Reasoning About Linear Combinations and Span in a Digital Game

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In this paper we analyze the reasoning of five students who were playing a 3-dimensional digital video game called Vector Unknown: Echelon Seas. The intention of Stage 4 of the video game is to give students experience with Numeric and Geometric representations of linear combinations of vectors in the context of a pirate throwing a grappling hook to reach an anchor. We illustrate three types of Numeric reasoning, and four types of Geometric reasoning students used to solve these puzzles. For each reasoning example, we discuss issues related to which vectors the player should choose so that the anchor is within the span of the vectors, and which scalars the player should choose so that the grappling hook ends precisely at the anchor.

Keywords: linear algebra, digital games, linear combinations, span

Introduction

The purpose of this paper is to characterize the reasoning of students playing a digital video game, Vector Unknown: Echelon Seas (VUES), created to engage students with *Numeric* and *Geometric* instantiations of linear combinations of one or two vectors in R^3. The larger goal of this analysis is (1) to add to the body of knowledge of the reasoning that is involved in understanding linear combinations of vectors and (2) to provide foundational information for future creators of digital environments and tasks for the teaching and learning of linear algebra concepts involving linear combinations of vectors.

Literature Review

Literature on student understanding of linear combinations of vectors and related concepts such as span and linear independence has taken several different forms. Some papers focus on applied examples created by the research team to promote student learning (e.g., Trigueros & Possani, 2013; Cáracamo et al., 2017). Other papers consider student generated applied settings or everyday examples for basis and hence how linear combinations of vectors serve to generate or describe other vectors in the space (e.g., Adiredja et al., 2020; Zandieh et al., 2019). Other settings have been created by instructors to have students engage with linear combinations of vectors as a way to travel through space (e.g., Wawro et al., 2012; Trigueros, 2019) or to explore locations in space using Dynamic Geometry Systems (e.g., Dogan, 2019; Turgut et al., 2022). More recently there are a few examples of digital games created to engage students in learning aspects of linear combinations of vectors (e.g., Nishizawa et al., 2013; Mauntel et al., 2021).

This paper adds to this body of literature by characterizing specific types of reasoning that are involved in students' game play and connecting those types of reasoning explicitly to aspects understanding linear combinations – even when the students did not themselves make these connections. In this way, we are suggesting ways that a game such as this may be leveraged by an instructor to help students build a more nuanced, experiential understanding of linear combinations and related concepts such as the span of a set of vectors.

To this end, our Research Question for this paper is: What are the ways that students reason with linear combinations of vectors in the context of playing the 3-dimensional (3D) grappling hook puzzles of Vector Unknown: Echelon Seas (VUES)?

Methods and Theoretical Framework

Context of the Game

Our work follows the philosophy of Freudenthal (1983) that mathematics is a human activity. In this way we are interested in how students engage in tasks and the ways that their reasoning emerges through this engagement. The tasks for this study are solving the puzzles in the digital game VUES that involve a pirate using a grappling hook in a 3D environment. The goal of each of these puzzles is for the pirate to send the grappling hook from his location to the anchor. This corresponds to creating a linear combination that equals the numeric vector goal in the upper right corner of the screen. Figure 1 shows a screenshot of the VUES 3D grappling hook environment. For this paper, we have added to Figure 1 black boxes outlined in purple with the names that we will use to refer to various game elements.

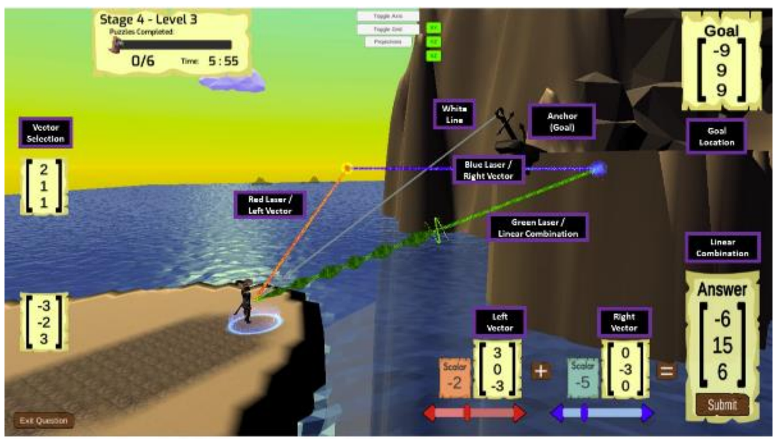

Figure 1: Illustration of Level 3 of Stage 4 (S4L3) of Vector Unknown: Echelon Seas.

As seen in Figure 1, the game provides both *Numeric* and *Geometric* information about the linear combinations of vectors. In addition, there are two main choices that the player can make in solving the Puzzle. The player can (1) select one or two vectors to move into the Left Vector and/or Right Vector slots and can (2) move the sliders underneath the Left or Right Vectors in order to change the scalar in front of the vector. Making these changes will automatically update the Answer vector and cause the Red, Blue, and/or Green Lasers to appear. We refer to these choices as *Which Vectors* and *Which Scalars*, respectively.

Data Collection and Data Analysis

To obtain information towards answering our research question, we conducted interviews with five students as they engaged in playing VUES. The fourth author interviewed each of the five students individually for at least one hour. The five students each chose their own pseudonym: Anh, Athena, Kyah, Patchy, TB. The intention of the interviews was to allow students to explore Stage 4 of VUES in which the pirate character's goal is to extend a grappling hook to connect to an anchor. For this paper we focus on the 3D level (Stage 4 Level 3, S4L3).

The video data for each student playing S4L3 was analyzed by at least two members of the author team and then discussed across team members. The S4L3 data consisted of three Puzzles completed by Kyah and six Puzzles completed by each of the other students. An initial analysis of the data focused on *Numeric* and *Geometric* reasoning and the gameplay strategies highlighted in Mauntel et al. (2021). These strategies overlap issues of *Which Vectors* and *Which Scalars*. As pointed out by Bernier and Zandieh (2024), some of the Mauntel et al (2021) strategies focus more on the choice of vector and others focus more on the choice of scalar. For this paper we emphasize issues of *Which Vectors* and *Which Scalars* and extend these types of analyses to a 3D setting. This extension also allows us to draw out additional connections possible between digital gameplay and reasoning with linear combinations of vectors.

Conceptual Framework

Mathematically, we can interpret *Which Vectors* as referring to which vectors should be selected so that the Goal will be in the span of those two vectors, and once selected, *Which Scalars* refers to which scalars should be used as the weight of each vector in the linear combination to create the Goal vector. In this way, the questions of *Which Vectors* and *Which Scalars* point to (1) game mechanics (allowable moves a player can make in the game), (2) statements about mathematical objects and relationships (at least to an expert) and (3) in-the-moment choices a player can make in terms of their focus or attention. Our conceptual framework interprets student reasoning with linear combinations of vectors through the choices students make in the digital game (*Which Vectors* and *Which Scalars*) and their focus of attention (*Numeric* or *Geometric*) in anticipating or interpreting the results of these choices.

Results

In analyzing our data, we found multiple ways that students reasoned about linear combinations in the digital game setting. Here we highlight three *Numeric* examples and four *Geometric* examples that illustrate different ways students navigated deciding *Which Vectors* and *Which Scalars* would solve the puzzle. The examples are chosen to illuminate variety in student reasoning as well as mathematical connections across these differences.

Numeric Reasoning: Two 0 Entries

We saw multiple instances of students trying to initially align one or two entries in the Answer vector (result of the linear combination) with the matching entries in the Goal vector. In this type of reasoning a student focused on initially aligning two entries and then adjusting to align the third entry as well. The more straightforward version of this reasoning occurred when the Vector Selection included a vector with two 0 entries. Each of the four students who had a puzzle including a vector with two 0 entries were able to solve such a puzzle numerically.

Anh prioritized *Numeric* aspects of the game, including leveraging 0 entries, when solving most of the puzzles of Level 2, and this continued in Puzzle 1 of Level 3 (L3P1; see Figure 2). In terms of *Which Vector*, Anh noticed that three of his vector cards had zero-valued entries saying, "Yeah, I'm thinking which one I can use because we have a lot of zeros here." To try and match the Goal vector, Anh selected <-3, -2, 3> and increased its scalar to 3 (*Which Scalar*) to match the x and z entries of the Goal vector (Figure 2(a)). After aligning these two entries, Anh used the <0, 3, 0> vector to change the y value of the Answer vector to match the y value of the Goal vector, as shown in Figure 2(b). Note that although Anh was focused on the *Numeric* features of the game, the *Geometric* effect of his gameplay was to position the tip of the Red Laser at the same height (z-value) as the Goal (Anchor) and then use the Blue laser parallel to the y-direction to reach the Goal (See Figure 2).

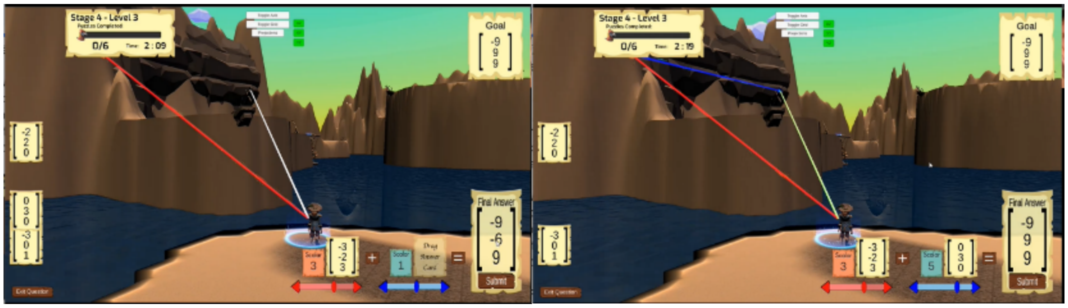

Figure 2: (a) Anh's use of the <-3, 2, 3> vector to match the x and z entries of the Goal. (b) Anh's use of the <0, 3, 0> vector to match the y entry of the Goal.

Numeric **Reasoning: No 0 Entries**

Athena's first two Puzzles of Level 3 each had a vector with two 0 entries, and she was able to solve them by reasoning in a *Numeric* way similar to Anh. However, in L3P3, Athena did not have any vectors with 0 entries. In terms of *Which Vector*, Athena initially used a single vector with a scalar of -4 (*Which Scalar*) to match two of the entries of the Goal vector (see Figure 3(a)). Eventually, she chose a second vector with $x = 2$ and $y = 3$ (*Which Vector*) which allowed her to match the x and y entries of the Goal vector, i.e., $x = -8$ and $y = -12$ (See Figure 3(b)).

Figure 3: (a) Athena's attempt to match the x and y entries with one vector. (b) Athena's initial Answer vector before she begins to incrementally decrease z by 1.

With support from the interviewer, Athena alternated between increasing the left scalar and decreasing the right scalar to decrease the z value of the Answer vector. Athena continued to alter the scalars in this manner until she finally matched the z entry of the Goal vector, decreasing the z value from 2 all the way to -10, as shown in Figure 4(a-c).

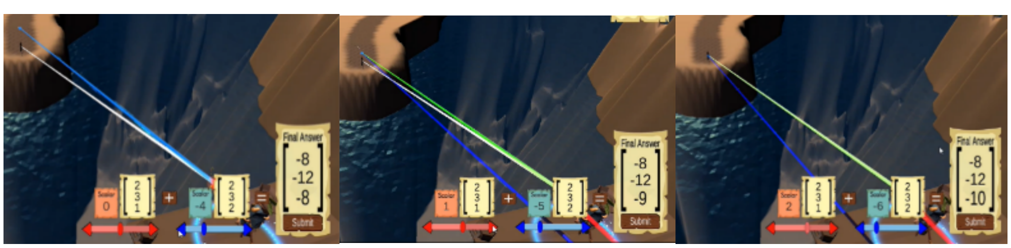

Figure 4: Examples of Athena's incrementing of the Left and Right scalars to decrease the **z***-entry.*

Athena noticed that in changing the scalars this way, "these two [x and y entries] don't change every time, but this one does." Note that in this example, Athena chose the vectors <2,3,1> and <2,3,2>, so her strategy of adding 1 to the left scalar and subtracting 1 from the right scalar is equivalent to adding (1)*<2,3,1> + (-1)<2,3,2> = <0,0,-1> to the resultant vector. This allowed her to keep the first two entries of her resultant vector -8 and -12 while she adjusted the third entry from 2 to the *Numeric* Goal of -10. Although neither she nor the interviewer mentioned it, the *Geometric* effect of adding <0,0,-1> repeatedly was to move the endpoint of the resultant vector down vertically until it reaches the anchor (see Figure 4).

Geometric **Reasoning: Green Laser onto White Line**

Patchy was one of the few students who used primarily *Geometric* reasoning about the linear combination in the first two puzzles of L3. This was because a Microsoft Teams window covered Patchy's Goal vector for most of his interview. At the start of his L3 gameplay, Patchy spent about nine minutes making sense of the lasers and vector cards. Once he noticed that his Answer vector (see Figure 5(a)) resulted in the Green Laser landing on the White Line, he said "we're onto something" multiple times. Because the Green Laser landed halfway along on the White Line Patchy said, "it was like halfway there, so if like double the scaling, that should do it." Patchy increased the Left scalar to 4 and the Right scalar to 2, reaching the Anchor. Patchy answered the questions *Which Vector* and *Which Scalar* almost simultaneously. Once the Green Laser landed on the White Line, he knew he could reach the Anchor by doubling the length of the Green Laser and scaling the Answer vector <-4, 5, 3> by a factor of 2, (see Figure 5(b)).

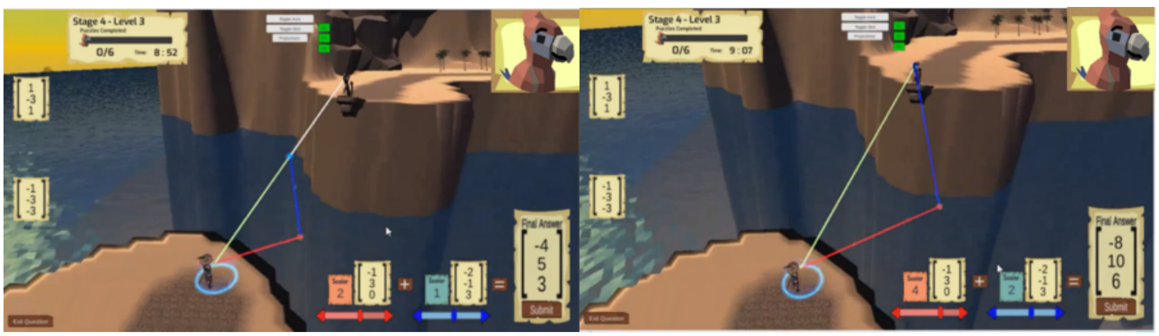

Figure 5: (a) Patchy noticed the Green Laser on the White Line, halfway to the Anchor. (b) Patchy doubled the value of each scalar to double the length of the Green Laser and reach the anchor.

Numeric **Reasoning: Scalar Multiples**

Previously Patchy used *Geometric* reasoning about the linear combinations because an application was covering the *Numeric* Goal vector on his computer screen. After recognizing that he could now see the *Numeric* Goal vector, he said "Oh! I haven't been seeing the Goal! This changes everything!" Patchy then said, "You can now just do it by addition." He then selected the <-1,1,-3> and <-1,-3,1> cards, and noticed that the *Numeric* Goal vector was a scalar multiple of the Answer vector, e.g., <-2, -2, -2> (see Figure 6(a)). Patchy said he wanted to "scale it up," and increased the left and right scalars each by 1 several times until he matched the Goal vector <-10, -10, -10>. Visually, Patchy was continually moving the Green Laser off and back onto the White Line each time his Answer vector resulted in a scalar multiple of <-2, -2, -2>. However, in this puzzle, Patchy was focused on making the Answer vector a scalar multiple of <-2, -2, -2> (see Figure 6(b)). Patchy's *Numeric* reasoning here can be seen as related to his earlier *Geometric* reasoning where he focused on scaling the length of the Green Laser.

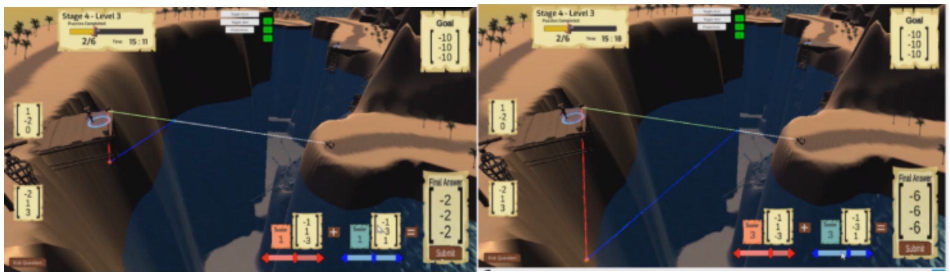

Figure 6: (a) Patchy's Answer vector <-2, -2, -2>. (b) Patchy's scalar multiple of the <-2, -2, -2> vector.

Geometric Reasoning: White Line is Not in the Span

In L3P1, Kyah moved the Green Laser onto the White Line for a fixed camera view as shown in Figure 7(a). Notice that even though she moved the Green Laser onto the White Line, her Answer vector does not match the Goal vector. After moving the camera to the view pictured in Figure 7(b), Kyah noticed that the Green Laser was not, in fact, on the White Line.

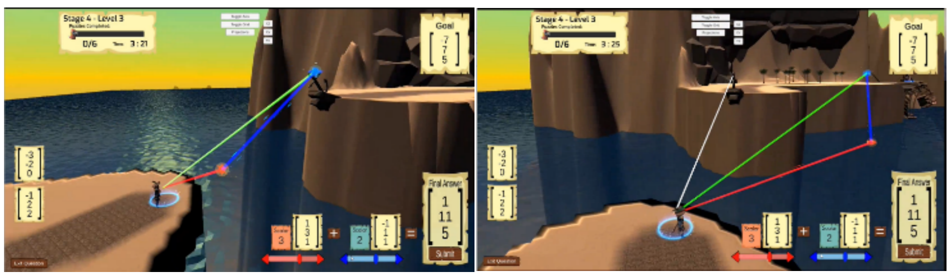

Figure 7: (a) Kyah's sees Green is on White. (b) Kyah realizes Green is not on White.

Kyah continued to leverage various camera views to determine whether the Green Laser was actually on the White Line, which helped her answer the *Which Vectors* question. From an expert point of view, this *Geometric* reasoning could be used to determine if the Anchor lies in the span of the selected vectors.

Geometric Reasoning: White Line _is_ in the Span

In L3P4, TB began to answer the question *Which Vectors* using different camera views to consider the relationship between the Lasers and the White Line. As shown in Figure 8(a), TB used a top-down camera view to demonstrate how the vectors <-3, 3, -1> and <-2, 2, 2> overlapped the White Line. From an expert point of view, this *Geometric* reasoning can be seen as demonstrating that the Anchor location, as well as the White Line, lie within the span of the selected vectors. In Figure 8(a) the Green Laser appears to be on the White Line but it is actually above it. TB changed the camera view to the perspective pictured in Figure 8(b) and decreased the right scalar to move the Green Laser through the Anchor.

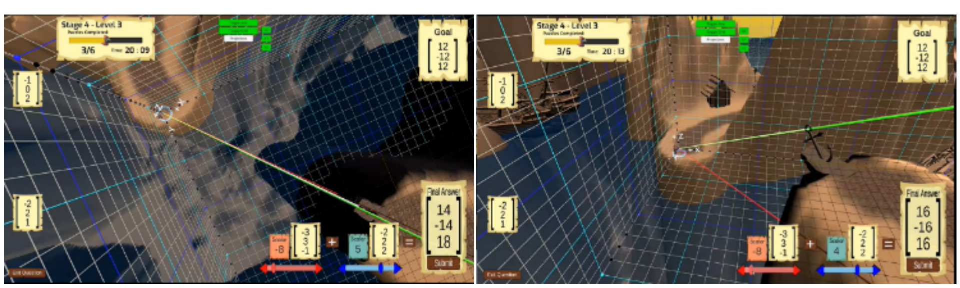

Figure 8: (a) TB's top-down view of Green above White. (b) TB's side camera view of Green through the Anchor.

Geometric Reasoning: Sweeping the Scalars

Earlier in his L3P4 gameplay, TB became convinced that the Lasers could be used to determine if a specific pair of vectors could be used to reach the Anchor. TB began L3P4 by continually increasing and decreasing the Left and Right scalars to determine possible places the Lasers could reach. He eventually turned the camera to the view pictured in Figure 9, noting that the lasers were at "a fixed angle," which would never move towards the anchor location. The narrow angle of the Lasers TB used to geometrically reason about the possible locations that could be reach by his selected pair of vectors allowed TB to consider a wedge of the plane spanned by his selected vectors. Though TB did not mention this plane, as experts, we recognize this type of *Geometric* reasoning about the linear combinations as nascent reasoning related to demonstrating that the Anchor is not in the span of TB's selected vectors, and therefore cannot be reached no matter how TB sweeps the scalars. This type of *Geometric* reasoning about the Lasers allowed TB to answer the question *Which Vectors* in the sense he recognized he needed to swap vector cards. TB summarized this by saying, "So using the line I can definitely see this combination of cards is never going to reach the right answer.

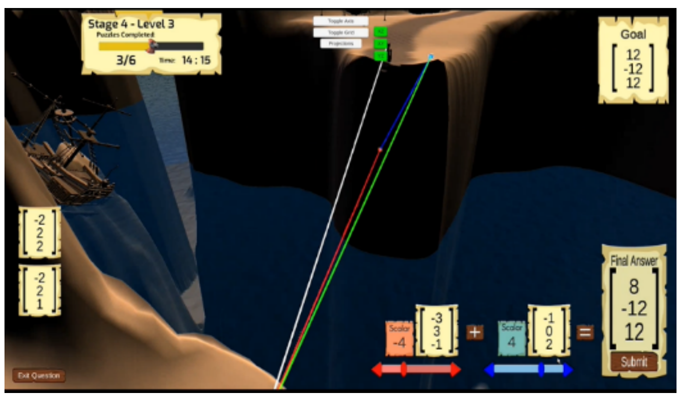

Figure 9: TB's camera view of the three Lasers to demonstrate why his selected vectors had the wrong angle and could therefore not reach the Anchor.

Discussion and Conclusion

In this paper we presented several ways (three *Numeric* and four *Geometric*) that students reasoned about linear combinations of vectors while playing S4L3 of VUES. These ways of reasoning about linear combinations were compatible with reasoning about span (*Which Vectors*) and reasoning about individual linear combinations for a specific resultant vector (*Which Scalars*) across both *Numeric* and *Geometric* conceptualizations of linear combinations.

We see this analysis as highlighting reasoning with linear combinations that adds to the body of literature in this area. In terms of *Numeric* reasoning, we see students working to coordinate three different but interconnected calculations (sum of products) to make all three numeric values align. Students must also consider issues related to orthogonal projection to consider situations in which two coordinates align and the third entry needs to be brought into alignment.

In terms of *Geometric* reasoning, we highlight the visual and experiential nature of interacting with a character that is moving in a digital 3D space and using changes in camera angles to gain deeper insight into the relationship between vectors and linear combinations in this space. We see students having a sense for what portion of R^3 is spanned by a set of linearly independent vectors and how one vector may lie within or outside of that. We also see students using both *Numeric* and *Geometric* reasoning that illustrate that if a vector v is in the span of a set of vectors (i.e., in a subspace) that a multiple of v will be in the span (subspace) as well.

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