Undergraduate Student Conceptions of Mathematical Syntax Meanings

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This research investigates students' structure sense in algebra, focusing on how they conceptualize and interpret mathematical syntax. Here we focus on how students parse symbolic representations in algebra, in particular, how and why students recognize certain substrings to be unified subexpressions. We use prescriptive conceptions (desired mental images) to guide our noticing of students' descriptive conceptions (actual mental images) as they engage with conceptual algebra tasks from the Algebra Concept Inventory. The findings offer implications for instruction and curriculum, suggesting approaches that may better support students in developing a deeper understanding of algebraic structures and symbolic reasoning.

Keywords: Algebra concept inventory, student conceptions, syntactic structure, mathematical syntax, symbolic representations

Research in mathematics education shows that students often struggle with structure sense, or the ability to understand, recognize, and manipulate the underlying structure of algebraic expressions (Hoch, 2003; Linchevski & Livneh, 1999). Working with symbolic representations as objects (e.g., transforming expressions and equations), requires understanding of the structure of these objects. Existing research focuses on what students *do* with symbolic representations (e.g., Hoch & Dreyfus, 2006, 2010; Kirshner & Awtry, 2004; Vincent et al., 2017), but includes only limited analysis of *why* they perform specific manipulations, and what *conceptions* students have of syntactic structure in algebra. This research addresses this gap by investigating how students conceptualize the parsing and chunking of mathematical syntax.

Understanding specific student conceptions related to structure sense can help instructors develop effective teaching strategies (Hoch & Dreyfus, 2010), and address diverse learning needs (Tomlinson, 2001). It is also key to building curricula that enhance students' mathematical communication (Sfard, 2008), strengthen their foundation for advanced math (Taban & Cadorna, 2018), and improve problem-solving skills (Linchevski & Livneh, 1999).

We focus on two key aspects of mathematical syntax: (1) the extent to which students are able to conceptualize certain substrings of symbolic representations as unified objects (subexpressions), which relates to operational vs. structural views of syntax; and (2) which criteria they use to determine that structuring of symbolic representations, which relates to student definitions around syntax. More simply put, we are interested in *how* students parse symbolic representations, and *why* they parse them in the particular way that they do. Through student written work on algebra tasks and interviews with students in post-secondary math classes, we analyze their conceptions, particularly regarding how they parse or chunk mathematical syntax.

Literature Review

Existing research on structure sense has focused more on what students do with symbolic representations than on how they conceptualize them. Additionally, current literature is underspecified regarding the meaning of "structure" and which structures are most important. Our research seeks to clarify what structure sense is by framing it as a set of conceptions.

In this research we focus on one of the most basic components of structure sense: *syntactic meaning*, which we define as reading the intended grammatical meaning of symbolic representations. Syntactic meaning refers to the literal interpretation of symbols based on conventions (e.g., order of operations), without considering broader concept images or the ability to simplify or transform the expression. For instance, the syntactic meaning of $(2x + 1)(x - 7)$ refers to two subexpressions being multiplied, but it does not include concept images like multiplication or real-world meanings of x.

Research shows that students often assign syntactic meaning based on surface features, like spacing (Erlwanger, 1973), which Kirchner (1989) calls "visual salience," rather than on mathematically-salient features. For example, students may give precedence to operations that are visually closer together (Landy & Goldstone, 2007) or interpret spacing and grouping based on familiar visual cues from previous experiences (Nathan & Koedinger, 2000). Visual salience can overshadow mathematical salience, leading students to group elements visually rather than apply mathematical properties. Helping students distinguish between visual and mathematical salience can guide curriculum development and instructional strategies to focus on meaningful relationships rather than superficial visual cues.

Well-defined definitions allow students to interpret algebraic expressions consistently, using mathematically-salient criteria, rather than relying on surface features (Edwards & Ward, 2004). For instance, a student may have an extracted definition of syntax, where they identify subexpressions based on what "looks right" (visual salience) rather than on mathematically salient features. A mathematically well-defined, stipulated definition would use the order of operations and other syntactic conventions to determine how the representation should be "chunked" into subexpressions.

Process-Object Theories and Subexpressions

Research shows that students often struggle to recognize substrings of symbolic expressions as unified objects. This ability is crucial for algebraic manipulation, such as applying the distributive property or identifying forms like $a^2 - b^2$ (Hoch & Dreyfus, 2004). This is related to how and whether students conceptualize representations as processes vs. objects. For example, students might interpret $2(x + 1)$ as a process of adding 1 to an unknown value and then multiplying by 2—an operational view (Sfard & Linchevski, 1994); or they may be able to treat it as an object representing the theoretical result of calculation without needing to perform the operations—this is classified as a structural view. Pseudo-structural views (Sfard & Linchevski, 1994a, 1994b) occur when students treat symbols as objects that have not been reified from processes.

We extend this theory to focus specifically on what we call *subexpressions*, or substrings of symbolic representations that can be treated as unified objects without altering the syntactic meaning of the representation. For example, in the expression $3x^2$, students may conceptualize $3x$ as a unified object (subexpression) because of visual salience (3 and x are close together) or extracted experiences from instruction ("the instructor always treated $3x$ like a single thing in class"), or they may conceptualize x^2 as a unified object (subexpression) because the order of operations prioritizes exponents over multiplication. The latter is the necessary interpretation to understand the normatively-intended syntactic meaning of the expression $3x^2$. In existing structure sense literature, treating substrings as objects is often combined with other skills (see e.g., Hoch & Dreyfus, 2006), yet understanding this ability as a distinct conception is important, which is why it is the focus of this work. Identifying subexpressions is a critical precursor to almost all other structure sense skills because it is the first step to identifying the structure of expressions and equations, which is essential to transforming them.

Method

This research focuses on the following questions: What conceptions do tertiary students have of structure sense, symbolic representations, and mathematical syntax? Specifically, what conceptions do tertiary students have of parsing and chunking mathematical syntax?

To address these questions, we use data from a broader study generating an algebra concept inventory for post-secondary students (AUTHORS, 2024). In this study, we collected thousands of responses to multiple-choice items, including 138 cognitive interviews of student thinking while completing these items. It is these cognitive interviews that we focus on here, specifically analyzing items where students were asked to attend to syntactic structure.

Our research is grounded in prescriptive conceptions (conceptions that serve as the intended goals of instruction) of symbolic representations and mathematical syntax. We generate these conceptions using literature and conceptual analysis, providing criteria for how students should "read" syntactic meanings and view mathematical syntax as reifications of normative processes. These can be seen in Figure 1, where the rows and columns represent broader categories, and the two cells at middle and bottom right represent combinations of these broader categories.

	Well-Defined Syntax Conception: Syntactic meanings must be unambiguous—the same syntax can't dictate more than one possible operational precedence at once, and different members of the mathematics community can't choose different conventions and operational precedence for the same symbolic objects.	**(Normative) Operational Precedence Conception:** Syntactic meaning is dictated by the order of operations and other mathematical conventions/norms stipulated by the mathematics community (whether or not those calculations can be carried out directly).
Structured Syntax Conception (Object): Symbolic representations can themselves be thought of as objects that can be "chunked" into unified subexpressions (or other unified sub-objects).	***Do-first Conception (operational):*** *The operational precedence norms govern the order of arithmetic (and where possible algebraic) calculations (e.g., which operation should be "done first").*	***Subexpressions Conception (structural):*** *The operational precedence norms govern how algebraic symbolic representations are structured, or "chunked" into unified subexpressions. (This may be a reification of the "do-first" conception or might be learned separately and linked to operational precedence later.)*

Figure 1. Prescriptive Conceptions of Mathematical Syntax

These categorizations of student thinking offer a way to classify students' *in-the-moment* thinking, providing insights for improving instruction and curriculum.

We used the prescriptive conceptions from Figure 1 to guide our noticing of student work, but also allowed themes to emerge more organically by employing theoretical thematic analysis

(Braun & Clarke, 2006) to constructed rich, thematic descriptions of students' structure sense conceptions. Our analysis led to the framework given in Figure 2, where we see student conceptions as a cross between operational vs. structural views and the extent to which they use non-mathematically salient vs. well-defined mathematically-salient features to determine syntactic meanings.

Reading syntactic meaning of symbolic representations (mathematically-salience features of syntax)	Non-mathematically-salient features, without being able to link this reasoning to stipulated criteria	Stipulated, logically-consistent and well-defined, normative criteria
	pseudo-process view ⟷	**process view**
Operational View Conceptualizing syntax as an instruction to calculate, or as describing a process of calculation (process view)	"I have to add these two terms first because this just looks/feels right."	"I have to add these two terms first before I square them, because they are enclosed in parentheses and parentheses comes before exponents in the order of operations."
	pseudo-object view ⟷	**object view**
Structural View Conceptualizing syntax as a structure that contains sub-structures (e.g., an expression is made up of multiple [potentially overlapping] sub-expressions) (object view)	"I know that $2x$ is a unit because I have just developed an instinct of how this works."	"I know that $2x$ is unit because multiplication comes before addition in the order of operations. Because the 2 is being multiplied by the x, $2x$ has to be thought of as a unit that could then be added to the other terms."

Results and Discussion

Classifying student work was our first step towards understanding student thinking. We allowed students' conceptions to emerge through the research but a priori theories also guided our noticing of students' conceptions as we attended to the following: (1) what criteria students were using to "read" the syntactic meaning of symbolic representations, and (2) whether students were able to conceptualize the syntax as dictating a process or describing an object, including how they "chunk" symbolic representations that they conceptualize as objects. We now provide examples of student work in each category (pseudo-process, process, pseudo-object, and object conceptions of syntax).

Pseudo-process Conceptions

Consider the following item: In the expression $3x^2 - 2(x^2 + 1)$, which part is being subtracted, or taken away? One student who was enrolled in a math course with elementary algebra as a prerequisite originally chose $2x^2$. During their cognitive interview, the student expressed uncertainty about the choice, so the interviewer followed up and eventually, the student says, "Honestly, the answer in my mind is not in the options." When asked to explain what is in their mind, the student and interviewer have the following exchange (Excerpt 1):

Excerpt 1:

Interviewee: It should be this one, like I would reverse this one like x square- I don't know. So, x squared minus two minus x squared plus one. So, this has to be taken away [pointing to $x^2 - 1$].

Interviewer: So, okay, so you would be subtracting the x squared plus one, kind of like what's in the parentheses?

Interviewee: Right.

Interviewer: And why would you do that?

Interviewee: I don't know.

Interviewer: It's just a feeling you have about how it should be done?

Interviewee: Yeah, but why is the parentheses still, it's not subtraction... If I'm going to multiply this one 3x- Wait, I think I did it wrong. 3x squared this, this [draws lines connecting $3x^2$ and 2 to other terms in the expression].

Interviewer: So, now you're multiplying the 3x squared by both of the things in the parentheses and also the two times both of those things in the parentheses?

Interviewee: Yes, 3x squared, it's going to be 3x four, am I correct? And then plus 3x squared and then this to the other negative 2x squared negative two.

Interviewer: Okay, so then would you be subtracting this minus 2x squared minus two?

Interviewee: Yes.

Interviewer: Okay, I got it. So, you're multiplying first and then whatever you're getting as a solution that is what you think you should take away, but that answer is not there?

Interviewee: Correct.

This student has an operational rather than structural view because they appear to struggle with identifying what is being subtracted without simplifying the expression. The student attempts to "distribute" every term that is outside the parentheses, even though the $3x^2$ is not being multiplied, which may be an extracted procedure from instructional experiences with the distribute property (e.g., AUTHOR, 2023). Further, the student appears to view $3x^2 - 2$ and $x^2 + 1$ as separate expressions to operate on, and are inconsistent with, the operations they apply. The student's attention to simplification appears to get in the way of being able to identify what is being subtracted in the expression in its current form, which may require conceptualizing $x^2 + 1$ as a unified subexpression. This student seems to be capable of thinking structurally, but they do not have a concrete justification for interpreting the syntax of the existing expression in a particular way. Thus, they may view symbolic representations as instructions to simplify rather than as an existing syntactic structure in and of itself, representing a pseudo-process view.

Process Conceptions

A different student displayed a process view for the following item: Consider $2 \cdot x^3$ in its current form (don't rewrite it or do anything to it). Which part of $2 \cdot x^3$ is being multiplied by 2?

Students were provided two options: "Only x^3" or "Only x ". This student chooses the first option because, "It's just that three is only being multiplied, not anything else. So, since it's to the third, then it would be only being multiplied by to the third because it says it on the question. And plus, it says what's being multiplied by two and it's obviously to the third." After some back-and-forth with the interviewer in which the student does not provide further detail about *how* they came to their answer, the interviewer asks, "How does the order of operations help you understand what's being cubed in the expression?" The pair then has the following exchange (Except 2):

Excerpt 2:

Interviewee: Well, since I know that there's no parentheses only thing that I will say is that there's an exponent here. So, we should focus on that first. Then we could get to the multiplication part.

Interviewer: Okay, so you said you noticed that there's no parentheses. So, where would the parentheses be if there were parentheses?

Interviewee: Now if it was a parentheses it would be around the x and then it would cube that. And once you cube that it will become x cubed.

Interviewer: Okay, if the parentheses were around let's say the $2x$ the two and the x how would that change things if it does at all?

Interviewee: Well, actually it would change both of them to my opinion, it would change both of them. Because you have to cube the two and the x.

The student uses a well-stipulated definition of the order of operations in their justification by when they describe how the order of exponents in operational precedence impacts the meaning of the syntax. However, their justification points to an operational view because they talk about processes (e.g. "multiplication", "cube") rather than describing any substrings of the syntax in a way that suggests that they view them as subexpressions. Further, since the student relies on mathematical (justification through operations) rather than visual salience, we classify this as a process view.

Pseudo-object Conceptions

We now consider an example from a student responding to the following item: In the expression $-(4x)^3 + 7$, which part is being cubed (or raised to the third power)? The item provided the following choices: a. $-4x$, b. $4x$, and c. x . The interviewee explains their choice of $-4x$ (Except 3).

Except 3:

Interviewee: So essentially the minus, the negative sign in the beginning gets distributed into the $4x$ that is within the parentheses, which is why I chose minus four x, which would get cubed.

Interviewer: Okay, so you're saying that the negative gets cubed as well as the $4x$?

Interviewee: Yes.

Interviewer: And I notice also that you included the 4 and some of the other answers don't include a 4. Why did you include the 4?

Interviewee: Because the 4 is together in the parentheses with the x.

Interviewer: With the x, what do the parentheses mean here in this expression?

Interviewee: It's saying that this whole expression, $4x$ is being cubed, not just one specific number.

Interviewer: Does the order of operations help you to understand what's being cubed here?

Interviewee: That didn't come to mind when I was doing this.

This student refers to $-4x$ as a unified object ("whole expression"), suggesting a structural conceptualization. However, what they consider the subexpression is not consistent with the standard order of operations, and when asked directly, the student admits to not considering the order of operations when doing the task. It is also notable that the alternate procedure of distributing the negative sign before cubing may also be indicative of a pseudo-process view, and thus this student may be moving between a pseudo-object and pseudo-process view or may hold both conceptions simultaneously.

Object Conceptions

For our final conception, we consider a student looking at the following item: In the expression $(x-7)^3$, which part is being cubed (or raised to the third power)? This student chose $x-7$ (over the other provided options -7 or 7).

The student explains how the third power, being "outside of the parentheses on the right side" means the third power should apply to the subexpression in the parentheses. The interviewer follows up with the question, "Do you know what the order of operations is?", leading to the following exchange (Excerpt 4):

Excerpt 4:

Interviewee: Yes, parentheses, exponents, multiplication, division, addition, and subtraction.

Interviewer: So, does the order of operations help you tell what's being cubed in that expression?

Interviewee: Correct, it does. It shows that what's in parentheses is actually being cubed. So, I would again follow what's inside the parentheses if I cannot do that equation then I would work outside of the parentheses.

This exchange reiterates the student's view of the string inside the parentheses as being a unified subexpression as well as their normative definition of the order of operations. This structural interpretation of the expression coupled with the normative definition leads to the object conception classification.

Conclusion and Implications

In this research, we not only classified conceptions, but also found interesting and common patterns of holding multiple conceptions that may have important implications for instruction and curriculum. For example, many students had a process conception of syntax in the context of numerical expressions but a pseudo-object conception in the context of algebraic expression. This may happen, for example, when students experience algebra as a sequence of transformation algorithms, without clear, consistent and ongoing connections to algebra as generalized arithmetic. We also observed students who had both process and object conceptions together, and where the object conceptions appeared to be reifications of the process conceptions. This illustrates that at least some students are able to reify processes into more structural object views in normative ways—better understanding how that happens for these students could help us to improve instruction so that all students have access to these conceptions when working with algebraic symbolism.

Overall, student responses illustrated how they often relied on surface-level cues or procedural steps when interpreting mathematical expressions, rather than understanding how symbolic representations can be conceptualized as objects that have a particular structure that is dictated by straightforward rules and conventions like the order of operations. Instruction could help students recognize and rely on mathematical salience for interpreting symbolic structures rather than visual patterns by using activities that encourage students to articulate why expressions or equations "chunk" into subexpressions in particular ways, and how this is linked to operational precedence and other notational conventions.

Students often identify subexpressions for non-normative reasons, relying on surface-level cues, and their process conceptions, such as the application of the order of operations, influences their ability to develop object conceptions, such as understanding subexpressions as unified structures. Instructors could also assist students' transition from process- to object-oriented conceptions by explicitly connecting the two, for example by consistently engaging students in tasks that illustrate how algebraic expressions are generalized representations of arithmetic.

References

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis) use of mathematical definitions. *The American Mathematical Monthly*, *111*(5), 411–424.

Erlwanger, S. H. (1973). Benny's Conception of Rules and Answers in IPI Mathematics. *Journal of Children's Mathematical Behavior*, *1*(Autumn), 7–26.

Hoch, M. (2003). Structure sense. *Paper Presented at the Third Conference for European Research in Mathematics Education*.

Hoch, M., & Dreyfus, T. (2004). *Structure sense in high school algebra: The effect of brackets* (M. J. Høines & A. B. Fuglestad, Eds.; 3rd ed., pp. 49–56). PME.

Hoch, M., & Dreyfus, T. (2006). *Structure sense versus manipulation skills: An unexpected result* (J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková, Eds.; Vol. 3, Vol. 3, pp. 305–312). Charles University Prague.

Hoch, M., & Dreyfus, T. (2010). *Developing Katy's algebraic structure sense* (V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarella, Eds.; pp. 529–538). CERME.

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education*, *20*(3), 274–287.

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education*, *35*(4), 224–257. https://doi.org/10.2307/30034809

Landy, D., & Goldstone, R. L. (2007). Formal notations are diagrams: Evidence from a production task. *Memory & Cognition*, *35*(8), 2033–2040. https://doi.org/10.3758/BF03192935

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173–196.

Nathan, M. J., & Koedinger, K. R. (2000). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education*, 168–190.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A., & Linchevski, L. (1994a). Between arithmetic and algebra: In the search of a missing link. The case of equations and inequalities. *Rendiconti Del Seminario Matematico*, *52*(3), 279–307.

Sfard, A., & Linchevski, L. (1994b). *The gains and the pitfalls of reification—The case of algebra* (P. Cobb, Ed.; Springer, Dordrecht, pp. 87–124). https://dx.doi.org/10.1007/978-94-017-2057-1_4

Taban, J., & Cadorna, E. (2018). Structure sense in algebraic expressions and equations of groups of students. *Journal of Educational and Human Resource Development (JEHRD)*, *6*, 140–154.

Tomlinson, C. A. (2001). *How to differentiate instruction in mixed-ability classrooms*. Ascd.

Vincent, J., Pierce, R., & Bardini, C. (2017). Structure sense: A precursor to competency in undergraduate mathematics. *Australian Senior Mathematics Journal*, *31*(1), 38–47.