Using Intertextuality to Study (Student) Conceptions of Proofiness

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This work provides a theoretical grounding for proof education studies that utilize proof comparison tasks. We leverage the notion of intertextuality, borrowed from genre theories, to discuss mathematics education theory and methods. Intertextuality involves the implicit or explicit comparison of two or more texts with regard to a given genre. We discuss how this comparison can serve explorations of (student) conceptions of the genre of proof and argue that intertextual comparisons lead to novel insights into (student) conceptions of the genre. Specifically, we illustrate how someone's comparison of two (or more) justifications relative to the genre of proof can provide insights into how they view (a) proof as a genre, (b) which attributes of a justification are relevant to that justification's (potential) status as a (good) proof, (c) the relative importance of those attributes, and (d) the reasons for that relative importance.

Keywords: Proofiness, Justification, Intertextuality, Genre

Küchle and Dawkins (2024) argued that proof is a genre; consequently, genre theories might be helpful for mathematics education researchers as lenses for studying proof. They discussed the three major genre theory approaches, focusing on what these approaches have in common in their conceptualization of genre. We focus on one of those common threads, the notion of intertextuality. This construct has two competing definitions (Duff, 2002). The definition Küchle and Dawkins (2024) discussed involves texts referring to other texts. For instance, when the proof of a theorem references another proof/theorem or when a scientific paper references other scientific papers. Instead, we reference the other definition of *intertextuality*. That definition refers to examining genre via studying the (implicit or explicit) comparison of two or more texts. This definition has a rich history in studies of literary genres (Duff, 2002). When applied to proofs, the definition implies that conceptions of proof as a genre are explorable via comparison of two or more justifications with respect to their status in the genre of proof. This definition of intertextuality harnesses two key insights. Firstly, genre is a rhetorical category that expresses a relationship between texts. So, intertextual comparison is a relevant tool for studying it. Secondly, disagreements regarding which texts belong to which genre rarely resolve, so it is commonly more fruitful to further such debates by comparing two or more texts with respect to a genre.

For illustration, we discuss an example from film genre. Consider the 1988 action movie "Die Hard." Since the movie takes place on Christmas day, some have argued that this film belongs to the Christmas Movie genre. That interpretation of genre led to "Die Hard" being voted #7 in USA Today's list of 20 best Christmas movies (Truitt, 2023). However, most participating in the debate about its Christmas movie status would agree that, for instance, the movie "Scrooged" is comparatively more representative of the genre of Christmas Movie. Thus, two people who disagree on the genre status of Die Hard might still find fruitful common ground via comparison. Additionally, those people's comparison of the two films might lead to a discussion of which motifs and themes contribute to the Christmas movie genre, why they are relevant, and their relative importance. Thus, these comparisons can lead to more precise images of individual conceptions of the Christmas movie genre.

Some proof education studies have utilized proof comparisons (Brown, 2018; Mejía Ramos et al., 2021; Vroom & Alzaga Elizondo, 2024). However, these tend to focus on some properties a proof might have, such as explanatory value (Mejía Ramos et al., 2021) or its convincingness (Brown, 2018), rather than juxtaposing justifications with respect to the genre of proof as a whole. This work advocates for expanding this approach by considering the comparison of justifications with respect to the genre of proof. We argue that such comparisons can be particularly revealing. Specifically, intertextual comparison of justifications in relation to the genre of proof can reveal not just what students attend to but also the reasons for their attention. This attention need not be limited to the attributes that delineate the difference between proof and non-proof; it can also include attributes that contribute to the relative status of valid proofs, such as elegance or cleverness. Additionally, in cases where students attend to multiple attributes of a justification, comparison can also reveal the relative importance of the attributes they attend to, and the reasons for that relative importance. The phrase, "intertextual comparison of justifications in relation to perceived ideals and norms of the genre of proof," is quite onerous. So, it is helpful to create a construct to describe it.

Proofiness

Seife (2010) defines proofiness as manipulations of mathematical norms that "make falsehoods look like numerical fact" (p. 18). We borrow this construct's name (though not its definition) from Seife (2010). We instead use *proofiness* to refer to a person's intertextual conception of justifications in relation to perceived ideals and norms of the genre of proof. Our definition is naturalistic since it is consistent with student use of the word "proofy" in mathematics settings. For example, when participants in our studies were presented with multiple justifications simultaneously and asked to what extent they should be considered proofs, they responded with phrases like, "This one feels proofier to me" and "This one is not proofy." In doing so, they commented on proofiness via intertextual comparison (Authors, submitted).

Proofiness is intrinsically comparative. So, we discuss two kinds of intertextual comparisons. The first is to examine a person's conception of proof via their comparison of two (or more) (researcher-generated) justifications in terms of which is proofier. The second is to encourage participants to edit justifications to make them proofier. This editing allows them to (implicitly or explicitly) compare their edited version of a justification to the original and elucidate how those edits contributed to proofiness.

One crucial way the approach we describe differs from some past proof education studies is that we are treating the comparison of two valid and complete proofs with respect to the genre of proof as legitimate. In other words, one proof may be proofier than another, even when both are valid and complete. A consequence of this is that we explore not just the boundary between proof and non-proof but also what makes a justification a good/desirable proof. This approach is reminiscent of Paul Erdös' discussion of 'The Book,' where God keeps the perfect proofs for mathematical theorems (Aigner & Ziegler, 2004). That is, there is some (possibly unreachable) perfect proof for a theorem, and other proofs of that theorem are, using our terminology, less proofy.

The Utility of Proofiness

The remainder of this document discusses the utility of proofiness for exploring conceptions of proof. We begin by engaging the reader in an intertextual comparison of proofs. Later, we use excerpts from student interviews to illustrate insights about their conceptions of proof gained from this approach.

Your Conceptions of Proofiness

Imagine two people, Paul and Valentin, debating the proofiness of several proofs. Of the three proofs below, Paul states he believes that proof A is proofier than proof B. Valentin disagrees, claiming the opposite. Consider your stance on the matter. The three proofs are below:

Claim: $\sqrt{2}$ is irrational.

Proof A: Toward a contradiction, assume that $\sqrt{2}$ is rational. The definition of rational implies that there exists $a, b \in Z$ with $b \neq 0$ such that $\frac{a}{b} = \sqrt{2}$. Furthermore, we may assume that a and b share no common factors. Since $\frac{a}{b} = \sqrt{2}$ we know that $a^2 = 2b^2$, which implies that a^2 is even. Since a^2 is even, a is also even. This implies there exists a $k \in Z$, with $2k = a$. Substituting into the equation and simplifying, we get $2k^2 = b^2$. We then use a similar argument to conclude that b is even. Since we have shown that a and b are both even, they are divisible by 2. So, we have contradicted our assumption that they share no common factors. This means that $\sqrt{2}$ must be irrational.

Proof B: Suppose that $\sqrt{2}$ is rational. The definition of rational then implies that there exists $a, b \in Z$ with $b \neq 0$ such that $\frac{a}{b} = \sqrt{2}$. This means that $a^2 = 2b^2$ (*). The prime factorization of a squared integer has an even number of primes in its prime decomposition. Consequently, in equation (*), the left side has an even number of prime factors, and the right has an odd number of prime factors. This contradicts the fundamental theorem of arithmetic since prime factorizations are unique. So $\sqrt{2}$ is irrational.

Proof C: Let p be prime. Assume $\sqrt{p}$ is rational. By definition, there exists $a, b \in Z$ with $b \neq 0$ such that $\frac{a}{b} = \sqrt{p}$. This implies that $a^2 = pb^2$. Since the left side of the equation has an even number of prime factors, and the right has an odd number of prime factors, we have a contradiction. So, we conclude $\sqrt{p}$ is irrational. This implies that $\sqrt{2}$ must be irrational.

Consider the various stances that might emerge concerning Paul and Valentin's debate. Some might consider proof A proofier than proof B because of its similarity to the proof in their preferred textbook. Someone may believe proof B is proofier than proof A because its argument is more compact or makes fewer assumptions. Some might view proof C as proofier than proof A and proof B because it is more general. Alternatively, proof C leaves many warrants implicit compared to proofs A and B. So, some might base their judgment on that observation. Finally, a significant possibility is that one might avoid the debate entirely by asserting that if some justification is a proof, then it is no more (or less) a proof than any other. In that case, one considers relative proofiness as illegitimate. Regardless, the answers one gives and the reasons for them provide a lens into the values and norms they hold concerning proof. Both those who (1) believe the question is not legitimate, and (2) those who sighted some differences between the proofs to conclude some proof was proofier have sensible responses. The point we wish to highlight here is that it is reasonable for a person who answers (1) and a person who answers (2) to not attend to some of the differences in their respective conceptualizations of proof when discussing the subject (Author(s), submitted). That creates a potential mismatch, which can be especially problematic when, for instance, the former is a mathematics education researcher interested in student conceptions of proof and the latter is a student interviewed about their conceptions.

Student Conceptions of Proofiness via Comparison

We argue that student explorations of proofiness can provide helpful insights. Below is an excerpt from an interview with a student, Redmond, who was near to completing an introduction to proof course (Author(s), year). Additional information regarding the tasks he worked on are in Author(s) (Submitted), and the two justifications he examined can be found below in Figure 1.

Claim 1: For integers $a, b,$ and c, if a divides b and b divides c, then a divides c.

Justification 1: Since a divides b, there exists an integer k, such that $ak = b$. Similarly, since b divides c, there exists some integer l, such that $bl = c$. Substituting for b in the equations yields:

$$(ak)l = c$$
$$a(kl) = c$$

Since k and l are integers, so is kl. Therefore, a divides c.

Claim 2: 5 divides 10, and 10 divides 30, so 5 divides 30.

Justification 2: Since 5 divides 10, there exists an integer—in particular, 2—such that $5 \times 2 = 10$. Similarly, since 10 divides 30, there is some integer—namely, 3—such that $10 \times 3 = 30$. Substituting for 10 in the equation yields:

$$(5 \times 2) \times 3 = 30$$
$$5 \times (2 \times 3) = 30$$
$$5 \times 6 = 30$$

Note that 6 is an integer. Therefore, 5 divides 30.

Figure 1: Justification Comparison Task

From our perspective, justification 1 is a valid, complete proof of the divisibility property of integers, and justification 2 is a corresponding *generic proof* (Leron & Zaslavsky, 2014). In other words, Justification 2 goes through the same steps as Justification 1 but does so using specific integers. Raymond was asked to read both justifications and compare them in terms of how proof-like (proofy) they were—that is, compare their relative proofiness. The excerpt begins after Raymond reads Justification 1:

Redmond: I like it because that is precisely how we would have to do it if that was on a test.
Interviewer: Tell me a little bit more about that. Why does it remind you of that?
Redmond: Because, my professor, she's very, very picky about the words that you use: let, assume, suppose, thus, therefore. This looks like a 6 out of 6 points, for sure. Yeah, without a doubt.
[Reads Justification 2]
Redmond: This would be the same exact proof as Justification 1, but they have integers. So, I would consider this a 5 out of 6. The reason for it is just... There are certain things like terminologies. For instance, this part here, "in particular 2." My professor would have

liked, "Since 5 divides 10, there is an integer 2", or "Let integer 2", you know, or "there's just an integer, such as 2", just sort of things like that.

Redmond indicates that Justification 1 is proofier than Justification 2 by assigning them grades 6/6 and 5/6, respectively. He believes this is reflective of how his professor would have graded these. Redmond notices the generic proof approach to justification 2 when he states, "This would be the same exact proof as Justification 1, but they have integers." However, that is not what he attends to when articulating the reasons for the two justifications' differing proofiness. Instead, the reasons for this difference in proofiness are the 'terminologies' used in Justification 2, specifically the phrase 'in particular 2.' From our perspective, those terminology differences are necessary when moving between proof and generic proof. However, Redmond's conception differs from ours. Redmond indicates that if Justification 2 had instead used the kinds of wording that his instructor expects it would have been proofier.

We can learn about Redmond's conceptions of proof from his excerpt. For instance, he believes that using the language normatively found in proofs is relevant to proofiness. Additionally, given comparable language usage, he believes generic proofs are just as proofy as non-generic proofs. In other words, moving between generic vs. non-generic proofs has no effect on proofiness from his perspective, but minor deviations from the word choice he expects do. Had we presented justifications 1 and 2 as separate tasks, without the juxtaposition of the two, we would have just learned that he regarded both as proofs and learned less about his conception of proofiness. That discussion of proofiness allowed us to glean his views on the relative importance of genericness vs. language to his assessments.

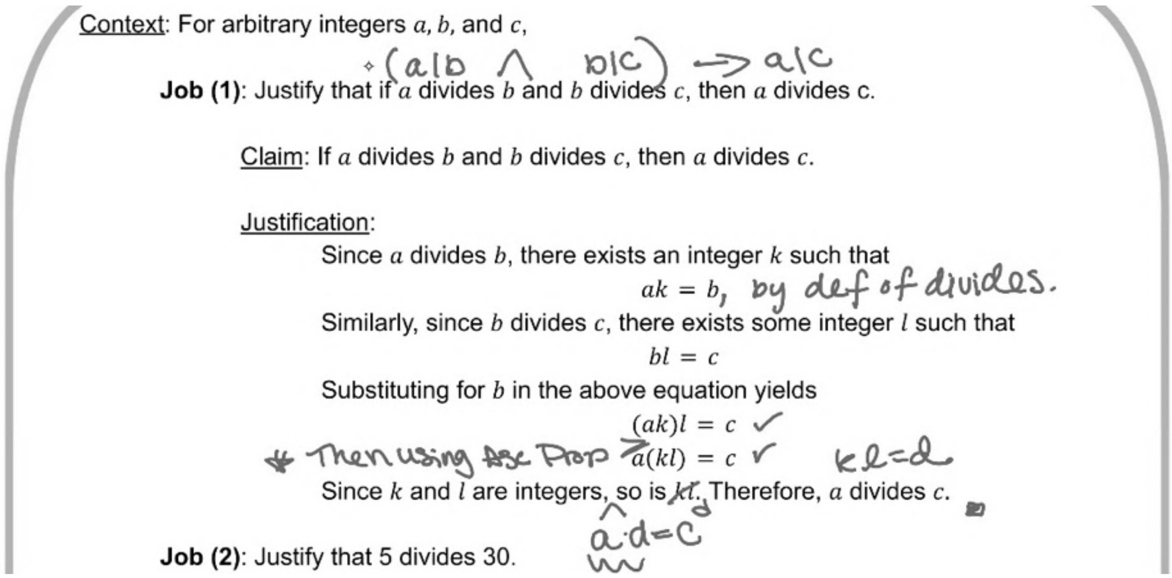
Student Conceptions of Proofiness via Editing

Next, consider the following excerpt from Claudette. When the excerpt begins, she has already participated in comparing justifications 1 and 2 (Figure 1). The excerpt is from the subsequent stage of the interview, editing. The transcript below begins after the interviewer encouraged her to revisit Justification 1 and edit it to make it more proof-like from her perspective. Her inscriptions are in Figure 2.

Claudette: ‘Since a divides b’. And then this [ak=b for some integer k] is just the definition of divides. Which in the words of [name of ITP instructor], 'sometimes you need to say that there's a definition. But also, it should always be assumed.' So here it looks there's more assuming that you know the definition of divides. In our class, most of the time you write it out… [Edits by adding ', by definition of divides.']

Interviewer: If you add that, does that make it feel more proof-like?

Claudette: I would say that it's very proofy, to begin with, but this gives more clarity. It's the idea, like if you are working with an 8th grader, whose never seen the definition of divides, then it's going to be like okay, by definition of divides, and if they have notes, they can look at the definition of divides and be like, Oh, if a divides b, ak=b for some integer k… definitely having the name of the definition so you can look it up makes it easier.



Context: For arbitrary integers a, b, and c,

$(a|b \wedge b|c) \rightarrow a|c$

Job (1): Justify that if a divides b and b divides c, then a divides c.

Claim: If a divides b and b divides c, then a divides c.

Justification:

Since a divides b, there exists an integer k such that

$ak = b$, by def of divides.

Similarly, since b divides c, there exists some integer l such that

$bl = c$

Substituting for b in the above equation yields

$(ak)l = c$ ✓

Then using Asc Prop $a(kl) = c$ ✓ $kl = d$

Since k and l are integers, so is $\cancel{kl}$ d. Therefore, a divides c.

$a \cdot d = c$

Job (2): Justify that 5 divides 30.

Figure 2: Claudette's inscriptions on Justification 1

Notice that Claudette indicates that her adding phrases and adjusting language is done to improve 'clarity' and 'make it easier.' So, in making these edits, she makes the justification more perspicuous from her perspective. She knows that the definition of divides implicitly warrants $ak = b$. So, that part of the justification is *transparent* to her. However, by adding 'by definition of divides' (Figure 2) she is 'making it easier' for someone else to make the same link. Additionally, she states that her intended audience is 8th graders. So, she is conscious of a potential audience for whom the justification should be clear. It is worth mentioning that proofiness facilitated our observation. By treating valid proofs as capable of being edited to be proofier, we were able to encourage Claudette to undertake such edits even though she viewed Justification 1 as initially 'very proofy.' This helped us glean what she viewed as the purpose of such edits: perspicuousness. Czocher and Weber (2020) discuss perspicuousness as one of the five properties mathematicians consider when determining whether a justification should be considered a proof. Claudette's excerpt shows that some students have similar considerations when editing proofs.

Both Claudette's and Raymond's excerpts reference their instructor. These references help illustrate that, especially in early proof courses, an instructor's discussions of proofs and how they should be communicated influence the conceptions of the genre students form. Specifically, their instructor spent significant time and effort emphasizing the importance of clarity and adherence to mathematical terminology. These themes were thus evident in their discussion of the proofiness of the justifications.

Discussion and Conclusion

Commonly, the themes studied concerning proofs are themes known to contribute to expert conceptions of proof. For instance, Czocher & Weber's (2020) expert cluster model of proof includes convincing, a priori, and sanctioned as properties mathematicians attend to when evaluating whether a justification is a proof. These three properties roughly correspond to the themes that Weber et al. (2022) argue that mathematics education researchers have historically prioritized (when studying students): 1) the psychological function of proof (Brown, 2018; Harel

& Sowder, 1998), 2) its logical or structural features (Hub & Dawkins, 2018), or 3) its social or institutional layers (e.g., Devlin, 1993). This correspondence between mathematicians' conceptions of proof and proof education researchers' priorities means that much of the current proof education research implicitly centers mathematicians' practice. The approach we described focuses on conceptions of proof as a genre. Thus, our approach leaves students open to articulate themes they believe contribute to the proofiness of a justification which may differ from themes identified by mathematicians.

Treating proof as a genre within the context of proof education points to the likely possibility that proof students and mathematicians view the genre very differently from each other. For instance some themes and motifs students view as central to the genre may be treated as surface by mathematicians (Selden & Selden, 2003) and vice versa (Chazan, 1993). Additionally, students are convinced by proofs in ways that differ from mathematicians (Healy & Hoyles, 2000; Weber, 2008). Thus, there are also differences in the role personal conviction plays in mathematicians vs. students' conceptions of the genre of proof. In this document, we have argued that intertextuality borrowed from genre theories provides one potential path for exploring student's conceptions of proofiness. This path is a potentially fruitful direction for proof education research, which does not center mathematicians' practice.

We describe two potential avenues for intertextual comparisons: comparison of researcher-generated justifications and student editing. For both, we demonstrated with brief excerpts of data that these intertextual comparisons of justifications revealed students' conceptions. The broader vision this work advocates involves using these tools to understand the themes that students view as relevant to the genre of proof, especially when those themes differ from those prioritized by mathematicians. Studying how students view the genre of proof and why can help the field better understand how students' experiences shape their conceptions of proof. Their conceptions of proofiness connect to how they approach proof construction and reading of proof. So, a better understanding of these conceptions, how and why they emerge, and their influence on reading and production of proofs can further proof education research and the design of proof-centered curricula.

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