardized guided notes in a coordinated classroom in first-year college mathematics affect student and instructor perceptions, expectations, and actions within their course?

This project focuses on first-year mathematics credit courses which have no prerequisites in which students are required to purchase student workbooks containing guided notes created by a former coordinator of the course. The study takes place at a large, public, research university in the Southern United States. Researchers utilized an inductive mixed methods approach.

To begin understanding the use of the guided notes, course coordinators were asked to provide basic course information to determine the intent of the guided notes and to determine the coordination structure, according to framework established by Couch et al. (2023). Next, the instructors completed an online Instructor Survey which asked about their experience teaching coordinated courses, their plan for using the workbook when teaching, and the survey included a Likert scale on Mathematics Teacher Self Efficacy, modified from Zuya et al. (2016). The instructors also participated in a focus group interview to provide qualitative context to their survey responses. Students enrolled in the course were asked to participate in a survey which included a Likert scale of student perception of guided notes (influenced by Feudal and Panse, 2022). Researchers observed a class meeting of two sections of the course, conducted by different instructors, covering the same topic of Systems of Linear Equations. A classroom observation rubric modified from Mustafa et al. (2014) was utilized to record instructor actions and the use of the workbook as well as student engagement. During the last five minutes of class, the researchers took pictures of student notes to determine the types of information recorded by students (Fukawa-Connelly et al., 2017).

Researchers piloted the study in Spring 2025, with the course sections of College Algebra for Economics and Business. The poster will feature preliminary results from the pilot study, highlighting instructor and student perspectives on guided notes. In Fall 2025, the researchers will expand the study to include two additional large multi-section first-year mathematics credit courses (Contemporary Mathematics and College Algebra). As the study is expanded to cover multiple courses, the researchers are also interested to learn how other universities utilize guided notes or student workbooks in their first-year mathematics courses.

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Exploring Covariational Reasoning Pathways in Undergraduate Mathematical Modeling

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Keywords: Covariational Reasoning, Mathematical Modeling, Differential Equations, Trajectories

Modeling plays a critical role in helping students make sense of and analyze complex biological systems, which is important for those pursuing life science or health professions (AAMC, 2009). Participants in this study previously completed a life science modeling course designed to improve their ability to use modeling as a tool for understanding the dynamics of biological systems. This work investigates students' covariational reasoning (CR) when engaging in tasks requiring movement across multiple representations, such as developing a trajectory from a time series and, conversely, generating a time series from a trajectory. A time series showcases how a single variable changes as a function of time, while a trajectory illustrates how two variables change in a space formed by the two variables that describe the system (Bennoun, 2024). Covariational reasoning refers to an understanding of how two quantities change with respect to one another (Carlson et al., 2002) and is foundational to understanding the concept of function, reasoning with multiple quantities, and modeling dynamic functional events (Thompson & Carlson, 2017), all critical skills for success in mathematics.

This work builds on Bennoun's (2024) work on examining the extent to which undergraduate students who completed a modeling course have learned how to sketch the trajectory associated with a time series and vice versa. Extending this line of work, we focus on how students reason through these representational shifts and how their covariational reasoning manifests across tasks. Our research question is: *How do students engage in covariational reasoning when working with time series and trajectory representations of mathematical models in biology?*

To investigate students' CR, we conducted 10 task-based think-aloud interviews (Zazkis & Hazzan, 1999) with students who had taken a mathematical modeling course at a research-intensive university in the southwestern United States. For the interviews, students were asked to sketch a time series associated with a given trajectory and compare their sketch to alternative solutions. To analyze the interview data, we deductively coded the data using Thompson and Carlson's (2017) six levels of covariational reasoning as a priori codes (Miles & Huberman, 1994), followed by reflexive thematic analysis (Braun & Clarke, 2006) to identify overarching themes that emerged from the data.

Our preliminary findings highlight CR's dynamic and flexible nature, as students shifted between reasoning levels across tasks, leveraging different levels depending on the demands of each task. We present three empirical cases that illustrate different CR pathways and discuss how classroom norms and teacher expectations may shape students' conceptions of what counts as "correct." For example, all students' responses for sketching task were categorized into the smooth, chunky, or coordination levels. When presented with solution comparison problems, they either remained at the same level across tasks or shifted to another level, sometimes moving from smooth to chunky, or from coordination to smooth, depending on the task. We then compare the three paths, noting that students often referenced classroom expectations, emphasizing the interplay between individual understandings and classroom norms.

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How Education Within Social Networks Shape Perceived Barriers to Applying to Graduate Mathematics Programs

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Keywords: Social Network, Graduate School Applications, Social Capital

Students rely on information from their social networks to navigate decisions about college and career-related choices (Canché et al., 2014). Beyond undergraduate education, enrollment in graduate programs is heavily influenced by a student's parent's level of education (Bryer, 2025; Mullen et al., 2003; Seay et al., 2011). The theory of social capital suggests that students learn information, norms, values, standards, and expectations for education from their family, peers, and others with whom they may form inter-personal relationships (Coleman, 1988). We use this framework to discuss our results. The following data was collected from 519 survey responses to a 57-question survey investigating student knowledge about applying to graduate school in mathematics. The following table shows selected results from one survey item about application barriers compared to the survey item: "What is the highest level of education completed by someone in your network (friends/extended family)?" We use the term "First-Generation" to describe students whose social network contains less than a bachelor's degree. Using Kruskal-Wallis tests we found there was an association where students whose social networks had people with post-secondary degrees were less likely to view obtaining letters of recommendation and relocating as potential barriers to applying to graduate school. When navigating the college application process, student peer groups provide knowledge which ultimately makes applying to college more accessible (Tierney & Venegas, 2006). Students who know someone with a graduate degree, can rely on them and/or their social network to know how to obtain proper letters of recommendation or prepare to relocate for graduate school. Thus, these students may find it easier to apply to graduate school compared to their peers due to their social capital.

Table 1. Kruskal-Wallis test results for selected items for the question "To what extent are the following factors a potential barrier to your pursuit of graduate school?" using the level of education of social network variable.

Item	Group	*N*	Mean	Mean Rank	*H*	*Df*	*P*	η^2
Obtaining letters of recommendation	First-Gen	44	3.07	323.69				
	Bachelor's	122	2.56	265.92	12.508	2	.002	.024
	Graduate	346	2.36	243.07				
Possibly having to move to a new place	First-Gen	44	3.41	324.30				
	Bachelor's	122	2.93	277.22	17.401	2	<.001	.034
	Graduate	344	2.57	238.97				

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Preliminary Design for Pilot: Bridging Conceptual Gaps in Orthogonal Projection in Applied Linear Algebra Through Principal Component Analysis (PCA) Visualization

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Keywords: Principal Component Analysis, Orthogonal Projection, Applied Linear Algebra Instruction, Visualization-Supported Reasoning, Representational Coordination

Students often encounter orthogonal projection as a formula-driven procedure, leading to disconnects between geometric intuition, vector decomposition, and symbolic manipulation. (Córdova-Esparza & Piñon, 2025). Leveraging PCA as a visual means, this study aims to bridge students coordinate image, vector, and symbol for deeper conceptual understanding. Research Questions: RQ1. How do PCA-based visuals support undergraduates in developing a robust understanding of orthogonal projection? RQ2. Which instructional features foster stronger conceptual explanations and reduce misconception?

PCA technique offers a visual framework links geometric intuition with algebraic operations, making orthogonal projection more accessible (Konyalioglu et al., 2011; Pandit et al., 2020). In undergraduate linear algebra, PCA connects dimensionality reduction, visualization, and real-world applications (Gewers et al., 2018; Greenacre et al., 2022), Its foundation in orthogonal projection enables students to visualize high-dimensional data on lower-dimensional subspaces (Espadoto et al., 2021; Görtler et al., 2019). PCA visualization and interactive activities connect theory and practice, supporting conceptual reasoning (Glen & Zazkis, 2020; Trigueros, 2018). Conventional learning creates gaps between symbolic manipulation and conceptual understanding (Kontorovich, 2020; Zhang, 2024). Few studies address how PCA-based activities coordinate geometric, vector, symbolic reasoning, an explicit gap this project addresses.

This study draws on Multiple External Representations (MERs) theory, which strengthens conceptual understanding by coordinating visual, geometric, and symbolic forms (Hahn, L., & Klein, P.2023). PCA-based visualization, vector decomposition, and symbolic projection activities foster representational fluency (Fonger et al., 2018). Guided by constructivism theory (Greenfield & Bruner, 1966), the design uses hands-on tasks to bridge image ↔vector ↔symbol. A conjecture map as a design-based research approach (Oktaç, 2019), systematically links instructional features to student reasoning and evidence of conceptual change. These frameworks guide activities to bridge conceptual gaps in orthogonal projection.

This activity applies PCA as a data-driven context to make orthogonal projection visible and discussable. Students work with three figures: a 3D projection to the PC1-PC2 plane, a 2D view with perpendicular drops, and a PC1-PC2 scatter plot to anticipate PC1 direction, mark orthogonal drops, decompose vectors ($v = v_{\parallel} + v_{\perp}$), and compute projections using the dot product. The sequence threads one idea across image ↔vector ↔symbol, with pre/post concept checks and figure annotations scored by a 0-3 rubric. Targeted prompts address recurring confusions (e.g. "along" vs. "onto," PCA vs. regression, and centering.)

At the preliminary design and pre-pilot stage, this approach is expected to bridge gaps and strengthen students' conceptual understanding of orthogonal projection by coordinating geometric, vector, and symbolic reasoning. It aims to reduce misconception, common errors, deepen explanations, and provide instructors with a portable toolkit-conjecture maps, rubrics, and repair prompts to support representational coordination. Pilot results will inform further refinement and broader instructional adoption.

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Can x Equal Zero? A Look at Two Calculus I Students' Reasoning About Variables

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Keywords: Calculus; Algebraic Manipulation; Variables; Errors

Introduction

When solving calculus problems, undergraduate students rely on variables and algebraic manipulation. However, students frequently view variables as symbols to be manipulated rather than as quantities to be related (LaRue & Infante, 2015; White & Mitchelmore, 1996). While algebra education research (e.g., Booth, 1989; Küchemann, 1981; Lucariello et al., 2014) addressed students' interpretation and manipulation of variables, more research is needed to understand how these processes manifest specifically among calculus students. This poster will address several facets of this need.

Context

This work is part of a qualitative research study conducted at a large R1 university on the East Coast of the United States. The study involved students enrolled in Calculus I courses for life and social sciences during the Fall 2024 semester (Abadir, 2025). This poster will highlight a task-based episode from the larger study in which two undergraduate calculus students reasoned about algebraic manipulation and the use of variables while identifying points of local maxima and minima for a given function. In calculating critical numbers, students mistakenly treated the variable as a constant and divided both sides of the derivative equation by it, which led to the omission of a critical number. However, during a debriefing session with the researcher, the two students explained and reflected on their solution process, allowing them to identify their mistake and the conceptual implications of their algebraic manipulation. The students were able to connect algebraic ideas and properties, allowing them to engage with error in a way that facilitated a deeper understanding of the significance of variables as placeholders.

Methodology

Guided by Borasi's (1987) framework "errors as springboards for learning," this poster presentation aims to answer the research question, "In what ways do students' reflections on their algebraic manipulation and errors serve as an opportunity to reason about variables?" This analysis examines how students' engagement with their own errors fosters a deeper conceptual understanding of the role and significance of variables. The primary data sources include students' written work and a video recording capturing both the problem-solving process and the subsequent debriefing session.

Results and Significance

This poster will present detailed data and systematic analysis of the students' algebraic manipulation and error, along with their reflections as they articulated the significance of variables and how they relate to algebraic properties. These findings contribute to the body of research that frames errors as opportunities for deeper learning, specifically by using students' algebraic mistakes to enhance their understanding of the role of variables.

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Grades, Learning, and Reattempts: How Students Experience Outcomes-Based Testing

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Keywords: alternative grading, outcomes-based testing, linear algebra

Recent years have shown an uptick in faculty interest in alternative grading practices as they question the effectiveness and meaning of traditional grading practices and points-based systems (Blum, 2020; Nilson & Stanny, 2023). According to Clark and Talbert (2023), the four pillars of alternative grading are (1) clearly defined standards, (2) helpful feedback, (3) marks indicate progress, and (4) reattempts without penalty. An alternative grading practice built on these four pillars prioritizes learning, feedback, iteration, and transparency over point-earning and early mistakes. Shifting the focus from grades to learning intends to make grades more meaningful and to support the learning process We are interested in how alternative grading can provide a supportive learning experience for our students and how the experience of reattempts on outcomes specifically affects them.

In this poster, we will present findings from a study on outcomes-based testing (OBT), a form of alternative grading, in linear algebra at a medium-sized public engineering institution. This work replicates and extends the research by Swanson et al. (2025) which reported on the benefits and drawbacks of mastery-based testing in linear algebra, and student beliefs about final grades as compared to other courses. In the present study, using qualitative and quantitative data collected from students across four different sections taught by two different instructors, we examine three key questions: (1) What benefits and drawbacks did students perceive in the OBT system?, (2) Did students believe that OBT grades more accurately reflected their learning compared to grades in other courses?, and (3) How did students' experiences differ based on the number of attempts they made to complete course outcomes?

Data for this study were collected during the fall 2024 and spring 2025 semesters. Students answered multiple 5-point Likert-scale rating items and open-ended questions about the perceived benefits and drawbacks of OBT and the meaning of grades through online, optional surveys administered at the beginning and end of the semester. Thematic analysis (Braun & Clarke, 2022) was used in coding the open-ended questions. Together with qualitative analyses, our results support the findings of Swanson et al. (2025) regarding benefits, drawbacks, and grades (RQ1, RQ2) with notable differences in the student experience based on outcomes attempt frequency (RQ3). Namely, we found that students perceived OBT to support their learning (e.g., deeper understanding) and to support self-regulated learning (e.g., flexible), and we found that aspects of OBT both reduced and provoked stress. Students rated grades in OBT as fairer and more accurate than grades in other classes. Some benefits of OBT were more pronounced for students who had less frequently reattempted outcomes; for example, while students reported less exam anxiety in the OBT-graded course than other courses overall, this benefit was greater for those who used fewer reattempts.

In this poster, we will include a brief description of our OBT system together with our full results on student-reported benefits, and drawbacks which are organized around how OBT supports both learning and self-regulation. We will conclude the poster by including implications for improving our OBT implementation and addressing equity in student experience.

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The Role of Context in Adult Undergraduates' Problem Solving

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Keywords: Problem-Solving, Role of Context, Adult Undergraduates

For this poster, we define adult learners as those who have had significant time away from school mathematics before returning to undergraduate mathematics learning. Based on the first author's prior work with adult learners, she became interested in how adult learners might leverage their experience in their adult professional and personal lives to better understand mathematical ideas. To pilot this type of exploration, she designed two tasks with similar mathematical structure but set in different real-world contexts. This design was informed by existing work, which shows students bring prior experiences to the undergraduate classroom, but often don't see their out-of-classroom experiences as relevant to their classroom work (Graham et al., 2000; Jameson & Fusco, 2014). Also, others have noted that the nature of the context can inform student reasoning in a task (diSessa, 2018; Thevenot, 2017). In light of this, our goal was to consider in what ways adult learners approached these similarly structured mathematical tasks, and in what ways the contexts of the scenarios influenced their problem-solving.

We conducted clinical interviews with two adult learners using missing value proportion tasks. We see these tasks as both mathematically fundamental, and likely to connect to adult learners' prior experiences, since proportional reasoning is one of the most common math applications in everyday life (Doyle et al., 2015; Hilton et al., 2016; Kontogianni & Tatsis, 2019).Our tasks were a Baking Task and a Traveling Task, which read, "Matthew is baking a cake. The recipe calls for 2 cups of flour and 3 teaspoons of vanilla. If a scaled-up version of the recipe calls for 7 cups of flour, how much vanilla should Matthew use?" and "As Buddy walks to school at a constant speed, he travels 8 feet in 3 seconds. How long will it take Buddy to walk 20 feet?" From an expert point of view, each task includes a given ratio and one value of another ratio, with the question asking for the missing value of the second ratio. We analyzed how two adult learners (1) solved each task, (2) thought about the similarities and differences across the two tasks, and (3) reflected on their solving processes.

For (1), we identified five different ways they solved the task. We characterized differences in terms of the main ratio the student leveraged and how they leveraged the ratio. By main ratio we mean whether (a) the student used the two unlike unit data (e.g., 8ft to 3 sec) as the main ratio and then tried to match that ratio or (b) the student used the two same unit data as the main ratio (e.g., 8ft to 20 ft) and then tried to match that ratio. Both participants had one solving method for the Traveling Task, with Travis leveraging a same unit ratio, and Jessica leveraging an unlike unit ratio. In contrast, on the Baking Task, both produced multiple solving paths and leveraged both unlike unit ratios and same unit ratios. With regards to (2) and (3), the students initially did not perceive the tasks as the same, primarily because they used different solving paths for the two tasks. However, upon reflection Travis noted that the same strategy would have applied to both saying, "I could have set them up the same way." With regards to the role of context, the Travel Task had similarities to problems in a math course both participants recently completed, and they both referenced school math solving methods in their reflections. In contrast, while solving the Baking Task, Jessica leveraged her prior cooking experience saying, "What if I was actually baking, what would I do?" We note that both the learner's solving approaches and their reflections seemed to be influenced by the context itself and their experience with the context.

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Student Perspectives on Effective Recitation Design in an Active Learning Calculus I Course

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Keywords: Active Learning, Calculus, Student Perceptions

Active learning has become a central approach to student-centered learning in undergraduate mathematics courses (CBMS, 2016; Rasmussen et al., 2019) and has been shown to improve student engagement and outcomes (Altindis et al., 2025; Kurepa et al., 2019). While research on instructor practices in active learning environments is growing, less is known about how students experience these classrooms. This study addresses this gap by examining students' perceptions of an active-learning model implemented in Calculus I recitations at a medium-sized university. Recitations were taught by Graduate Teaching Assistants (TA) and aided by Undergraduate Learning Assistants (LA).

To capture students' experiences, we analyzed responses to two open-ended questions on an end-of-semester survey: *If you were an instructor or a TA designing recitation: (1) What are at least two things you would keep? Explain why these are strengths. (2) What are at least two things you would alter or remove? Explain why those changes would be an improvement.* Data were collected during Fall 2023 and Spring 2024, with 238 students (of a total of 404 enrolled) completing both prompts. Using inductive qualitative thematic analysis and the constant comparative method (Patton, 2002; Strauss & Corbin, 1998), the authors collaboratively developed and refined a codebook with 23 distinct themes organized into five overarching categories: *Student Work Structures & Group Dynamics*, *Instructional Practices*, *Recitation Environment*, *Assignment-Related Aspects* and *Assessment & Review.* Interrater reliability was established through independent coding and reconciliation discussions.

Overall, students mentioned more aspects to *keep* in recitation (429 total codes; ~2 per student) than to *alter or remove* (240 codes; ~1 per student), indicating generally positive perceptions of the active-learning format. The most frequent "keep" themes involved *Instructional Practices* (42.7% of codes) and *Student Work Structures & Group Dynamics* (30.8% of codes). Within these themes, specific strengths were noted by students, with percentages reflecting the proportion of students who mentioned each: *group work* (~50%), valued for fostering collaboration, comfort, and peer explanation; *worksheets* (27%); *teacher-led problem solving* (18%); *introductions* (18%); and *conclusions or wrap-ups* (16%). In contrast, for the "alter/remove" question, *Assignment-Related* themes were most common (31.7% of codes), though most suggestions focused on refinements rather than removals (full removal accounted for 11% of codes). Common refinements mentioned by students included revising *worksheets* to clarify instructions and adjust length or difficulty (21% of students) and improving *group work* implementation, such as having clearer facilitation roles (13% of students).

This study reinforces that collaborative active-learning structures support student engagement (Kurepa et al., 2019; Nji & Chinyelogo, 2025) and adds insight into *how* students want these structures implemented. A novel contribution of this study is eliciting student feedback from the perspective of a TA/instructor, yielding actionable recommendations for strengthening recitations, such as worksheet design and group work facilitation. Incorporating student feedback into active-learning recitation design has the potential to improve the effectiveness of active learning implemented in introductory mathematics courses.

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Student Motivation and Reflection on Midweek Math Training

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Keywords: Student resources, exam readiness, math anxiety

The Midweek Math Training (MMT) program is designed to empower and prepare calculus students for exams via weekly peer-led practice exam sessions students can voluntarily attend. MMT draws inspiration from the well-documented positive effects of Supplemental Instruction (Skoglund, Wall, and Kiene, 2018) and repeated exposure to practice exams (Kulik, Kulik, and Bangert, 1984). In each session, students first individually attempt exam-level problems on recent course content. Trained peer facilitators then guide student discussion of appropriate strategies for these problems and test-taking strategies more generally.

In the spring semester of 2024, MMT was used by 61.84% of students in Calculus 1 & 2 before Midterm 1. Many students have expressed anxiety about the course. Math anxious students often avoid working practice problems, particularly practice exam problems (Jenifer, Levine, and Beilock, 2023). We want to know 1) What impact MMT has on student's math anxiety? 2) How do students believe MMT is impacting their understanding, behavior, and relationship with math? 3) Why do students come to MMT?

Methodology

Study participants were recruited through their Calculus classes and MMT. Each week student attendance is taken at MMT each week. Six times through the semester, study participants were sent the same select questions Abbreviated Math Anxiety Scale (AMAS) (Hopko et al, 2003). After each exam, study participant were sent the AMAS and a brief survey about MMT survey. Participants are invited to participate in a 30-minute interview to discuss their relationship with math and MMT. Data is currently in collection and will be ready for analysis spring 2025.

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Chemistry Graduate Students' Problem Solving With Quantum Mechanical Models: "What Do I Plug in for Delta?"

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In the chemistry curriculum, physical chemistry stands out as one of the most challenging given its reliance on calculus, particularly when using quantum mechanics to describe atomic and molecular structure. Though there is literature in chemistry education research that investigates students' conceptions of quantum chemistry, few of these studies focus on students' use of mathematics in this context. To this end, we interviewed graduate students (N = 23) about the "particle-in-a-box", one of the first quantum models discussed in physical chemistry. Preliminary analysis investigates phenomenographical categories of students' problem-solving strategies and their affective responses to the problem-solving task.

Keywords: chemistry, quantum mechanics, calculus, interdisciplinary

There are few studies in the chemistry education literature that specifically focus on students' conceptualizations of quantum mechanical models (McAfee et al., 2025), and though these articles provide a great foundation for further research, they do not explicitly investigate students' use of mathematics. This gap does not insinuate that quantum mechanics is taught in chemistry without mathematics, but rather that chemistry education research often borrows findings from physics education research to contextualize students' conceptualizations of quantum mechanics. To this end, the aim of our study is understanding how students apply quantum mechanical models when problem-solving. For our study, we conducted semi-structured interviews with chemistry graduate students (N = 23) who received $50 gift cards for participating in a two-hour-long, in-person interview.

Preliminary data analysis, guided by phenomenography, focused on two major components of the graduate students' problem-solving: (1) their performance on the problem, and (2) their affective responses to solving the problem in front of a peer. To better understand their treatment of the molecule in the problem, we generated a rubric with a normative application of the particle-in-a-box model in which points were assigned to the graduate students' work based on correct decision-making during the task. The variation in model application shone through in the students' treatment of variable *a*, in which they needed to calculate the length of the molecule. Notably, none of the students applied the model completely normatively which speaks to the immense challenge quantum mechanics poses to students.

During the interviews, we noticed many affective responses from the graduate students such as nervous laughter, screams, qualifiers, etc. Given this, we inductively coded these responses which we frame as observed indicators of stress. Taken together with their problem-solving, three groups of students have emerged: (1) high performers with low observed indicators of stress, (2) low performers with low observed indicators of stress, and (3) low performers with high observed indicators of stress.

For next steps, we are interested in expanding our analysis to further understand the affective responses we observed form the graduate students and further describe the emergent phenomenographical groups.

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Applying Process Tracing to the Development of the Indigenous Math Courses and Secondary Math Education Program at Turtle Mountain College

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Keywords: tribal college, indigenous math, process tracing, course development

Background and Motivation

Undergraduate math courses focused solely on Indigenous math are relatively new within tribal colleges/universities. Turtle Mountain College has developed a three-course sequence on Indigenous Math in conjunction with developing their bachelor's program in Secondary Math Education. The development process of the Indigenous Math courses led to the Indigenous Math Framework. The framework has four components situated within the medicine wheel. The four components are (1) Nature/Place/Land, (2) Community, (3) Heritage/Traditional Games, Arts, Engineering, etc., and (4) Language. The framework is not the only way to view Indigenous math but rather just a way that is used at Turtle Mountain College (Luecke 2023, 2025). The courses and framework build upon scholars who articulate the error and danger of Western perspectives of math being mythologized as pure, universal, culturally neutral, and culturally superior (Aikenhead, 2017; Bishop, 1990; Ernest, 2021; Kawagley, 1997; Medina et al., 2024; Stevens, 2021; Xu & Ball, 2024). Further, these courses stand on previous K-12 math education research demonstrating that math and local language/culture integration strengthens students' cultural identity and improves math exam scores (Lipka et al., 2007; Kisker et al., 2012).

Purpose of the Study

The development of the Indigenous math courses and the Secondary Math Education program focused on content and pedagogy (Luecke, 2023, 2025). This research will go beyond the end-product of what was developed and study the why and how of the development process. The expectation is that a greater understanding of how the Indigenous Math courses and the Secondary Math Education program were developed will afford opportunities to strengthen the courses and program as well as offer insights into supporting any math faculty interested in integrating more Indigenous ways of knowing and being into an undergraduate math class. The research question is "What strategies for developing math (and math education) programs with a greater emphasis on Indigenous STEM can be derived from a retrospective analysis on the development of the Indigenous Math courses?"

Methods

The research methods are situated within an Indigenous research paradigm with its emphasis on relationality, relational accountability, and storywork (Archibald, 2008; Kovach, 2009; Wilson, 2008; Windchief & San Pedro, 2019). Aligning with the research question, a process tracing method (Atcha, 2022; Beach, 2017; Montano et al, 2025) will be applied to the initial development of the Indigenous Math courses and Secondary Math Education bachelor's program to determine strategies in navigating tensions/decisions in development. A process tracing methodology affords a systematic, reflective analysis to see causal relationships on how the courses/program developed. Identifying moments of decision, struggle, and overcoming demonstrate one way to increase Indigenous Math content in all math courses/programs.

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Course Coordination and Student Success: Evidence From the Literature

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Keywords: course coordination; precalculus; calculus; student success

In the United States, Calculus I is extremely efficient at reducing student enjoyment and perseverance in mathematics (Bressoud et al., 2013). Coordinating gateway courses (e.g., precalculus, calculus) is a strategy that has been proposed as a way to improve students' experiences and outcomes (Bressoud & Rasmussen, 2015; Rasmussen et al., 2019). In an effort to assess that strategy, we conducted a literature review on course coordination to answer the question: *How is course coordination associated with changes in student success across institutions?* Our literature search consisted of peer-reviewed mathematics education journals and edited volumes that address coordination and institutional change efforts for student success. From this review, we identified 62 articles that explicitly mention or describe course coordination, with 23 discussing student success outcomes as a result of coordination.

Coordination is characterized as a system that synchronizes course content and instructor interaction under designated leadership, emphasizing collaborative design and ongoing instructor communication (Carney et al., 2025; Martinez et al., 2022; Rasmussen et al., 2019, 2021; Williams et al., 2021). Note that we define coordination beyond common syllabi, textbooks, and exams; it involves sustained structures that support the collaborative development of materials and pedagogy, with more robust coordination systems developing into *de facto* communities of practice (Apkarian & Rasmussen, 2020; Couch et al., 2023; Rasmussen & Ellis, 2015; Rasmussen et al., 2019; Williams et al., 2021).

Twenty of the 23 papers targeted passing rates as their measure of success and reported some level of improvement. These papers credited a variety of strategies, such as instructors forming a community to share in-class activities (Oliver et al., 2023) and addressing student concerns around grading transparency (Akin & Veil, 2020). Institutions that focused on research-based practices like active learning had larger improvements in their DFW rates, particularly when faculty were supported through learning assistants (LAs) or embedded peer tutors (e.g., Carney et al., 2020, Mingus & Koelling, 2021). Across these contexts, varied degrees of prescriptiveness had a similar pattern: shared infrastructure, ongoing instructor dialogue, and structured supports that align with improved pass rates and more equitable outcomes.

Eight of the 23 papers moved beyond pass rates to consider shifts in identity, belonging, and persistence in mathematics. Factors that contributed to students feeling a stronger sense of community within their classrooms included the integration of LAs (e.g., Chamberlain et al., Mingus et al., 2023), flipped classrooms that focused on student group discussion (e.g., Chamberlain et al., 2021; Diamond et al., 2021), and adoption of active learning activities (e.g. Hancock et al., 2021). Taken together, these student perceptions complement the achievement findings and align with broader accounts linking coordinated structures to enhanced confidence, interest, and persistence in STEM pathways. The poster will further elaborate on the benefits (and constraints) of different coordinated models for departments and mapping these design choices to student outcomes.

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Exploring Conceptions of Smartness in Mathematics: Connecting Student and Instructor Perspectives at a Hispanic-Serving Community College

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Keywords: Smartness, Equity, Community Colleges, Mathematics, Teacher Beliefs

In this poster, we build upon the work of (Hernandez, Wilkes, Camacho, & Pilgrim, 2025), describing how community college students' conceptions of smartness in math class can be categorized based upon a framework that accounts for both dominant (Boaler & Staples, 2005) and counternarratives (Reinholz, 2021), which are important for how learners see themselves and how classrooms reinscribe and perpetuate inequities (Leonardo & Broderick, 2011; Hatt, 2012). To build on this work, we leveraged the perspectives of the students' instructors to explore potential alignment between student and teacher conceptualizations of smartness.

The study takes place at a Southern California Hispanic-Serving Institution (SCHSI) a comprehensive two-year community college. Participants were students enrolled in math classes taught by five instructors who participated in professional development (PD) sessions. The students were enrolled in gateway mathematics courses. The data collected consisted of 103 open-response answers to the question: “What do you think it means to be smart in math at SCHSI?” In the first phase of analysis, the student data were analyzed using thematic analysis (Braun & Clarke, 2012) across three coders, resulting in a two-axis coordinate system and a heat map to visualize the distribution of student responses. The coordinate system ranged from process to outcome and from dominant to counternarratives. (Hernandez et al., 2025)

The next phase of analysis looked at 10 interviews (2 interviews per instructor) that focused on instructors’ beliefs, experiences, and practices around equity in mathematics. While the interviews did not explicitly focus on smartness, we argue that instructors implicitly communicated their conceptions of smartness. The analysis was done in two parts: a) disaggregate students’ placement on the axis based upon their instructor, b) leverage the coding scheme developed in the first phase to gain a broadened understanding of where instructors’ conceptualizations fall in the aforementioned coordinate system.

One major finding is that one instructor, Paul, has prominent placements along the tension of counterprocess and dominant outcome, as well as numerous counter outcomes. Nhung has equal amounts of dominant outcome and counter process, also demonstrating a tension. The second finding is that the instructors seem to describe tensions in the interviews that describe their own experiences against their desired practices. Nhung described striving for conceptual understanding but often reverting to procedural practices, such as limited wait time, relying on frequent participants, and narrowly structured lessons. Similarly, Paul noted the tension between wanting to meet students’ needs through varied supports and the reality that time and content demands constrain his ability to do so.

After analyzing interviews by instructors to understand their conceptualizations of smartness, we found that instructors had tensions between their implicit conceptions and their students’ explicit conceptions of smartness. While we do not make claims about the instructors’ impact on students' placement in this study, we believe it’s important to consider instructors’ conceptualizations of smartness as opportunities to disrupt the conceptualizations of smartness that students may hold, which might be unproductive.

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Development of the Undergraduate Mathematics Class Attributes Questionnaire

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Keywords: class environment, questionnaire development, learning environment

Understanding instructional dynamics in post-secondary mathematics requires tools that capture both classroom complexity and students' lived experiences. Existing instruments such as EQUIP (Reinholz & Shah, 2018), WIHIC (Dorman, 2003), and X-PIPS-M (Creagar et al., 2022; Street et al., 2021; Walter et al., 2016) have advanced research on teaching environments by assessing discourse, engagement, and collaboration in the classroom environment. However, these tools often prioritize researcher-defined metrics over student perspectives, leaving critical gaps in equity-focused analysis. Melhuish et al. (2025) caution that observation-based measures risk reinforcing dominant cultural norms, framing "good teaching" through whiteness and middle-class values unless marginalized voices are centered. They advocate for QuantCrit-informed redesigns that embed cultural responsiveness and intersectionality to ensure validity.

Additionally, observation protocols like EQUIP provide insights into classroom discourse but fail to capture out-of-class learning spaces such as homework and study groups, creating blind spots in understanding student experiences. To address this, we are developing the Undergraduate Mathematics Class Attributes Questionnaire (UMCAQ), grounded in the Instruction as Interactions framework (Cohen, Raudenbush, & Ball, 2003) and informed by Hiebert and Grouws' (2007) definition of teaching as interactions around content. By integrating participatory equity (Reinholz & Shah, 2018; Shah & Lewis, 2019), UMCAQ shifts from measuring what happens to exploring how it feels and for whom, positioning equity as a guiding principle in instrument design.

Pilot Study – Item Selection

In spring 2025, we conducted a pilot study with 197 students enrolled in four undergraduate mathematics courses (Calculus I, $n = 39$; Calculus II, $n = 54$; Differential Equations, $n = 76$; and Intro to Proof, $n = 28$), with the intent of exploring item sensitivity to section-level differences related to instruction. At the section level, groups ranged from small (2–8 students) to large (up to 17 students), providing variability to examine section-level differences.

For each of the 37 items included in the pilot study, which were derived from both extant classroom surveys and investigator developed items, we looked for the response patterns to find items with low variance. To examine section-level sensitivity in student perceptions, we employed cumulative link mixed models (CLMM) using the ordinal package in R, which is well-suited for ordinal data such as Likert-scale responses. We fit two complementary models for each UMCAQ item. The first model specified the course section as a fixed effect and the course as a random intercept, allowing us to directly test whether differences between sections were statistically significant while controlling for course-level clustering. The second model treated the courses and sections as nested random intercepts, providing variance components for both levels and enabling computation of Intraclass Correlation Coefficients adapted for ordinal outcomes. This hierarchical specification captured the proportion of variance attributable to course and section differences without imposing fixed-effect assumptions. This poster invites attendees to explore opportunities for collaborative development and adaptation of UMCAQ, ensuring that diverse student voices shape the future of mathematics education research.

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The Thinking Project: Supporting Metacognitive Growth in Calculus Students

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Keywords: Metacognition, Calculus, Higher Education Mathematics, Self-Regulated Learning

Despite extensive efforts to improve undergraduate mathematics instruction, many students continue to struggle, not only with mathematical content but also with the metacognitive skills necessary to monitor, regulate, and direct their own learning. Metacognition refers to individuals' awareness and control of their own cognitive processes (Flavell, 1979). Incorporating metacognitive strategies in mathematics classrooms can help students become more deliberate in planning, monitoring, and evaluating their learning, which can in turn positively influence affective outcomes such as confidence and anxiety (Roick & Ringeisen, 2018). Research consistently shows that students with higher metacognitive awareness tend to perform better in mathematics and problem-solving tasks than their less-aware peers (Schneider, 2010).

In this poster session, we will present *The Thinking Project* (TTP), a metacognitive support initiative implemented in multiple sections of an Applied Calculus course at a large R1 university. The project is designed to foster students' metacognitive awareness through structured, ongoing activities embedded throughout the course. These web-based activities promote self-reflection and help students become more aware of their understanding of course content, academic progress, and learning goals, while making the activities easily accessible and straightforward to complete. The activities were adapted from an educational intervention in which students identified self-reflective components as some of the most valuable aspects to their learning. Moreover, students who participated in that intervention had significantly lower mathematics anxiety and higher self-efficacy (Guy, 2024). TTP was first piloted in Summer 2025 in a single Applied Calculus course and expanded to additional sections in Fall 2025. The pilot highlighted the need to improve the technology used to administer the activities and to optimize the instructor feedback process.

The Metacognitive Awareness Inventory (MAI) was administered at both the beginning and end of the term to assess changes in students' metacognitive awareness (Schraw & Dennison, 1994). Additionally, qualitative feedback was collected at the end of the semester to further explore students' perceptions of the activities. During the poster session, we will share how the project was designed and facilitated, along with a QR code linking to the activity materials for instructors interested in adopting or adapting them. Preliminary findings will be presented, and we will discuss implications for future research and instructional practices.

This session highlights a practical, research-informed approach to fostering metacognition in undergraduate mathematics courses, an area that remains challenging for many students. By presenting TTP, participants will explore how structured, course-embedded activities can support students' self-reflection, awareness of their learning processes, and strategic engagement with mathematics content. This poster emphasizes both the instructional design and the theoretical rationale, offering educators and researchers a model for promoting metacognitive growth across diverse mathematics learning contexts while also showcasing the ongoing mixed methods study to investigate the project's effectiveness.

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Building Bridges Through Mentoring: Graduate Student Instructors' Pedagogical Development

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Keywords: active learning, boundary crossing, community of practice, professional development

This action-research project examines how peer mentoring supports novice precalculus graduate student instructors (GSIs) in developing as reflective, collaborative teachers. Grounded in *communities of practice* (Wenger, 1998) and the notion of *boundary crossing* (Akkerman & Bakker, 2011), we explore how mentoring can help GSIs navigate between the communities of academic mathematics and mathematics education (see also Darragh, 2022; Fried, 2014). We conceptualize mentoring as a structured, dialogic process (Roberts, 2000) to promote reflection, pedagogical growth, and participation in professional learning communities. Our team, comprised mostly of graduate students serving as mentors, investigated the following: (1) what motivates GSIs to engage in mentoring, (2) how do GSIs perceive mentoring's purpose, and (3) how does participation shape GSIs' teaching reflection and active-learning (AL) strategy use.

At a Midwestern public university, five GSIs teaching precalculus for the first time were mentored by a third-year PhD student ("Morgan"), who had completed a mathematics education mentoring PD program (Yee et al., 2024). During a 15-week semester, Morgan facilitated seven bi-weekly small group meetings emphasizing reflection and active learning, and Morgan observed each novice teaching twice. After Week 5, GSIs completed a three-question survey identifying motivations for participation and desired supports. Their feedback guided revisions to subsequent sessions, including the addition of a collaborative AL activity design task. Data included survey responses, observation notes, and post-observation conversations. We analyzed these data thematically to identify patterns in motivation, reflection, and pedagogical change.

Survey results revealed novices primarily valued *sharing experiences* but less often identified *improving teaching* or *learning AL strategies* as motivators. Few novices articulated an explicit desire for pedagogical development. In response, mentoring sessions were restructured to foreground collaborative design and AL implementation. After this change, novices exhibited greater engagement and reflection: one novice transitioned from individual practice problems to structured group work with conceptual questioning; another novice displayed increased confidence in facilitating AL activities; and a previously disengaged novice remained for a full session to design lessons. Although overall engagement increased, inconsistent attendance across meetings also shaped participation patterns and represents a limitation in this study.

Individual post-observation conferences revealed novices' growing awareness of instructional challenges and their need for pedagogical support. These shifts suggest that intentional, practice-oriented mentoring can help GSIs cross boundaries between disciplinary and pedagogical communities, fostering deeper instructional reflection. In conclusion, peer mentoring, when explicitly framed as both community-building and pedagogical development, can motivate GSIs to engage more meaningfully in teaching reflection and AL design. The process also served as professional growth for mentors, who refined their facilitation and research practices. This project highlights the potential of peer mentoring as a means for theory-informed, sustainable professional development that bridges the math and math ed communities.

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Chunky or Smooth? Students' Covariational Reasoning When Analyzing Phase Plane Trajectories

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Keywords: differential equations, modeling, covariational reasoning, time series

Over the last thirty years there has been a trend in the teaching of differential equations to move away from analytic techniques and to focus on graphical and numerical methods (e.g., Rasmussen, 2001). In parallel to this movement, mathematical modeling has gained a lot of importance in biology and the life sciences. Leading professional organizations have called for a reform of the mathematics instruction of biology students in order to include the study of modeling and dynamical systems (e.g., American Association for the Advancement of Science, 2009). The notions of time series and phase plane trajectories are central to the study of modeling and dynamical systems. A pivotal skill is the ability to navigate between these two types of representation of solutions to differential equations.

Previous research has analyzed students' conceptions of solutions to differential equations as well as strategies used to sketch their representations (e.g., Rasmussen, 2001; Trigueros, 2001; Bennoun, 2024). In particular, Bennoun (2024) used the covariational reasoning framework developed by Thompson and Carlson (2017) to analyze students' reasoning when sketching time series and trajectories. Bennoun (2024) observed that students reasoned in majority at the smooth continuous or chunky continuous levels. An important open question is: How do students explain why they use either of these levels when working with time series and trajectories?

To investigate this question, we conducted ten think-aloud interviews with students who had completed a mathematics course focusing on modeling biological systems. Students were given a textbook exercise showing the (smooth) time series of an epidemiological model that depicted the evolution of healthy and infected populations. Then, they were presented with two possible solutions, one chunky continuous trajectory (i.e., with clear corners) and one smooth continuous trajectory. They were asked which one of the two trajectories best corresponded to the given time series and why. We conducted a reflexive thematic analysis (Braun & Clarke, 2006) of the interviews focusing on the reasons the students used to justify their choice as well as the level of covariational reasoning.

Our preliminary findings reveal that while all students picked the smooth trajectory as being the one that best corresponded to the given time series, their reasons varied. Several students reasoned that since the time series given was smooth, the associated trajectory should also be smooth. For example, one student explained that "because these [two functions on the time series] are curved graphs, […] the actual [trajectory] would also be a curved graph." He added that if the functions on the time series were chunky, then the trajectory would also be chunky. Other participants mentioned Euler's method, that they had learned to approximate trajectories, and how a smaller time step means that the trajectory is more accurate. They explained that as one decreases the time step the trajectories look more and more curvy and that therefore the smooth trajectory was more accurate. Some students used arguments relying on more superficial features such as the smooth graph having axes that are easier to read or the chunky graph feeling confusing to them. On the poster, we will describe these themes in more depth as well as discuss implications for practice.

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From the Polar to the Pole: The Enactive and Embodied Choreomathematics of the Polar Coordinate System

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Keywords: Pre-calculus, Polar Coordinate System, Enactivism, Embodied Cognition

The polar coordinate system (PCS) is typically introduced in Pre-calculus, where it provides a foundation for undergraduate coursework in mathematics, physics, engineering, and other STEM fields. Regrettably, students' engagement with it is often limited to procedural tasks, such as plotting points, rote memorization of the relationship between the shapes of graphs and the parameters of their polar equations, and converting these equations between Cartesian and polar forms. And too little attention is paid to how radius and angle measure co-vary in the graphs of polar equations (e.g., Moore et al., 2014; Paoletti et al., 2013). This project seeks to remedy the anemia of this conventional approach by discerning the features of an enactive educational ecosystem for emergent learning that enables learners to graph and reason within the PCS.

The enactive theory of cognition (Maturana & Varela, 1987; Thompson, 2007) and the theory of inclusive materialism (de Freitas & Sinclair, 2014) frame our investigation of mathematical thinking as a complex and emergent phenomenon constituted in embodied and material engagement, and in the affect and aesthetics of relational interactions between learners and their environments. Owing to its attention to the essence of in-the-moment, lived experience, these theories take inspiration from the philosophy of phenomenology (Husserl, 1977; Merleau-Ponty, 1962). Prior research has demonstrated the viability of such an approach, which enabled young learners to make the sophisticated distinctions between qualitative and quantitative change (Greenstein et al., 2024a, 2024b).

At the onset of design-based research (Brown, 1992; Collins, 1992; DBRC, 2003) into what might be the preconceptual (Varela, 1992), enacted, and embodied groundings for understanding the PCS, we found ourselves in need of an actual "pole" as well as the use of gestures and other bodily movements to enact the roles of radius and angle measure through movement. This experimental and exploratory experience led to the decision that pole dance could situate our approach to emergent learning, and that we would seek to move in new ways (Abrahamson & Sánchez-García, 2016) that might enable us to predict the shapes of polar curves. As soon as we moved forward, though, an obstacle we encountered involved the reliance of the PCS on references to trigonometric functions in the Cartesian coordinate system (CCS; e.g., substituting $x = rcos\theta$ and $y = rsin\theta$ to move from Cartesian to polar). And as we recursively moved between the two worlds of the PCS and CCS in order to coordinate simultaneous changes in the covarying quantities, we got the sense-ation that these parametrized experiences (Dimmel et al., 2025) might illuminate the roles of radii and angle measures in dynamic traces of polar curves. This proved to be a surprisingly joyful aesthetic experience, the essence of which we hope to illuminate through phenomenological inquiry (e.g., van Manen, 2023). Our next step is to determine what pole-dance choreographies and other practices make this interworldly movement productive. We will share these and other findings, which lend credence to our emergent learning approach to teaching the polar coordinate system.

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Self-Reported Student Experiences in a Combined Pre-Service Elementary Education Teacher and STEM Major Quantitative Reasoning Course

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Keywords: quantitative reasoning course, pre-service teachers, student experiences, self-regulated learning, reflective portfolio assignment

We are interested in the self-reported student learning experiences based on written reflections in end-of-semester portfolios. Previous work has looked at the effects of reflective notebooks (Salinas, 2004) or end-of-semester portfolios (Hanush, 2020) on perceptions of prospective elementary teachers in math, both of which had positive outcomes. Others showed increased performance in classes like precalculus (Burks, 2010) due to portfolios, or that portfolios were seen as valuable for pre-service teachers and for departments (Johnson & Koirala, 2013). Keith (1996) highlighted the ability to gain summative evaluation on assignments that involve self-reflection on student learning. Hauk (2005) and John et al. (2020) both analyze student writing to capture students' past math experiences. These studies lay the foundation that portfolios are helpful tools to gauge student perceptions and experiences. Other relevant work includes the mathematical experiences of pre-service teachers (Szydlik et al., 2003); and a look at mathematical beliefs of students (Ward et al., 2010), including comparing majors (Szydlik, 2013). Dias and Santos (2016) found promising self-regulated learning (SRL) in a mathematical portfolio of one high school student but SRL has not been investigated in college level math portfolios. We seek to answer the research question: how does a reflective portfolio assignment in a college level mathematics class reveal students' SRL?

The study context is a quantitative reasoning course that the first author teaches that is intended for elementary education and STEM majors with a focus on students being competent and confident in mathematics, learning not "what to see" but "how to see," and developing the skill of seeing patterns in multiple ways. In this course, students collaborate to build their own understanding, develop solutions using multiple representations of a function, and practice communicating their approaches to a diverse class of peers.

We use SRL (Zimmerman, 2002) as our theoretical framework to examine our research question. We collected data from the final reflective piece of 25 student portfolios, approximately half education majors and half STEM. We performed thematic analysis (Braun & Clarke, 2006) and categorized our codes into different phases of SRL. The Forethought phase (self-motivation beliefs) revealed in student work included codes related to enjoyment, renewed interest, and mindset shifts; the Performance phase of SRL (self-control processes) included codes about problem-solving creativity, approaches, and exploration; and the Self-Reflection phase (self-reaction processes) consisted of codes regarding perseverance, slowing down, and productive struggle. For our poster, we will share the comprehensive list of codes and representative student responses.

We believe this work is beneficial for instructors of similar quantitative reasoning courses at other institutions (e.g. mathematics for the liberal arts or for pre-service teachers) who may be able to implement course features (e.g. reflective portfolios, "menu"-style assignments) into their own courses in order to facilitate SRL processes for students, as well as for scholars who study student writing or SRL. Next steps include comparing student experiences from different majors.

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Developmental Math Course for TCUs: Standardizing Pathways to College-Level Success

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Keywords: Tribal Colleges and Universities, pathways, developmental math, online learning

Background and Motivation

At many Tribal Colleges and Universities (TCUs), students are seeking to achieve dreams rooted in community and cultural responsibility. Yet placement exams and prerequisite requirements often place them into multiple developmental math (DevEd Math) courses that delay degree progress, add tuition costs, and offer no credit toward graduation. In Montana, where TCUs largely serve Indigenous, first-generation, and non-traditional students, these barriers can lessen confidence and momentum, especially when courses are inconsistently offered due to low student enrollment or faculty availability.

This study builds on existing literature addressing developmental education redesign (Boatman, 2021; Ngo, 2019; Vandal, 2014) and culturally responsive frameworks such as Tsosie's (2022) Six Rs of Indigenous Research. It explores how Montana TCUs might implement a unified, online developmental math course that (1) strengthens academic readiness, (2) sustains Indigenous ways of knowing, and (3) promotes equity and access in higher education. This research seeks to answer the research question, What are the opportunities and challenges in developing a unified, online developmental math course accepted across TCU campuses?

Methods

A qualitative case study was used to examine current DevEd Math structures across the seven Montana TCUs. Data sources include faculty, staff, and administrator interviews, and a review of course syllabi, institutional placement practices, and success data collected in Fall 2025. Data were analyzed thematically using an inductive approach to identify recurring ideas and experiences across institutions. Coding and interpretation were guided by Tsosie's (2022) Six Rs of Indigenous Research, which provided a culturally grounded framework for understanding how the values of respect, reciprocity, and responsibility shape the design of developmental mathematics. The analysis attended to each institution's unique context while revealing common barriers and collective strengths across Montana's TCU network, centering the voices of faculty, staff, and administrators.

Preliminary Findings and Next Steps

Early findings reveal widespread recognition of the need for a consistent, culturally responsive DevEd Math pathway. Common themes emerging from participants include: access and equity, faculty capacity, cultural relevance, and collaborative potential. Participants described a unified course as a way to reduce redundancy, enhance TCU collaboration, and provide equitable opportunities for students to begin their college-level mathematics journey with confidence and belonging. Next steps will focus on: course logistics, instructional design, curriculum development, institutional agreements, and a pilot implementation at Montana TCUs. This ongoing collaboration represents an important step toward standardizing equitable math pathways, strengthening inter-institutional partnerships, and honoring Indigenous approaches to mathematics and learning.

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Precalculus Students' Perceptions of Instructor Actions and Intentions via Stimulated Recall

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Keywords: participative math identity, stimulated recall

The results in this poster contribute to a larger dissertation study aiming to understand how students' perceptions of their interactions with their instructor relate to their math identity, or their view of themselves as a "doer of math." This is a participative approach to math identity, influenced by the figured worlds framework (Holland et al., 1998) and Wood's framework of micro-identities (2013), which both contest that the 'kind of person' one enacts within specific environments relates to the 'kind of person' they generally identify themselves as. Studies conducted in undergraduate STEM courses show how instructors can set classroom expectations that influence the STEM identities students are permitted to enact (Jackson & Seiler, 2017), and that students are aware of these different expectations that may be imposed on them (Lim et al., 2023). This poster will discuss results on students' varied interpretations of instructor actions and intentions in an undergraduate math context.

Data for this poster comes from observations of an interactive precalculus class in the southwestern U.S. during Fall 2025, and early- and late-semester interviews with seven focal students. Observations involved video-recording each class session, providing video clip data used for stimulated recall (Lyle, 2003) in both interviews. The interviews were designed to understand students' beliefs about their math ability and have them reflect on their roles while interacting with their instructor. Analysis for these results was completed via open coding (Strauss & Corbin, 1998) of interview transcripts, involving refining codes along three dimensions: (1) *emotions* and *beliefs* expressed by the student about the interaction; (2) perceptions of the *emotions* and *beliefs* of the *student in the video clip* (sometimes themselves); and (3) perceptions of *instructor actions* and their *intentions*. These codes have been organized into various categories of meaning and are still actively being explored. For this poster, two initial findings will be presented: (a) *variability of an interaction*, and (b) *the typical interaction*.

For (a), the students had variation in both the instructor actions they noticed in these clips and their interpretations of these actions (and their intentions). For a particular early-semester clip, some students noted a vocal inflection used by the instructor when responding to incorrect answers, while others noted the instructor's physical gesturing. With the vocal inflection, students also had varied interpretations for the instructor's intent, including to indirectly guide the students to the correct answer or to push back against student responses. For (b), the students overall described a typical student-instructor interaction as the instructor probing for student understanding and avoiding directly giving answers to students, which students typically interpreted as the instructor encouraging them to think deeply about the material.

These findings suggest that students may be consistent in interpreting the 'typical' interactions in their classroom, but may have varied perceptions of more unique interactions. Further work needs to be done to understand how these different perceptions might be impacting students' math identities. There are future plans to analyze the math identity items from the interviews to understand the students' relationship to math throughout the semester. In doing so, I will be able to better connect the different ways these students interpret instructor actions to the different math identities they enact (and believe they enact), which will lead to a deeper understanding of the complex ways instructors can influence student math identity development.

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Needs, Success, Race, and Equality: A Q Methodology Study of Community College Algebra Students

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Keywords: college algebra, community colleges, diversity, equity, inclusion, Q methodology

While prior studies have explored students' experiences in college algebra (Gardner, 2020) and examined undergraduate students' sense of inclusion and belonging in mathematics at predominantly white institutions (Leyva et al., 2021), there remains a gap in research focusing on how community college students understand diversity, equity, and inclusion or how these understandings develop over time. Our study seeks to address this gap by exploring the question: What are community college students' understandings of diversity, equity and inclusion in their college algebra classrooms?

We present preliminary findings from a Q methodology study exploring community college students' perspectives of equity in college algebra. This study is part of a grounded theory analysis (Charmaz, 2006) to build a conceptual framework of this construct from students' perspectives. Community colleges serve students from diverse local contexts, and thus, their interpretations of equity are shaped by both institutional and community factors. We collected data from 41 students from eight colleges located in four regions of the United States—Northwest, Southwest, Central, and Midwest. The participating colleges were selected for the demographic diversity of their student populations. Our recruitment efforts intentionally focused on including students representing varied racial, gender, ethnic, physical, and neurodiverse identities.

Each participant completed four semi-structured interviews over one year. During these interviews, students (1) discussed their personal and mathematical histories and how these shaped their current understanding of diversity, equity and inclusion; (2) provided commentary on short classroom videos, identifying moments they associated with equitable or inequitable practices; (3) engaged in a Q-sort activity (Rieber, 2020) using 33 statements derived from student interviews and literature to express their viewpoints on equity in mathematics classrooms; and (4) are currently completing interviews that allow participants to reflect on changes in their perspectives throughout the process.

In the Q-sort activity, 25 of the 41 participants placed statements on a scale ranging from most disagree to most agree. We used a scale in the shape of a normal distribution for the use of a forced sort structure allowed us access to individual students' unique perspectives and to directly compare student responses to identify common themes. Following the Q-sort activity, we asked the participants to explain how their unique perspectives were formed, focusing on the four statements they placed into "most agree" and "most disagree". The Q-sort data was analyzed using the KADE desktop application (Banasick, 2019), which identifies shared and divergent perspectives among participants. Our analysis yielded four distinct factors (Needs, Success, Race, Equality), each representing a different way community college algebra students conceptualize equity in mathematics classrooms. In this poster, we present findings from the Q-sort analysis, highlighting the themes and perspectives that emerged from these four factors.

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Making Sense of Earth's Climate Using Iterative Algebra

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Keywords: Sensemaking, Iterative Algebra, Climate Model

Differential equations are fundamental to many models of dynamic systems. Earth's climate, for example, is a dynamic system often modeled using differential equations (Kaper & Engler, 2013). Differential equations thus provide a promising avenue to incorporate pressing topics such as Earth's shifting climate into the mathematics classroom; however, not all students study differential equations, which necessitates an alternative method to introduce students to mathematical relationships governing a complex system's evolution (Bressoud, 2015; Bressoud, 2021). One method proposed to teach students ideas similar to differential equations without the use of calculus is through *iterative algebra* (Coles, 2013). Using an iterative algebra approach, students are given an initial state and an equation that defines how the system changes for each future timestep. The change is always based on the system's current state and is repeated iteratively to model how a system changes over time.

Despite the potential of iterative algebra in mathematics classrooms, few studies have determined how students can understand dynamic systems using iterative algebra. Our study addresses this gap in research through investigating how students—who mostly do not have a calculus background—make sense of the mathematics and science behind a climate system using iterative algebra. In particular, we ask, *How do students make sense of the mathematics and science of equations modeling aspects of Earth's climate?* To answer this question, we studied students *sensemaking*—how individuals make meaning of new situations based on prior experience and knowledge (Odden & Russ, 2019). We used an even more specific framework, the Sci-Math sensemaking framework (Zhao & Schuchardt, 2021), as a tool to analyze the types of mathematics and science sensemaking students engaged in while working with a mathematical model of climate.

In a college algebra classroom in a large Midwestern university, students were tasked to develop and interpret recursive equations (i.e., engage with iterative algebra) representing the dynamics of the North Atlantic Current. The activity took place across three consecutive 75-minute class sessions of college algebra. We collected audio recordings and written responses from each participating group. Using the Sci-Math Sensemaking, the sensemaking processes of three different groups of students (each composed of 2-3 students) were studied as they worked on the mathematical climate model activity.

Students engaged in an array of mathematics and science sensemaking types. We found that students understood variables and rates of change as mathematical and scientific constructs. For example, students recognized that certain parts of their equations represented change, either by discussing the quantity or direction of change (mathematics sensemaking) or by describing the science context of the change (science sensemaking, e.g., "This part of the equation represents the temperature of the water decreasing over time"). Furthermore, when developing equations, students often began by making sense of the science context, completing a series of calculations, followed by generalizing their calculations in equation form. These results suggest that students, with primarily no calculus background, can engage in interdisciplinary sensemaking of a dynamic system. This provides promise for teaching more students dynamic systems through iterative algebra.

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Class Size, Room Setup, and Use of Group Work: Undergraduate STEM 2019-2025

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Keywords: instructional practice; class size; STEM; teaching change

Despite recommendations to engage in interactive pedagogy, didactic lecture remains the primary experience of most undergraduate students in introductory STEM courses (e.g., Apkarian et al., 2021; Stains et al., 2018). Often-cited barriers to adoption of alternatives, such as group work, include large class size and auditorium-style setup, though the empirical evidence is somewhat inconclusive across instructors' experience and contexts (e.g., Ake-Little et al., 2020, Apkarian et al., 2021; Yik et al., 2022ab). We endeavor to shed further light on how these factors impact instructors' pedagogical decisions, leveraging survey data from the same individuals at two points in time: Spring 2019 and Spring 2025. Specifically, we explore the research question: *to what extent do changes in course enrollment and classroom setup relate to changes in the percentage of class time instructors report spending in group work?* For brevity, this proposal discusses only the former.

Data come from two NSF-funded projects which surveyed postsecondary instructors of general chemistry, single-variable calculus, and introductory physics across the USA. The surveys deployed in each project used web-based surveys with many identical questions; the latter additionally included items related to instructors' experience during the pandemic response teaching period, although that is not the focus of this analysis. The same roster was contacted for both projects, with a smaller response to the follow-up: $N_{19} = 3769$; $N_{25} = 933$.

Our outcome variable is the reported percentage of in-class time, during a normal week, that students are expected to work on mathematics problems in small groups. We also considered their course enrollment (binned as: 0-19; 20-29; 30-39; 40-59; 60-99; 100+) and classroom setup (easily accommodates group work; does not). Instructors' responses to these items were paired across their 2019 and 2025 responses. In total, 551 instructors' responses were available for pairwise pre-/post-analysis of these items; we admit appropriate caution about generalization in light of response bias and the complexity of the intervening years. Instructors were assigned to one of three groups based on a comparison of their class size in 2019 and 2025: increase, decrease, or no change. Three paired *t*-tests, with Bonferroni-corrected *p*-values to account for repeated testing, were conducted for class time spent in group work between 2019 and 2025. The results are in Table 1. In short, we find no differences in the amount of time spent using groupwork, regardless of whether, and how, their class size changed. Our poster will include additional details of this analysis and a similar analysis related to changes in classroom setup.

Table 1. Results of paired t-tests on instructors' reports of percentage of class time spent in group work, 2019 compared to 2025, disaggregated by changes in their course enrollment.

	Percentage of class time spent in group work: M (SD)					
Enrollment Change	2019	2025	Δ	*df*	*t*	Adjusted *p*
Increase (N = 89)	20.51 (19.96)	17.98 (18.80)	-2.53	88	-1.736	0.258
No change (N = 324)	22.24 (20.82)	24.58 (21.51)	2.34	323	2.372	0.055
Decrease (N = 138)	20.83 (18.74)	21.48 (20.43)	0.65	137	0.479	1.000

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Research Outline of Preparation of Potential STEM Majors for College Calculus

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The initial college mathematics placement of Science Technology Engineering and Mathematics (STEM) majors may influence the direction and duration of their college studies. This study aims to explore how secondary mathematics preparation of potential STEM majors may influence their initial mathematics placement. For this study, we consider mathematics preparation to consist of secondary mathematics courses and college preparation advice.

Literature Review

Secondary mathematics course sequences may be accelerated or hyper accelerated. These practices can be advantageous for some students yet push others too quickly through the mathematics curriculum (NCTM, 2012). Oftentimes, accelerating mathematics ensures the student has access to a high school calculus course. However, sometimes in a Rush to Calculus, students may complete the highest secondary mathematics course available to them before their senior year of high school; this can create a gap in rigorous mathematics coursework prior to beginning college mathematics.

First year college students who understand the expectations of college mathematics typically have a smoother transition from high school to college mathematics (Clark & Lovric, 2008). In high school, potential STEM majors may obtain advice from their secondary community about college preparation; however, the community may prioritize secondary calculus for its perceived role in college admissions.

Research Question and Theoretical Framework

Guiding this study is the following research question: How are undergraduate STEM majors' high school mathematics experiences and college preparation connected to their initial college mathematics placement?

The theoretical framework is The Theoretical Model of the Secondary to Tertiary Transition in Mathematics by Clark & Lovric (2008) where the researchers mapped the Rites of Passage from the field of anthropology onto the transition from high school to college mathematics. The theoretical model considers not only the time required to learn mathematics but also the community's role in preparing students for college mathematics.

Methodology

This observational study is a sequential mixed methods design. In the first phase, factors that may influence initial mathematics placement are analyzed quantitatively. The factors being the timing of participants' secondary mathematics courses including any gaps in rigorous mathematics learning prior to beginning college mathematics. After completion of the quantitative data analysis, a survey will be distributed to collect qualitative data in the second phase. The survey questions participants about their experiences transitioning from secondary to tertiary mathematics and the advice they received. Once the qualitative data is analyzed, the results will be compared with the quantitative findings to uncover any meta inferences.

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Mathematical Dispositions of Pre-Service Elementary Teachers

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Keywords: Eye tracking, math anxiety, mathematical dispositions, pre-service teachers

Dispositions include an individual's behavioral (i.e., things they say & do), cognitive (i.e., attention & memory), and affective (i.e., emotions & beliefs) tendencies (APA, 2020). In mathematics education, researchers discuss productive mathematical dispositions as a "...habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy…" (NRC, 2001, p.116). In contrast, unproductive dispositions (e.g., anxiety, avoidance) do not reflect a lack of ability but rather behaviors that can hinder students' engagement with mathematics (Feldhaus, 2014). Supporting the development of productive dispositions is therefore critical for improving students' mathematical knowledge and achievement (Awofala et al., 2022; Cobb & Hodge, 2002).

Pre-service elementary teachers (PSETs) often exhibit unproductive mathematical dispositions (Cornell, 1999; Patkin & Greenstein, 2020), including low confidence and avoidance behaviors, which may negatively influence their learning of mathematics, their future teaching, and their students' attitudes toward math (Beilock et al., 2010; Phillipp et al., 2007). Addressing these mathematical dispositions early in teacher preparation is essential. This three-year study examines PSETs' mathematical dispositions during their first mathematics content course, focusing on four key constructs: mindset, self-efficacy, anxiety, and avoidance. Data were collected via pre- and post-semester surveys, including validated scales for math anxiety, self-efficacy, growth mindset, and content knowledge in all three years. During the second and third years, students completed targeted in-class interventions guided by their instructors. Additionally, a subset of students (volunteers) participated in an eye-tracking study at the beginning and end of each semester. Eye-tracking technology (Eyelink 1000+) was used to assess implicit visual behaviors during arithmetic tasks, offering insights into math-related anxiety and avoidance.

In the baseline year (Fall 2023), 58 students provided matched pre- and post-data; in the intervention year (Fall 2024) there were 59 matched ones. Quantitative results revealed that students in the intervention semester showed a significantly greater decrease in math anxiety (mean change = –2.62 vs. –0.57, $p = .048$) and significant gains in growth-oriented mindset (mean change = 1.58 vs. 0.21, $p = .027$) and math content knowledge (mean change = 1.42 vs. –0.07, $p = .0095$). Changes in self-efficacy and avoidance were not statistically significant. Preliminary eye-tracking data suggest that students with higher math anxiety exhibit lower fixation rates and durations during arithmetic tasks, indicating potential visual avoidance. These findings underscore the importance of targeted interventions to reduce math anxiety and foster productive dispositions among future educators. This research and its findings on interventions in mathematics content courses offer valuable contributions to the RUME community and to faculty teaching lower division mathematics courses for non-math majors. Moreover, it provides insights into the potential of multimodal data collection methods, such as eye tracking, for studying students' mathematical dispositions and learning processes.

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Mathematical Modeling Experiences in Nonformal Learning Environments for Preservice K-8 Teachers

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Keywords: Pre-service K-8 teachers, mathematical modeling, nature of mathematics

Because of the many purposes mathematical modeling serves, several organizations have called for the incorporation of modeling in K-8 mathematics education (AMTE, 2017; CCSSM, 2010; GAIMME 2nd edn., 2019; NCTM, 2000). Engaging in modeling as learners can help teachers more effectively implement modeling experiences in their own classrooms (Anhalt & Cortez, 2016; Burroughs, 2021; Wickstrom & Jung, 2024). This project seeks to explore the following questions:

1. How is mathematics used by a business owner in her work and how can her experiences be used to create tasks that make connections between elementary school mathematics and activities outside of school explicit for preservice K-8 teachers (PSTs)?
2. What are preservice PSTs' perceptions of the nature and purposes of mathematics?
3. How does an off-campus mathematical modeling experience in a nonformal learning environment impact PSTs' perceptions of mathematics?

Literature Review

Out-of-class environments can provide authentic contexts for students to engage in mathematical modeling (Boaler, 1993; Eshach, 2007; Martínez-Jiménez et al., 2022; Vale et al., 2019). Non-formal learning, which is usually structured, mediated with formal education, and teacher-led, has the potential to impact the affective domain of student engagement (Eshach, 2007), through positive, long-lasting memories of the experience (Falk, 1983).

Methodology

During Fall 2025, a business owner was interviewed regarding her use of mathematics in her work running a farm. This interview informed the design of out-of-school mathematics experiences for a math content course for PSTs. Students visited a farm to bake bread, estimate areas of fields, and engage in various profit estimation activities. Students engaged in tasks that required the application of arithmetic, proportions, percentages, fractions, estimation, and area. Interviews with students were collected focusing on their perceptions of the nature and contexts of mathematics.

Results

Data were analyzed using reflexive thematic analysis (Braun & Clarke, 2006) to identify themes regarding (a) connections between school mathematics and activities outside of school and (b) PSTs' perceptions of mathematics and its contexts and the purposes of learning mathematics. Themes identified from the businesswoman's interview include using elementary mathematics for profit estimation (including cost estimation and pricing goods), recipe development, and plant management (including diluting chemicals and estimating seed purchases). Results from this interview were used to create mathematical tasks for students during the field trip to the farm. Analysis of student interview data is still in progress at the time of writing. The results from this project will inform activity revisions and research design in future semesters.

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Exploring Math Identity and Cultural Perspectives Through Narrative Math Games

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Keywords: game-based learning, math identity, narratives in mathematics, cultural perspective

Students at a Tribal College or University (TCU) can be said to have a "home and family culture that does not fully resonate with that of school and the wider society (Bishop, 1991)." Frequently, attempts to improve math education for these students look at improving access to better math instruction in order to increase achievement. This is the "dominant axis of equity (Gutiérrez, 2012)." Is it possible that using the critical axis, identity to power, math instruction can help TCU students leverage their cultural identities?

Purpose of the Study

The purpose of the study is to examine the mathematical identities of students in a mathematics class at a Tribal College. The mathematics class employs a culturally relevant technology game based on the units of the mathematics reasoning course. The mathematical identities of college students shape their perspectives on their capacities as mathematics learners and persistence in STEM fields. (Cribbs, Mantone, & Piatek-Jimenez, 2015) These mathematical identities are tied to students' sense of belonging in the mathematics class, and to the field of mathematics itself. Researchers have found that mathematics instruction that is culturally relevant, in this case to indigenous culture, increases students' sense of belonging and positive mathematics identities. (Matthews, 2014)

The mathematics class under study aims to provide mathematics students with a culturally relevant experience in part through a technology game. The game is to provide students with mathematical experiences that are embedded in a narrative around activities that are specific to the Mi'kmaq tribal communities. The game is set in the historical period of the late 1500s, and the activities include building a birch bark canoe and learning navigation skills. (Bahar, 2019) As the students engage in the games, they interact with certain tasks along the way that include both mathematics and cultural content. They are also prompted to reflect on their experiences as they play the game, including their relationships with the mathematics and with the cultural activities.

Methods

This study is being conducted at a tribal college in the upper Midwest. The students are taking a math reasoning class and part of the curriculum is the game. The study employs a qualitative methodological approach to study the students' engagement in the mathematical games and the kinds of mathematical identities they are developing as a result. Students will be asked to complete a survey that asks about their attitudes and expectations in a math setting. We will also interview a sampling of the participants to get more open-ended responses.

Potential Impacts

The study aims to counter existing narratives about what math is and who can do it that tend to limit the participation of certain communities in mathematics. For the tribal college community, this research will reveal the importance of culturally relevant mathematics instruction. The resultant technology will be made available for other TCU's to review and potentially adopt.

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Expanding What We Notice When Observing Mathematics Instruction: Operationalizing the 5Rs Framework for Use in Undergraduate Mathematics Instruction

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Keywords: Lesson Study, Observation Protocols, 5Rs Framework, Noticing.

Our Lesson Study (Cerbin, 2011; Lewis et al, 2019) group of mathematicians and mathematics educators adapted and operationalized the 5Rs framework for promoting equitable norms and culture in mathematics (Sears, 2021) in the context of studying undergraduate geometry courses (Boyce et al, accepted). The 5Rs —Responsibility, Relationships, Relevance, Responsiveness, and Rigor (Sears, 2021)—refer to critical aspects of equitable mathematics teaching that instructors of prospective teachers should model to inform their future practice. This research aims to understand how instructors can adapt and use a 5Rs observation instrument to promote equitable norms and culture. In this poster, we describe our process of operationalizing the framework for observations of courses that serve a variety of audiences (e.g., PSTs and STEM majors) and instructor backgrounds.

The observation instrument includes two components for each R: a description of the construct and protocol questions to guide observations. The process of creating and revising these components consisted of three cycles, each consisting of (1) coming to shared understandings of individual Rs' meanings and relationships amongst Rs, and (2) posing and testing construct descriptions and observation protocol questions to determine the extent to which they align with our shared understandings of the 5 Rs. In the first cycle, we maintained the original descriptions of the Rs' meanings from Sears (2021) and developed a set of 20 questions, some of which attempted to simultaneously address multiple Rs. Upon testing that protocol with a lesson observation, we experienced multiple tensions, such as the degree to which we maintained fidelity with the original descriptions; we felt that it was necessary to allow for clarifications and ways to embrace common understandings associated with the terminology. For example, for "Rigor," we noted that the Dana Center (2019) has written about different interpretations of that word, and scholars have identified connections between Rigor and Relevance (Daggett, 2014). In the second cycle, we revised the descriptions of the Rs and created new questions to address only a single R at a time, attempting to resolve the tensions we experienced after the first round. In the third (ongoing) cycle, we have been more deliberate about group members' roles and responsibilities in the testing of the protocol and analysis. Pairs of group members asynchronously analyzed a single lesson, focused on a single R. Subsequently, other group members met synchronously to identify commonalities, themes, and learning opportunities from their peers' observations. The poster will present the current form of the protocol as well as highlights of our learning from creating it.

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A Bustling Inquiry-Oriented Math Environment: Who is Present and What is Happening?

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Keywords: inquiry-oriented instruction, new materialism, post-representationalism

Inquiry-oriented (IO) instruction enables learners to discover and explore mathematics, actively engage in the classroom, ask questions, and construct their own knowledge (Kuster et al., 2018; Rasmussen & Kwon, 2007). Grounded in constructivist theories, inquiry-based mathematics education emphasizes the construction of knowledge through learners' interactions with their prior knowledge, peers, and instructor (Laursen & Rasmussen, 2019). These theoretical foundations are crucial for realizing IO's transformative potential for students' cognitive growth. However, we argue that these alone are insufficient given that IO environments and learners are dynamic and thus warrant careful consideration to inform thoughtful classroom design. In this poster, we explore a researcher's (Author 1) relationship with classroom data to consider how the material world participates in learning within an IO classroom. Drawing on new materialist and post-representational approaches, we aim to evoke new wonderings about IO instruction.

We drew on MacLure's (2013) proposal to encounter data that "glows," analyzing bodily responses and spontaneous interactions from two videotaped sessions of an undergraduate IO Calculus II course. By focusing on interactions among different agents in an IO class, we moved beyond traditional transcriptions and coding and instead identified significant material entanglements in the IO classroom that captured our interest. We decentered the teacher and students as the sole agents in the IO instruction, and acknowledged the agency of non-human entities that actively shout their presence in the classroom and contribute to facilitating learning (De Freitas & Sinclair, 2014). Kimmerer (2017) explained "…just because we don't think of them as humans doesn't mean they aren't beings" (p. 57). This idea, stemming from indigenous knowledges, prompts us to envision the possibilities of rich interactions that can exist between these "beings or becoming" (De Freitas & Sinclair, 2014) within the classroom environment.

We identified four "beings or becoming" that brought life into the IO classroom: (1) teaching and learning resources, (2) classroom practices, (3) students, and (4) teacher. Together, these emphasize that the bustling atmosphere is not only contributed to by the students and the teacher, but also by other non-human agents. As an example, a particular teaching and learning resource, Euler maps, served to engage students in conjecturing about sequences (Vroom et al., 2025). When observing the recording, we were struck by the unique invitation the maps extended to students, encouraging them to draw closer and approach learning with their own expressions, bodies, and physical movements. We saw some groups of students becoming more animated and using hand gestures on the Euler maps, while others maintained silence before interacting with their maps, and some students wrote on the adjacent blackboard. We could hear the bustling energy in the classroom through the intensity of the conversations happening around the maps. This leads us to wonder – in what ways can the IO classroom provide different opportunities for students' embodiments? In our poster, we will present our observations and corresponding wonderings, which could lead to reformulating theories for IO classrooms.

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Development and Change in Motivational and Self-efficacy Beliefs in Calculus: Relations With STEM Retention

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Retention in science, technology, engineering, and mathematics (STEM) disciplines continues to be an important focus in higher education, with many institutions seeking to enhance student persistence in these disciplines. Previous studies have shown that a higher sense of belonging, self-efficacy, and perceptions of the usefulness course curriculum (e.g., utility value) relate to higher rates of persistence (Tinto, 2017). However, high rates of attrition among students who initially declare a STEM major still exist (e.g., Brewer et al., 2021; Cromley et al., 2016; Doerschuk et al., 2016; Sithole et al., 2017). Previous studies have noted that students' personal characteristics such as motivational and self-efficacy beliefs, academic backgrounds, and institutional factors such as policies related to course sequencing, registration, and grading are factors that play a part in STEM retention (Cromley et al., 2016; Green & Sanderson, 2018). Students who do leave STEM degrees often cite negative experiences in introductory level courses, such as calculus, as their primary reason for leaving (Ellis et al., 2014; Rasmussen & Ellis, 2013; Seymour & Hewitt, 1997). These early courses are often framed as being gateway or "weed-out" courses (Lee et al., 2024). Previous research has noted that students often experience declines in motivation in these courses (Perez et al., 2014), with students enrolled in calculus classes reporting significant declines in their confidence, enjoyment, and desire to take additional mathematics classes (Bressoud et al., 2013; Sonnert et al., 2015).

Given that calculus is a foundational course supporting many STEM majors, it is important to investigate the nature of students' experience in calculus, changes in motivational processes and beliefs in calculus, and the role of such changes in predicting students' intention to remain in STEM. The goal of the present study is to conduct such an investigation through a longitudinal analysis of the utility value, complexity, and general and task-based self-efficacy beliefs of students enrolled in a first-semester calculus course. A survey was implemented across four time-points consistently after completion of common course exams. Our research is framed by the following questions:

- How do students' motivational and self-efficacy beliefs change over time?
- How does the nature of students' change in motivational and self-efficacy beliefs relate to their intent to remain in a STEM degree?
- How do students' motivational and self-efficacy beliefs predict their academic achievement?

Data analysis is ongoing. However, the analyses will provide insight into several key areas addressing student retention and support in STEM. By understanding the nature of change in students' motivational and self-efficacy beliefs in a first semester calculus class along with the nature of students' experiences, we can gain insights into how best to support students. Pedagogical approaches that foster and bolster motivational and self-efficacy beliefs and promote positive learning experiences may be beneficial in promoting persistence.

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Justification of Exponent and Logarithm Rules From an APOS Perspective

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Investigations have been conducted into how students conceptualize exponents and logarithms, utilizing APOS theory, particularly regarding how students justify exponent rules. The thesis by Williams (2011) presents a list of codes to identify the conceptions that students hold regarding logarithms. However, a gap in the literature remains on how students justify the corresponding logarithm rules. In this poster, I compare the thinking of two students to extend her work.

Keywords: Algebra and Algebraic Thinking, Undergraduate Education, Precalculus

This poster draws on data from my dissertation regarding how students in a complex analysis course leverage their knowledge of real-valued exponents and logarithms to reason about the complex-valued analogs. Part of my dissertation aims to describe the conceptions that college students have for real and complex exponents and logarithms, and to compare the similarities and differences in terms of APOS theory (Dubinsky, 1991; Asiala et al., 1997; Dubinsky and McDonald, 2001). In my literature review, I have encountered studies that examine how students justify various logarithm rules, particularly the work of Williams (2011). Part of Williams' (2011) claims is that responses to questions about logarithm rules can be used to characterize their process or object conceptions about logarithms. However, her thesis does not report on responses to such questions. The findings in this poster contribute to a larger goal of extending Williams' (2011) framework on exponents and logarithms from an APOS perspective.

Discussion

There was evidence to indicate that both May and Jim's schemas for exponents and logarithms were unified under a broader schema due to the inverse relationship between the two operations. Another similarity between their two schemas was the combinatorial interpretation of exponential expressions, where a^b would be considered shorthand for the number a multiplied by itself b times when b was a natural number. There was more substantial evidence that their schemas diverged when operationalizing an interpretation of fractional exponents. While it is likely that both May and Jim would use roots when calculating fractional exponents, there was definite evidence that such interpretations were not operationalized in proof contexts.

The differences in proofs indicate some variations in their respective schemas. Jim relied solely on the interpretation of iterated multiplication to justify exponent rules. For example, he attempted to reduce the justification of fractional exponent rules to the case of integer exponent rules. He likely operationalizes definitions when justifying properties of exponents and logarithms. He will likely explain the properties of the complex exponent and logarithm by referring to the definition of the complex exponent and the inverse relationship between the exponent and logarithm. In contrast, May's use of logarithm rules to justify exponent rules suggests that in the complex setting, they are less likely to refer to exponents when justifying properties of the complex logarithm.

These phenomena suggest that a similar conception for exponents does not necessarily imply similar conceptions of logarithms. More broadly, among college students, it is likely that the schema for exponents and the schema for logarithms are unified. However, conceptions relating to exponents are likelier to be homogeneous compared to conceptions for logarithms.

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Mentoring, Belonging, and Feedback Loops: A System Dynamics Model of Foundational Mathematics Course Coordination

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Keywords: Course coordinators, system dynamics model, active learning, instructor belonging

Coordinated collegiate mathematics courses are complex systems wherein social dynamics, mentoring structures, and institutional factors shape how graduate student instructors (GSIs) approach teaching. We focus on eight faculty members who coordinate and teach foundational math courses (FMCs) taught by GSIs (Rogers & Long, 2025), including College Algebra, Precalc, Quantitative Reasoning, Intro Statistics, Math for Architecture & Construction, Math for Elem. Teachers, & Calculus. We hypothesize how implementing professional development (PD) focused on teaching math through active-learning (AL) strategies (Freeman et al., 2014; Laursen & Rasmussen, 2019) for FMC coordinators can enhance their sense of belonging and increase their use of AL in teaching and course coordination. We assess sense of belonging with a modified version of the survey developed by Sidney et al. (2025). Our study considers: (1) How do course coordinators' orientations toward humanistic growth (Martinez et al., 2022) shape their GSI mentoring and engagement with AL within PD contexts? (2) How does course coordinators' participation in PD influence their sense of belonging and relate to evolving approaches to mentoring and instructional change? Together, these RQs lead to: *What interrelated mechanisms among coordinators' humanistic orientations, mentoring structures, and experiences of belonging support emergence and maintenance of meaningful instructional change in FMCs?*

To explore these foci, we developed a qualitative system dynamics model using Vensim simulation software (Mohaghar et al., 2017). By thematically analyzing course coordinators' dialogue and surveys, we found 12 key constructs shaping the FMC coordination landscape; we list the six most prevalent: *Shared Time*, *Resource Accessibility*, *Communication Channels*, *Recognition Events*, *Workload Policy Support,* and *Barriers to AL*. We organized the 12 constructs into five key areas of influence, i.e., system stocks: *AL Use in Coordination*, *Humanistic Growth (HG) Coordinator Orientation*, *Mentoring Structure*, *Instructor Belonging*, and *Exclusion Among Instructors*. We then mapped feedback loops for how these areas interact:

- *Mentoring Loop*: how shared time, trust, and modeling reinforce effective mentoring
- *HG Orientation Loop*: how values and coordinator identity sustain reform
- *Exclusion Brake*: structural barriers that disrupt coherence and inclusion

Our model highlights how mentoring and coordination are not isolated interventions, but dynamic systems shaped by institutional conditions and relationships. When policies support HG mentoring goals and create a collaborative space, FMC instructors are more likely to use AL strategies and experience a stronger sense of community. Conversely, when access, time, or recognition are constrained, exclusionary pressures can dampen change efforts. Rather than presenting coordination as a fixed structure, this model conceptualizes it as a responsive system whose trajectories can shift depending on feedback dynamics, institutional levers, and interpersonal trust. By using system dynamics modeling, we visualize how instructional change emerges through interconnected loops of mentoring, belonging, and institutional structure. The framework makes visible both leverage points and tensions within departmental systems and offers a way to design mentoring structures that are equity-minded, sustainable, and context-sensitive; insights that can inform future PD programs and coordination policies.

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Characterizing Generative AI Policies in a University Mathematics Department

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Keywords: Generative AI; Course Policy Analysis; GenAI in Math Education

Background and Literature Review

For the 2025–26 academic year, the university examined in this study required instructors to include a generative AI (GenAI) policy in their syllabus but provided only sample language, not specific directives. Prior research shows mixed effects of GenAI on student learning, enhancing engagement and confidence in some contexts (Serhan & Welcome, 2024; Kestin et al., 2025; Wang & Fan, 2025) but reducing cognitive effort and independent reasoning in others (Stadler et al., 2024; Krupp et al., 2024; Kosmyna et al., 2025). While a few studies have examined course-level policies directly (Ali et al., 2025; Kaufmann et al., 2025) none have focused specifically on mathematics courses. This study aims to provide information in this gap by answering the research question: how are math instructors framing policies about GenAI in their syllabi?

Methods

We conducted an inductive thematic analysis (Braun & Clarke, 2006) of 14 course-level GenAI policy statements from math classes at a single institution. This institution serves as a case study and allows us to see what variation exists while controlling for the culture and environment of the department and university. Each policy was authored by an individual instructor or coordinator, reflecting personal perspectives in the absence of established disciplinary norms. Seven policies came from the calculus–linear algebra sequence and seven from proof-based courses. We read and discussed the documents to identify recurring language, rationales, and contrasts in how instructors framed AI use (Bowen, 2009). The resulting descriptive themes capture emerging approaches to GenAI in a university math department.

Results and Discussion

Across the calculus sequence, all courses adopted a moderate approach to GenAI: allowing use for conceptual understanding but prohibiting it for completing homework. Most framed GenAI within broader discussions of learning resources; about half provided explicit examples, and five required students to acknowledge or document their use.

Policies in proof-based courses were more varied. One referred only to general academic integrity, two fully prohibited GenAI, one was completely open, and three mirrored the moderate stance of the calculus sequence. Learning-focused rationales were most common, though some also cited unreliability, integrity, or broader social concerns.

Overall, departmental policies reflected diverse individual perspectives but a shared trend toward moderation, acknowledging that GenAI use is likely and difficult to monitor while clarifying appropriate versus inappropriate use. Several emphasized metacognitive learning rather than unreliability alone, echoing calls to frame GenAI as a learning tool rather than solely a risk (Walkington, 2025). One open policy noted that grades were based mainly on closed-book exams, implicitly justifying limited concern about GenAI access. However, contradictory language, banning GenAI while requiring citation, suggests ongoing ambiguity. Clarifying such tensions and supporting students in ethical, informed use remain central challenges as disciplinary norms continue to develop.

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Bridging Mathematics and Science: Task Design and Blended Sensemaking

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Keywords: mathematics sensemaking, science sensemaking, task

Mathematical equations are a key part of undergraduate science instruction (Li & Schoenfeld, 2019). Students in science classes have both science and mathematical resources they can use to make sense of mathematical equations (Hammer, 2000). Students who make connections between these resources (engage in blended sensemaking) are better able to solve novel or more complex quantitative problems (Bing & Redish, 2009; Schuchardt & Schunn, 2016; Taasoobshirazi & Glynn, 2009). Little research has been done on affordances and constraints for blended sensemaking of different types of tasks involving mathematics in science classes. **RQ:** How do different tasks on mathematical equations in biology affect the types of blended sensemaking discussed by students?

Incoming biology students participating in a session on mathematical modeling of population growth were recorded as they worked in pairs on a sequence of two mathematical tasks. Varying by session, the first task consisted of one of four variants: DM) developing mathematical equations, CM) comparing two mathematical equations, VR) describing relationships between equation variables, and PS) solving problems following an example. The second task was always the same, modifying an equation from the first task. Recorded discussions for each task (4-6) were coded for types of mathematical and science sensemaking and for blended sensemaking using the Sci-Math sensemaking framework (Zhao & Schuchardt, 2021) by two independent coders (Krippendorf's alpha=.77). Counts were standardized for number of groups, time on task.

Across both tasks, the DM/DM-Modify sequence had the most instances of blended sensemaking (DM=100, DM-Modify=169) while the PS/PS-Modify sequence had the fewest (PS=22, PS-Modify=64). Blended sensemaking was categorized into levels based on the types of connected science and mathematics sensemaking (Kaldaras & Wieman, 2023). Lower level indicated connections between sensemaking about mathematical procedures and rules and identifying variables as components of the scientific system. Higher level indicated connections between sensemaking about the structure/syntax of the equation, the quantitative relationships between variables, or represented mathematical concepts and the scientific mechanisms or patterns of the system. Transitional represented connections between lower and higher levels of sensemaking (one from science and one from mathematics). The modify task in each sequence contained high levels of higher level sensemaking (DM=24, DM-Modify=113; CM=31, CM-Modify=76; VR=24, VR-Modify=67; PS=10, PS-Modify=35) mostly between mathematical structure/syntax and patterns students observed in the scientific system. The DM task had many instances of transitional sensemaking (64), most frequently between sensemaking about arrangement of the variables in the equation (mathematical structure/syntax) and what they represent in the scientific system.

These results indicate that different tasks elicit different levels of blended sensemaking indicating different affordances and constraints for helping students make connections between mathematical equations and the scientific system they represent. These findings have implications for which tasks instructors design to leverage student resources and meet learning objectives not only for science classes where students are working on mathematical tasks but also mathematics classes where students are working on modeling scientific phenomena.

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Students' Positive and Negative Experiences in Proof Courses

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Keywords: Proof Courses, Student experience, Undergraduate mathematics courses

Existing research on students' experiences in proof-based courses has often focused on examining insights students gain from lectures (Krupnick et al., 2018; Lew et al., 2016) and their beliefs about the processes of reading, writing, and interpreting proofs (Guajardo & Lew, 2024; Lew & Mejia-Ramos, 2019; Weber & Mejia-Ramos, 2014), with much of this work highlighting students' struggles with proof. In their transition to proof-based mathematics, students are often required to conform more deeply than they would have previously towards linguistic, logical, and authoritative standards that trickle up from and symbolize larger mathematical cultures (Lew K. & Mejía-Ramos, 2019). Such standards can be alienating, especially when they suggest that individuals have struggled or made mistakes, or when they receive unclear or confusing responses (Dawkins & Weber, 2017). Therefore, it is important to understand, from the students' perspective, what students have found supportive or unsupportive throughout their experiences with proof courses. This poster aims to address the research question: What do students perceive as helping or hindering their experiences in proof-based courses?

This poster will use data from interviews with five undergraduates who all had taken at least three proof-based courses. These students were recruited after they had taken a survey regarding their experience with proof courses and had responded "yes" to an interview invitation. The interviews took place over Zoom with the second author and an additional researcher. Students reflected on their experiences in proof-based courses and proof culture. They shared experiences that shaped their views on what is valued in the proof community.

To answer our research question, we used a thematic analysis to identify themes related to what students noted as helpful or not in relation to their experience in their proof-based course.

From our initial analysis we saw a few things come up right away. Grading and expectations were huge issues for students. Some students couldn't figure out what their professors wanted. They lost points for writing too much, or for using shorthand. One student said, "I thought I did everything I needed to. But I didn't know what he wanted." Another said, "Proofs aren't supposed to be assignments… they're explanations for why something is true." Suggesting that disparities in grading expectations hindered their learning experience as it was unclear what was "right" and what was "wrong". This shows students wanted proving to feel more authentic and less rule-driven. The students also pushed back against the idea of blindly accepting answers just because they came from a textbook or a professor. "Most of the class would agree with a proof just because it came from the professor. But nobody's perfect. Anyone can have a small error." This suggests that providing a more authentic proving experience that allows mistakes and errors can help students' learning experience. Students also noted that they wanted more chances to work with classmates, more flexibility. In their eyes, proving did not have to be a solo job, it could feel more like a group puzzle or even a lab, suggesting collaborative work in their proof courses was helpful. This poster will share the themes that came out of these interviews. By identifying these themes, we will gain insight into small changes like better feedback or more chances for group work that can help students feel more supported in proof-based math.

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Power Evasiveness in the Graduate Texts in Mathematics Series

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Keywords: power, critical discourse analysis, graduate texts

Texts have an interpersonal function that establishes relationships between authors, readers, and the content of the texts (Morgan, 1996), and, thus, can be exclusionary. For example, examining the interpersonal function of middle school mathematics texts, Herbel-Eisenmann (2007) found readers are rarely positioned with mathematical authority despite the curriculum's intention. Similarly, texts can valorize masculinity (McBride, 1989), individualism (McBride, 1994), and passively maintain the status quo (Dowling, 1991). Despite the demographic homogeneity of postsecondary mathematics education (Blair et al., 2018), little research has critically analyzed postsecondary mathematics texts. We extend Herbel-Eisenmann's analysis to investigate the interpersonal function of the Graduate Texts in Mathematics (GTM) series with a particular focus on the use of pronouns by textbook authors. We pursue the research question: What do pronouns in the GTM series convey about interpersonal power?

Our analysis is grounded in theories of power evasiveness. Power evasiveness emerged from critical whiteness studies to characterize how racial power can be enacted in implicit or invisible ways (e.g., Frankenburg,1993; Haviland, 2008). For example, white people's common avoidance of racially direct language can be understood as power-evasive, as it deflects recognition of the roles of racism and white supremacy while still allowing them to benefit from their ongoing enactment. Although our analysis does not explicitly center race, we recognize whiteness operates by creating factions to suppress collective resistance (Tochluk, 2010) and structures oppression across many dimensions (e.g., gender; Robertson & Hairston, 2022). We use power evasiveness to describe any enactment of power that is shrouded by power-neutral framings.

We selected 40 books from the GTM series, prioritizing books currently used by graduate mathematics students. Leveraging critical discourse analysis (Fairclough, 2013), we analyzed the opening pages of these texts through the first proof. We used an open/axial/selective coding scheme (Straus & Corbin, 1998) with two sets of open codes: one for pronouns and another to characterize the reader's mathematical authority. In line with prior critical textbook analyses (Herbel-Eisenmann, 2007; Morgan, 1996), we then categorized coded excerpts as inclusive when the reader was an actor in the sentence and exclusive otherwise.

Authors tended to use pronouns that signify the readers' inclusion while simultaneously constraining their agency. For example, "There are topologies on X that do not respect the vector operations in this way, but they are of no interest to us" (Clarke, 2013, p. 4) uses inclusive language to refer to the author and the reader while imposing the authors' mathematical agenda. Similarly, "We agree to write $\mathrm{ord}_p 0 = \infty$" (Koblitz, 1984, p. 2) includes the reader as an actor but forces their agreement with the author. These excerpts demonstrate power evasiveness through their seeming inclusiveness while exerting power over the reader.

While the postsecondary mathematics community has made important strides towards inclusion, this analysis illustrates that inclusion is insufficient to reconstruct mathematics as a liberatory domain (Martin, 2019); power continues to constrain mathematical agency and epistemic diversity (Gutiérrez, 2017). Power evasiveness in pronoun use symbolizes a need to reconstruct the very ways we think, write, and talk mathematically.

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Mathematics Support Structures for Calculus and Precalculus: What are the Impacts on Student Outcomes?

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Keywords: Undergraduate Education, Policy

A long-standing body of evidence (e.g., Bressoud, 2014; Whitcomb et al., 2020) suggests that Calculus I often serves as a gatekeeper course – meaning that a relatively high percentage of students are unsuccessful in Calculus I, and that a failure in the course often causes students to leave STEM degree programs. Some research suggests that Precalculus can play a similar role (e.g., Cooper & Woodward, 2011, Wang et al., 2017). Given the need to maintain a strong STEM pipeline and sustain student career aspirations, there is significant need for university interventions to support student success in courses such as Calculus I and Precalculus. Research indicates that peer-to-peer interactions and mentoring, paired with curricular support, can be effective structures to help students be successful in their early mathematics courses. For instance, co-requisite models have demonstrated success at the Developmental Mathematics level (e.g., Childers et al., 2021; Ngo & Melguizo, 2024), though there is less research on the role of co-requisite courses at the level of Precalculus and above.

This research aims to add to the body of knowledge about co-requisite and peer-led supports by evaluating the success of two mathematics support options for undergraduates at a large public university: an intensive summer "Bootcamp" immediately prior to the start of the Fall semester, and "Companion Courses" which run throughout the semester on a weekly basis. Both structures are supervised by faculty members and implemented by Undergraduate Learning Assistants and incorporate just-in-time mathematics practice, conceptually-oriented review, and study skills support. The Bootcamp was aimed at incoming first-year students who demonstrated readiness for Precalculus or Calculus I, while the Companion Courses are offered to all students enrolled in Calculus I. To understand the effects of these structures, we analyzed data related to student course outcomes (passing grades of ABC vs. repeating grades of DFW and course GPA) for students who participated in these two structures, compared to the general student population.

Results suggest that both structures promoted student success, though the impacts were different for the two programs. For instance, 83% of students who participated in the Bootcamp (N=54) earned passing grade in their first mathematics course, and they earned an average grade of 2.93 (on a four point scale). In comparison, for non-attending students in those same classes (N=431), only 65% earned a passing grade, with an average grade of 2.30 (these differences are statistically significant with $p < .05$). The Companion Course showed more limited evidence of impact, with 63% of enrolled students (N = 40) earning a passing grade in their Calculus class, compared with 62% of non-enrolled students (N = 402; not statistically significant). These results should be interpreted with caution, as selection bias is a significant complicating factor. Our poster will share additional data and explore implications for university policymakers.

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An Examination of the Implementation of Inquiry-Based Learning in a Developmental Mathematics Course

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Keywords: inquiry-based learning, developmental, college algebra

In order to address the college-readiness of students entering colleges and universities, many universities have developmental mathematics courses to help prepare students for later mathematics courses. To help the students in these courses, active learning is often implemented in the classroom, where students are generally engaging in problem solving and using higher-order thinking skills (Bonwell & Eison, 1991), such as analyzing, evaluating, and creating (Krathwohl, 2002). Active learning has also shown generally positive results (e.g. Freeman et al., (2014), Theobald et al. (2020)). A specific type of active learning model is known as Inquiry-Based Learning (IBL), where students are given investigative tasks that promote the engagement of students in the creation of important mathematical concepts (Laursen & Rasmussen, 2019). Generally, this involves students creating and testing conjectures, making generalizations, providing justifications, and reflecting upon their conclusions and reasoning (Mesa et al., 2020). In addition, IBL often involves student collaboration, where students work together to build their knowledge, while the instructor facilitates the learning, in contrast to being the sole source of knowledge. IBL has generally seen positive effects on academic achievement, as well as reducing learning gaps between groups of underrepresented students (e.g. Kogan & Laursen, (2014); Laursen et al., (2014)). However, the use of IBL varies by practitioner, without a consistent definition, although it is consistent with and supported by education research (Laursen and Rasmussen 2019). Because of this, how IBL gets implemented in the classroom may differ across instructors or universities. My project focuses on how IBL is implemented in a developmental mathematics course by five different instructors.

This project uses classroom observations to see how IBL is being implemented in a developmental mathematics classroom covering college algebra. The classes were taught by five different instructors, who were given a few days of training in IBL as well as activities and worksheets they could implement in the classroom. To measure the level of inquiry, the Inquiry-Oriented Instructional Measure (IOIM; Kuster et al., 2019) is used, which is a well-defined measure of inquiry for Inquiry-Oriented Instruction. In addition, to measure how often the students and instructors were engaged with different activities, the Toolkit for Assessing Mathematics Instruction – Observation Protocol (TAMI-OP; Hayward et al., 2018) was used. I examine the IOIM scores as well as the TAMI-OP data, across four content units, to see any similarities and differences in the IBL implementation. This poster will present the data collected and the resulting analysis, to answer the questions: (1) What are the differences in the implementation of IBL by instructors? (1a) What is the frequency of activities and questions students and instructors engage in? (1b) What are the differences of the levels of inquiry in the classroom?

For five different sections, there was a wide variety of ways IBL was implemented. The percentage of groupwork and lecturing varied between the sections. In addition, the frequency and type of questions varied drastically across sections. The IOIM scores also showed differences across sections, both in terms of the mathematics being completed, as well as the ways the instructors elicited and utilized student contributions.

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Beyond a Surface-Level Understanding: Developing a Conceptual Assessment of the Derivative

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Keywords: Calculus, Derivative, Assessment, Concept Inventory, Productive Understanding

Background and Literature

Many examinations in mathematics given to students often focus on the students being able to perform procedures such as calculating derivatives. Even when questions are not designed around students invoking specific procedures, problems are often routine enough that students can perform well in the examinations without necessarily having a good grasp of the concepts involved in why their responses are valid. (Tallman et al., 2016, 2021). In contrast, a *concept assessment* or *concept inventory* is an examination which is dedicated to determining how well students understand the concepts associated with a given subject area. Several concept inventories exist for branches of science and mathematics (Hestenes, Wells, & Swackhammer, 1992; Carlson, Oehrtman, & Engelke, 2010), and for calculus the first such concept assessment was Epstein's Calculus Concept Inventory (Epstein, 2013). However, this assessment has been criticized by being not research-based as well as not holding up to tests of validity (Gleason, 2019), leading to additional projects aiming to create improved concept assessments for calculus (Hall et al., 2023; Lee et al., 2025).

Research Aims

This research is part of a larger project, tentatively called Developing CALC, which aims to create a set of smaller, modular concept assessments focused on specific areas within calculus. The first section of the project focused on developing an assessment for definite integrals (Hall et al., 2023), whereas this portion focuses on developing a similar assessment for derivatives. This research is based around much of the literature surrounding student understanding of derivatives, and what makes such an understanding productive (e.g. Zandieh, 2000; Carlson, 2002; Jones, 2018; Reed et al., 2023). Our research question is: "What kinds of questions about derivatives assess whether students have a productive, conceptual understanding of the derivative as opposed to merely surface-level or procedural understandings?"

This second portion of the project is currently in its preliminary stages, with the current goal being the development of the questions themselves. The questions being developed are based on tasks found in previous research, questions in common calculus textbooks, as well as our own teaching experiences. Our aim is for these questions to require students to have good conceptual knowledge of the corresponding topic to be answered correctly, and then the questions will later be given to students to determine if the questions do in fact assess the students' conceptual understanding. The primary goal of this poster will be to serve as a space of discussion, where experienced researchers can critique the tasks we have chosen, come with their ideas for what kind of questions could best serve our aims, which additional topics may be important to ask in an assessment such as this, or to otherwise give feedback on the project at this early stage.

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Exploring College Algebra Students' Attitudes Toward Mathematics in Community Colleges

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Keywords: college algebra, community colleges, mathematics, emotions, attitudes

Attitudes toward mathematics play a critical role in shaping students' learning experiences, academic performance, and long-term engagement with the subject. In the context of community college education, where students often face diverse academic and personal challenges, understanding these attitudes becomes especially important. College Algebra, a foundational course for many degree programs, frequently serves as both a gateway and a barrier to academic progress for STEM fields. For a significant number of community college students, this subject also evokes complex emotions - ranging from anxiety and frustration to curiosity and resilience. Research links positive attitudes to increased motivation and academic success, while negative attitudes, particularly math anxiety, can significantly hinder performance and persistence (Barroso et al., 2021). Students' beliefs about their ability to succeed in mathematics, often shaped by their mindset and goal orientation, influence both their engagement and achievement (Bostwick et al., 2017). As such, exploring how students perceive and emotionally respond to mathematics and the learning of it in college algebra classrooms can offer valuable insights for improving future instruction and support.

As part of a broader research project involving eight community colleges in four states, this study seeks to explore the following research question: What are college algebra students' general feelings and attitudes toward mathematics in community college? Participants engaged in a series of four interviews within a year. During the interviews, participants (1) discussed their personal and mathematical histories related to their current understanding of equity, diversity, and inclusion in the learning of mathematics and the learning environment; (2) provided commentary on short videos of classroom instruction noticing specific characteristics that can be associated with of equity, diversity, and inclusion in the learning of mathematics and the learning environment; (3) engaged in a Q-sort exercise of student statements from the first interview supplemented with statements from the literature that were geared to elicit their perspectives on equity in their learning of mathematics and their learning environment; and (4) reflected on how their understandings of equity, diversity, and inclusion in the learning of mathematics and the learning environment had evolved over time.

This poster will focus on the first interview, where questions were asked to get students' affective connection to mathematics, and gain insight into how their experiences in college algebra classes at community colleges impact their relationship with mathematics. We will share our preliminary findings on college algebra students' general feelings and attitudes about mathematics in community college classrooms around the themes of participants' (1) affective relationship with mathematics, (2) observations about learning mathematics, and (3) beliefs about the nature of mathematics.

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Keeping Time: Utilizing Non-Normative Axes in Graphing Tasks to Support Covariational Reasoning

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Keywords: covariational reasoning, graphs, graphing, task design

Covariational reasoning—reasoning about the relationship between simultaneously varying quantities—can support students' sense-making about dynamic real-world or abstract mathematical phenomena. Covariational reasoning has been shown to be challenging for undergraduates (Carlson et al., 2002), preservice mathematics teachers (Moore et al., 2019), and in-service mathematics teachers (Thompson et al., 2017). However, there is also evidence that students as young as middle school can engage in covariational reasoning (Ellis et al., 2015; Paoletti et al., 2023). Tasks in which students are prompted to construct a graph representing the relationship between covarying quantities in an animation of motion can support students in reasoning covariationally (Johnson et al., 2020; Paoletti et al., 2023). However, researchers must pay careful attention to how particular features of the task support or constrain students' opportunities to engage in such reasoning (Stevens et al., 2017).

Covariational reasoning entails constructing quantities (i.e., identifying attributes of a situation that are measurable; Thompson, 1994), imagining those quantities changing in some way (Castillo-Garsow, 2012), and then coordinating those changes (Thompson & Carlson, 2017). When graphing tasks prompt students to coordinate changes in a non-temporal quantity with changes in the quantity of time, students often reason variationally by keeping elapsing time as implicit and something they sense passing by as the other quantity changes (Janvier, 1998; Johnson et al., 2020; Paoletti, 2015; Stevens et al., 2017). As a result, researchers have recommended avoiding using time as a quantity in graphing tasks that are intended to promote covariational reasoning (Paoletti et al., 2022; Stevens et al., 2017). However, many quantitatively rich dynamic situations are time-dependent, particularly in courses that are preparing students for future work in science and engineering fields. In those cases, adhering to the design recommendation would create other curricular tensions.

In this poster, I address the question: *How do students reason about the relationship between time and distance in a dynamic situation when engaged with a graphing task that utilizes non-normative axes?* I use data collected from a series of task-based interviews (Goldin, 2000) with undergraduate Calculus 1 students. Each student participated in two 90-minute interviews which consisted of a series of animation-based graphing tasks designed to elicit covariational reasoning. I coded for the level of covariational reasoning (Thompson & Carlson, 2017) that I inferred students drew on as they engaged with the graphing tasks. I compared students' covariational reasoning on distance-time graphing tasks that utilized normative and non-normative axes (i.e., time on the vertical axis). I found that students engaged in much more sophisticated levels of covariational reasoning when engaged with the non-normative graphing tasks. This finding indicates that designing graphing tasks with non-normative axes is a strategy for supporting students in reasoning covariationally even in contexts where time is one of the quantities. This has implications for curriculum design and research on covariational reasoning in the context of graphing tasks.

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Collaborative Approaches for Developing Teaching, Learning, and Stewardship

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Keywords: teaching-focused professional development; leadership; collaboration-based learning

The research in undergraduate mathematics education (RUME) community has amassed insights into features of high-quality learning opportunities for students and what instructors need to know and do to create such opportunities. What does it look like if we leverage what is known from that literature to consider the question: How do we use RUME findings to design and study development of those who sustain and connect professional learning across roles, contexts, and time? Such leaders function as *Stewards* of a discipline, not just expert researchers; they are guardians, teachers, and innovators who ensure that the discipline thrives and remains meaningful for future generations (Bass, 2006).

Powerful classroom-level practices include student collaboration on group-worthy, authentic tasks. RUME has examined ways instructors build the knowledge, skills, and dispositions to support such teaching and learning of mathematics. RUME has also explored how those who are Providers of teaching-focused professional learning create opportunities for instructional growth through purposeful design (e.g., Akin et al., 2023; Braley et al., 2023; Deshler, et al., 2015; Smith et al., 2021; Yee et al., 2023). Together these contributions reveal interconnected layers of subsystems in learning and teaching.

Math-Students-Instructor interactions occur in Instruction. Instruction is a subsystem, an object of study, in Teaching-focused Professional Development (TPD). TPD, as an object of study, includes interactions among Providers and Instructors as they work on Instruction. TPD is, in turn, a component in an even broader system of Leadership Development where leaders facilitate learning by Providers to offer TPD. Human interactions include:

- Students engaging in learning mathematics via collaboration, guided by an Instructor,
- Instructors learning to teach mathematics, guided by a Provider,
- Providers learning to teach about teaching mathematics, guided by a Facilitator.

In this exploration of theory, we utilize prior work (Hauk & Speer, 2023; 2024) to expand on the model. We offer and extend a visual representation of the above. The aim is to capture the work Stewards do as they draw on broad and deep knowledge to guide efforts across (sub) systems across time. We explore the use of authentic group-worthy tasks for professional learning (i.e., tasks relevant to the work of teaching and leadership) and what is necessary for those who orchestrate such learning to know and do. Outcomes will inform designs for learning across multiple layers of the system and leverage the robust base of literature on authentic tasks, collaborative groupwork, and scaffolded learning in mathematics.

Theoretical contributions are two-fold: (1) synthesizing work across several fields to inform the nascent theory-building about leader and Steward development, and (2) exploring how practices used by instructors can be beneficial both for undergraduate student success in the learning of mathematics and in supporting the learning of stewardship. Poster-side conversations will focus on gathering stories about Provider, Facilitator, and Steward development to stress-test the model. Just as the mathematics community has benefited from both theory- and practice-focused research on students and their instructors in the context of learning mathematics, we contend that analogous work focusing on Stewards offers valuable insight into leadership development in post-secondary mathematics education.

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Observation Protocols as Research Tools for Video-Based Analyses of Undergraduate Calculus Recitations

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Keywords: Observation protocol, active learning, classroom observation

Active learning is broadly recommended as an instructional approach for improving student outcomes in undergraduate mathematics classes (Smith & Funk, 2021). However, some research suggests little active learning is being implemented, leading to questions about how time is spent in these classrooms (Ellis et al., 2014). To address this, observation protocols have been developed that categorize instructor and student actions over time. Classroom Observation Protocol in Undergraduate STEM (COPUS) (Smith et al., 2012) and Toolkit for Assessing Mathematics Instructors – Observation Protocol (TAMI-OP) (Hayward et al., 2021) are two such protocols meant to describe actions taken in classrooms, especially to identify those using active learning methods. While these tools are typically utilized as a means of professional development and in evaluation processes, less is known about how they can function as analytic tools within the research process. We address the application of these observation protocols to mathematics education research, asking: What are the affordances and challenges of applying the TAMI-OP and COPUS protocols to undergraduate mathematics classroom video recordings from a research perspective?

This data comes from a larger NSF-funded study on active learning in introductory STEM courses at a medium Northeastern University. Here, we focus on Calculus recitation sessions which are led by a graduate teaching assistant (GTA) and supported by an undergraduate learning assistant (LA). The classroom recordings were selected from two semesters of recordings from the same GTA, used for a case study on the GTA's changes in classroom actions (Sager et al., 2024). The recordings were then coded using both COPUS and TAMI-OP by both authors. Any discrepancies were resolved by discussing until a consensus was reached. In COPUS and TAMI-OP, codes can be analyzed by calculating the percentage of 2-minute time intervals in which the code was marked, or as the percentage of a specific code is marked out of the total codes marked. We provide both computations to illustrate the different perspectives each of these protocols brings to the same dataset.

From our analysis, we show that COPUS and TAMI-OP can both be valuable research tools, but each has strengths that highlight distinct aspects of classroom interactions. TAMI-OP emphasizes questioning and answers, with multiple options to count types of questions and answers. We suggest it is advantageous for research aiming to capture the type of conversation or instruction in a classroom. Alternatively, COPUS tends to show an outline of actions taken in the classroom, making it more suitable for research settings focused on the overall actions occurring. For example, if an instructor begins class with a 10-minute mini-lecture, the first 5 intervals in COPUS may be coded with *lecturing* and *posing questions* for the instruction, while TAMI-OP additionally provides a number and type of questions and answers in each interval, thus giving some information regarding the nature of student participation during this time.

Our work illustrates how these observation protocols can be appropriately adapted as research tools and how each provides different perspectives on active learning in undergraduate mathematics classrooms. We encourage future researchers to apply these protocols to additional contexts in order to further explore the role they can play in the research process.

Acknowledgments

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Doing Math Instead of Watching Math: The Impact of In-Class Activities and Written Feedback in a Prerequisite Mathematics Course

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Keywords: active learning, instructor feedback, student success, D/F/W rates

Research Aim

This study investigates how integrating frequent in-class mathematical problem-solving with individualized written instructor feedback influences student success in prerequisite postsecondary mathematics courses. Specifically, we examine whether redesigning daily instruction around structured engagement and descriptive feedback is associated with improved D/F/W rates and patterns of persistence.

Context and Rationale

Research shows that active learning and constructive feedback support deeper understanding and confidence in undergraduate mathematics (Cirillo et al., 2024; Vroom et al., 2022; Grove & Good, 2020; Bressoud & Rasmussen, 2015). Yet most studies examine calculus or upper-level courses, leaving open how these practices function in prerequisite mathematics courses where attrition is high. Few studies consider their combined influence or link instructional features to institution-level outcomes such as D/F/W rates and persistence. This study addresses that gap by analyzing success patterns following a redesign integrating in-class problem-solving, individualized written feedback, and scaffolded reflection, thereby extending and replicating prior findings in a critical gateway context.

Methods & Research Question

RQ: How does the integration of frequent in-class engagement and individualized written feedback relate to student success in prerequisite postsecondary mathematics as measured by patterns in D/F/W rates and persistence? Institutional outcomes from semesters before and after an instructional redesign, featuring structured problem-solving, individualized feedback, and reflective prompts, are compared using descriptive and comparative analyses.

Anticipated Findings

We anticipate that the redesigned instructional model will correspond with lower D/F/W rates and improved persistence, suggesting that prerequisite mathematics courses may benefit from greater emphasis on daily mathematical engagement and meaningful written feedback. By analyzing both instructional features together and focusing on a high-attrition gateway course, the study offers insight into how to move prerequisite mathematics from a barrier to a pathway for student success.

Implications for Practice

Findings may inform institutional reform efforts by highlighting how active learning, individualized feedback, and reflective practice can be integrated into daily instruction to promote engagement, confidence, and persistence. This work also demonstrates the value of replication studies in mathematics education and underscores the importance of systematically examining instructional redesigns in prerequisite mathematics courses, spaces where small pedagogical changes may yield large institutional impacts.

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Multiplying Fractions with Sunlight: A Scaling Model

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Keywords: fractions, units, scaling, ratio

Physical models are frequently used in elementary mathematics classrooms to support students' understanding of multiplication. These models are discrete, and many represent multiplication as repeated addition, area or an array. Thompson and Saldanha (2003, p. 104) found that "thinking about multiplication as repeated addition leads to severe difficulties in later grades" and Hurst (2015, p. 11) suggested that "multiplication as scaling…is a key aspect of thinking that needs to be developed." McLoughlin and Droujkova (2013) developed a diagrammatic definition of multiplication using parallel lines to scale the length of one segment by the length of a different segment. We developed the SunRule, an interactive device that harnesses parallel rays of the sun to perform multiplication. Beams of light cast shadows from gnomons onto a measuring plane; by varying the tilt and height different multipliers, multiplicands, and products can be generated.

This proposal extends research with K-8 preservice teachers reported in (Dimmel et al., 2021) where participants made sense of multiplication as a scaling, continuous operation. They used informal language to describe the relationship between changes in the multiplier, multiplicand, and product - that they were growing (or shrinking) in similar ways. While not specifically mentioning ratios, the preservice teachers described the shadows as changing in ways that were multiplicative or proportional in nature.

The study will investigate the use of the SunRule specifically to explore fractions, using a new model that zooms in to numbers between 0 and 1. In this version the gnomons and the shadow plane are partitioned into fractions of a whole (typically eighths or twelfths, but it could be any increment). The unit gnomon is scaled down to fractions between 0 and 1 by tilting the device, and the multiplier is scaled between 0 and 1 by changing the height of the gnomon. We ask the question: "How do preservice teachers make sense of multiplying fractions between 0 and 1 when using a dynamic, continuous, scaled model?" One significant change/improvement of the new model is that every rational product falls on the shadow plane, even if it doesn't fall on a specified increment. We also ask: "How do preservice teachers make sense of rational products cast as shadows that do not fall on a ruled increment?"

Figure 1. 3 x 4 = 12

Figure 2. 1/2 x 2/3 = 1/3

Figure 3. 1/2 x 3/8 = 3/16

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Analyzing How Course Modality Impacts the Learning of Mathematics of Students in a Hispanic Institution

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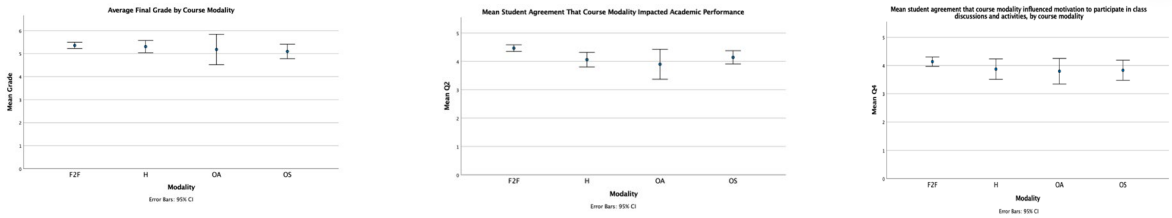
Keywords: Mathematics Education, Course Modality, Motivation, Engagement, Achievement

Universities have recently adopted the use of various course modalities to accommodate diverse types of students (e.g., distance-learning). It is important to investigate how the different course modalities affect students' achievement and affect (e.g., motivation, willingness to participate, confidence). Building on prior research that explores the relationships between course modality and student success (e.g., Fischer et al., 2020; Helms, 2014; Kundu et al., 2023), this study investigates how different instructional modalities–Face-to-Face (F2F), Hybrid (H), Online Synchronous (OS), and Online Asynchronous (OA)–affect the learning and motivation of undergraduate mathematics students. This study addresses this research question: How does the modality of undergraduate mathematics courses affect students' academic success?

Data were collected through an anonymous survey administered at the tutoring center of a majority Hispanic institution. This institution offers undergraduate mathematics courses in four modalities: F2F, H, OS, and OA. Survey participants were students in classes ranging from college algebra to ordinary differential equations. A total of 249 responses were analyzed over eight different courses; these responses included Likert-scale questions regarding how course modality affects their motivation to participate, the impact of their final grade due to modality, as well as their final letter grade for each respective class and modality. The data were analyzed using SPSS to compare means and confidence intervals of the survey scores across the four modalities. The means and confidence intervals are shown in Figure 1.

Across all measures, Face-to-Face and Hybrid students outperformed online students (see Figure 1a) and expressed a stronger perception of learning motivation (see Figure 1c).



(a) (b) (c)

Figure 1. (a) Average final grades across modalities with a 95% confidence interval. (b) Mean agreement that modality impacted academic performance. (c) Mean agreement that modality influenced motivation to participate.

The findings suggest that environments that promote active interaction, such as F2F and H, enhance students' achievement and motivation when it comes to the mathematics classroom. These results align with Kundu et al. (2023), who found that hybrid learning promotes satisfaction and engagement. For struggling students, immediate feedback, collaboration, and guided instruction appear to be crucial for both success and motivation.

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How is Gender Discussed? A Textual Analysis of RUME Proceedings

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Gender has long been studied within math education, evolving from deficit and assimilationist models, positioning girls' lower participation as a problem to be fixed, to feminist and sociocultural frameworks that conceptualize gender as relational and intersecting with other axes of identity (Leder, 2019). The "leaky pipeline" continues to be well-documented (Blickenstaff, 2005) and persistent disparities in representation and persistence remain in math-intensive fields (Brown & Leigh, 2021). Despite this history, little is known about how the undergraduate mathematics education research community itself conceptualizes and engages with the construct of gender. Because gender functions as both an empirical tool and a critical lens (Leder, 2019), examining how it has been theorized, operationalized, and represented in RUME proceedings sheds light on the field's evolving commitments to equity and inclusion. *This gap motivates our landscape analysis of RUME proceedings from 2018 to 2024 to map how gender is being operationalized.*

Our gender textual analysis draws from previous work analyzing race (Contreras et al., 2025), and began by performing pattern-based text retrieval on the corpus, retrieving a 40-word list of gender-related terms. The list included terms related to binary identities (e.g., "woman", "male"), non-binary and fluid identities (e.g., "non binary," "two spirit"), transgender experiences (e.g., "trans"), related social and biological concepts (e.g., "cis gender," "pronoun"), and terms for gender expression (e.g., "masculine," "gendered"). We used the results of the text retrieval to assign each paper to one of several categories based on the terms they used, refining this categorization through regular team discussions. Papers were coded based on the role of gender in the research: *Core* for a central focus (and a summary noted); *Peripheral* if gender was substantive but not central (like positionality or an analytical tool component); *Non-Focal* if gender was only mentioned in passing (e.g., unanalyzed demographics or quotes); and *Unrelated* if gendered terms appeared only within the references or embedded within math problems.

Our preliminary analysis indicated that of the 1,426 reports and papers published in the proceedings from 2018-2024, 438 papers (30.7%) contained a keyword. Most of the keywords were found within contributed (50.7%) and preliminary reports (23.7%), with 17.8% of posters and 5.3% of theoretical reports containing keywords. 13.5% of all proceedings containing a keyword were identified as Core while 11.2% were identified as Peripheral, totaling 24.7%. This left 75.3% of papers identified as Non-focal or Unrelated (45.4% and 29.9% respectively). Therefore, a majority of the papers which contained at least one keyword used those keywords only in reference to participant demographics, other authors' work, or in the context of a mathematical problem. These counts demonstrate that less than a quarter of studies which reference gender incorporate it meaningfully in their work, with around one-eighth of papers utilizing it as a central focus to their study. Our poster will further analyze the papers coded as Core and Peripheral to understand the ways that gender is being captured and analyzed (e.g., focus on a specific group, binary comparison, critical intersections). We intend to expand this work to examine the ways in which gender topics are utilized within RUME, to identify the theoretical and methodological tools employed, and to elevate authentic ways of humanizing gender research in mathematics education.

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Student Perspectives on the Mathematical Formalism in an Operator-Based Quantum Mechanics Course

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Keywords: Quantum Mechanics, Operators, Student Perspectives, Semi-Structured Interviews

Quantum mechanics represents a uniquely interdisciplinary learning environment where student success depends on navigating the boundary between mathematical abstraction and physical interpretation. Students must coordinate the symbolic reasoning (e.g., linear algebra, Dirac notation, operators) with conceptual quantum ideas like measurement. Prior research has shown how this intersection often creates cognitive and epistemological tension for learners, particularly in quantum (Schermerhorn et al., 2019; Wawro, Watson, & Christensen, 2020).

Within both the mathematics education and physics education research (PER) communities, scholars have noted that students' epistemic framings, such as their views on mathematics in constructing physical meaning, are critical to developing a deeper understanding. Wawro et al. (2025) found that students often interpret eigenequations differently across linear algebra and quantum mechanics contexts, revealing gaps in how mathematical symbols acquire physical meaning. Complementary work in PER highlights that representational and epistemological struggles manifest as affective and identity challenges, where learners continually renegotiate what "counts" as math or physical reasoning in their STEM courses (Schermerhorn et al., 2019).

Broader studies in undergraduate STEM education report that students' success in advanced mathematics and science courses is influenced by both their prior preparation and their self-efficacy in applying mathematics to novel contexts (Korpershoek, Kuyper, van der Werf, & Bosker, 2011). Moreover, Matthews, Adams, and Goos (2009) argue that many science curricula treat mathematics as a procedural prerequisite rather than a tool for sense-making, thereby limiting opportunities for interdisciplinary reasoning.

The data were collected through a series of semi-structured, think-aloud interviews conducted with undergraduate students enrolled in a junior-level quantum mechanics course. Each participant would provide their attitudes, opinions, and perceptions on a variety of topics pertaining to their quantum mechanics course. Responses were coded and analyzed, revealing a range of attitudes, from appreciation and confidence in mathematical precision to frustration with abstraction and a perceived lack of physical grounding.

This poster will focus on findings suggesting a correlation between students' sense of ownership of mathematical tools and their ability to conceptually bridge quantum measurement and experimental practice. The analysis emphasizes the pedagogical significance of explicitly linking symbolic operations to interpretive and experimental contexts. These findings align with the cross-disciplinary disconnects reported by Matthews et al. (2009) and Wawro et al. (2025), suggesting that challenges in integrating mathematics and physics extend beyond cognitive to include epistemological and emotional aspects. Importantly, this study provides a student-centered representation of these dynamics, revealing how learners articulate their own mathematical understandings within the context of quantum mechanics.

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A Proposed Extension to the Productive Struggle Framework

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Keywords: Productive Struggle, Student Thinking, Undergraduate Developmental Mathematics

Struggling with mathematics can be beneficial for students (Hiebert & Grouws, 2007). However, we need to ensure that student tasks are developmentally appropriate (Hiebert & Grouws, 2007). Productive struggle (PS) is "the space within which a student's struggle is likely to be productive" (Hiebert & Grouws, 2007, p. 388). PS hinges on the participant reaching a productive resolution, maintaining the intended level of cognitive demand, and completion or moving forward with the task (Warshauer, 2015a). PS also involves the risks of vulnerability and self-reflectivity, for both student and teacher (Murdoch et al., 2020).

The research question related to this poster asked: What pedagogical approaches could be used to support undergraduate developmental mathematics students as they work toward constructing fraction understandings that will be useful in college-level mathematics courses? Specifically, what pedagogical approaches support an environment such that a student productively struggles in a difficult fraction task during a teaching episode? By undergraduate developmental mathematics students, I mean undergraduate students who do not test ready for college-level mathematics.

In this poster, I share insights from a larger dissertation study in which I used a productive struggle (Warshauer, 2015a; 2015b) approach while conducting clinical interviews (Ginsberg, 1997) and teaching experiments (TE; Steffe & Thompson, 2000) to explore how undergraduate developmental mathematics students understood fractions. Vivian (pseudonym) was recruited from a randomly selected undergraduate developmental mathematics class at a large research university in the southern United States. Her TE was during fall 2023 and spring 2024. Data were analyzed through ongoing and retrospective analysis (Steffe & Thompson, 2000). Using subtasks as my unit of analysis, I applied the PS framework (Warshauer 2015a; 2015b) to identify instances of Vivian's struggles. Vivian had to express a struggle in some form, for a subtask to be classified as containing a struggle, consistent with Warshauer (2015a). Using Warshauer's (2015a; 2015b) framework, I identified Vivian's productive struggles, then classified the teaching response(s) and move(s) used during the tasks. Multiple codes were applied when present.

Through the coding process, I found that Vivian experienced PS during 24 of the 31 subtasks in her TE where she struggled. When coding Vivian's PS for teacher responses (Warshauer, 2015a) I found that one subtask received a telling response. Vivian received directed guidance during four subtasks, where she was given probing guidance during 17 subtasks. Vivian received an affordance response, such as acknowledgement or giving time, during subtasks. The results for teacher moves (Warshauer, 2015b) used during Vivian's PS revealed that she was asked questions during 20 subtasks. She received encouragement during six subtasks. Vivian was given time to work through 11 subtasks, and her mathematical thinking and effort were acknowledged during 3 subtasks. Unexpected results included the telling and directed guidance moves, but they were productive when Vivian was reiterating the task prompt or being redirected back to the task goals. Additionally, Vivian had two unproductive struggles that remained unresolved when she chose to shift the focus of her solution strategy for a task, showing a potential extension to the PS framework (Warshauer, 2015a; 2015b) by adding an additional ending to an episode of struggle.

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Finding Joy in the Abstract

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Keywords: Mathematics education, Rehumanizing mathematics, Mathematical play

"This is the most fun I have ever had in a math class," exclaimed one of the students during a collaborative homework session. A collective revelation followed - none of us had had this much fun in mathematics before. Our abstract algebra class, which started as a space of insecurity for some, had morphed into a community of joy and playfulness that encouraged us to explore mathematics in ways we had never considered before. This study aims to describe the characteristics of a joyful mathematics classroom through the lived experiences of researcher-participants in this course, which can be applied to secondary and post-secondary mathematics settings.

The Need for Joy and Play in Mathematics

Su (2020) declares, "We would teach math so differently if we thought of it as a playful sport, not a performance sport" (p. 62). Joy and playfulness lead students to develop more nuanced understandings of mathematics and encourage them to pursue it further (e.g., Takeuchi et al., 2016). Experiences of joy and play also contribute to rehumanizing mathematics and pushing the field towards much-needed culturally and historically responsive work (Gutiérrez, 2018). Research on joy and play in mathematics spaces has increased; however, while there are some exceptions (e.g., Ellis et al., 2024), most studies have predominantly focused on early education or primary school settings, overlooking the potential for joy and play as we mature into adulthood (McLennan, 2020; Solarski, 2018). There is little work on rehumanizing mathematics in postsecondary settings (Gutiérrez et al., 2025; MacArthur et al., 2025).

Methods and Preliminary Findings

This study examines our experiences as five doctoral students at a large mid-Atlantic university as we participated in an Abstract Algebra seminar class. Our group included four female students and one male student studying mathematics or physics education, two of whom had previously taken an abstract algebra class. One student identifies as Latina, one as MENA, one as Black, and two as White. We met weekly to explore topics in abstract algebra, including sets, groups, rings, logic, and fields. Homework was assigned weekly, and collaboration was strongly encouraged. We also read and discussed papers that defined algebra and its relevance to K-12 mathematics education. We applied a mixed-methods concept mapping analysis (Windsor, 2013; Trochim, 1989) to collaboratively synthesize our diverse ideas into a shared framework, identifying the characteristics of a joyful and playful mathematics classroom. Preliminary themes of structured flexibility, safety, exploration, and community have begun to emerge as among the most imperative characteristics of our joyful and playful mathematics class.

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Instructor Moves in an Inquiry-based Undergraduate Mathematics Course for Prospective Secondary Mathematics Teachers

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Keywords: Secondary Preservice Mathematics Teacher Education, Functions, Instructor Moves, Inquiry-based Learning

To support the integration of inquiry-based learning practices in mathematics teaching, many studies have explored instructor moves and their effects on student outcomes (Deshpande & Guglielmo, 2019). In addition, on the topic of functions in mathematics, Breidenbach et al. (1992) apply Action-Process-Object-Schema (APOS) theory to describe the development of the process conception of function. Dubinsky and Wilson (2013) also use APOS theory and ideas of Quine (p. 89) to explore students' understanding of the function concept. Representations of functions (Even, 1998) and covariation (Thompson & Carlson, 2017) play important roles in the understanding of functions. This poster explores the research question: What are the key instructor moves used to facilitate student learning of functions in an inquiry-based undergraduate mathematics course for prospective secondary mathematics teachers (PSMT)?

This study is part of a larger study conducted at a large, diverse, urban university in the southwestern United States. Participants (n=27) worked in groups on an inquiry-based lesson on covariation in a mathematics course for PSMTs (Beach, et al., 2020) over three eighty-minute class periods. Student and instructor interactions were video recorded, focusing on two to three groups at a time. After audio transcription, open coding of instructor moves identified the following codes: Promoting, Probing, Redirecting, Revoicing, Mathematically Precise Speech, Relating to Other Concepts, or Vague/Non-Response. In addition, continuous periods of instructor engagement with a group, typically involving several instructor moves, were clustered as instructor move episodes (IME). Student discourse before and after each IME was assessed on a four-point scale, informed by instructor lesson logs. Level 0 indicated hesitant discourse while Level 2 characterized moderate, on-topic discussion of functions by most group members. Level 1 and Level 3 indicated minimal and maximal discourse, respectively. For each IME, the initial and final level of student discourse and instructor move frequencies within the IME were noted. The data analyzed consisted of four IMEs of the same student task worked by three groups.

Probing was the primary instructor move identified in episodes that shifted discourse about functions from Level 0 to Level 2 and Level 1 to Level 2. However, for the two episodes where discourse shifted from Level 2 to Level 3, Prompting surpassed Probing and frequencies of Mathematically Precise Speech, Relating to Other Concepts, and Revoicing increased.

Preliminary results suggest that the effectiveness of using a particular instructor move may be dependent upon the state of student discourse at the time of use. The move from Level 0 or 1 to Level 2 may be attributed to asking students direct questions (Probing) to facilitate engagement. In contrast, a shift towards a deeper level of discourse may require instructor moves that focus on the introduction of more specific information and precise language. Findings suggest that the shifts in student discourse after each IME also may indicate enhanced conceptual understandings of functions.

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Approaching Zero: Exploring Calculus Students' Conceptions of Zero

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Keywords: Zero, Calculus, Student Conceptions, Mathematical Understanding

Although it is perceived as trivial, zero plays a crucial and nuanced role in many topics of mathematics that often goes unnoticed by students. Zero's roles include that of a placeholder and an identity, but for most students zero is perceived simply as a number representing nothing. Previous studies have suggested that a conception of zero rooted in nothing is quite nuanced (Seidelmann, 2004; Vest & Alibali, 2024), but it may not align with all situations involving zero. Studies have also shown that teacher conceptions of zero vary greatly but are very similar to those of their students (Wheeler & Feghali, 1983). Ultimately, researchers have suggested that students' and teachers' understanding of zero is limited or incomplete (Inhelder & Piaget, 1969; Tsamir, Sheffer, & Tirosh, 2000; Seidelmann, 2004). These studies, of which there are very few, have yet to consider how one's conception of zero may be a fragmented combination of many different conceptions. By reframing conceptions of zero this way we can consider how students operationalize different elements of their conception of zero based on problem types and context.

In this study, I consider the many conceptions of zero held by calculus students and how they employ those conceptions based on different tasks. Calculus students have been chosen as the population of study because of the central role of zero in calculus and the fact that students have undergone many mathematical domains in which zero played an important role. Studying calculus students also presents an opportunity to uncover how early or traditional conceptions of zero do or do not align with calculus understanding and procedures. In this work I am guided by the following research questions:

1. In what ways do calculus students conceptualize the concept of zero and its many roles in mathematics?
2. In what ways do calculus students employ varying conceptions of zero to navigate tasks and mathematical situations?

To answer these questions, I have designed a two-stage mixed-methods study with high school and undergraduate calculus students as the population of interest. Students from two high school classrooms and two undergraduate classrooms will respond to a questionnaire that is meant to uncover how students conceptualize zero as an object as well as in the context of traditional arithmetic problems and calculus situations. I will conduct a thematic analysis (Braun & Clarke, 2006) with a series of a priori frameworks from existing literature as a guide to establish students' conceptions of zero. A subset of five to seven students with diverse conceptions of zero will be selected from each population to participate in a follow-up interview. The interview will consist of mathematical tasks and reflection questions that expand upon the topics covered in the questionnaire. A thematic analysis (Braun & Clarke, 2006) of the interview data will enrich my understanding of the conceptions of zero held by students and the ways in which they navigate tasks with these conceptions.

Although the primary data collection is set to take place in the spring, the pilot data suggests that students are comfortable using zero in mathematical situations but may not have a strong understanding of why zero behaves as it does. When some students describe zero, they find that their conception does not align with how they use zero to operate in domains such as multiplication, division, and limit calculations.

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Calculus Students' Mindsets are Not So Simplex

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Keywords: mindsets, growth mindset, metacognition, calculus, topological analysis

In Psychological Meaning System Theory, *mindsets* are attitudes that people hold about themselves (Dweck, 2000). Traditional studies on mindsets are rooted in psychological methodologies, relying on surveys, questionnaires, and psychometric scales. While these tools give researchers initial insight into the mindset phenomenon, due to their nature, they also prime participants to think about specific mindsets in isolation. To study mindsets from a different methodological perspective, Hensley & Gilroy (2025a, 2025b) analyzed the differences in how Calculus I students' mindsets manifest over a semester depending on students' first exposure to a Calculus course (high school or university) using data collected from short reflective prompts during an actual Calculus course. Results suggested that students who took Calculus in high school tend to exhibit different mindsets than those taking it in college for the first time.

In Topology, an intuitive definition of a *simplicial complex* is a space with a triangulation (Weisstein, n.d.), which is useful for modeling the structure of multi-dimensional combinations of objects. Each simplicial complex contains a variety of sub-dimensional objects; a vertex is zero-dimensional, an edge is one-dimensional, a filled-in triangle is two-dimensional, a solid tetrahedron is three-dimensional, and so on. The purpose of this study is to revisit the coding results from Hensley & Gilroy (2025a, 2025b) and to conduct a novel analysis of the mindsets exhibited by students who were "successful" in Calculus I. We defined "success" as those earning a final grade of "A" ($n = 31$) or "B" ($n = 28$). For each student, we assigned a "mindset simplex." For each of Units 1, 2, 3, and 4, we constructed a simplicial complex that modeled each student's combinations of mindset codes. Then, we tabulated the frequency of simplices in each unit whose vertices represented productive mindsets (growth, mastery, strategic, and curious), non-productive mindsets (fixed, helpless, non-strategic, and non-curious), and mixed mindsets (at least one productive and at least one non-productive mindset). We note that the mindset simplices are similar to the Code Co-Occurrence Model produced by software like MAXQDA (*Code Co-Occurrence Model*, n.d.). Viewing these combinations of mindsets through a topological lens allows for them to be viewed in higher dimensions.

We calculated the relative frequency of the types of simplices (productive, non-productive, and mixed): first, on the aggregated data, then disaggregating initial frequency tabulations by gender, final grade earned, and first setting of Calculus, and comparing between groups. We saw the largest differences between students who first saw Calculus in a high school (HC) setting versus a university setting (CC). CC students had higher proportions of strictly productive mindsets than their HC counterparts.

This provides further evidence of the conclusions in Hensley & Gilroy (2025a, 2025b) that these two groups of students exhibit different mindset profiles, implying that they experience a typical general Calculus I course very differently. Furthermore, it may be advantageous for instructors to consider these differences when teaching the course. An extended version of this writing with figures can be accessed at https://tinyurl.com/MindsetSimplex.

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A Covariational Lens for Proofwriting in Undergraduate Mathematics

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Keywords: Proofwriting, Chunky and smooth framework, Undergraduate Mathematics

Literature Review

We examine how undergraduate students navigate proof writing through two complementary images of activity: chunky assembly and smooth development, and how students transition between these modes while producing a final proof write-up. While the chunky/smooth framework was originally developed to characterize students' images of change in covariational reasoning, Castillo-Garsow et al. (2013) emphasize that these distinctions also describe how mathematical activity itself is organized, independent of domain.

In this study, we adapt the chunky/smooth distinction as an analytic lens for describing modes of engagement during proof construction. Proof writing often involves alternating between discrete exploratory episodes without attention to the larger argument (chunky) and more continuous episodes in which students attend to the overall logical throughline (smooth). Existing literature documents related distinctions, such as exploration versus validation (Weber, 2008) and dimensions of proof comprehension and coherence (Mejía-Ramos et al., 2012), but offers limited language for characterizing how students move between these modes during authentic proof writing.

By framing proof activity in terms of chunky and smooth episodes, we aim to provide a descriptive vocabulary that complements existing proof frameworks and enables fine-grained analysis of how students organize, refine, and translate exploratory work into coherent written proofs.

Research Questions

RQ1. What episode types (e.g., exploratory "chunky," throughline "smooth," resource-use) arise during students' proof writing, and with what typical sequences?

RQ2. What triggers students' commitment from exploratory "chunky" work to a sustained "smooth" throughline, and what triggers stalls or switches?

RQ3. How do ideas generated during exploration survive (or not) into the final written proof, and how does resource-use (text/web/notes) inform that translation?

Study Design

We conduct exploratory think-aloud interviews (40–60 minutes) with undergraduates enrolled in proof-based mathematics courses. Participants work through two multi-route proof tasks, selected strategically from Hammack's *Book of Proof* (2018), with access to resources as in an authentic homework setting. Data sources include audio recordings, scratch work, and final written proofs.

Episodes will be coded to identify chunky and smooth modes of activity, transitions between them, and their alignment with features of the final proof write-up. Following the interviews, participants complete Likert scale items probing their beliefs about proof writing and what constitutes a "good" proof, allowing us to situate observed activity patterns within students' stated views of proof.

We are beginning the interview phase and anticipate initiating episode coding in January.

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Examining Undergraduates' Conceptions of Series Convergence via Script Writing

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Keywords: calculus, script writing, series convergence

Students typically encounter the definition of series convergence as the limit of a sequence of partial sums in second-semester calculus. However, students struggle coordinating series and partial sums (Martin, 2013; Martínez-Planell et al., 2012), which Martínez-Planell et al. attribute to a weak limit schema. This supports claims by McDonald et al. (2000) that students require a "a rich sequence schema … to fully understand series as the limit of a sequence of partial sums" (p. 98) and that students encounter difficulties in applying their understanding of sequences to subsequent mathematical concepts. Other research on student understanding of series includes how students think about series convergence (e.g., Eckman & Roh, 2022; Martínez-Planell et al., 2012) and how students interpret or develop series notation (e.g., Eckman & Roh, 2024a, 2024b; Larsen et al., 2024). Researchers recognize the challenges students encounter in learning series and the need for meaningful tasks that enhance student learning about series (e.g., Lindaman & Gay, 2012; Torres Manzanarez & Álvarez, 2025; Vroom & Alzaga Elizondo, 2025).

This exploratory study centers on examining students' understanding of series convergence using both non-scripting and scripting-based tasks to gain insight into the affordances of script writing for assessing understandings in this context. Script writing requires student engagement in imagined role-play by writing a script that extends a prompt that begins a dialogue between a teacher and student or between two students. In 2018, Zazkis issued a call-to-action for researchers to use script writing to explore or assess mathematical understandings. This poster addresses the following research question from a larger study: What is the nature of possible conflict factors with regard to students' concept definition and concept image of series convergence that arise in student responses to non-scripting-based tasks and a scripting task?

In a departmentally-coordinated second-semester calculus course, over a 15-week semester, 105 students across ten different sections granted researchers access to their second midterm exams and scores. Participants (n=18) scoring in the third and fourth quartiles consented to engage in two interview sessions. They completed two interview sessions which included: (a) a non-scripting task-based interview, (b) a script writing training activity, (c) a scripting task, and (d) a semi-structured interview about their script. Half engaged in the non-scripting task in the first interview session and the scripting task in the second interview session. The order was reversed for the other participants. Qualitative data analysis entailed coding interview responses using a priori codes on student conceptions of series identified in the research literature and using the theoretical lens of concept image and concept definition (Tall & Vinner, 1981). In addition, emergent codes from the data were identified.

Aligning with Martin (2013) and Martínez-Planell et al. (2012), participants encountered difficulties relating the sum of a series and the limit of the series' sequence of partial sums. However, the scripting task revealed aspects of participants' understanding of series in relation to partial sums even for participants who were unsuccessful demonstrating this on the non-scripting tasks. This exploratory work underscores the potential viability of scripting tasks as a tool for investigating student understanding of series convergence. Furthermore, this study shows promise for future work in developing and using new scripting tasks on series in second-semester calculus courses and tracking their effectiveness for improving student learning.

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Towards a Tribal College and University Mathematics Education Research Agenda

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Keywords: tribal colleges and universities, developmental math, math pathways

Background

There is a vast underrepresentation of contexts of Indigenous learners and from the contexts of TCUs. Further exacerbating this underrepresentation is the general lack of presence of research in developmental mathematics education in mathematics education research spaces. Research in undergraduate mathematics education focusing on developmental math and research on mathematics education in TCU contexts are each severely lacking. Research at the intersection, developmental mathematics education in TCU contexts, is scant if not nonexistent. In order to advance knowledge in these areas, there is a need to highlight the work being done at TCUs.

TCU faculty are tasked with aligning mathematics education with TCU missions to revitalize language and cultures, but there are limited existing mathematics resources that are ready to implement in the classroom in ways that directly align with TCU values and missions (Marshall, 2018; Crazy Bull & Guillory, 2018; Crazy Bull et al., 2020). Furthermore, TCU faculty are often working with extensive teaching and service duties in small mathematics departments. This results in little time and few resources to engage in mathematics-specific collaborative learning opportunities (Al-Asfour & Young, 2018). While TCUs work to provide innovative STEM education to Indigenous communities, what is known and published about STEM education in TCU contexts is limited. With such limited research on developmental math education in TCU contexts, there is rich potential that such research can contribute to the RUME community by taking perspectives rooted in TCU missions and values.

Current Trends in Mathematics Education at TCUs

This project provides snapshot of current contexts for developmental mathematics education and mathematics pathways in TCU contexts, with a particular focus on developmental mathematics and math pathways, consisting of: a literature review that provides an analysis of trends in the literature pertinent to mathematics education at TCUs, targeting trends and needs in research and practice in the areas of developmental math and math pathways; a landscape analysis consisting of summary of qualitative and quantitative data on mathematics programs, in particular developmental math and math pathways at TCUs; and a Promising Practices report summarizing innovative and promising practices in developmental math and mathematics pathways as shared by TCU math faculty participating in the project.

Towards a TCU Mathematics Education Research Agenda

Given the current trends in mathematics education at TCUs, we call not only for more research and opportunities to highlight innovative mathematics education at TCUs. We also outline a research agenda that situations math ed research in the context of TCU's roles of empowerment of Indigenous communities and tribal nation building.

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Can Prerequisites Be Predictors?

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Keywords: Student Outcomes, Prerequisites, Prior Knowledge, Curricular Coherence

According to constructivist and cognitivist theories, students learn through the reconstruction of new knowledge using related, existing knowledge (Rach & Ufer, 2020). The greater the depth of a learner's prior knowledge, the easier it is to integrate new information into existing cognitive structures (Kollar et al., 2014). Prerequisite courses provide access to "domain-specific prior knowledge, [which] has been found to be the most important variable influencing student achievement" (Hailikari et al., 2007, p. 59).

This study examines two prerequisite pathways, Quantitative Literacy (QL) and College Algebra (CA), leading into the target course, Elementary Mathematics for Teachers (EMT), at a public, state university in the United States. QL courses are typically taken by non-STEM majors and focus on building data literacy, while CA courses are typically taken by STEM majors and focus on building procedural fluency. This university's QL courses are assessed non-traditionally, opting for projects and article analyses, whereas the CA courses are assessed conventionally, using tests, quizzes, and homework.

Our exploratory analysis was guided by the following question: How, if at all, does a student's prerequisite pathway impact their final course grade in EMT? We began with institutional course-tracking data across the past two academic years (Fall 2023 to Spring 2025) to compare final course grades (4.0 GPA scale) between students entering EMT from these two pathways. After cleaning the data (e.g., removing duplicates), we generated visualizations (e.g., grouped bar chart) and used both descriptive (e.g., comparing median grades) and inferential (e.g., Kruskal-Wallis test) statistical methods to conduct our analysis.

Preliminary findings indicate that students entering from the CA pathway tend to earn higher final grades, on average, than those entering from the QL pathway. This result in isolation, however, obfuscates the complex nature of student outcomes; to claim that one pathway is stronger than the other inherently dismisses the multifaceted nature of students' mathematical experiences and learning outcomes. Instead, it encourages further questioning of the pathway's underlying mechanisms. Are certain skills emphasized within the CA pathway more easily transferable to EMT? Do similarities in assessment modalities between the EMT course and courses within the CA pathway positively facilitate this transfer? Could pedagogical choices within the CA pathway be positively influencing affect?

Future work will extend this analysis by incorporating additional contextual variables, such as class size and start time (Hurdle & Mogilski, 2022), to identify potential sources of student performance variation. If information about students' past experiences can offer insight on their future experiences, then a student's outcome in a pre-requisite course could indicate a positive experience to be had in the following course.

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Modeling the Relationship Among Classroom Climate, Math Self-Concept, and Math Interest in College: A Proposed Humanizing Framework

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Keywords: classroom climate, math self-concept, math interest, African philosophy of Ubuntu

Mathematics self-concept and interest are key predictors of student achievement and future career choices (Geisler et al., 2023; Pietsch et al., 2003). Studies have demonstrated that the two psychological processes of motivation can be related to the psychosocial environment in the classroom, also termed as classroom climate (Erdem & Kaya, 2024) for students at all educational levels in mathematics courses (Sekone, 2025; Wang et al., 2017; Yin et al., 2020). However, those studies were mostly grounded in Eurocentric sociocultural theories including Moos's (1979) framework, Bronfenbrenner's (1979, 2005) ecological systems, Vygotsky's sociocultural (Walshaw, 2016), and the four-phase model of interest development (Hidi & Renninger, 2006) to either describe the classroom climate and/or examine its contribution to self-concept or interest in mathematics (e.g., Stone-Johnstone, 2023; Yang, 2015; Yin et al., 2020).

To gain a comprehensive and sociopolitical understanding of the mechanism of the causal relationships among the three constructs, this study proposes a framework that integrates sociocultural Eurocentric theories with the African philosophy of Ubuntu (Blackwood, 2025; Tutu, 1999) which translates to "I am, because we are" (Mbiti, 1989, p. 106). Ubuntu emphasizes the interconnectedness and shared humanity of all people and in education, Ubuntu pedagogy is considered "a learner-centered pedagogical approach that promotes democratic atmospheres where students feel respected, cared for and have the freedom to co-learn in an environment where power relations are grounded in humanism." (Ukpokodu, 2016, p. 155).

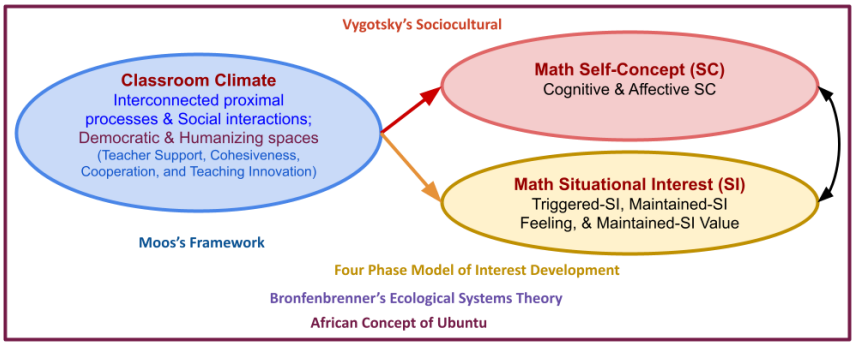

Figure 1. Hypothesized Model of Causal Relationships Among Classroom Climate, Self-Concept, and Interest

In the poster, I include the modeling process that resulted in the hypothesized model (see Figure 1) depicting how the humanizing African philosophy of Ubuntu can provide a sociopolitical lens that enriches the descriptive and/or explanatory power of the four Eurocentric sociocultural theories to gain a comprehensive understanding of the influence of classroom climate on self-concept and interest in mathematics. From this study, RUME researchers could identify avenues for using this model to inform classroom practices and curriculum design aimed at fostering mathematics motivation. I also discuss how my future research will collect quantitative surveys and qualitative data in Calculus I courses to test the proposed humanizing framework.

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Where and to Whom Should Students Talk About Math: Mathematicians' Contemplations of Student-Instructor and Student-Student Interactions

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Keywords: Active learning; changing pedagogical practice; linear algebra; university mathematics instruction

Aims

In this poster, we focus on discussion or communal problems-solving to promote higher-order thinking, aiming to identify where instructors are open to employing instructor-student and student-student interaction in teaching and learning of abstract linear algebra.

Literature review

Despite effective alternative research-based practices showing positive effects on student learning outcome (see Freeman et al., 2014), most university mathematics courses are taught predominantly by lecture (Fukawa-Connelly et al., 2017; Johnson et al., 2018; Johnson, 2019). In response, Artigue (2016) recommends "collaborative projects, building and negotiating, jointly with mathematicians … [finding approaches] … that make sense for all … and meet their respective … needs" (p. 12). As proposed changes to instructional practices occur in environments where there are obligations to mathematics as a discipline, to students, and to their institutions (Charzan et al., 2016), considering adoption of active learning strategies involve managing sometimes-conflicting obligations.

Methods

We utilize data from a collaborative project between mathematics education researchers and mathematicians where together they attempted to integrate active learning pedagogy into mathematicians' lectures in a proof-based linear algebra course. We used the method of constant comparison through the grounded theory approach (Strauss and Corbin, 1994) to form a preliminary codebook using Atlas.ti software. Next, we focused on the interactions where instructors were considering in class and out of class activities. The table of code was further developed to code the instructors' responses to active learning strategies which focused on two different interlocutor pairs, namely student-student and student-instructor.

Preliminary Results and Discussion

We found instructors cited time and student preference as the two major constraints to implementing discussion in the classroom. Overall, the participating instructors were more open to student-instructor interaction (coded 119 times) than student-student interaction (coded 21 times), though not in all cases (12 student-student coded by a single instructor, who had facilitated active learning as a graduate student).

Outside of class activities were raised as an alternative, by the instructors when discussing both time, and student preference obligations that constrained their use discussion in the classroom. In these cases, 7 were coded student-instructor interactions and 5 were coded student-student interactions. Outside of class discussion activities overcame the in-class time constraint and were viewed as beneficial. Furthermore, they were student initiated, and so respected and encouraged student autonomy.

Acknowledgments

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Investigating the Alignment of Instructors' and Students' Perceived Self-Efficacy Levels and Sources in an Introduction-to-Proofs Course

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Keywords: self-efficacy, proving techniques, alignment

This study investigates whether there is alignment between an instructor's and students' perspectives on students' self-efficacy levels and sources for using proving techniques in an Introduction-to-Proofs (ITP) course. It uses Bandura's self-efficacy theory (1977) and its four sources of self-efficacy (performance accomplishments, vicarious experience, verbal persuasion, and emotional arousal) as a basis for understanding where students' self-efficacy levels may originate. This builds on an established interest in understanding Bandura's four sources with respect to mathematics self-efficacy levels (Gainor & Lent, 1998; Gao, 2020; Parsons et al., 2011; Regier & Savic, 2020; Usher & Pajares, 2009; Watkins & Middleton, 2021).

Klassen and Lynch (2007) investigated the perspectives of Grade 8 and 9 students with learning disabilities and their teachers regarding the students' self-efficacy levels and sources. They concluded that the students' and teachers' perspectives of the students' self-efficacy levels and sources were misaligned. My study investigates if there is a similar case of misalignment between student and instructor perspectives in an ITP course, as "the differences in teacher and student perceptions and understandings of self-efficacy may result in teachers' providing interventions that are not valued by students, and possibly even counterproductive to their academic performance" (Klassen & Lynch, 2007, p. 504). This study is therefore guided by the following research question: Do teachers' and students' perspectives regarding students' self-efficacy levels for using proving techniques and the classroom experiences that influence students' self-efficacy levels for proving techniques align?

Data collection is ongoing from an ITP course that is running during the Fall 2025 semester by observing and recording each class session and conducting interviews with six students and the course's instructor. There will be, in total, three rounds of student interviews (SI) and two rounds of instructor interviews (II) that take place in the following order: SI1, II1, SI2, II2, SI3. Each interview consists of three phases. The first phase focuses on each participant's recent experiences in the course. The second phase's questions focus on two proof-based problems that I give each participant. Note that SI1 and II1 have the same problems, as does SI2 and II2. In the third phase, I show participants four classroom recordings and ask them questions about their recollection of each moment and each moment's possible impact on students' self-efficacy levels. This phase is being conducted similarly to Lew et al.'s methods (2016) that showed a real analysis course's students and instructor the same classroom recording to investigate alignment between what the instructor intended to convey during the lecture and students' understandings of the lecture's material.

This poster will focus on the study's preliminary results. The preliminary findings suggest there is some alignment between the instructor's and students' perspectives on students' self-efficacy levels and sources for using proving techniques. For example, the perspectives of the instructor and several students agreed that students were more confident in their ability to prove a statement directly than in their ability to prove a statement by contradiction. Additionally, the instructor and most of the students said that the in-class example of proving $\sqrt{2}$ is irrational by contradiction did not positively impact the students' self-efficacy levels for using contradiction.

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Exploring Student Engagement With Concept Maps

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Keywords: concept maps, connection making, student engagement, multivariable calculus

Successful students see mathematics as connected ideas (OECD, 2013). Concept maps can be used to visualize these connections (Jeong & Evans, 2021). Research suggests that concept mapping can lead to high-quality learning (Schroeder et al., 2018); however, there is little recorded usage of them in mathematics education (Jeong & Evans, 2021). Between 2019 and 2024, only 4 out of 50 studies in a meta-analysis on concept maps with measurable outcomes related to learning and/or engagement were explicitly about mathematics (Kefalis et al., 2025), underpinning the notion that research is needed to examine the efficacy and engagement of students with concept maps in mathematics. We will answer: how effective are concept maps at eliciting student engagement with the derivative?

In a supplemental instruction session (SIS), 16 multivariable calculus students created concept maps using up to 23 terms from a word bank (derived from known derivative frameworks (Likwambe and Christiansen, 2008; Zandieh, 2000)) and provided reasons for each connection they identified. We analyzed connection reasons using the Relational Scoring Method created by McClure and Bell (1990) with context-specific modifications by the authors (Figure 1). Procedural (P) and conceptual (C) relationships were the focus because P and C knowledge of the content is needed for student success (Rittle-Johnson, 2017). Scoring was chosen because it is one of the most common analysis tools for concept maps (Ekinci & Şen, 2020).

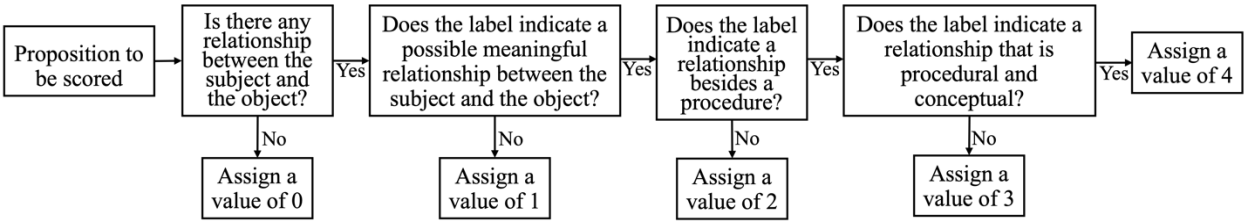

Figure 1. Modified scoring protocol.

As seen in Figure 2, the median number of words from students was 62. The median score of the concept maps was 66.5. More details about scoring and its implications will be discussed in future work. This study suggests that current approaches to analyzing concept maps make them an ineffective tool for deeply understanding student reasoning about the derivative. However, in future work, we will explore novel approaches to analysis that might bring new life to concept mapping for research.

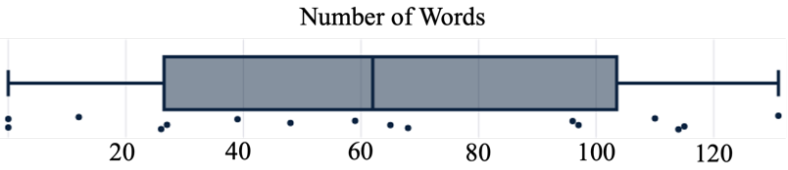

Figure 2. Number of words box plot.

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Preliminary Analysis of Problem Solving in Introductory Environmental Engineering

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Keywords: problem solving, environmental engineering, water systems, flow diagrams

Strong computational and algebraic skills are commonly understood to be a prerequisite for success in engineering. In a 2021 review of the literature on mathematics in engineering education, Pepin, Biehler, and Gueudet discussed a need for "bridging" between school mathematics and the mathematics needed in university engineering courses. This study homes in on aspects of problem solving in engineering that may be challenging for students.

Written data were collected, including homework assignments, midterm exams, and final exams, from consenting students in an introductory environmental engineering course on designing small water systems at a mid-sized university in the Midwestern USA. This poster will focus on a preliminary analysis of the midterm exams of 17 students from this course.

A modified version of Carlson and Bloom's (2005) Multidimensional Problem-Solving Framework was used to code the data. This framework consists of four main phases: orienting, planning, executing, and checking. However, the nature of our data prevented us from being able to identify instances of students' planning and checking in their problem solving. We therefore modified this framework and used an open coding process (Strauss, 1987) to identify instances of students *orienting*, *setting up a problem*, and *executing a solution strategy*. Codes were applied at the level of a question-part. This was then used to determine in which phase of the problem-solving framework students most frequently committed their first error in multi-part problems.

Data analysis is ongoing; in this proposal we present preliminary findings from the midterm exams of five students. Among these five students, no student successfully identified all system boundaries when given a flow diagram for a small water system (Question 2). This task requires only engineering knowledge to complete. However, the subsequent three questions required students to use their knowledge of these reaction boundaries to write symbolic mass-balance equations, and three of the five students were able to write correct equations for all three of these questions. This suggests that these three students did not recognize the significance of the reaction boundaries and their relationship to these equations. On Question 1, students were given information and asked to draw and label a flow diagram for a small water system. All five students drew the correct structure, but two mis-identified masses as volumes and a third produced an incorrect number for a variable that was not given. Both students who incorrectly labeled masses missed follow-up calculation questions because of this mislabeling, suggesting an understanding of mathematics that was undercut by a misunderstanding of engineering content.

Though preliminary, these findings suggest that the engineering students in our sample had the mathematical skills necessary to solve problems related to the design of water systems, but their ability to solve these problems successfully was inhibited by gaps in their understanding of engineering content. In the completed poster, we will present results from the midterm exams of all 17 students in our data corpus, and future research will analyze the final exams and homework assignments of all students in this sample. We will identify the problem-solving phase in which students encounter the most difficulty and examine students' trajectory throughout the course to identify transient and persistent obstacles to their successful solving of engineering problems.

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Examining a GTA's Knowledge for Teaching Statistics via Scripting Tasks

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Keywords: Statistics Education, Graduate Teaching Assistants, Knowledge for Teaching

According to the Conference Board on Mathematical Sciences survey in the Fall of 2021, approximately 1 in 4 (24%) introductory statistics instructors in statistics departments at 4-year institutions were graduate teaching assistants (Blair & Kirkman, 2025). Despite reliance on graduate teaching assistants (GTAs) in providing introductory statistics instruction for undergraduates at a time where data science and statistics programs are growing in popularity, research into statistics GTAs' knowledge for teaching is sparse. Previous work highlights GTAs' desire for support in multiple areas including technology, content, and pedagogical practices (Green, 2010) and an inconsistency in preparation for GTAs' across institutions (Justice et al., 2017). This leaves many GTAs to construct their knowledge and beliefs about teaching statistics independently which may result in their practice being rooted in their own classroom experiences and beliefs rather than recommended educational practices.

We conceptualize knowledge for teaching statistics using aspects of the Technological, Pedagogical, and Content Knowledge model (TPACK; Mishra & Koehler, 2006), Ball et al.'s (2008) domains of mathematical knowledge for teaching, and Groth's (2007) model of statistical knowledge for teaching. In this way, we draw upon the important aspect of technological knowledge in the realm of statistics while also utilizing Ball et al.'s subcategorizations of content and pedagogical-content knowledge in our exploration of GTAs knowledge for teaching statistics. The incorporation of Groth's model of statistical knowledge for teaching allows us to explore some of the non-mathematical aspects of knowledge for teaching statistics which differ from knowledge for teaching mathematics. In this poster, we aim to show how the use of scripting tasks allows us to explore GTAs' knowledge for teaching and how their knowledge may affect their pedagogical decisions, which may position them as helpful activities for use in the preparation of GTAs to be statistics educators.

In this project, five GTAs participated in a two-part sequence of one-on-one interviews centered around a scripting task. Scripting tasks (Crespo, 2018) are a type of activity designed to engage future educators in thinking about the process of teaching by providing a hypothetical, written scenario and asking them to continue the scenario in a script format where they control all actors in the scene. Our scripting task modeled a discussion between the GTA and two students while working on an activity exploring bootstrapping and confidence intervals in the large-enrollment introductory statistics course at the GTAs' home institution. In their first interview, each GTA was asked to "prepare for class" by reviewing the activity students would be working on in their hypothetical class. Then, they were given the scripting task and asked to continue the script, talking aloud about their decision-making process. For their second interview, GTAs' worked through a set of alternative scenarios based on their original script that were designed to present common alternative conceptions students may have in the context of the activity, again discussing their thought process and decision making.

This poster will focus on the interviews with one GTA, Norman, with a focus on identifying areas where aspects of his knowledge for teaching statistics were made visible through the use of the scripting task. Additionally, we'll discuss the potential of scripting tasks for engaging GTAs in thinking about students' alternative conceptions in statistics.

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Redesigning Assessment for Growth in Mathematics for Elementary School Teachers

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Keywords: Alternative grading, redesign, preservice teachers

Our institution is increasingly concerned with the success of students in their first-year courses, especially underrepresented minority students. As part of an institution-wide push to redesign courses with higher failure rates, we redesigned the assessments and grading scheme in our two-semester sequence of courses for future elementary teachers.

Literature and Redesign

The redesign efforts were aligned with the four pillars for alternative grading (Clark & Talbert, 2023), which include clearly defined standards, helpful feedback, marks indicating progress, and reassessment without penalty. Working backwards from revised course standards, we created methods of assessment that fell into four categories: skill checkpoints (in-class, written assessments), videos (short recordings showcasing a student's knowledge of how to teach math with manipulatives), and projects (longer, multi-week homework targeting broad skills).

Each assessment is graded as either a "pass" or a "not yet", with written feedback provided for each "not yet". Additionally, each assessment can be either revised and resubmitted or reattempted with no penalty to the grade. Letter grades were assigned based on the number of assessments passed, with cutoffs set for each letter grade in each category.

To assess the effectiveness and reception of this redesign, we asked the following questions: (a) is there a statistically significant difference in DFW rates between redesigned sections and non-redesigned sections, and (b) what are students' opinions about the new grading system?

Methods

Since 2023, the redesigned grading system has been used in 7 sections of the first course and 3 sections of the second course. Grade distribution data for sections of the course taught by the authors since 2022 was analyzed computing the pass rate for each section, defined as the number of students with a C- or better divided by the number of students enrolled.

In the fall of 2024, focus group interviews were conducted with 5 participants. The interviews were conducted by a third party. The participants were asked to reflect on their experiences using the grading system. End-of-semester course evaluations for the redesigned sections were also included in the analysis in order to cross-reference student sentiment.

Results

Comparing the pass rates with a t-test gave the following results: the mean of the redesigned sections is 0.617, the mean of the traditionally graded sections is 0.662, which was not a significant difference (p=0.36).

Qualitatively, students' opinions were divided. Some were positive about the reduction in anxiety enabled by reattempts. Others experienced more anxiety, as falling behind led to an ever-growing number of reattempts. Some praised the redesign for its focus on growth and understanding, while others felt that an all-or-nothing "pass" marking system was preventing them from succeeding. Students' receptiveness was impacted by the instructor's experience with the redesign.

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Integrating Computational Thinking in Undergraduate Statistics Projects: A Comparative Analysis

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Keywords: computational thinking, statistics

Computational thinking (CT) is a way of approaching problem-solving by breaking the problems into steps or algorithms that can be executed by humans, computers, or both (Aho, 2012; Li et al., 2020; Wing, 2006, 2008). In statistics education, CT may be employed as a problem-solving strategy (Postelnicu, 2025a, 2025b; Postelnicu & Postelnicu, 2020) by guiding students to decompose complex problems or datasets in manageable parts, recognize patterns, abstract key features, and design algorithmic processes for data analysis (Weintrop et al., 2016; Wing, 2006). Recent studies (Astuti et al., 2025; Irawan et al., 2025) pointed out to the benefits of CT integration in the teaching and learning of statistics in high school and college, and the effect of the type of learner and the statistics curriculum on student performance across the CT core components.

We present here the findings from a pilot study, part of an ongoing study, conducted with college students from two undergraduate courses – an elementary statistics course addressed to students with various majors (Group 1), and another course addressed to future early childhood and elementary education teachers (Group 2). The statistics curriculum for Group 2 students included only descriptive statistics. Students from both courses completed a statistics project designed to incorporate CT features and reflected on the CT approach. The following research question was addressed: What similarities and differences emerged in CT integration between the two project sets?

Data was analyzed across the CT core components (Astuti et al., 2025; Irawan et al., 2025; Postelnicu, 2025a, 2025b; Postelnicu & Postelnicu, 2020): i) *decomposition*- dividing problems into manageable parts; ii) *pattern recognition* – identifying patterns, trends, correlations; iii) *abstraction* – simplifying the problem, focusing on essential variables or characteristics; iv) *algorithm desig*n – constructing an algorithm or a sequence of computational steps.

Across both groups, decomposition consistently achieved the highest scores. This was expected, given the structure of the projects, containing the following: title and research question, data collection/ dataset description, descriptive statistics and/or inferential statistics, visual representations, analysis and interpretation, CT reflection, references, data source documentation. Some of the projects from Group 2 demonstrated strong pattern recognition, especially those projects requiring comparisons between various populations with respect to some characteristics. The projects from Group 1 achieved higher scores in abstraction and algorithm design. Overall, Group 1 reflections showed stronger CT integration, connecting each CT component to the design and execution of the project. Group 2 reflections were more of a practical nature, focusing on how to perform calculations or use the software.

The CT integration is promising, but we need to account for its variability across educational contexts (Astuti et al., 2025; Grover & Pea, 2018; Irawan et al., 2025; Postelnicu, 2025b; Shute et al., 2017).

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Exploring Calculus Students' Technology Use With Dynamic Graphing Technology

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Keywords: Technology, Graphs, Conceptual Understanding, Cognitive Load

Technology is commonly used in mathematics education to communicate ideas and support understanding of complex concepts. It can assist students by providing automated feedback on practice problems, such as in online homework environments (Dorko, 2020), and by individualizing problem sequences based on students' responses to prior questions (e.g., ALEKS). Drijvers (2012) describes students' use of technology as either doing mathematics or learning mathematics, with the latter including practicing skills and developing conceptual understanding. Roschelle et al. (2017) expand on this by considering productive and transformative perspectives of students' purpose to situate to describe the broader literature on educational technology.

Technology can support student understanding in several ways, such as reducing cognitive load by offloading calculations to technological tools, allowing students to focus more on conceptual reasoning (Paas, Renkl, & Sweller, 2004; NGA & CCSSO, 2010). Interactive technologies can also be programmed to follow formal mathematical rules, thereby supporting students' understanding of different representations (Heid & Blume, 2008). In this poster presentation, we explore how this framework helps us understand students' engagement with dynamic graphing software (Desmos) in a calculus context using Roschelle et al.'s framework.

Our data comes from one-on-one, task-based interviews with two undergraduate students enrolled in first-semester calculus. Students were asked to reason about function values, rates of change, and how these quantities vary when presented with a graph. They first interacted with a static image of the graph and were then given access to a dynamic version in Desmos.

We found that the students interacted with the technology in distinct ways and with seemingly different purposes. Roschelle et al.'s framework was useful in capturing this variation, as both students used Desmos in productive and transformative ways. Their engagement included doing mathematics, practicing skills, and developing concepts. For instance, one student used Desmos to practice a skill in a transformative way by alternating between reasoning with the printed static graph and the dynamic version. Notably, one student's understanding of changing quantities and slope evolved after leveraging Desmos features such as continuous sliding and zooming. We interpreted this as evidence of a transformative shift in their conceptual understanding. One student had taken a new understanding of an output $f(a)$ during the interview, and the task designer used the Desmos software to remind and reinforce that understanding– reducing cognitive load– by visually depicting the naming convention in the graph, supporting the student in maintaining that understanding throughout the task. Overall, we found the transformative categories to be more applicable to the data, possibly due to the nature of the graph interpretation task, which emphasize estimation from visualized measurements rather than numerical calculation. Future work will explore different tasks that incorporate other technologies (e.g., matrix calculators), which may yield different categorizations within Roschelle et al.'s (2017) framework.

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Identifying Barriers to Success in First Year College Mathematics Courses

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Keywords: first year mathematics, barriers to success

For first year students, their first semester GPA is a strong predictor of college persistence, that is, whether they return to university in the fall of their sophomore year (Steward, Lim, & Kim, 2015). This study was conducted at private, nonprofit university in the northeastern United States. Approximately 75-80% of the roughly 900 first year students enrolled in the Fall 2025 registered for a mathematics course. While many of these students are successful in their mathematics courses, too large a number of them struggle. This poster presents on a study designed to identify these barriers and suggests ideas of future work in designing interventions to help overcome some of the barriers.

For this study, the researcher went to 35 sections of what are traditional first year mathematics courses (e.g., elementary and college algebra, precalculus, calculus 1, intro statistics, math for elementary teachers, intro discrete mathematics) to ask students to fill out a preliminary survey discussing their high school mathematics experiences. 464 of the approximately 700 students enrolled in these courses filled out the survey in its entirety. The questions on the preliminary survey asked mainly about students' experiences in their prior math courses as well as their expectations for their current mathematics course. Some preliminary results include approximately 44% of respondents identifying as first-generation college students and nearly 22% of students not having taken a mathematics course in the previous year. 36% of the students reported taking online tests in their previous mathematics course, with 20% of those students reporting that they had no exams on paper. (3% of the students reported not having exams at all.) Nearly 92% of the students at least somewhat agreed that they had the knowledge and skills necessary to succeed in their current mathematics course with a similar number anticipating that they would earn a grade in the A or B range in their current course. When asked what they anticipated would be their biggest barriers to succeeding in their mathematics courses, the answers reported by the most students were "I don't do well on tests/have test anxiety" (46%); not being fully prepared for their current course (40%); and "I am not a math person" (37%).

At the end of the semester, students will be asked to complete a post-survey with similar questions regarding their current mathematics course. Then the researcher will use students' final course grades to try identify a profile of students who were least successful in their mathematics courses, as well as those who were most successful. Using final course grades, attendance records, and pre and post surveys, we hope to be able to determine factors that contribute both to students struggling and succeeding in their first year mathematics courses. We will also look to see if strong differences exist in reported experiences in high school versus college courses, as differences between the way that high school and college mathematics courses were structured and taught can lead to students struggling in their STEM courses (Meaders et al., 2019).

After gathering and analyzing all of the data, future work will entail designing interventions to help students overcome the identified barriers, such as review modules for missing content areas and test preparation advice for studying and taking college math exams.

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An Exploratory Analysis of a Reading-Based Introduction to Proof Course Compared to Business-as-Usual Introduction to Proof

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Keywords: Introduction to Proof, Sense of Belonging, Proof Reading

Introduction to Proof (ITP) is a pivotal point in the trajectory of mathematics majors. The Reading and Appreciating Mathematical Proofs (RAMP) project is a multi-year study that developed a series of proofs and accompanying author biographies for use in ITP courses (Contreras et al., 2025). Mathematicians each created three proofs designed to be both accessible and engaging for students at this level. The mathematicians were purposefully selected to broaden the kinds of mathematical stories students encounter challenging narrow, stereotypical images of "who does mathematics" and supporting a stronger sense of belonging in mathematics, factors linked to women's underrepresentation and attrition in the field (Good et al., 2012; Piatek-Jimenez, 2008).

Data thus far has been collected from nine sections (four using the curriculum; five not using the curriculum) of introduction to proof from two institutions. The final round of data collection is occurring this year. Students in the classes took a pre- and post- survey covering proof (Logic; Roh & Lee, 2024) and Sense of Belonging (SB; Good et al., 2012): 56 students in (15 women; 41 men) RAMP classes and 47 students (20 women; 27 men) in non-RAMP classes. In this poster, we will present some preliminary analyses of this data, including additional data collected from classes this semester. Our base model (women as reference group) for each survey is $\text{Post} = \beta_0 + \beta_1(\text{Pre}) + \beta_2(\text{Treatment}) + \beta_3(\text{Gender})$. Assumptions were checked and reasonably met. Pretest scores were standardized to allow meaningful comparison. In our initial exploration, the pre-surveys and gender were substantial predictors of logic post-surveys. For sense of belonging, pre-surveys were the only substantial predictor of post-surveys. A one standard deviation increase in the pre-survey score predicts a 3.57-point increase for the logic post-survey and a 5.34-point increase for the sense of belonging post-survey. Women are predicted to score 1.87 points higher on the logic post-survey.

Table 1. Regression Model Predicting Logic and Sense of Belonging Post-Test Scores.

Logic			Sense of Belonging		
	Estimate (SE)	*p*-value		Estimate (SE)	*p*-value
(Intercept)	21.44 (0.83)	<.001***	(Intercept)	55.30 (1.59)	<.001***
LogicPre	3.57 (0.44)	<.001***	SBPre	5.34 (0.87)	<.001***
Treatment	1.17 (0.88)	.19	Treatment	-0.38 (1.77)	.83
Gender	-1.87 (0.92)	.045*	Gender	0.83 (1.84)	.65

Note. * $p < .5$, ** $p < .01$, *** $p < .001$. Adjusted $R^2 = 0.41, 0.29$ respectively.

By the time of the poster, we will have additional student background data and more data to allow for better addressing moderators. For example, we hypothesize that the RAMP classes may have an impact that is moderated by pretest (e.g., those with lower sense of belonging may see more substantial gains.)

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Degrees of Access: The Role of Family Education in Knowledge of Applying to Graduate School

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Keywords: First Generation Students, Graduate School Applications, Social Capital

An important factor in attaining a graduate degree in mathematics is an applicant's familiarity with the graduate school application process—a form of knowledge that is not always formally taught but instead passed through familial networks. Research shows that first-generation college students frequently encounter unique challenges in navigating higher education, including limited access to guidance, financial barriers, and unfamiliarity with academic systems (Cragg, 2009; Davis, 2010; Engle & Tinto, 2008; Holley & Gardener, 2012; Pascarella et al., 2004). We apply the theory of social capital to explore how family members influence first-generation students' understanding of the graduate school application process (Coleman, 1988). This study surveys 519 undergraduate mathematics students across 181 institutions to examine how the highest level of education attained by an immediate family member impacts familiarity with the graduate application process. A Chi-squared test found that participants with family members who have a bachelor's degree or higher were more likely to report familiarity with the application fee waivers and that doctoral programs do not always require a master's degree. This finding reinforces the theoretical framework of social capital, which holds that educational knowledge is often passed down informally through familial and relational networks. This allows some applicants to navigate the admissions process with greater confidence and insight, while others are left to figure it out alone. These findings highlight the need for early advising on the norms of graduate school in mathematics, transparent funding guidance, and mentorship programs to support first-generation students.

Table 1. Chi-squared test results for selected items for the question, "Please rate your familiarity with the following concepts about the graduate school application process."

Item	Education level	Never heard this before	May have heard this before	Definitely heard this before	Values	*p*	*w*
Some programs offer ways to reduce or eliminate application fees	First-Gen Bachelors Graduate	24 43 36	43 49 72	36 53 133	15.631	.004	0.17
Some doctoral programs do not require applicants to get a master's degree first	First-Gen Bachelors Graduate	37 28 38	28 27 50	38 67 160	24.382	<.001	0.22

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Learning to Learn: Enhancing Student Success in General Education Math Courses Through Metacognition, Social Learning, and Resource Awareness

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Keywords: Metacognition, Resource Awareness, Social Learning, Learning Strategies

Students entering college face an educational landscape that is vastly different from their high school experience. Many first-year college students encounter challenging mathematics courses that demand new forms of self-regulation and require adaptation of existing and developing new productive learning strategies. To aid students with these challenges in mathematics classes, we have integrated three dimensions of learning: metacognition, resource awareness, and social learning. Research shows that students who engage in metacognition and self-regulated learning (e.g., Zimmerman, 2002)—systematically analyzing and refining their learning approaches—perform better academically. However, learning is not an isolated process, so social learning plays a crucial role in shaping students' understanding. These dimensions encompass three key ideas of one's own learning: helping students learn about different learning strategies and discover their own learning strategies in the process.

The objective of this project is twofold. First, we aimed to design adaptable instructional materials that provide a systematic approach to integrating three mutually reinforcing dimensions of learning: (1) developing metacognitive skills, (2) increasing awareness and use of university learning resources, and (3) fostering social learning. Examples of such activities include weekly reflective journals, a scavenger hunt for locating campus learning resources, and a variety of out-of-class collaborative assignments. Second, we aimed to assess the effectiveness of these instructional materials, in particular, in terms of students' abilities to develop and articulate productive learning strategies. As such, our central research questions are: 1) What is the effectiveness of the proposed activities in developing metacognitive skills, fostering social learning, and increasing awareness of resources? 2) What learning strategies do students develop as a result of their experiences with the proposed activities?

To answer these research questions, we implemented the designed instructional materials in one section of general education Calculus 1 taught by the third author. Using a mixed-method approach (Johnson & Onwuegbuzie, 2004), we collected data through surveys and semi-structured clinical interviews (Clement, 2000). The survey items were adapted from existing inventories (e.g., Brown et al., 1996; Schraw & Dennison, 1994) and supplemented with our own items targeting various aspects of social learning. Clinical interviews provided richer, open-ended insights into students' perceptions of the activities and learning strategies they developed.

The data analysis is ongoing. During the poster presentation, we will share our instructional materials, preliminary findings on their effectiveness, and students' professed learning strategies.

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Undergraduate Students' Bijective Reasoning about Fibonacci Numbers

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Keywords: Bijective Reasoning, Combinatorial Reasoning, Combinatorics Education

Bijection is an important topic in undergraduate mathematics. Although researchers have studied whether students use bijections to compare infinite sets (Hamza & O'Shea, 2011; Tirosh & Tsamir, 1996; Tsamir & Tirosh, 1992; Tsamir, 1999) and how students understand group isomorphisms (Larsen, 2009; 2013; Leron et al., 1995; Melhuish et al., 2020; Rupnow, 2017; Weber & Alcock, 2004), there is little direct research on how students understand and use bijections. One content area where researchers have looked explicitly at bijections is combinatorics (Lockwood & Ellis, 2022; Mamona-Downs & Downs, 2004; Muter & Uptegrove, 2011). These studies have shown mixed results, suggesting that although students are capable of constructing bijections, it is a challenging topic. More research is needed to understand how students think about and reason with bijections. The research question guiding the current study is: *How do undergraduate students' ways of understanding and thinking about bijections support their combinatorial reasoning and vice versa?*

To investigate this question, I conducted a small-group teaching experiment (Steffe & Thompson, 2000) with six undergraduate engineering majors. I collected 16 hours of data across 12 teaching episodes (each episode was 60-90 minutes). Each episode was video- and audio-recorded, and each student had an iPad which captured screen recordings of their written work. The teaching experiment was broken into three phases: 1) Developing students' combinatorial reasoning, 2) Developing students' bijective reasoning, and 3) Refining students' bijective reasoning. For this poster, I will discuss data from four teaching episodes conducted during phase 2 where students were given a list of written descriptions of sets counted by the Fibonacci numbers and asked to construct bijections between the different sets. Prior to engaging in this series of tasks, students constructed a definition of bijection as a whole class and at the end of these teaching episodes, I asked the students to each write a personal description/definition of what a bijection is. I am currently analyzing this data set to understand how students think about bijections and how they drew on their combinatorial thinking to reason about bijections. To frame my analysis, I draw on Lockwood's (2013) model of student's combinatorial thinking and Harel's (2008) notions of *ways of thinking* and *ways of understanding* to capture students' thinking about bijection

I am currently in the early stages of analyzing this data set to identify students' ways of understanding and thinking about bijections and how these connect to their combinatorial thinking. By the time of the conference, I expect to share emerging findings that illustrate how students reason bijectively when working with Fibonacci-related combinatorial tasks. These findings will contribute to understanding how undergraduate students construct and reason with bijections and can inform instructional approaches for supporting students' bijective reasoning.

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Analysis of the Conditions and Constraints Shaping a Modelling-Based Course of Life Sciences

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Keywords: mathematical modelling, anthropological theory of didactic, didactic transposition

Teaching mathematics to students in non-mathematics undergraduate programs remains a persistent challenge because of difficulties in engaging students with genuine problems and models from the disciplines they study (Barquero et al., 2013). Life Sciences is one of those disciplines where the role of mathematics in relation to genuine problems is often questioned (Cozzens & Roberts, 2020, p. 87; Garfinkel et al., 2022) despite its increasing relevance to biology, physiology, and other disciplines (Cozzens & Roberts, 2020). The sense of *applicationism* (Barquero et al., 2013) that runs through mathematics courses for life science majors is amplified by the perception that mathematics is not a tool for understanding biological phenomena. This situation calls for the creation of a course that can engage students from life sciences more meaningfully with mathematics, where mathematics takes on a role as an integrated discipline (Bosch et al., 2020).

One of the approaches to creating such a course was to focus on the biological themes that "resonate deeply with life science students" through "a mathematical framework for making simulating and analyzing dynamical models" (Garfinkel et al., 2022, p. 42). With a greater focus on modeling phenomena, this course stressed concepts like feedback loops, dynamical systems, and bifurcations through differential equations. It also included computer programming, defining derivative as a linear approximation to a point, and analyzing the stability of equilibrium points of single-variable differential equations (Garfinkel et al., 2017). To enroll in this course, life science students do not need calculus as a prerequisite.

A course is an empirical object. The Anthropological Theory of Didactic (ATD) allows us to question, model, describe, and analyze the didactic transposition of the knowledge of the course (Bosch et al., 2021). Didactic transposition is the process of selection, adaptation, organization, and declaration that shapes the knowledge to be taught within school institutions, including universities. The reasons that shape this knowledge can be located at the levels of codeterminacy (Chevellard, 2002). This brings us to our research question: *What conditions and constraints for didactic transposition contribute to shaping the modelling-based course? At what levels of the scale can these conditions and constraints be identified?*

To address this question, we present the genesis and shape of this modelling-based course for life sciences developed by a quantitative biologist at a large public university in the USA. Our account is informed by the course developer's retrospective interpretations of the process of production of the course. We use praxeological analysis to generate an in-depth account of mathematics in the modelling-based course. Comprising a practical (praxis) block and a theoretical (theory) block, praxeology for teaching (didactic praxeology) of the modelling-based course enables us to account and analyze the practical and theoretical nuances of the course. The reasons for the conditions and constraints that affect the production of the course are obtained from the levels of codeterminacy, providing empirical evidence for the dependence of the course on specific institutional factors (Bosch et al., 2020).

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Examining Graduate TAs' Mathematical Knowledge for Teaching in a Teaching Seminar Context

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Keywords: Graduate Mathematics Teaching Assistants, Teaching Assistant Preparation, Mathematical Knowledge for Teaching

Graduate mathematics teaching assistants (GMTAs) carry a substantial portion of the responsibility for teaching undergraduates (Speer, Gutmann, & Murphy, 2005), yet many enter their graduate programs with little to no prior teaching experience (Speer et al., 2005). With limited guidance, GMTAs are expected to help undergraduates bridge the gap between high school and college-level mathematics. Although GMTAs often possess extensive and nuanced content knowledge (Sarkar et al., 2024), content knowledge alone is not sufficient for effective teaching (Grossman, 1990).

Researchers in K-12 mathematics have long examined ways to conceptualize what knowledge is required besides content expertise. One of the most influential frameworks is mathematical knowledge for teaching (MKT), which is defined as the specialized understanding of mathematics required to support students' learning (Ball et al., 2008). MKT is strongly linked to instructional quality and student outcomes (Ball et al., 2008; Hill et al., 2005). More recently, studies have sought to expand MKT definitions to secondary and post-secondary teaching contexts, highlighting that both student populations and institutional contexts influence how MKT is defined and enacted in practice (Speer et al. 2015; Ko et al., 2021).

Building on this research, our study aims to answer the overarching research question: *How does participation in a graduate mathematics teaching seminar shape TAs' Mathematical Knowledge for Teaching?* We conducted a mixed-methods case study of a semester-long teaching seminar course for a group of thirteen beginning GMTAs at a large, public, research university in the Southern United States. Our data collection methods included observations of the seminar, semi-structured interviews with the instructor and GMTAs, and document analysis of course materials, which were triangulated to ensure reliability (Stake, 1995). Additionally, we administered a condensed and altered version of the Mathematical Knowledge for Teaching College Algebra Community College Version (MKT-CCA) instrument (Lim, Duranczyk, & Akoto, 2024) to examine GMTAs' mathematical knowledge for teaching in practice. As many graduate students at this university will eventually serve as instructors of record for college algebra, we administered this instrument at the beginning and end of the seminar to measure growth and compare quantitative findings with TAs' interviews, class interactions, and teaching.

Early evidence suggests that GMTAs enter their programs with highly variable MKT, but structured seminars can enhance this knowledge. A particularly influential mechanism for this development is facilitating increasingly legitimate peripheral participation.

The goal of our poster presentation is to engage RUME conference participants in discussing how we can support GMTAs in their development of MKT. Building on this work, we will pursue a design-based research project to iteratively refine the seminar curriculum and further investigate how structured support shapes GMTAs' growth in MKT.

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PRIME-ing Teaching Assistants for Inclusive and Effective Instruction in Gateway Math Courses

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Keywords: Teaching Assistants, Student Belonging, Mindset, Professional Development

Preparing Responsive and Inclusive Instruction for Meaningful Mathematical Experiences (PRIME) builds on research demonstrating the critical role of instructor beliefs, expectations, and pedagogy in shaping student outcomes in mathematics (Boaler, 2015). In this novel, in-service training and professional development community, the role of the graduate teaching assistant (GTA) is leveraged to implement evidenced-based practices for math learning in gateway college algebra, precalculus and calculus courses with historically high DFW (drop, fail, withdraw) rates.

Recent work has identified the critical and impactful role of GTAs in freshman and sophomore-level math courses (Bressoud et al., 2015; Pilgrim et al., 2025; Speer et al., 2005). We posit that GTAs are uniquely positioned to impact student belonging and mindset, two conditions critical for math learning (Fitz, 2025). Furthermore, GTAs who lead discussion sections in large-enrollment math courses self-report as being under-supported. Thus, this pilot program was developed to support and prepare GTAs to enact responsive pedagogy grounded in academic mindsets and growth-oriented instruction.

PRIME GTAs were supported with a $1500 stipend and a semester-long series of working sessions intended to build a community of learning and support. The primary learning objectives were (1) identify a capacity to learn in every student (2) foster meaningful mathematical discourse and (3) recognize and develop the elements of an effective math discussion section. Curricular components were informed by Farrington's work (2012) which identifies academic mindsets—including students' sense of belonging, growth orientations, and students' perceived value of their work—as foundational to deeper learning.

Survey data from GTAs and undergraduates, short participant reflections, and end of course grades were collected to investigate two primary research questions:

1. How does participation in the PRIME program impact GTA instructional practices in gateway math courses?
2. What effect does GTA participation in PRIME have on undergraduate students' sense of belonging, engagement, and learning in mathematics?

We consider the efficacy of PRIME in terms of the GTAs' behaviors & self-reflections and their undergraduate students' reports of belonging, learning and persistence. However, this work also examines the praxis of instructional reform within the context of our institution. The value of collaborative efforts between teaching and learning centers and disciplinary departments to implement sustainable change in teaching practices is well-documented (Condon et al., 2016; Finelli et al., 2014). Typically departmental change is initiated internally, but PRIME was designed and proposed by staff from the university's teaching and learning center. Still, over the course of this project, positive, collaborative relationships with mathematics faculty and department leadership have grown in ways that may pave the way to further pedagogical and instructional change with the math department, and this endeavor may later serve as an institutional case study for implementing inclusive and effective instruction across other STEM departments and gateway courses.

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Mathematical Knowledge for Teaching in Undergraduate Contexts: A Literature Review

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Keywords: Mathematical knowledge for teaching, literature review, undergraduate mathematics

Understanding what teachers must know to teach effectively has consistently been a central focus in educational research. The Mathematical Knowledge for Teaching (MKT) framework is one of the foundational models to explore professional knowledge of mathematics teachers (Ball et al., 2008). The MKT framework comprises two groups—subject matter knowledge and pedagogical content—each containing three domains of knowledge, for a total of six knowledge domains. Subject matter knowledge includes common content knowledge, horizon content knowledge, and specialized content knowledge. Pedagogical content knowledge (PCK) includes knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. Originally, the MKT framework was designed and has been extensively applied for elementary education (Hoover et al., 2016). Over time, the framework has been utilized for higher grades. This review asks: What patterns emerge regarding the theoretical frameworks utilized, methods employed, purposes, and participant selection in MKT studies focused on university-level teaching? The review results are: (1) conceptualization of MKT domains, (2) diversity in participants and instructional contexts, (3) methodological innovations. Only the first two of them are briefly discussed below due to space constraints.

Several studies in the MKT literature focus on defining and making sense of the MKT knowledge domains in typical or new undergraduate contexts (Hauk et al., 2014; Speer et al., 2015; Johns & Burks, 2023). For example, Speer et al. (2015) argued that the distinction between CCK and SCK became blurred as we move beyond elementary teaching to higher educational contexts. Many tasks identified as SCK (e.g., evaluating student-generated solutions) for mathematics instructors appear in everyday research practices of mathematicians (e.g., evaluating a proof in a manuscript for publication). Future studies in the undergraduate context need to clarify what SCK means.

This review highlights diverse participants type: from university mathematics faculty (Speer et al, 2013), community college faculty (Hauk et al., 2014; Nabb & Marawska, 2020), dual enrolment in high school and community college (Beckford et al., 2023), and graduate students as tutors (Johns & Burk, 2023). An interesting trend emerged in many studies involving non-university mathematics faculty participants, which focused on relational and socio-cultural aspects (e.g., caring, trust, affective, cultural, practical, and contextual) of teaching, and suggested that these be incorporated into MKT.

Many studies draw heavily on self-reported data such as interviews (Nabb & Murawska, 2020), online working group sessions (Kim & McCrackin, 2024), and surveys (Beckford et al., 2023), which focus on what these participants say they know or do. Two gaps are noted concerning the limited ability to connect MKT and student learning. Firstly, few studies draw on real classroom practice (Johns & Burks, 2023; Speer et al., 2013). Secondly, none of these studies reviewed include how MKT translates into classroom impact or student learning data. Future research needs to include classroom transcripts, student work, and evidence of how students respond to certain MKT domains in instruction.

The poster will present all the patterns with gaps and future steps. I look forward to discussions with conference attendees about more results mentioned in the poster.

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Using AI to Adapt Secondary Mathematics Discourse Frameworks for Graduate Teaching Assistant Development

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Keywords: Discourse, Teaching Assistants, TA Training, AI

Graduate teaching assistants (TAs) play a pivotal role in undergraduate mathematics instruction, often leading recitations that shape students' early university experiences. However, while the literature on TA training highlights the need for pedagogical preparation (Shannon, Twale, & Moore, 1998; Speer, Gutmann, & Murphy, 2005), few resource materials have been developed specifically for this audience. This gap exists, for example, in the area of classroom discourse, and contrasts with the robust set of discourse-based professional learning tools available for secondary mathematics teachers. As a result, TA training often lacks the concrete, discipline-specific examples necessary to help novice instructors develop productive discourse practices in settings where their assistantships are situated (e.g., calculus recitations). This is important because in mathematics, a recitation is not intended to replicate the lecture but to provide a smaller, discussion-based environment where students actively engage with problems and reasoning. Ideally, in this context, TAs facilitate collaboration, questioning, and sense-making rather than delivering additional content. Such active-learning structures have been shown to enhance students' conceptual understanding and engagement in undergraduate mathematics (Freeman et al., 2014; Laursen & Rasmussen, 2019).

Upon learning that students in my first-semester TA-training course were unhappy with being asked to learn about engagement, classroom discourse, and teacher discourse moves using materials for secondary instructors, I used AI (i.e., Chat GPT 5), to generate new transcripts that made use of discourse frameworks (e.g., Freeburn and Arbaugh's (2017) communication moves for productive struggle and Herbel-Eisenmann et al.'s (2017) Teacher Discourse Moves. Thus, this project builds on two research-based established frameworks and analysis of classroom roles and participation structure to generate Calculus Transcripts that adapt secondary-level discourse cases for the graduate TA context. The new materials were designed for use in a semester-long TA training course at a research-intensive university. The cases were created to simulate authentic challenges TAs face—such as encouraging student reasoning and sustaining mathematical discussions—while maintaining fidelity to the original pedagogical constructs.

The poster shares the adaptation process, examples of the resulting Calculus Transcripts, and reflections from pilot implementation. Preliminary observations suggest that ChatGPT did a reasonably good job of generating these transcripts with the following exceptions - the AI: (a) consistently had students providing correct, polished answers with precise vocabulary, (b) had the instructor continuously praise student contributions, and (c) continuously went into telling mode after the vignette to point out all of the things in the case that I wanted the TAs/students to notice about the discourse. Ultimately, this project contributes to addressing the need, identified by Speer et al. (2005), for providing a mechanism to generate research-based pedagogical tools and resources that explicitly bridge the K–12 and postsecondary mathematics education communities. By expanding discourse-based case study resources to the calculus classroom, this initiative supports a more coherent trajectory of teacher learning across educational levels and offers a model for developing reflective, discussion-oriented TA training in mathematics.

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How Cultural and Familial Ideologies Shape MENA Students' Mathematical Identities

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Keywords: mathematical identity, cultural ideologies, MENA students

Research in mathematics education has increasingly recognized that learning and identity are intertwined, and what mathematics means to people is socially produced (e.g., Nasir, 2002). These meanings differ across settings and are mediated by race, gender, and culture (Walker, 2012). Students' racialized experiences and sociohistorical contexts shape how they interpret mathematics (Martin, 2000). Yet, little research has examined how students' cultural ideologies, belief systems that help people make sense of the world (Shah, 2017), influence their perceptions of mathematics. This study examines how Middle Eastern/North African (MENA) students, a group largely absent from education research (Modir & Kia-Keating, 2018), negotiate their mathematical identities and perceptions of mathematics with respect to their cultural and familial ideologies. The question guiding this study is: How do MENA students' cultural and familial ideologies shape their mathematical identities and perceptions of mathematics?

This study draws on research on mathematical identity to explore how MENA students' ethnic and cultural backgrounds influence how they view and value mathematics. I define mathematical identity as how students see themselves and are seen by others as math doers, developed through social practice and recognition within communities (Martin, 2000; Cobb et al., 2009; Hall et al., 2025).

Participants were eight undergraduate students who self-identified as MENA and had completed at least Calculus I or equivalent. Semi-structured interviews explored (1) participants' feelings about mathematics, and (2) the influence of cultural and familial ideologies on their mathematics trajectories. Data analysis followed an inductive and iterative coding process informed by thematic analysis (Corbin & Strauss, 2014).

Findings suggest that many participants experienced familial pressures to perform well in mathematics, pursue STEM degrees, or follow similar paths as relatives with mathematics backgrounds. Most expressed that these pressures stemmed from MENA cultural ideologies which conflate mathematics and STEM achievement to success and socioeconomic stability. They felt that because of their culture they "have the potential to be engineers and doctors" and that "the expectation to succeed is linked to mathematics." For many participants, success was often equated with prestigious fields, many of which required mathematics.

Participants' narratives reflected the complex negotiation of self, family, and culture with respect to mathematics identity, revealing how recognition and expectation shaped how they saw themselves in relation to mathematics. Sentiments shifted from obligations tied to family expectations and cultural ideologies, to appreciation when they "found meaning and connections" to their majors or when family members framed it as beautiful. While some family members upheld and enforced cultural ideologies, others encouraged participants to make sense of mathematics in a positive light. This communal support fostered more positive feelings and meaning-making in the mathematics that they encountered. Although most participants described familial pressure as motivating, a few felt that these expectations were limiting. These findings align with Martin's (2000) argument that mathematical identities are shaped by community and broader cultural meaning, underscoring the need to examine how identity development in mathematics is influenced by social and cultural forces that may not be immediately visible.

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Modules for Developing Statistics Graduate Teaching Assistants' Capacity for Teaching

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Keywords: graduate teaching assistant, introductory statistics, resources for teacher education

Although statistics graduate teaching assistants (GTAs) face the difficult task of teaching undergraduate students with a broad range of backgrounds, they often do so with little to no preparation in statistics teaching. There are resources developed for mathematics GTAs (e.g., http://collegemathvideocases.org); however, there is a lack of resources that can support *statistics* GTAs in addressing the disciplinary need of teaching and learning statistics. Based on a design-based approach, we developed four modules for **L**earning to teach **E**quitably with **A**uthentic **D**ata (LEAD modules; https:/sites.ced.ncsu.edu/lead-modules/), building on existing frameworks of teaching mathematics and recommendations for teaching statistics. We introduce LEAD modules and examine the research question: What problems of practice emerged when GTAs participated in LEAD modules at two GTA communities?

The first module, *Teaching Statistical Thinking*, introduces the distinctive features of statistical thinking compared to those of mathematical thinking: context, variability, and inference (GAISE Committee, 2016; Smucker et al., 2024; Wild & Pfannkuck, 1999). The second module, *Launch, Explore, Discuss*, provides a framework for GTAs to structure a lesson centered on exploring statistical concepts (Lappan et al., 2009). This module responds to the persistent pattern of narrowly focusing on procedural aspects of doing statistics (Justice et al., 2017; Noll, 2011). The third module, *Teacher Discourse Moves*, offers a set of discourse moves for GTAs to facilitate classroom discussions during the Discuss phase of the lesson (Herbel-Eisenmann et al., 2017). Lastly, the fourth module, *Promoting Participation*, reflects the challenge of eliciting student participation commonly experienced by GTAs. The module asks GTAs to examine initial patterns of student participation and consider opportunities to broaden participation across the classroom (e.g., Byun, 2024).

Our preliminary findings show multiple problems of practice (hereafter, problems) emerged as GTAs worked toward implementing teaching approaches included in the modules. Informed by Cohen et al. (2003), we broadly categorized the problems by student-, teacher-, content-, and environment-related problems. Student-related problems included the perceived lack of students' prior preparation and participation, and their negative disposition toward the subject. Teacher-related problems highlighted difficulties in adopting new teaching strategies and uncertainty about how to respond to a wide range of student contributions. There were fewer problems related to content and environment, but implementing new strategies within the time constraint was salient.

Without adequate teacher learning opportunities, GTAs are likely to fall back on more familiar teaching approaches that they have experienced as mathematics learners, which often entail attending lectures, taking notes, and doing homework problems (Findley, 2022; Justice et al., 2017). The identified problems of practice offer valuable insights for further iterations of LEAD modules so that modules can better fill the void of resources when course coordinators and instructional mentors work with GTAs to reshape undergraduate teaching.

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Teaching the Teachers: What AI Student Agents Reveal About Preservice Teachers' Noticing in Statistics

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Keywords: Pre-service teachers, Artificial Intelligence, Statistics, Data Science

The growing importance of data science and AI in the United States has sparked a need to prepare students for these emerging fields (National Council of Teachers of Mathematics, 2024). One natural place is working with pre-service teachers to develop core ideas in statistics and data science. In particular, the ability of a teacher to notice and interpret student work is essential. To address this need, we created a series of instructional activities, which were designed to help pre-service teachers (undergraduate and graduate students in mathematics teacher education) understand and notice statistical thinking of K-12 students.

In this activity, pre-service teachers interact with an AI student. An AI student is a custom agent powered by generative AI that is given a profile and told to act as a student in the classroom. The AI students are given mathematical conceptions related to a task about mean, median, and mode as developed by research on real students. They are also given personalities, dispositions to mathematics, and preferred forms of communication. Our first trial of virtual students (Authors, under review) resulted in pre-service teachers being able to attend, interpret, and respond at high levels. The AI students also produced a wide variety of student thinking for the preservice teachers to analyze due to their nature of being powered by generative AI. We extend this work with this poster by designing a new set of AI students that will be used in a classroom of preservice teachers covering the topic of centers of a distribution. The chosen task is the screentime task (Virginia Department of Education, 2025) which asks students to explore centers of a distribution of a data set to remain under a screen time limit imposed by parents.

Data was collected includes video recordings of the students while they interacted with the AI students (Figure 1) over Zoom while they shared their screen, their chatlog, and a post-activity reflection. For this poster, we present our preliminary analysis of the AI student's reasoning through descriptive coding and thematic analysis (Saldana, 2013). In order to analyze the preservice teacher's noticing, we combined the FAIR framework (Louie et al., 2021) with the five types of representations (NCTM, 2014) We discuss some analysis of the preservice teacher's noticing with regards to the AI students including what mathematics they attend to and how they respond to the AI students. This study has implications for the pedagogical design of undergraduate mathematics education courses by providing tools and activities to make training practices low-stakes, sustainable, and scalable.

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Mathematics Identity and Belonging in an Undergraduate General Education Terminal Mathematics Course

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Keywords: Inclusive course design, inclusive instruction, mathematical identity, sense of belonging

This study directly follows from Homp et al., where inclusive instructional practices, like Open Educational Resources and biographies from mathematicians currently doing research from historically minoritized backgrounds, were implemented in a general education terminal undergraduate mathematics course from a large, predominantly White, land-grant based, public university in the midwestern United States. Homp et al. found a statistically significant increase in students' sense of belonging, illustrated from pre- and post- course survey data. Gutiérrez (2012), developed a framework for conceptualizing equity in mathematics through the axis of identity and power, and the axis of access and achievement. Building off this research, Riley (2022) utilized the Gutiérrez framework to perform thematic analysis on *Testimonios* (2022). In this study we thematically analyze student artifacts from one out of five sections of the course in Homp et al.'s research to understand why the feeling of belonging improved within the students across the duration of the semester.

This poster explores artifacts collected from students after being shown biographies of current, historically minoritized mathematicians three times throughout the semester, analyzed using a codebook derived from Gutiérrez's framework. The themes presented in the two axes, power, identity, access, and achievement were coded both as they related to the chosen mathematician and to the student. The frequencies of each code within each of the three artifacts were compared. Students at the beginning of the semester primarily focused on the achievement of the mathematician in the chosen biography, with little to no mention of the mathematician's identity, access, or power. As the semester went on, students chose to write about more than just the mathematician's success– focusing on the identity of the mathematician, the power they have to make societal change, and the access they have to education. Students also incorporated more aspects of their own identity within the later artifacts.

Many students indicated they had little to no knowledge on the various topics mathematics researchers study. Perhaps by introducing the students to mathematicians that share aspects of their own identities, students are able to relate more to complex topics like disease spread and gerrymandering, which many do not associate with mathematics. This analysis appears to show that students were able to form a connection to mathematics beyond the presented course material– allowing them the ability to feel they could belong in the mathematical community.

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Through the RME Looking Glass: Utilizing Analogies to Bridge the Gap for Beginners

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Keywords: RME, Analogies, Numerical analysis, Embodied Cognition

According to Gentner (1983), an analogy maps a source domain (the unfamiliar) to a target domain (the familiar). Thus, the objective of an analogy is to capture the essence of the concept and demonstrate that idea in a familiar setting. In our poster, we will introduce an example of a carefully constructed analogy and how it can be a useful tool for beginners that are still gaining intuition in unfamiliar concepts. Consequently, the audience for the analogy assumes only comfortability in college algebra skills. The goal of our research is to demonstrate an example of the resourcefulness of analogies as a teaching tool and the notion of translating complex mathematical concepts into unconventional contexts.

The teaching topic in question is introducing a specific numerical method for solving linear variable-coefficient ordinary differential equations (ODEs) in a two-point boundary value problem (target) by utilizing an analogy involving a cube of varying density restricted to a ruler (source). Variable-coefficient ODEs are equations that are defined on an infinite domain and thus often must be restricted to a set interval to become manageable when solving. For a boundary value problem, the ODE is simply paired with the given solution at its boundary.

Our analogy relates an ODE and cube in the following ways: their restriction to a set interval (two-point boundary and ruler), dependence on varying characteristics (variable coefficients and varying density), solutions (function $y(x)$ and total weight of cube), boundary conditions (given say $y(1)$ and $y(4)$ and weight of cube at 1 in. and 4 in.), and method for solving (steps of numerical method explained in target and source domain). Along with the poster itself, we include manipulatives that represent the physical components in the source domain: a cube, ruler, tape measure, thumbtacks, paper slices acting as slices from the cube, etc.

Through our analogy, we introduce an approachable example that can engage students and, ideally, create a fun and memorable learning experience. These qualities are at the core of Realistic Mathematics Education (RME) and research in embodied cognition. RME urges that mathematical instruction should be realistic and rooted in real-world context. Gravemeijer & Doorman (1999) define "realistic" in RME to situations that students can imagine and comprehend and thus are relevant in the student's reality. Our analogy (or "the model") meets students at their level of understanding and thus allows them to reason and gain intuition in multiple unfamiliar concepts (Gravemeijer, 1999). Additionally, our analogy provides visual and interactive components that encourage embodied cognition. When students can engage their bodies in physical actions, gestures, object manipulation, and so on, this allows them to form a deeper understanding and have a more memorable learning experience (Antonides & Battista, 2022; Lakoff & Núñez, 2000; Nathan, 2021).

Overall, RME and embodied cognition education research tend to stray from the "traditional classroom" that involves no student engagement and only instructor-based lectures. Thus, with the support of RME and embodied learning, the addition of analogies to the classroom can hopefully open the doors for math to be more fun, memorable, and engaging—not only for children, but also for students in higher education. Feedback is welcomed on the proposed use of analogies as a teaching tool, the notions of unconventional math translations, and on the analogy itself.

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Factors That May Guide the Design, Execution and Research of Professional Development

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Keywords: Professional Development, Factor Analysis, Descriptive Statistics

There is considerable research on effective professional development (PD) in both undergraduate and K-12 educational settings, but less on how the characteristics of PD can influence teacher uptake and retention of PD resources and lessons (Zhang et al., 2024, Hill et al., 2013, Henderson et al., 2011). The TADD 2.0 (NSF DRL No. 2405432) project focuses its research on teachers' perceptions of critical characteristics of PD impact on sustainably affecting teacher instruction and student learning. Using the preliminary findings and analysis of the survey of 157 teachers who participated in 1 of 6 chosen large-scale NSF-funded research projects within the last 1 to 5 years can shed light on the characteristics that could also be emphasized in STEM undergraduate PD.

The initial survey was designed to utilize Likert-scale items and follow-up open-ended questions, allowing participants to elaborate on their numeric responses based on six characteristics of PD. Upon factor analysis and Cronbach's alpha tests, researchers used a five-factor model to define the characteristics. These new categories included administrative support, resources and materials, community, classroom practices, and facilitation. The results of the factor analysis reveal that teachers' perceptions of what is thought to be well-defined in the research community may not be as explicit as thought. The same could be said about feedback in undergraduate PD, as the variety of training and experience can vary greatly in a department (Postareff, 2008).

Upon redefining and analyzing the categories, researchers determined the average scores of participants across each factor and conducted descriptive statistics, along with ANOVA tests, which showed that the factors were statistically significant across all PDs. Results showed that all PDs were of high quality, with no median value falling below 5. Fortunately, administrative support (median = 8.5-10) and facilitation (median = 9.2 - 10) were rated very highly. Sustained administrative support and facilitation are not always common in undergraduate PD and often inhibit scaling and uptake from professors (Chasteen, 2015, Henderson et al., 2011). Factors with lower medians were community (median = 7.5 - 10), resources and materials (median = 5.8-9.8), along with classroom practices (median = 6.5-9.2). Classroom practices surrounding pedagogy and content ratings being lower aligns with findings in K-12 spaces as it is more difficult to implement and takes longer to see sustained changes (Kennedy, 2016, Koellner & Jacobs, 2015). Pedagogical change is understandably challenging to support in undergraduate spaces, particularly due to culture and community (Reinholz & Apkarian, 2018, Chasteen et al., 2015, Postareff, 2008).

Each PD served different communities and had varying goals towards teacher pedagogical change, but the factors appear to show important attributes of PD. While K-12 schools have different contexts, nuances, and challenges than undergraduate teaching spaces do, these survey results may provide guidance when designing, implementing, and conducting research in undergraduate communities.

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Impact of Reflective Practices and Videos on Graduate Teaching Assistants' Pedagogy

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Keywords: professional development, novice instructors, TAs, decentering, reflecting

The development of instructional practices, skills, and pedagogical content knowledge is a key aspect of training Graduate Teaching Assistants (TAs) (Dehsler et al., 2015). Current work like the CoMinDS resource (2021), shared at RUME conferences aim to understand and support TAs as they develop these professional teaching practices. This study was conducted in a TA seminar for an inquiry based precalculus course at a large research university to investigate the impact of guided reflection, alongside video recording, on graduate teachers' pedagogy and how it supports decentering (Teuscher et al., 2016), a process advocated for and supported in previous literature (Hauk & Speer, 2023).

This semester, we introduced a self-reflection assignment following observation and recording by one of the graduate co-instructors. This included rating their practices twice: once immediately following teaching, and again while watching a recording of themselves and discussing their findings with one of the co-instructors. Comparing the ratings students gave themselves (on a scale of 1-3) before and after, showed no significant change in overall rating. However, most of the students realized they weren't initially using student centered practices. One TA noted:

> I really thought… I was engaging students… After I watched myself, I realized that I ask questions but don't give them enough time to think through it… I need to give them enough time to think about it. But, then again, when I do that I'm just pressed for time.

The narrative of 'Too much material and not enough time' is a common barrier for instructors in using student-centered teaching practices (Petersen et. al., 2010). During this teacher's second observation, it was clear they took wait time into account as they intentionally engaged with table groups and waited for the group to come to a conclusion before moving on. In the second interview when asked how they felt before and after watching their lesson:

> I felt way better than I did before. There was more engagement and I used a new approach…I gave them some time to think about the question and then if I wasn't getting my answer I tried to lead the students to answer the question or rephrased it.

This TA recognized the change in their practice and even increased their self-efficacy, which was an unintended consequence, but not unheard of (Chiu et al., 2019). From the observation, the co-instructor noted that the TA also repeated student explanations and asked students to explain what expressions represented, two decentering practices used on the reflection form.

Using reflection forms that intentionally prioritize decentering techniques, and problem-solving strategies can help graduate teacher assistants come to conclusions about their pedagogical approaches. Using recordings to reflect helps them gain specific insights into how they teach and ways they can individually improve their teaching. This poster will provide additional quotes from TAs to support the current literature, which highlights video and guided reflection as tools to improve instruction and move towards student-centered teaching practices.

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"Here, students write math differently": An Exploration of Indian and USA Students' Non-proof Mathematics Writing

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Keywords: Mathematical Writing, Calculus, International Comparison

As undergraduate instructors both in the USA and India, we observed that Algebra and Calculus students write mathematics differently across the contexts. We suspected that the differences related both to presentation and the student's purpose of mathematics writing. However, exploration into the literature suggested that RUME community has attended primarily to proof-based writing in mathematics (Guce, 2017; Nachowitz, 2019; Pugalee, 2010; Teledahl et al., 2024). This leaves open the question of how undergraduate students write in courses like Calculus where one needs to provide reasons, justifications and perform computations.

Students' written mathematical communication is a critical competence emphasized worldwide, serving as an essential part of mathematics. Defining and measuring quality in mathematical writing remains challenging (Guce, 2017; Kline & Ishii, 2008; Morgan, 2001; Pugalee, 2001; Teledahl et al, 2024). Students' mathematical writing in assessments revealed that there is a fundamental disconnect between teachers' expectations for evidence of conceptual reasoning and students' reliance on imitative or algorithmic solutions driven by perceived assessment demands (Boesen et al., 2010; Guce, 2017; Lithner, 2008; Miller et al., 2018; Morgan, 2006).

We employ Garofalo and Lester's (1985) framework to capture the metacognitive complexity by identifying the orientation, organization, execution, and verification behavior in students' written descriptions of their answer-writing and problem-solving process. We further analyze its relationship with the familiarity of the task (Boesen et al., 2010). In this study, we explore: *What forms of metacognitive behavior do students engage in while writing mathematics in an exam setting? What impact does familiarity of the task have on students' metacognitive behavior?*

Our poster will present the theoretical framework, methodology and the results obtained from to pilot task-based interviews with students. The pilot is intended to generate responses on four types of tasks which are classified based on their relatedness to textbook/reference material.

Results are expected to provide us with a viable range of cognitive monitoring acts (Garofalo and Lester, 1985) that Indian and American students engage in while writing solutions to obtain maximum points on familiar and unfamiliar tasks (Boesen et al., 2010). Not only do we expect this exploration to shed some light on the factors that caused us to hypothesize but we also expect that it would provide some base for further exploration of institutional factors affecting students' mathematics writing – both as a product and a process.

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A Model of Epistemological Justification for Teaching and Learning Linear Algebra Proofs

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Keywords: Linear algebra education, proofs, epistemological justification

Research on proofs has long been a topic of interest for the mathematics education community (see Melhuish et al., 2022). Despite this, linear algebra has not received substantial research attention in proof and topics in second courses (Stewart et al., 2019, 2025). Some of the conversations between studies in proof based linear algebra have been on how students should think about the proofs. For example, Stewart and Thomas (2019) observed that a group of students preferred to reason with familiar objects, and some desired to use symbolic or embodied examples to aid their thinking. Meanwhile, Harel (2019) argued that building too strong a geometric reasoning for students may cause difficulties in proving in abstract or unfamiliar cases. Regardless of the approach, there is evidence that a primary goal of instructors is for students to have a strong ability to prove (e.g., Rensaa et al., 2020; Johnson et al., 2025). Furthermore, some studies offer insight into how students perceive proofs in linear algebra. A common theme that has emerged is that students struggle with the interplay between concepts in linear algebra proofs (e.g., Stewart & Tran, 2022; Harel, 2024b; Stewart & Thomas, 2019). There is developing research into combating these challenges for students and fostering a strong proving ability. For example, Meyer et al (2024), examined an instructor's textbook where he promoted student-created proof and exploration. Altogether, more research on examining the instructor's decisions on promoting critical thinking about proofs in linear algebra is needed.

Capturing this idea of thinking critically about proving is something that needs to be defined. The aim of this study is to build a model explaining how students think about the formation of their knowledge and proofs in linear algebra. Harel (2024a) outlines a theoretical framework for analyzing student understanding in relation to both proofs and broad mathematical knowledge. He introduces the notion of epistemological justification, "a way of thinking that manifests itself through perturbation-resolution cycles revolving around the question why and how was a piece of mathematical knowledge conceived?" (Harel, 2024a, p. 1488). Epistemological justification has specific criteria for its existence, those are the intellectual need condition, the elicitation condition, and the awareness condition. Intellectual need itself comprises five different categories according to Harel, those being: certainty, casualty, computation, communication, and structure. Further, there are two types of observed epistemological justification, discerned and embedded (Harel, 2024a). He poses some open questions to continue building on the framework of epistemological justification. One of particular interest is "What instructional practices promote the habit of seeking epistemological justification among students?" (Harel, 2024a, p. 1499). To that end, we hope to apply Harel's (2024a) theory of epistemological justification to students' thinking about proof and identifying what instructor decisions may have promoted this thinking.

The research questions guiding this study are: What are some instances of students' epistemological justification, and what instructor decisions promote them? This proposed case study aims to bring attention to a gap in linear algebra education research on how instructors interact with students during the proving process. This model will be used in a second course on linear algebra.

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Studying Solidarity Between Dalit Mathematics Faculty and Dalit Students: A Literature Exploration

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Keywords: caste, Dalit teachers, Dalit students, classroom pedagogy, equity

African American students and teachers in the USA and Dalit students and teachers in India share similar sociopolitical struggles in education, despite different historical trajectories (Goodnight, 2017). Dalits are historically positioned as the lowest group of caste communities in the caste hierarchy. They have been subjected to severe forms of segregation by "upper" castes, with Brahmins on top of the hierarchy, controlling educational access. Brahmanical ways of knowledge production have entrenched harmful stereotypes in mathematics (Subramanian, 2025) and higher education, while negating Dalit intellectuality (Kawade, 2019). I am investigating how Dalit mathematics faculty support their students in and out of the classroom, drawing parallels with existing mathematics education research that center Black teachers' contributions in supporting mathematics and affective needs of their students (e.g. Ortiz, 2024). In this poster, I share a literature review describing the state of research on the experiences of Dalit students and teachers in mathematics and higher education to inform my investigation.

My review started with a few publications recommended by Dr. Jayasree Subramanian, a mathematics education scholar working in anti-caste work. I used a snowballing method and general Google scholar search to identify other publications that are focused on the experiences of mathematics Dalit students and faculty in and out of the classroom. I found a limited number of published works in the area. So, I expanded my search to include works in higher education generally, and K-12 mathematics education. In my review, I identified two primary themes that play significant roles in shaping experiences of Dalit individuals in educational spaces: classroom pedagogy and curriculum, and institutional norms and practices.

In K-12 mathematics settings, some work is done to understand the impact of classroom pedagogy and curriculum on Dalit students. Subramanian and Visawanathan (2024) studied how mathematics is inaccessible for Dalit students by studying 'purity' of language in mathematics textbooks, while Sadafule and Berntsen (2017) studied how differential classroom practices negatively impact mathematical learning of Dalit students. In non-mathematics higher education research, Chand (2025) found that dominant caste teachers hypervisibilized and stigmatized Dalit students, segregated them during group work, and excluded Dalit icons from the curriculum. Campus climate pressurizes Dalit students to be discreet, impacting their "internal solidarity" resulting in isolation (Lum, 2019). When this aspect of being discreet is snatched by professors in instances like oral exams, there is hardly any support for Dalit students nor scrutiny for the discriminatory practices on an institutional level (Chand, 2023). At the same time, Dalit professors in universities are subjected to scrutiny by non-Dalit peers and students and are often the last option to collaborate (Ovichegan, 2015). In this literature review, I found that there has been no work done so far on building solidarity amongst Dalit teachers and students in higher education. There is hardly any work that discusses the caste questions in mathematics, and no work that discusses Dalit mathematics teachers' practices in and out of the classrooms that support Dalit students across grades. I aim to bridge this gap by studying how Dalit university mathematics teachers and students build solidarity and support each other in and out of the classroom to counter the brahmanical supremacy in mathematics spaces (Stinson et al., 2024).

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Factors Influencing the Attitudes of Multilingual Learners in Undergraduate Mathematics Classrooms

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Keywords: multilingual learners, mathematics attitudes, undergraduate education, identity

As U.S. undergraduate classrooms grow more linguistically diverse, attending to multilingual learners' experiences is vital for equity. From 2009 and 2019, undergraduate enrollment of Latino students, many of whom are multilingual, increased by 48%, even as total enrollment declined by 5% (NCES, 2021a, 2021b). Attending to this growing linguistic diversity is essential for building equitable mathematics learning environments at the undergraduate level. Prior research suggests that language identity influences mathematics engagement (Rios, 2024), yet few studies examine how multilingual students' identities shape their attitudes specifically within undergraduate mathematics contexts. We define mathematics attitudes as a blend of cognitive (value, confidence, self-concept), affects (enjoyment, anxiety), and behavior (willingness to engage with mathematics).

Our research aims to identify key factors influencing multilingual undergraduate students' attitudes toward mathematics, explore the role of language, culture, and identity in shaping these attitudes, and examine how students interpret and respond to instructional practices in mathematics classrooms, highlighting how inclusive, asset-based approaches can enhance mathematics learning experiences for multilingual learners.

We employed a mixed-methods design integrating qualitative interviews and quantitative ranking activities. Participants were purposefully selected multilingual undergraduates who had completed at least one year of education in the United States, ensuring that their reflections were grounded in undergraduate mathematics experiences. Five Hispanic undergraduates (ages 18–23) enrolled in Psychology, Pre-Nursing, and Spanish programs participated. Semi-structured interviews explored personal mathematics learning histories, language use in mathematics classrooms, and attitudes toward mathematics. Participants were also asked explicitly about instructional practices, such as teacher encouragement, use of language, and classroom climate, that shaped their confidence and engagement in mathematics. This was followed by a ranking activity identifying perceived influence factors. Data were analyzed thematically and supported by descriptive statistics from rankings.

Preliminary findings show personal drive and teacher support as the most influential factors shaping attitudes toward mathematics. Participants associated bilingualism with greater confidence and self-perception of ability when instructional environments acknowledged and valued their linguistic resources. Teacher encouragement, or lack thereof, was more influential than familial or peer perspectives. Students' conceptions of 'math people' shifted from focusing on grades to valuing conceptual understanding. Negative attitudes were often linked to deficit-based assumptions about language or identity.

We plan to expand this study by interviewing additional multilingual undergraduates across disciplines who are currently enrolled in, or have recently completed, undergraduate mathematics courses and conducting a deeper thematic analysis using MAXQDA. The goal is to identify specific strategies, drawn from students' reported experiences, that strengthen self-efficacy, value students' linguistic assets, and foster equitable participation in undergraduate mathematics.

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