Graduate Student Instructors' Mathematical Knowledge for Teaching Inverse Functions and Their Goals for Student Learning

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The mathematical meanings that instructors hold necessarily impact the ways that they teach and their goals for student learning. This study examined five graduate student instructors' (GSIs') mathematical meanings related to inverse function with the goal of exploring links between their meanings and the meanings they hoped for their students to construct. I found that the GSIs had multiple meanings for inverse functions that were evoked in different contexts, and that they had trouble drawing connections between these meanings when asked about relationships between them. Similarly, their goals for student learning did not include connections between meanings and were additionally constrained by factors such as the departmental curriculum used and the need to adhere to mathematical convention that students were accustomed to.

Keywords: Mathematical Knowledge for Teaching, Inverse Functions, Graduate Student Instructors

Researchers and teacher educators have long called for more attention to the mathematical meanings that teachers hold (Thompson & Thompson, 1996), arguing that without a coherent web of strong mathematical meanings, teachers are not best positioned to effectively work to foster strong understandings in their students (Tallman, 2021; Thompson, 2013). Graduate student instructors (GSIs) teach large numbers of undergraduate students in introductory math courses every semester (Blair et al., 2018), and often have little to no training (Belnap & Allred, 2009; Lewis & Tucker, 2009). Although there has been a tacit assumption that the mathematical content they teach is unproblematic for GSIs (e.g., Ellis, 2014) recent research focused on GSIs' mathematical meanings calls this into question and indicates a need for more attention to GSIs' mathematical meanings (Musgrave & Carlson, 2017; Rocha, 2022).

This proposal reports on a study designed to investigate the mathematical meanings related to inverse functions held by GSIs teaching Precalculus, and to gain insight into aspects of their mathematical knowledge for teaching (MKT) in this context. It is well known that students generally struggle with inverse functions (e.g. Bayazit & Gray, 2004; Brown & Reynolds, 2007; Engelke et al., 2005; Even, 1992; Paoletti et al., 2018; Vidakovic, 1996), but less is known about how teachers understand these ideas, especially at the postsecondary level. The broad goals of the study were to answer the questions:

1) What are graduate student instructors' meanings for inverse functions?
2) How do these meanings impact their goals for student learning related to inverse function?

Literature Review and Background

The knowledge required for teaching has been studied for several decades, beginning with quantitative analyses of teachers' education and their students' achievement (e.g., Ferguson & Womack, 1993; Guyton & Farokhi, 1987; Monk, 1994). When these studies began to shed doubt on the assumption that teachers with more mathematical knowledge would have higher student achievement, mathematics education researchers sought new methods for investigating the link between teacher knowledge and student outcomes. Several frameworks for conceptualizing mathematical knowledge for teaching (MKT) have been proposed, with some focusing on measuring and categorizing that knowledge (e.g., Ball et al., 2008) and others working to provide explanatory mechanisms for how MKT might be developed (e.g., Silverman & Thompson, 2008). I draw on Silverman and Thompson's conceptualization of MKT for this proposal, as it focuses specifically on the mathematical meanings that teachers hold with respect to a given concept.

Silverman and Thompson's (2008) framework for mathematical knowledge for teaching is based in a constructivist perspective, where "knowledge" and "meaning" are thought of differently than how they are generally used elsewhere. Rather than knowledge referring to things that someone believes to be true, it instead refers to assimilation to a scheme, which entails assigning meaning to the object of assimilation by fitting it into an already developed cognitive structure and includes inferences based on this meaning (Thompson, 2013). These related inferences are critical in understanding someone's meanings, because they establish what it is that particular knowledge will allow someone to do. Silverman and Thompson (2008) argued that the first step in developing MKT is to first have a strong personal understanding of the mathematical concept. This understanding includes relationships between the concept in question and other content knowledge, creating a web of connections between seemingly disparate mathematical topics. Next, a teacher must transform this knowledge into knowledge with pedagogical power by understanding how students conceptualizing mathematical ideas in a similar way to the teacher will enable them to learn related ideas, and by imagining the actions that the teacher might engage in to help support students in developing this understanding. Silverman and Thompson (2008) claimed that this transformation of knowledge into knowledge with pedagogical power requires a teacher to consciously reflect on their understanding, and to reflect on how students might develop compatible understandings.

The productivity of instructor's mathematical meanings can be defined by how productive these meanings would be for students' long-term learning if they were to also hold those meanings (Byerley & Thompson, 2017). From this perspective, meanings that help a student to answer questions about the given topic correctly but that are not easily built upon to develop future mathematical ideas are less productive than meanings that provide a foundation for further learning. In addition, the purpose of knowledge is to help the knower maintain a state of equilibrium as they engage with the world (von Glasersfeld, 1995), and so meanings that are useful in a wider variety of contexts can be considered to be more productive than those that are useful only in limited contexts (Byerley & Thompson, 2017). Similarly, those meanings that provide important foundations for reconciling future perturbations that occur and engender learning are considered to be more productive.

Methods

This study was conducted at a large public university in the southeastern United States, and five graduate student instructors participated. The GSIs ranged from second to sixth year mathematics PhD students and had all taught Precalculus for at least one semester prior to the commencement of the study. At this university, inverse functions are taught in Precalculus as a prelude to exponential and logarithmic functions. Students are expected to determine whether a function is one-to-one algebraically and by examining the graph of a function, and they are presented with the standard composition definition of inverse functions. Students learn to find the inverse of a given function algebraically by switching x and y and solving for y (the "switch and solve" technique), and graphically by reflecting over the line $y=x$.

I conducted a series of clinical interviews (Clement, 2000) with participants individually. The first two interviews focused on the GSIs' goals for student learning in relation to inverse function, and the remaining interviews were task-based interviews aimed at developing models of the GSIs' mathematical meanings for inverse functions (Steffe & Thompson, 2000). The GSIs participated in between 6 and 9 interviews each, with the number of interviews determined by how long it took to develop stable models of each participant's mathematical meanings. I engaged in ongoing analysis between each task-based interview to develop and refine my models for how my participants understood inverse functions and developed tasks for future interviews to test these hypotheses. I then compared the goals that participants had for the meanings their students would construct to the models of my participants' own meanings that I developed to look for areas of agreement or disagreement.

Results

Throughout the task-based clinical interviews, it was clear that the GSIs held meanings that allowed them to productively engage with inverse function tasks. However, most of them drew on meanings that were disparate (from my perspective) to solve different types of tasks, and they often had trouble describing coherence or connections between their different meanings. Isabelle, a sixth year graduate student, consistently thought of inverse functions as functions that describe the same invariant relationship from two perspectives. With this meaning, one function is viewed as giving one quantity (Q_1) in terms of another quantity (Q_2), while the inverse function describes the same relationship between Q_1 and Q_2 but instead gives Q_2 in terms of Q_1. She described this as being one of the big ideas that she wanted students to understand about inverse functions, saying,

> So let's say if we have cosine theta is *a*, we can have cosine inverse *a* is theta. And what it tells us is this is theta, uh, *a*, 1, right? [pointing at the triangle in Figure 1] So both of them tell me the same story. So I have the same, uh, object or same thing in the real life and I can say that in a different way. And I can use whichever I want to use or whichever is useful for me in that situation.

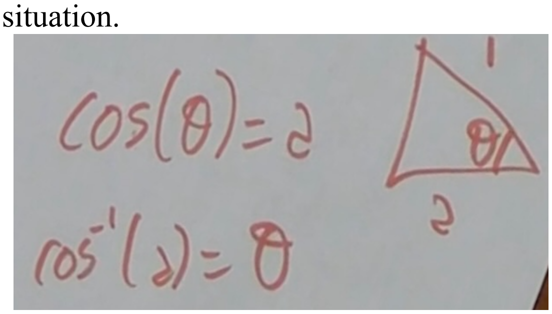
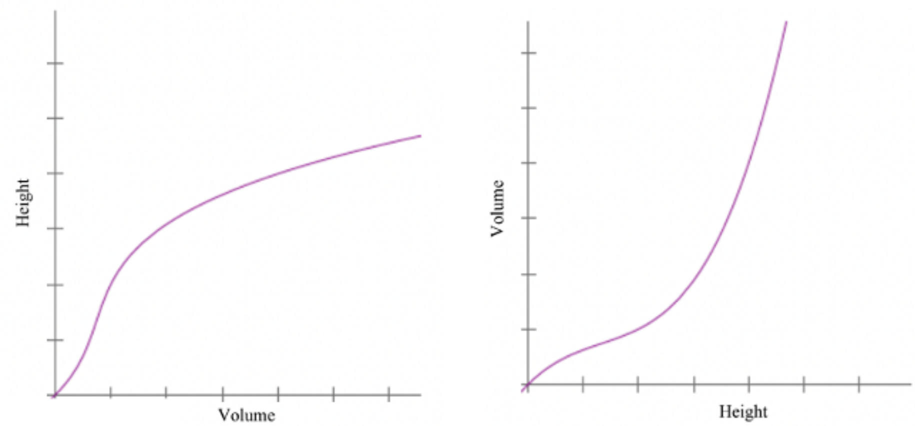


Figure 1. Isabelle 's work while discussing intended student meanings for inverse function.

At other times throughout the interview sequence, Isabelle also talked about wanting students to understand that inverse functions undo the original functions via function composition, and to think about inverse functions involving inverse operations from the operations in the original function. I was curious about the extent to which these two meanings were related for her, because she talked about undoing and composition in relation to algebraically defined functions without context, whereas she discussed the invariant relationship meaning when discussing trigonometric functions or functions with a real-world context. As an example of the latter, when presented with the following task (Figure 2), Isabelle almost immediately identified that the two graphs could be thought of as representing the same glass, and that they could be seen as inverse functions. To explore the connections between her meanings for inverse functions, I asked her to clarify how a student might understand how these two ways of thinking about inverse functions might be related in this context.



The following two graphs represent relationships between the height of water in a glass and the volume of water in the glass. Describe the two glasses and how they compare.

Figure 2. A task used in Isabelle 's interview.

Interviewer: What about if a student was like, "I thought inverse functions, like, undo each other, um, but these are just, like, kind of representing the same relationship. How do, how do I, like, make sense of that?" How would you respond?

Isabelle : For the undoing thing, I would say they are undoing in algebraic way.

Interviewer: Okay, what do you mean by that?

Isabelle : Like, when I'm adding two, for one of them, I'm subtracting two for the other one.

Interviewer: Okay.

Isabelle : But they represent the same thing in the real life. They represent the same glass.

She continued to describe how undoing can be seen algebraically when finding the inverse of a function, and I pushed her to explain conceptually how a student might connect the idea of undoing with the physical context of the glass.

Interviewer: If we're thinking about, like, the relationship between volume and height, or height and volume, is there a way of thinking about the undoing, in terms of, like, that relationship?

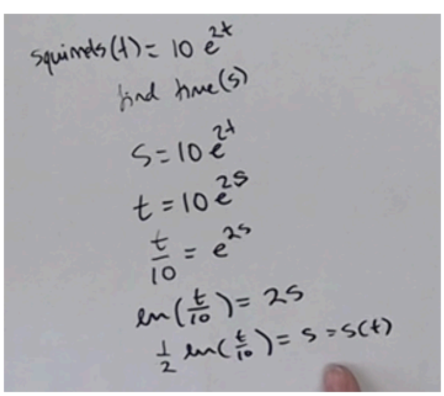
Isabelle : No, because I feel like they are doing the same thing. Like, again, I will just look at one part here [pointing to one graph], or here [pointing to the corresponding section on the other graph], and for both parts, when I look at, like, volume and height, let me say it like this, when I think, this is the height, this is the volume, this is the height, this is the volume here, so they give me this [pointing to her drawing of the associated glass]. They give me the same object. So because of that, in the real life explanation, there is no undoing. Because height is going, well, let's say, two units, but volume is going eight units in both of them. So, no. There is no undoing part.

I interpreted this to mean that Isabelle thought about composition and "undoing" in the context of algebraically defined functions, but that these meanings were not explicitly related to her meaning for inverse functions in real-world contexts (i.e., a function and its inverse capturing an invariant relationship). When I asked whether she wanted students to see these different ways of thinking as related, she did not express a desire for them to make any connections. While I have only presented an example of a lack of connections in one participant's meanings due to space constraints, other GSIs demonstrated similar differing meanings for inverse functions across various contexts, and these inconsistencies were reflected in the lack of specific connections between the ideas that they wanted students to develop.

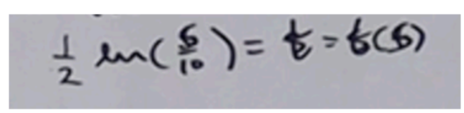
However, the GSIs' teaching goals were not only constrained by their personal mathematical meanings, but also by their perceived constraints due to the curriculum used and mathematical convention that they perceived their students to be accustomed to. All the participants explicitly described a disconnect between finding the inverse of a given function in a real-world context and solving the same type of problems without context. When finding the inverse of a function expressed algebraically, participants generally used the "switch and solve" technique. When solving problems in context, however, participants did not swap the variables and instead simply solved for the other quantity in the problem.

One participant, Nadia, described a problem that she used in her class, in which the population of squirrels in someone's backyard was growing according to the relationship $S = 10e^{2t}$ where *S* represented the number of squirrels and *t* represented time. Students were tasked with finding a function that would represent the time elapsed as a function of the population of squirrels in the yard. When doing this problem in an interview, Nadia first employed the switch and solve technique and ended up with a function that was again in terms of *t* (see Figure 3a). Once she finished, she perceived that her solution no longer accurately described the original relationship and switched the variables again to fix it (Figure 3b), stating "if it were *x*'s and *y*'s, it would have worked out with inputs and outputs. Um, but I shouldn't have switched these because I wanted to keep time and squirrels different." To her, the "switch and solve" technique was appropriate to use when dealing with *x* and *y* because it maintained the convention that *x* is the input of a function and *y* is the output of a function, but was not appropriate to use when the variables in a function represented real-world quantities. When I questioned how she helped students understand when to swap variables versus when to leave them, she stated,

> I think it would have confused them more if I had talked about that, because then it would be like, why are *x*'s and *y*'s special where we want to switch them? And, and I mean, the answer to that is just because it would be more confusing for the students to be trying to graph, like, if you were, if you had a function *x* with *y* as the variable, and you were trying to graph that on your regular plane, that would just be more confusing for the students. As opposed to saying *y*'s are on the *y* axis, *x*'s are on the *x* axis.



(a)



(b)

Figure 3. Nadia's work for the squirrel population problem.

Another participant that identified the same disconnect discussed that he would prefer to approach the topic differently than the given curriculum did in order to address the perceived disconnect. Jake reflected on how he would prefer to teach students to find the inverse of a given function, saying he would like them to solve for x in terms of y instead of swapping the variables. He noted that,

> Somehow that makes more conceptual sense to me, because y, kind of, you know, in this sort of picture [pointing to Figure 4], it's like, the x's lived over here [pointing to the circled x], the y's lived over here [pointing to the circled y], and then there is the function, which takes you from there to there [motioning from left to right]. And so this is a recipe which tells you, given an x, how do you get a y? So it's like, how do you go this way? And if you solve for x in terms of y, that's telling you given a y, how do you get an x?

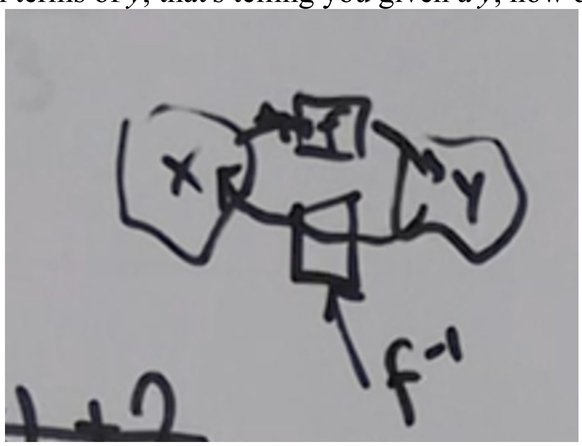


Figure 4. Jake's function diagram.

He went on to describe why he does not teach in this way, saying,

> I've tried that in like once or twice and I think that it's another inertia momentum thing where, you know, just solving, they're just so used to solving for y. Yeah, ideally, I would not do the swap. But, yeah, why do they do the swap in the process? I think it's just somehow to make it, um, yeah, because, like, they look at this kind of thing [pointing to the equation $y = 3x + 2$] and they're, like, not sure what to solve for somehow. Um, so once you make the swap, then it becomes clear. Like, uh, I wanted y equals some stuff, and now I don't have that situation anymore, so now I have to do algebra to get, you know, y equals some stuff in the process.

I interpreted this to mean that Jake prioritized maintaining the procedural habits that his students were accustomed to over teaching in a way that aligned with his personal meanings for inverse functions. He had tried to teach using his own meanings in the past, but was, in his opinion, unsuccessful, which could indicate that he had not transformed his personal meanings into meanings that were pedagogically useful in this context. While he had an image for the meanings he wished for his students to construct, he did not know how to support them in constructing these meanings. Nadia and Jake both compromised the coherence of their instruction to maintain habits that students were used to (such as graphing x on the horizontal axis and always solving equations for y), instead of working to help their students construct mathematical meanings that were compatible with their own.

Discussion and Conclusion

Through this study, I sought to investigate the coherence of participants' meanings for inverse functions because, in accordance with my theoretical perspective, coherence in an instructor's mathematical meanings is a necessary condition for them to be able to intentionally design instruction to support students in constructing powerful, coherent meanings. I found that the meanings for inverse functions that the GSIs' drew on in different contexts were distinct from each other, and that they struggled to draw connections between these differing meanings. Some, like Isabelle , did not have a desire to draw these connections, while others actively searched for ways to connect their meanings during our interviews. One participant expressed her frustration when trying to understand how a function and its inverse can simultaneously represent the same quantitative relationship and can be seen as undoing one another, saying "I feel like I'm so close to getting this right. Um, like this connection between what we saw here, that like, it's, it's really the same relationship, and it being some inversion of the relationship. It's still not there." She sought to reconcile these ideas over the course of the interview sequence, and wanted to find a way to help her students understand the ideas as compatible as well.

The curriculum that the GSIs used in their courses centers actions for students to do, rather than meanings for them to construct. If student meanings for inverse function are limited to actions such as "I should compose functions to show that they are inverses", "I switch x and y and solve for y to find an inverse function", and "I reflect the graph over $y=x$", then each of these meanings is only useful in very specific contexts. If the productivity of meanings is determined by the range of situations that they allow a learner to handle, then these meanings are extremely limited. As the GSIs pointed out, the "switch and solve" technique has the potential to be conceptually limiting for students. The GSIs themselves understood why the technique works in some cases but not others, but had not operationalized these understandings in ways that made them useful pedagogically. Jake and Nadia, along with other participants, described their reasons for teaching this technique despite its limitations, which centered around wanting to match what students had seen in previous classes, wanting to align with the textbook and course materials, and not wanting to confuse students by altering the convention that x is always an input and y is always an output. These are all entirely valid reasons and underscore that shifting to teaching for mathematical meanings is incredibly difficult at the level of the individual instructor, and requires a broader scale effort at the level of course coordinators and curriculum designers.

It should not come as a surprise that the GSIs struggled to describe central meanings behind the concept of inverse functions when the curriculum they use has no such underpinnings. It would be unreasonable to expect GSIs to come into their graduate programs with fully developed sophisticated understandings of all the mathematical ideas they will teach, and often the materials that they are given to teach offer them no opportunities to further develop their understandings. While this study uncovered worrisome findings about GSIs' mathematical knowledge for teaching, I do not intend for it to be a critique of the GSIs, but instead of the systems within which they are expected to operate. The GSIs in this study are clearly mathematically capable and yet they hold limited meanings for these mathematical concepts and meanings that are not pedagogically powerful. This is an indication that they have not had opportunities that would lead them to develop stronger mathematical meanings, not that they are incapable of it. This has implications for both professional development programs for GSIs and curriculum design, as developing stronger meanings in GSIs will be of limited value if they are still constrained by action-focused curriculum. Future research should investigate how to help GSIs develop more coherent meanings, and how to help them transform their meanings into meanings that are pedagogically useful.

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