An Analysis of one Calculus Student's Understanding of Derivative Through Gestures

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This preliminary report focuses on investigating the range of gestures students use to explain the concept of derivative in different contexts. Students were asked to explain their meaning of derivative and how they make sense of the formal limit definition of derivative. In addition to taking an embodied cognition perspective in our analysis, we also draw from Zandieh's Derivative Framework to make meaning of the range of gestures we observed. For this preliminary report, we analyzed a video recording of one student during a 30-minute assessment interview for a first semester calculus course wherein he displayed over 40 different gestures. Our findings focus on five representational gestures that were salient in the student's explanation of derivative. We discuss methodological implications about analyzing gesture alongside a cognitive framework of a mathematical concept, as well as how our findings connect with existing themes in research on mathematical gestures.

Keywords: calculus, derivative, gesture, embodied cognition

Embodiment in mathematics education research has contributed much to a more robust conceptualization of students' sensemaking of mathematics (Abrahamson et al., 2020). Introductory Calculus is often seen as a barrier to gaining access to STEM-related college majors and hence, STEM-related careers (Bressoud, Mesa, Rasmussen, 2015). This report attends to the variety of gestures used by a student, Andy, while explaining his conceptual understanding of derivative. This work may support extending current frameworks for analyzing students' understanding of derivatives (e.g., Zandieh, 2000) and promoting the value of attending to gestures by instructors when assessing and advancing student thinking.

Literature Review and Research Questions

Two common phenomena observed in studies of gesture in mathematics education are representational gestures (Alibali & Nathan, 2012) and gesture-speech "mismatches" (Alibali and Goldin-Meadow, 1993). While Alibali and Goldin-Meadow (1993) suggest mismatches indicate students' knowledge in transition or on the cusp of verbalizing it, Abrahamson et al., (2020) argue that "non-redundant gestures" may be an expression of action-based experiences.

In undergraduate mathematics education research, gesture has been studied in content areas such as complex analysis (e.g., Oehrtman et al., 2019), abstract algebra (Soto et al., 2024), and mathematical proof (Kokushkin, 2021). Prior studies of derivative have focused on experts' reasoning through gesture (Oehrtman et al., 2019) and within classroom settings (Arzarello et al., 2009). Individual analyses of Calculus students' understanding of derivatives have frequently drawn from Zandieh's (2000) Derivative Framework, which provides a deconstruction of the concept as layers of process-object pairs woven through various contexts or representations.

Yu and Uttal's (2022) analysis of pairs of Calculus students' as they solve real life rate of change problems about changes in volume of a water bottle found that some of their participants may have developed a Calculus-based way of reasoning about the problem through their gestures. For example, some used a "disk" shaped hand gesture representing a cross section of the water bottle. Alibali and Nathan (2012) refer to these as representational gestures--or those that "depict aspects of their meaning" (Alibali & Nathan, 2012, p. 252). Thus, gestures may serve as a tool for abstracting complex ideas (such as the formal definition of derivative). The current preliminary research report examines the research question, *what are the range of gestures students use to convey their understanding of the concept of derivative in calculus?*

Theoretical Framework

Theoretically, we situate this study within embodied cognition research, which examines aspects of mathematical thinking and understanding that are "bodily-grounded, *embodied* within a shared biological and physical context" (Núnez, Edwards, and Matos, 1999, p. 46). Contributing to the larger research program of embodiment in mathematics education (Abrahamson et al., 2020), this report further broadly investigates how gestures can "ground formalisms and abstractions to the physical environment, support simulated action of mathematical ideas, and invoke conceptual blends and metaphors" (p. 3). Also supported by findings from current research on gestures, we do not assume that students' speech would always be consistent with the gestures they produce (see also "mismatching gestures" in Hostetter & Alibali, 2019). Indeed, gestures reveal information that students do not or have yet verbalized.

Drawing on Zandieh's Derivative Framework (2000) we connect our analysis of gestures to the landscape of conceptual understanding of the specific mathematical content. This framework helps us to consider gestures in multiple contexts for student understanding of derivatives (primarily graphical, verbal, physical, and symbolic), related to its different aspects (ratio, limit, and function), as well as ways that students produce the gestures as dynamic processes and/or as static objects. Zandieh (2000) argued that the concept can be represented *graphically* as the slope of the tangent line, *verbally* as the instantaneous rate of change, *physically* as speed or velocity, and *symbolically* as the limit of the difference quotient.

Methods

Data Source

The case we consider comes from an introductory Calculus course taught at a large public university in the midwestern U.S. The instructor for the course (first author) heavily embedded Desmos Activity Builder (DAB) to the design of the course to leverage students' voice in classroom discussions, and to facilitate intuition building activities focused on conceptually understanding topics in a first-year calculus course. The data collected for this report comes from a 30-minute interview assessment about derivative for students in the course. It took place approximately two months into the semester. From these meetings, video recordings and written inscriptions created by the participants were gathered as data. Three students in the course consented to the analysis of their interview data and we selected Andy, a white male student, as a case where a wide variety of gestures were used. We focused our current analysis on less than eight minutes of video data where Andy is answering the two assessment questions: (1) what is a derivative to you, and (2) how do you make sense of the formal definition of a derivative?

Analytical Methods

All three authors reviewed the video data together and stopped to discuss moments of interest. Upon observing an abundance of gestures occurring simultaneously with Andy's speech, we rewatched the video with specific attention on his gestures. We then rewatched the video with the sound muted to focus attention on potentially mathematical gestures and selected possible gestures. We watched these selected moments once again with the sound on to guide our discussions on what counts as a mathematical gesture. We considered momentary pauses as natural separation points between dynamic gestures. When it appeared that multiple gestures occurred in a short amount of time, we watched, paused, and discussed each gesture, frame-by-frame, until we came to an agreement of the timestamps where the gestures started and ended.

We focused our analysis on gestures that demonstrate a mathematical concept (see also "representational gestures" in Alibali & Nathan (2012)), those that simultaneously occurred with speech about mathematics, or those previously identified as mathematical but occurred with different words (e.g., Gesture 3 in Findings). We noted but did not analyze gestures that are more conversational in nature (e.g., Andy's gesturing with his hand to himself while saying "that's the easiest way to explain it…at least for me."). Once agreement was reached, we placed selected gestures within Zandieh's (2000) Derivative Framework. The three authors selected the cell in the table that they feel best fit the mathematical nature of the gesture. We discussed and resolved disagreements and arrived at the placement for the five gestures we are highlighting from Andy.

Findings

During the interview, we found that Andy made over 40 gestures as he described what a derivative is and how his understanding connected with the formal definition. We highlight five gestures that we interpret as particularly salient in Andy's explanation of derivative. We organize the presentation of gestures chronologically corresponding to Andy's responses to the two questions in the Data Source section above. We present the gestures as consecutive images in each figure. We then connect them to Zandieh's (2000) Derivative Framework.

Gesture 1: "Instantaneous rate of change"

On multiple occasions, Andy makes variations of a gesture in which his hands move closer together and sometimes meet each other. The first instance of this occurs immediately after being asked to explain what a derivative is. Over the interval :52-:59, Andy states, "umm, the derivative is like the instantaneous rate of change, at any given, like on[e]--single point," and makes the following sequence of gestures (see Figure 1).

Figure 1. Andy's "instantaneous rate of change" gestures

Gesture 2: "A graph" as a context for instantaneous rate of change

After his initial response to the first question (see Gesture 1 section above), he continues "the instantaneous rate of change…of like [*gesture*]…I know it's more specific than a graph, but a graph [*gesture*]." Note that the gesture is basically the same each time, a positioning of his hand and forearm at an angle in front of himself.

Figure 2. Andy's gesture as he says "of like" and "a graph," respectively

Gesture 3: “velocity graph” and “distance, time”

We refer the following as Andy’s “velocity graph” gestures (see Figure 3 below). At 1:20, he states, “a graph that would be measuring velocity, so velocity graph.” He elaborates what he means by “velocity graph,” and makes two gestures that appear to trace out coordinate axes in the air in front of him (see Figure 3, second and third images) as he says, “so you got distance [*traces horizontally*], speed [*traces up*]. Or distance [*traces horizontally*] time [*traces up*]…”

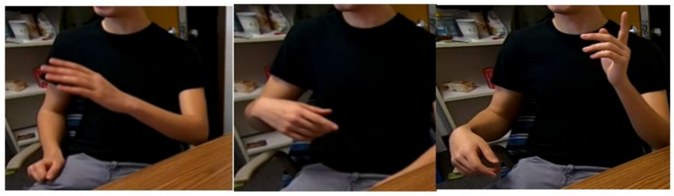

Figure 3. Andy's "velocity graph" gestures

Gesture 4: “h gets closer and closer to zero”

The next two gestures occurred after Andy’s response to the second interview question about the formal definition. Prior to being asked explicitly what the h in the formula means, Andy made a similar gesture to that when initially asked what a derivative is (see Figure 1). The gesture here differs in that the emphasis is on the hands becoming closer as opposed to his hands coming to one point as is the case in Gesture 1. Over the interval 4:19-4:31, Andy states, “And then the limit as h approaches zero means the—the two points get closer and closer together, the distance between the two of them becomes less and so h gets closer and closer to zero…” while making the gestures shown in Figure 4.

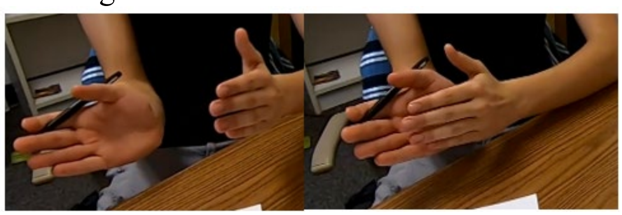

Figure 4. Andy’s “h gets closer and closer to zero” gesture

Gesture 5: “the h”

After being asked to explain what the h in the denominator of the formal definition means, Andy states, “uhh, the h is the distance between the first x and the second. The first input and the second.” As he states this, he makes the gestures below (see Figure 5), tapping the table with each hand as he says, “first x” and “second,” defining a space between his hands.

Figure 5. Andy’s gesture for “the h”

Andy’s gestures and the Derivative Framework

We first placed Andy’s five gestures in different cells of the Derivative Framework based on the context as described by his speech when he displayed the gesture: “instantaneous rate of change” (verbal), “velocity” (physical), and limit definition (symbolic). For the process-object layer, Gesture 2 focused on the whole velocity graph so we associated it with the function layer, whereas Gesture 3 focused on velocity as a ratio of the two “axes” of distance and time. Gesture 5 focuses on the static interval *h*, as the denominator of the difference quotient, whereas Gesture 4 was dynamically illustrating the limiting process as well as the limit. Gesture 1 emphasizes the “single point,” without additional explanation. In this way, we also characterize Gesture 1 as a closed circle as the student’s gesture displayed a “pseudo-object understanding” of that cell and Gesture 4 as an open circle as the gesture also displayed the limiting “process” involved (Zandieh, 2000, p. 114). There was no graphical gesture because Andy never spoke of “slope” in this explanation of derivative despite sharing gestures of graphical representations.

Table 1. Andy’s Gestures within the Zandieh Derivative Framework

<table>
<tr><td></td><td colspan="5">Contexts</td></tr>
<tr><td></td><td>Graphical</td><td>Verbal</td><td>Physical</td><td>Symbolic</td><td>Other</td></tr>
<tr><td>Process-object layer</td><td>Slope</td><td>Rate</td><td>Velocity</td><td>Difference Quotient</td><td></td></tr>
<tr><td>Ratio</td><td></td><td></td><td>Gesture 3 (_|)</td><td>Gesture 5 (| |)</td><td></td></tr>
<tr><td>Limit</td><td></td><td>Gesture 1 (^)</td><td></td><td>Gesture 4 (|→←|)</td><td></td></tr>
<tr><td>Function</td><td></td><td></td><td>Gesture 2 (\)</td><td></td><td></td></tr>
</table>

Discussion

The Derivative Framework proved productive in interpreting the contexts represented by Andy’s gestures. Moreover, it helps us interpret the gesture in terms of the process-object layers that add meaning to Andy’s gestures beyond what they represent. This exercise suggests that incorporating a cognitive framework in thinking about gestures can be fruitful. In the other direction, examination of gestures students used to explain the concept has the potential to bolster the framework. The Derivative Framework has largely been used to analyze students’ understanding of derivatives via students’ verbal and written explanations.

We found Andy’s gestures varied widely, may serve different communicative purposes, and became more precise as he attended to nuances of understanding derivative (e.g., the h in the formula). We observed non-redundancy between Andy’s gestures and his verbal explanation. Given the value of non-redundant gestures to reasoning (Abrahamson et al., 2020; Alibali & Goldin-Meadow, 1993), instructors’ attention to these moments can be important to gaining a deeper understanding of student thinking and might suggest the student is on the cusp of verbalizing something that relates to action-based gestures they make.

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