Student Meanings for $\hbar/2$ in a Quantum Mechanics Expectation Value Problem

Julia Truman
University of Colorado Boulder

Megan Wawro
Virginia Tech

In modeling physical phenomena and solving associated quantitative problems, students reason about constants in various ways. These symbols represent elements of mathematical structures with embedded algebraic operations and are imbued with contextual physical meaning. In this paper, we report on what meanings for the constant $\hbar/2$ students conveyed in a quantum mechanics expectation value problem from a spin-½ context. Through analysis of interview data with 12 students from two different universities, our analysis documents eight different meanings for $\hbar/2$: part of expression in problem setup, part of calculation, result (either consistent or not consistent with unit interpretation), eigenvalue, measurement, average, and bound. This analysis highlights both the cognitive complexity of reasoning with symbols and the impressive ability of students to flexibly and fluidly leverage multiple meanings for symbols at this content level.

Keywords: linear algebra, quantum mechanics, student thinking

How symbols are used in mathematics and physics can differ. For example, in contemporary mathematics the term "constant" often refers to numbers of interest such as π, $\sqrt{2}$, or the golden mean φ (Wolfram, 2024). In physics, "constant" is additionally used in ways not necessarily needed in pure mathematics, such as for fundamental physical constants that have stable values derived from experimental measurement (e.g., the gravitational constant G). As Redish (2005) states, "We have many different kinds of constants – numbers (2, e, π, …), universal dimensioned constants (e, h, k_B, …), problem parameters (m, R, …), and initial conditions" (p. 2). We note that Redish cleverly included the literal symbol "e" twice, with the first referring to the number and the second referring to elementary charge[1]; we note that e can also be considered as the unit of electric charge. The constant of interest in this study is $\hbar/2$. Planck's constant h describes the behavior of particles and waves on the atomic scale; the reduced Planck's constant[2] $\hbar = \frac{h}{2\pi}$ is the quantization of intrinsic angular momentum, or spin.

In this study, we investigate students' meanings for the constant $\hbar/2$ as they engaged with an expectation value problem from a spin-½ context. For instance, at times students were engaged with $\hbar/2$ as the measured value of spin along an axis, as a quantity with units, or as an eigenvalue of the operator associated with a spin observable. This work is part of a larger project that investigates students' understanding, symbolization, and interpretation of eigentheory and related concepts in quantum mechanics. In particular, we investigate the following research question: *what meanings do students demonstrate for* $\hbar/2$ *in a quantum mechanics expectation value problem from a spin-½ context*?

Theoretical Perspective and Literature Review

In this paper we analyze students' *meanings* for $\hbar/2$. We view an individual's mathematical meaning for a concept not to be inherent or intrinsic to a concept itself but rather be personal to the individual. Furthermore, an individual's meaning for a concept is dynamically formed through the individual's continual interaction and interpretation within their given cultural

---

[1] $e = 1.602176634 \times 10^{-19}$ coulombs
[2] $\hbar = 1.0546 \times 10^{-34}\ J \cdot s$ or $\hbar = 6.5821 \times 10^{-16}\ eV \cdot s$

settings (Blumer, 1969). Particular to research on students' meanings at a given time, Thompson (2013) offers that "the meanings that matter at the moment of interacting with students are the meanings that students have, for it is their current meanings that constitute the framework within which they operate" (p. 62-63). In this paper, we analyze the meanings in the moment that students are operating with for $\hbar/2$ during their work on a particular expectation value problem within an interview setting. Because we cannot access a person's meanings directly, we infer students' meanings from what they say and write during the interviews. In this way, we "delimit categories of responses according to particular mathematical meanings that we discern from them," and we categorize students' responses "based on meanings we believe might underlie the response based on the best available evidence of interviews" (Yoon & Thompson, 2020, p. 4).

The idea of particular symbols like $\hbar/2$ having a variety of meanings for students has been examined in the literature with other more prominent symbols, such as the equal sign (e.g., Carpenter et al., 2005; Falkner et al., 1999; Kieran, 1981; Knuth et al., 2005). There has also been research that considers the importance of student interpretations of the equal sign in physics contexts. For example, Alaee et al. (2022) break down the relational interpretation of the equal sign into multiple physics-specific categories depending on the relationship described, such as definitional, causality, and assignment. This illustrates a variety of student meanings for this same symbol including ones that may not appear in purely mathematics-focused research. Based on this, when considering the meaning of symbols in physics, it follows that examining and categorizing not only the field-specific meanings of standard mathematical symbols but other symbols which are entirely field-specific could provide some of this same importance.

For such an examination, we began with Arcavi's (1994, 2005) notion of symbol sense. For our purposes, we particularly focus on four of Arcavi's aspects in the specific context we are analyzing: reading symbolic expressions involving $\hbar/2$ in different ways, consideration of the meaning of $\hbar/2$, recognizing the meaning of $\hbar/2$ during intermediate (both calculational and conceptual) steps, and how it has different meanings for the students across different contexts. In Rodriguez et al.'s (2018) analysis of students' mathematical reasoning in chemical kinetics, they state, "Symbols mediate the connection between the physical world and how we think about phenomena" (p. 2115). In our study, the ways that the symbol $\hbar/2$ mediates the connection between physical interpretation and mathematical calculation are central to how students work through the expectation value problem and explain their reasoning.

Methods

The data come from individual, semi-structured interviews (Bernard, 1988) with students enrolled in a Quantum Mechanics course from two different public research-active doctoral universities. All interview participants were recruited during class, gave informed consent before being interviewed, and received compensation for their time and effort. Four interviewees (students labeled A#) were from a junior-level course at a large university in the northwestern US; the course met for seven hours a week, and interviews occurred one month into the semester. Eight interviewees (students labeled C#) were from a senior-level course at a medium-sized university in the northeastern US; the course met for three hours per week, and interviews occurred two months into the semester. Both courses used McIntyre et al. (2012) as the textbook.

This study is part of a larger research project focused on student use of and student thinking around the concepts of basis and eigentheory, as well as their use of and meanings for various notation. In this paper, we consider student responses to the following interview question: "Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system. Calculate the expectation value for the measurement of S_x.” Students were encouraged to think aloud as they engaged in the problem, with the interviewer asking clarifying questions along the way as needed.

The results presented in this paper started as one portion of a broader analysis. That analysis considered the correctness of students’ solutions, what problem-solving approaches were used, what alternative approaches were discussed (if any), whether certain meanings and terminology were observed in the data, and whether certain concepts were discussed (e.g., basis, eigenvalue). One element we noticed in broader analysis was that some students had written their responses in forms like $\frac{7}{25}\frac{\hbar}{2}$, which prompted us to wonder about their meanings for $\hbar/2$. We then focused on how they interacted with $\hbar/2$, considering all their mentions of “h bar” and what meanings they seemed to correspond to. For our initial analysis, we focused on every instance of the students saying “h bar” in any context (or verbally referring to where they wrote it, but not if they wrote it and did not talk about it). We used open coding (Miles et al., 2013) to classify the students’ statements. One of the first apparent divisions was between the use of $\hbar/2$ in a purely mechanical or calculational way, its use in a more conceptual way, and its use in a statement of a result. This led to further categorical divisions, primarily in the conceptual categories based on what was discussed by each student. The meaning categories were developed by considering each of the students in turn and either making new categories (describing what student meanings seemed to be) or grouping them into existing ones. The meaning categories were then cross-checked, combined or refined, discussed, and used to recode the data until we reached agreement.

Results

Our analysis revealed eight meanings for $\hbar/2$ in students’ work with the expectation value problem: as part of expression in problem setup, as part of calculation, as part of result consistent with unit interpretation, as part of result not consistent with unit interpretation, as eigenvalue, as possible outcome of an operation or a measurement, as average measured value or weighted average, and as an aspect of positional bounds. In Table 1, we give examples of each of these meanings. In the remainder of the Results section, we share detail regarding the meanings for $\hbar/2$ as a unit with a dimensioned constant, as well as an outcome. We then conclude with an example of student reasoning that utilized many of the meanings for $\hbar/2$ in quick succession.

Table 1. Summary of the 8 meanings for $\hbar/2$ with examples.

Category Name	***Examples***
[Setup] As part of expression in problem setup	(C11) This is the h bar over two times 0 1 1 0 is the spin operator in the x direction. [The matrix used is incorrect here.] $= \begin{bmatrix}-4/5 & -i3/5\end{bmatrix} \frac{\hbar}{2} \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \begin{bmatrix}-4/5\\ i3/5\end{bmatrix}$
[**Calculation**] As part of calculation	(A8) “...so S sub x acting on plus x is equal to h bar over 2, include both vector, x, so we’re just gonna use that, so then this, is equal to negative four fifths h bar over 2 plus, plus, minus i h bar over 2 three fifths minus.”
[Result-Unit] As part of result with unit interpretation	(C6) “And now I like my answer. Um, then it’s h bar over two 7/25, um, is the expected value of S sub x.” (C5) “...you'd get zero h bar over two as your expectancy value. So, yeah.”

[Result-Nonunit] As part of result not with unit interpretation	(C8) "...7, h bar out over 50 is the expectation value for spin x." (A25) "So your expectation value ends up being, uh, 7 h bar over 50, I think."
[Eigen] As eigenvalue	(C1) "And so you'd expect the average, overall the measurements to be halfway between the two eigenvalues, which are h bar over 2 and negative h bar over 2."
[Outcome] As possible outcome for an operation or measurement	(C5) "We only know this, because if I'm already in the x state, and I'm operating on x, I'm going to get an h bar over two or a minus h bar over two with these probabilities, so." (A30) "Um, so, basically when you measure psi for Sx in the Sx direction you can either get h bar over two or minus h bar over two and this is like the average of the probabilities of one or the other."
[Average] As average measured value or weighted average	(C2) If the sample was large enough for you to take, um, the average value you would get would be 7 fiftieths, 7 h bar fiftieths, would be your average measured value, [Int: Okay.] which is an impossible value for you to get with, as, or for a, which is an impossible value for you to actually measure on each individual one [...]"
[Bound] As an aspect of positional bounds	(C1) "And so the average over all measurements would be, uh, positive. It'd be something between zero and h bar over two, and that's not what I'm seeing here."

Student meanings for ħ/2 as a unit

Two of the meanings found in our overall analysis for ħ/2 are distinct but related: as part of a result consistent with a unit interpretation [Result-Unit] and as part of a result not consistent with a unit interpretation [Result-Nonunit]. The Result-Unit meaning is consistent with considering ħ/2 as a universal dimensioned constant (Redish, 2005), but also with considering ħ/2 as the unit itself. As stated by Norton (2024), "(extensive) quantities rely on a unit for measuring them. Whatever we take as a unit becomes 1, and when 1 becomes an iterable unit, every other number in the number sequence becomes some measure of 1s" (p. 466). In our context, a Result-Unit meaning of ħ/2 would be consistent with treating $\hbar/2$ as a 1 for which other numbers – in this case, the expectation value – are measures of this contextual 1.

A notable example comes from C5, who, after completing the question in Figure 1, was asked to change the state vector from Figure 1 to have the same coefficients but now in terms of the z-basis and calculate the expectation value of measurement of S_x. Immediately after correctly getting 0 as their answer, C5 stated, "So, hey I suppose, that's equal to zero h bar over two. And that's a good um, way to write that because it's well, zero with respect to what? With respect to h bar over two …" We note that C5 keeps the $\hbar/2$ in their solution even though it is not necessary in a strictly algebraic context. C5 emphasizes this explicitly by pointing out their solution can be interpreted as no $\hbar/2$'s, implying C5 can hold meaning for $\hbar/2$ as a type of object, namely a unit. By contrast, other students do not give evidence that they were treating $\hbar/2$ with a unit meaning, as seen in the two examples in Table 1.

Student meanings for $\hbar/2$ as an Outcome

In the data set, we documented instances in which students seemed to assign $\hbar/2$ a meaning of being an outcome of either a mathematical operation or the outcome of a physical measurement. If it was unclear from the context of the utterance in the interview which of these two meanings for outcome was aligned with the student utterance, we only assigned it the category of general outcome. Overall, this was the second-most common meaning for $\hbar/2$ in our data set, behind only the Calculation meaning. Furthermore, all but two of the students made utterances that we coded as conveying an Outcome meaning.

Almost all utterances coded as Outcome were consistent with an Outcome-Measurement meaning for $\hbar/2$. For example, as C12 was beginning their solution process, they considered the measurement values associated with spin up and spin down, stating: "this measurement [points to $|+\rangle_x$] is going to be h bar over two, and this [points to $|-\rangle_x$] is minus h bar over two." A25, also while explaining how they were approaching the problem, stated "you're measuring the x component of the angular momentum and you can either get positive h bar or minus h bar over 2." In both of these examples, the students discussed the act of measuring $\hbar/2$ and $-\hbar/2$ as the result of those measurements. An example in which it was unclear if the student meant the outcome of a mathematical operation or the outcome of a measurement comes from A9: "...it's just the, like, if you did, a histogram result, right, you tabulate your results to be nine twenty-fifths of the time you got negative h bar over 2, and h bar over 2." The student uses the words "results" and what you "got," but the student does not explicitly mention either a measurement or a mathematical calculation. In the context of when the Outcome meanings occurred, namely, as students were explaining why their answer to the expectation value problem was sensible to them, the frequent Outcome-Measurement meaning is appropriate.

Only one student, C5, had meanings in the Outcome category that were consistent with an Outcome-Operation meaning. One example is included in Table 1: "We only know this, because if I'm already in the x state, and I'm operating on x, I'm going to get an h bar over two or a minus h bar over two with these probabilities, so," which was given when C5 shared their problem-solving approach to the question in Figure 1. Although the Outcome-Operation meaning is less appropriate in this context, there is precedence for it in the literature in that students may sometimes conflate mathematical operation and measurement (e.g., Gire & Manogue, 2012; Wawro et al., 2024). Gire and Manogue (2012) offer an explanation for why this could be an understandable conclusion that students might make as they are learning: "In mathematics, operators transform vectors; in physics, measurements change the state of a quantum system; but these are two different uses of the idea 'to change'" (p. 1), and that "it is common for textbooks to say that observables are represented by operators" (p. 4).

Examples of Multiple Codes Together

We conclude our Results section with a vignette highlighting how multiple meanings of $\hbar/2$ are employed in concert with each other by students to convey their problem-solving and/or explanation of an expectation value problem. While all students employed multiple meanings of $\hbar/2$ in some way, this vignette with A30 is a point where they are particularly intertwined. After reading the question, A30 writes $\langle\psi|S_x|\psi\rangle$ (see Figure 1), and says:

A30: I think I'm going to keep this in ket notation because I know what Sx does to the x, uh, kets [...] so the bra so [mumbling] sixteen over twenty, twenty five h bar over two plus nine over twenty-five which is h bar over two!

Interviewer: Okay so, what does that mean physically? What's going on?

A30: Um, so, Sx can either be, oh whoops I made a mistake [laughing] uh this should be a minus x ket and this should be a minus h bar over two which makes this minus … [corrects their answer to 7/25 $\hbar/2$]

A30: Um, so, basically when you measure psi for Sx in the Sx direction you can either get h bar over two or minus h bar over two and this is like the average of the probabilities of one or the other.
Interviewer: Okay. Okay. Do you have any way to look at that and like that seems about right? [...] I don't know if when you got h bar over two you changed your mind if something seemed wrong about it or how'd you see that mistake?
A30: Uh, well, I just I looked up here and I saw that, mm, how I put Sx on the minus x operator was h bar over two rather than minus h bar over two. I was like, oh. that's not correct.
Interviewer: Gotcha. And why - where did the minus come from?
A30: Oh, cause the eigenvalue of Sx times the minus x ket is minus h bar over two.
Interviewer: Okay. Okay, so if you look at the seven h bar over twenty five times two is there any way to look at that and think, oh that seems about right? [...]
A30: Well, I know it's - it can only be between h bar over two and minus h bar over two and so, uhh yeah if it's anything outside of that I know oh that's wrong or, I guess if you have something like this and you're measuring Sx you should not have, solidly one h bar over two or minus h bar over two.

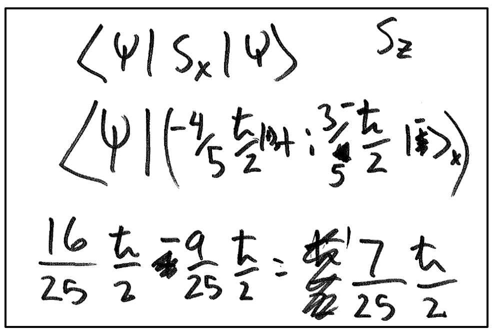

Figure 1. A30's work on the question.

In these 24 lines of transcript, A30 had 13 utterances of $\hbar/2$ or $-\hbar/2$ that show a variety of different meanings in a short span of time. First is one use of $\hbar/2$ with a straightforward Calculation meaning (line 2), where A30 substitutes in the linear superposition of psi in terms of $|+\rangle_x$ and $|-\rangle_x$. Next is A30's statement of their solution (line 3), which has a Result-Unit meaning. After explaining their error and correcting it with a Calculation meaning (line 6), we observe them interpret their solution using an Outcome - Measurement meaning twice (line 9) and an Average meaning (with "this," line 9). When asked if there was a way they knew their first answer was incorrect, they mentioned their error via a Setup meaning for $\hbar/2$ (line 15). When asked about why the negative sign was needed in that setup, A30 uses an Eigen meaning (line 18). Finally, when asked again about evaluating the plausibility of their solution, A30 uses $\hbar/2$ twice in a way consistent with a Bound meaning (line 21), and finally goes back to Outcome - Measurement (line 24). In this case, we observe A30 fluently switching between a variety of different meanings for the same symbol, including multiple procedural and conceptual ones.

Conclusion

In this paper, we report on what meanings for $\hbar/2$ students conveyed in a quantum mechanics expectation value problem from a spin-½ context. Through analysis of interview data with 12 students from two different universities, our analysis documents eight different meanings for $\hbar/2$: part of expression in problem setup, part of calculation, part of result (either consistent or not consistent with unit interpretation), eigenvalue, measurement, average, and bound. This analysis highlights both the cognitive complexity of reasoning with symbols and the impressive ability of students to flexibly and fluidly leverage multiple meanings for symbols at this content level. We illustrated this with data with from A30, an exemplar of a student quickly and seamlessly moving between several different meanings.

Additionally, the use of $\hbar/2$ in ways that algebraically and conceptually work like a unit is a notable result. Further research could point to a difference in the type of understanding held by students who use $\hbar/2$ as a unit and those who do not. We note that this split in the Result-Nonunit and Result-Unit codes has some similarities to the broader calculational/conceptual dichotomy, given the difference is whether the students' statements simply report a calculational result or suggest a related conceptual interpretation of the numerical value of the result. Accordingly, it also corresponds to Arcavi's (1994, 2005) symbol sense category of the ability to read symbolic expressions in different ways, here more focused on the consideration of meaning. Further questions could not only examine this, but also use it in contrast to other instances of units in order to gain a greater understanding of students' concepts of various units in math and physics.

Our results regarding two meanings for $\hbar/2$ related to outcome, either outcome of a measurement or of an operation, are consistent with prior research (Gire & Manogue, 2012; Pina et al., 2024; Wawro et al., 2024). We see this as further evidence that emphasizes the complexity encountered when introducing new symbols with new meanings. This variation in developing or desired meanings is also consistent with the findings of previous work related to the meaning of other symbols, such as the equal sign (e.g., Carpenter et al., 2005; Kieran, 1981; Knuth et al., 2005). In terms of symbol sense (Arcavi, 1994; 2005), developing a conceptual understanding of the mathematical equations, calculational steps, and results in terms of the related physical referents involves ability to understand the different kinds of meaning that different symbols can have across various contexts.

Limitations to our study include focusing our analysis on students' verbalized instances of $\hbar/2$. Our eight meanings for $\hbar/2$ were drawn exclusively from analysis of students engaging in one expectation value problem in a spin-1/2 setting. Future research could analyze student meanings of $\hbar/2$ from other problem contexts (other than expectation value) or settings (such as spin-3/2, which would introduce additional multiples of $\hbar/2$), which could further refine the existing meanings or introduce additional meanings.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Number DUE-1452889. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

References

Alaee, D. Z., Sayre, E. C., Kornick, K., & Franklin, S. V. (2022). How physics textbooks embed meaning in the equals sign. *American Journal of Physics 90*(4), 273-278.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics 14*(3), 24-35.

Arcavi, A. (2005). Developing and using symbol sense in mathematics. *For the Learning of Mathematics 25*(2), 42-47.

Bernard, H. R. (1988). *Research methods in cultural anthropology*. Newbury Park, California: Sage Publications.

Blumer, H. (1969). *Symbolic interactionism: Perspectives and method.* Prentice-Hall.

Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *Zentralblatt für Didaktik der Mathematik 37*(1), 53-59.

Falkner, K., Levi, L., & Carpenter, T. (1999). Early childhood corner: Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*(4), 232-236.

Gire, E., & Manogue, C. (2012). Making sense of quantum operators, eigenstates and quantum measurements. *AIP Conference Proceedings, 1413*(1), 195–198. https://doi.org/10.1063/1.3680028

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*(3), 317-326.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence & Variable. *Zentralblatt für Didaktik der Mathematik*, *37*(1), 68-76.

McIntyre, D. H., Manogue, C. A., & Tate, J. (2012). *Quantum mechanics: A paradigms approach* (Vol. 192). Pearson.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2013). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Sage Publications.

Pina, A., Topdemir, Z., & Thompson, J. R. (2024). Student understanding of eigenvalue equations in quantum mechanics: Symbolic forms and sensemaking analysis. *Physical Review Physics Education Research, 20,* 010153.

Thompson, P. W. (2013). In the absence of meaning…. In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57-93). Springer, New York, NY. https://doi.org/10.1007/978-1-4614-6977-3_4

Wawro, M., Pina, A., Thompson, J. R., Topdemir, Z., & Watson, K. (2024). Student interpretations of eigenequations in linear algebra and quantum mechanics. *International Journal of Research in Undergraduate Mathematics Education.* https://doi.org/10.1007/s40753-024-00241-7

Yoon, H., & Thompson, P. W. (2020). Secondary teachers' meanings for function notation in the United States and South Korea. *The Journal of Mathematical Behavior*, *60*, 100804.