Eliciting Students' Informal Ideas About Series: The Partial Sum Sequence Game

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Series play key roles in calculus and other disciplines, yet research highlights several challenges with the topic, underscoring the need for curricular innovations to support conceptual understanding. In this paper, we introduce the Partial Sum Sequence Game and argue that it, with the guidance of a teacher-researcher, supported a pair of undergraduate students to make important connections between sequences, sequences of partial sums, and series convergence.

Keywords: Calculus, series, sequence of partial sums, guided reinvention

Series play essential roles in calculus, computer science, physics, and other disciplines. However, several reasons are cited in the research literature for why this topic can be challenging for learners. González-Martín (2010) found that their participating instructors (and the textbooks they used) focused on decontextualized, algorithmic procedures rather than promoting conceptual understanding of series. Additionally, learners may need support in distinguishing between sequences and series (Akgun & Duru, 2007; Kung & Speer, 2013) or viewing them both as particular functions with natural numbers as their domains (Martinez-Planell et al., 2012). Some learners describe series as running totals without coordinating them with index values (natural numbers) (Eckman & Roh, 2022; Martinez-Planell et al., 2012). Such a view explains students' difficulties with series convergence (Eckman & Roh, 2022) and solving problems that require viewing series as a sequence of partial sums (Martinez-Planell et al., 2012).

Because series is a challenging topic, there is a need to improve instruction to promote conceptual understanding of series. Toward this goal, scholars have recommended supporting students in obtaining the first few terms of the sequence of partial sums with different representations (Martínez-Planell et al., 2012) and using concrete examples (Morrel, 1992). This paper discusses a task (game) that aligns with these recommendations. We argue that the game, with the guidance of a teacher-researcher, supported a pair of undergraduate students to make important connections between sequences, sequences of partial sums, and series convergence.

Guided Reinvention and the Partial Sum Sequence Game

We drew on Realistic Mathematics Education's heuristic of guided reinvention (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) to frame the instruction during our experiment. In particular, we aimed to support "a process by which students formalize their informal understandings and intuitions" (Gravemeijer et al., 2000, p. 237). The "guided" in "guided reinvention" emphasizes that students are not expected to formalize their informal understandings on their own; the students are guided by strategically designed tasks that feature a well-chosen context problem as well as the instructor's moves. Gravemeijer and Doorman (1999) explain that this context problem should "offer opportunities for students to develop informal, highly context-specific solution strategies" that can "function as [...]catalyst for curtailment, formalization or generalization" (p. 117). The context problem does not necessarily have to draw on "everyday-life contexts" but should be meaningful or "experientially real" for students (Gravemeijer, 1999, p. 158). The Partial Sum Sequence Game, is the context problem that we designed to elicit students' informal ideas about series and series convergence (see Figure 1). We intended to design the game to promote informal thinking of series as sequences of partial sums and series convergence/divergence. In particular, we engineered the game so that students could be supported to determine that a sequence $\{x_n\}$ won if and only if the series $\sum x_n$ converged.

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Game:
We are conducting an experiment in which we move an object a certain number of feet north per day. A sequence $\{x_n\}$ will tell us how many feet x_n to move the object north on any given day n. To win the game, accurately predict the object's location if the experiment continues indefinitely!

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Playing the game with $\{n^2\}$:

$\boldsymbol{n}$	1	2	3	4	5	…
$\boldsymbol{n^2}$	1	4	9	16	25	…
Total distance from starting point	1	5	14	30	55	…

$\{n^2\}$ <u>loses</u> the game since we cannot accurately predict the object's location

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Figure 1. Partial Sum Sequence Game.

As mentioned, instructors' task facilitation also provides essential guidance (Kuster et al., 2017; Rasmussen & Marrongelle, 2006; Stephan et al., 2014). A crucial part of the instructor's role is eliciting and building on students' informal ideas. Kuster et al. (2017) explain that teachers "determine which [elicited] ideas (correct or incorrect) are important and relevant to the development of the mathematics, and which ideas can be leveraged to build student thinking that is commensurate with the mathematical goals of the lesson," and importantly, "use these contributions to inform the lesson" (p. 6). Hence, when considering the implementation of the task, we consider not just features of the task that support student thinking but also the instructor's guidance.

Methods

The data for our study comes from a teaching experiment (Steffe & Thompson, 2000) with two first-year undergraduate students - Lara and Stella (pseudonyms), a teacher-researcher - the first author, and an observer/camera operator. Lara was enrolled in Calculus 2 and Stella in Statistics at the time of the study. Both students had passed Calculus 1 the previous semester. There were 11 teaching experiment sessions that were each 1.5 hours. The sessions were recorded in such a way that synched the students' collaborative digital work with a video recording of the students' discussion and gestures.

We focus this report on Sessions 8 and 9 when the students played the Partial Sum Sequence Game with various sequences that they had previously created and explored. In particular, before these sessions, the students generated different sequences and organized them in an Euler diagram with different sequence properties such as increasing, bounded, and convergent (Vroom et al., 2024). As they used this diagram and examples of sequences to make general conjectures about sequences, they also defined sequence and the various sequence properties. Stella and Lara considered a sequence a function with the natural numbers as the domain, defining it as "a list of only natural numbers as inputs, where there is always an output, and the relationship can be defined as some mathematical function." After Sessions 8 and 9, the students wrote a "cheat sheet" for playing the game, which included several conjectures related to different tests for divergence, such as the comparison test and the divergence test (Davis & Vroom, 2024; Vroom & Davis, 2024).

We began our data analysis for this paper by re-watching the video and re-reading the transcripts. As we did so, we noted key moments, which we took to be instances when students articulated an idea that seemed connected to ideas cited in the research literature as important or challenging. For instance, instances when students' discourse suggested seeing series as sequences of partial sums (Martinez-Planell et al., 2012). We then identified how (if at all) we thought the task and/or the teacher-researcher's moves guided the students to develop these ideas. We present some of these key moments in the following section.

Results

The game with the teacher-researcher's guidance elicited students' informal ideas about series, and in particular, the students' informal ideas about (1) sequences of partial sums, and (2) the convergence or divergence of sequences of partial sums. We discuss these next.

Eliciting Informal Ideas about Sequences of Partial Sums

Playing the game involved Lara and Stella creating various sequences of partial sums, which the students termed total distance sequences. The students (with the teacher-researcher's guidance) (a) identified these as particular types of sequences and (b) represented the mathematical relationship between the daily distance sequence and the total distance sequence in a couple of different ways. A key moment that led the students to identify the total distance sequences as particular types of sequences occurred when the teacher-researcher introduced the students to the game and suggested playing it with $\{n^2\}$. As she asked the students how far the object was from the starting point on various days, she recorded their responses in a table tracking days with the corresponding total distance. After recording a few days of total distances, the teacher-researcher asked if this table could represent a sequence. The students responded yes, and referenced their definition for sequence that they constructed (see Methods for their definition), checking that it could be "modeled by a mathematical function" and the inputs were natural numbers. The table, along with the teacher-researcher's question, supported students in making explicit connections between their work in the context of the game and the concept of sequences. As they continued to play the game, they further emphasized this connection by using various function representations (e.g., tables, graphs) to represent total distance sequences.

The game context also seemed to support the students in representing the mathematical relationship between the daily distance sequence and the total distance sequence in a couple of different ways. A key moment occurred immediately after Lara and Stella decided that the table of total distances for $\{n^2\}$ could represent a sequence. Lara indicated that the mathematical function representing the total distance sequence was $\{n^2\}$. After the teacher-researcher pointed out that on the fifth day, they were 55 feet from the starting point rather than 25 like n^2 would suggest, the pair continued to brainstorm how to represent the relationship. They focused on how each term was the sum of all the previous terms of $\{n^2\}$, as Stella noted: "So we need to add together all the previous days, squared. I don't know how to write that." To further support the students in articulating their idea, the teacher-researcher asked them to write out how they found different total distances on given days. The students then indicated that they added previous terms, such as $1^2 + 2^2 + 3^2 + 4^2$ for the total distance on the fourth day. To further promote connections between the daily distance sequence and the total distance sequence, the teacher-researcher continued the conversation:

Teacher Researcher: I thought you were doing something a little bit different whenever you were actually calculating. If we were like looking at the eleventh day [it] didn't seem like you were doing $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \ldots$ all the way up to, you know, [11] days. What were you doing?

Lara: Adding the previous total distance?

Stella: Because the previous total distances are all the previous days added together. So, we technically still like are adding all the numbers, it's just shorter.

This led the students to write this strategy as $t_n = t_{n-1} + n^2$. Here, the teacher-researcher gave the students important guidance by (a) supporting them to see that the daily distance sequence representation (i.e., $\{n^2\}$) did not match the values of their total distance sequence and (b) pointing out how their representation did not reflect the strategy they used to find each value of the total distance sequence. After several rounds of the game, the students generalized their idea and noted that the total distance sequence could be represented by $t_n = t_{n-1} + x_n$ where x_n represents terms of the daily distance sequence.

Eliciting Informal Ideas about Sequences of Partial Sums Divergence or Convergence

For Lara and Stella, determining if a sequence won or lost the game was equivalent to determining whether the sequence of partial sums converged. With guidance, the students explained that (a) if a total distance sequence diverged, then the daily distance sequence lost the game, and (b) if a total distance sequence converged, then the daily distance sequence won the game. One key moment that led the students to develop the idea that a sequence loses the game if its total distance sequence diverged occurred as they played the game with $\{n^2\}$. After the students discussed how the total distance sequence was a sequence (see previous section), the teacher-researcher asked if $\{n^2\}$ won the game. Initially, Stella responded, "Yes," but Lara pushed back, saying, "So yeah, wait, no? Well, it's gonna be infinity," and the object was going to be "beyond space." Stella clarified, "So like given a day, yes, we can. I feel like that's what I meant. So, like if they were on day 300 we could figure it out." The students then decided that if the experiment continued indefinitely, then $\{n^2\}$ would lose. Here, we see the students made a key connection: winning or losing the game required them to predict the object's end behavior, not just the location of a given day. The experiment continuing *indefinitely* was an important game feature that supported this idea.

As the students continued to play the game, they articulated that, in general, a sequence loses the game if its corresponding total distance sequence tends to infinity. A key moment that led to this generalization occurred when the students, in response to the teacher-researcher asking the students which sequence they wanted to play the game with next, indicated that they did not want to play with $\{1, 2, 3, 4, 5, 15, 15, 15, 15, \ldots\}$. The students explained:

Stella: Yeah, I don't think we can win this one because you're adding 15 on every time, because it would be going… it would be infinity.

Lara: Yeah, that would be divergent.

Stella: It would be infinity again.

Here, we see the students suggesting that they shouldn't play the game with a sequence that has a total distance sequence that diverges to infinity. The teacher-researcher then continued to support the students in generalizing this idea further, specifically, that a sequence would lose the game if its corresponding total distance sequence diverged in general, not just by tending to (positive or negative) infinity. To do so, she suggested that they play the game with the sequence defined by n^2 if n is odd and $-n^2$ if even. Before calculating the first few terms of the total distance sequence, Stella conjectured that it would lose the game because she expected the total distance sequence to "decrease infinitely." After the teacher-researcher suggested that they calculate the first few terms, Stella noticed that the terms were alternating between positive and negative values and stated, "I take back what I said." She indicated that she still thought the sequence would lose the game because they "would not be able to predict where it's gonna end up" however, it was not because it tended to negative infinity, as Lara elaborated, it was "because it's oscillating, and it's not oscillating by [a smaller amount] … we can't predict it". Here, we see that by supporting the students to play the game with sequences that would have various types of divergent total distance sequences (e.g., tending to infinity, oscillating), the teacher-researcher reinforced the notion that a sequence loses the game when its total distance sequence diverges.

The students were also guided to see that if a total distance sequence converged, then the daily distance sequence won the game. A key moment that led to this occurred after determining the sequence $\{n^2\}$ lost the game. The teacher-researcher requested that they play the game with $\{½^n\}$. After calculating the first ten terms of the total distance sequence, Stella said, "Dang it, we lost again," presumably noticing that the total distance sequence would not obtain its limiting value of one. The teacher-researcher pushed back, saying "You think so?" and Lara responded, "I don't think so because we're still like converging to one, I guess. So… eventually…" The teacher-researcher connected convergence back to the game, saying, "So if the experiment continues indefinitely, would you be pretty confident to say…" and Stella finished her sentence by saying, "It's just like one." Here, we see that by pushing back, the teacher-researcher supported the students in realizing that a total distance sequence did not need to obtain its limiting value to win the game; it only needed to converge. The part of the game that indicated that the experiment ran *indefinitely* also seemed to support this idea.

The students continued to attend to whether the total distance sequence converged to determine if the daily distance sequence won. For instance, as the students played the game with $\{1/n\}$, the corresponding total distance sequence caused them to pause. The teacher-researcher asked, "What are you trying to notice?" and Stella responded, "If it's going to a certain number," as she tried to determine if it "flattened out." Additionally, when they played the game with the sequence defined by n^2 if n is odd and $-n^2$ if even, the students noticed that they could have a winning sequence that had an oscillating total distance sequence, as Lara explained, "so as long as it narrows into a number." In other words, the oscillating sequence converged. Similar to how playing the game with various losing sequences supported the students in seeing that they generated divergent total distance sequences, playing the game with various winning sequences supported them in seeing that they generated convergent total distance sequences.

Discussion

In this paper, we presented a task that we argue supported, with the teacher-researcher's guidance, a pair of students to make important connections between sequences, sequences of partial sums, and series convergence. We shared how, in the context of the game, the students developed informal ideas related to concepts that have been cited as particularly challenging for students in the research literature. In particular, the students developed informal ideas related to sequences of partial sums and their convergence/divergence.

Further, we highlight how the features of the task and/or moves by the teacher-researcher supported the students in developing these ideas. For instance, to support students in seeing sequences of partial sums as a sequence (a function with the natural numbers as its domain), the teacher-researcher strategically recorded the students' ideas in a function table; Rasmussen and Marrongelle (2006) have conceptualized moves like these as transformational records. We also found that the game experiment ran indefinitely as an important feature to elicit informal ideas about series convergence and divergence. The wording supported the students to connect winning (losing) the game as equivalent to the sequence of partial sum converging (diverging).

The data we presented here are promising first steps towards promoting conceptual understanding of series. In the future, we aim to implement the task in a whole-class setting to continue revising it and suggesting implementation.

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